Short Note

Not peer-reviewed version

An Algorithm for Calculating Terms of the Stirling's Formula Remainder

Fabio Dorini * and Leyza Dorini

Posted Date: 18 September 2025

doi: 10.20944/preprints202509.1468.v1

Keywords: stirling formula; remainder; approximation

Short Note

An Algorithm for Calculating Terms of the Stirling's Formula Remainder

Fabio Antonio Dorini [1,*] and Leyza Baldo Dorini [2]

[1] Department of Mathematics, Federal University of Technology - Paraná, Curitiba 80230-901, PR, Brazil
[2] Department of Informatics, Federal University of Technology - Paraná, Curitiba 80230-901, PR, Brazil
* Correspondence: fabio.dorini@gmail.com

Abstract

In this note, we deal with Stirling's approximation to N!. We employ elementary mathematical techniques to provide an algorithm to calculate many terms of Stirling's formula remainder. Our algorithm, based on a recursive scheme using polynomials, is an alternative to some methodologies already proposed in the literature.

Keywords: stirling formula; remainder; approximation

MSC: 41A80; 68W30

---

1. Introduction

In this note, we study Stirling's approximation to $N!$, a fundamental result in asymptotic analysis and number theory. Originally derived in the 18th century, the approximation provides an accurate estimate of the factorial function for large values of N. Over the centuries, various forms and refinements of this approximation have been developed. A particularly elegant and instructive derivation, based solely on elementary transformations of integrals and term-by-term integration of a series, was introduced by Marsaglia [1].

Our first objective is to briefly derive Stirling's formula. We begin with the representation of $\ln N!$ as a sum of logarithms:

$$\begin{aligned}\ln N! &= \ln 1 + \ln 2 + \ldots + \ln N = \\ &= \int_1^N \ln x \, dx + \sum_{k=1}^{N-1} \frac{\ln(k+1) - \ln k}{2} - \sum_{k=1}^{N-1} D_k,\end{aligned} \tag{1}$$

where each term D_k measures the discrepancy between the trapezoidal rule and the exact integral of $\ln x$ over the interval $[k, k+1]$. That is,

$$\begin{aligned}D_k &= \int_k^{k+1} \ln x \, dx - \frac{\ln k + \ln(k+1)}{2} = \\ &= (x \ln x - x)\Big|_k^{k+1} - \frac{\ln k + \ln(k+1)}{2} = \\ &= (k+1)\ln(k+1) - k \ln k - 1 - \frac{\ln k + \ln(k+1)}{2} = \\ &= \left(k + \frac{1}{2}\right) \ln\left(1 + \frac{1}{k}\right) - 1.\end{aligned} \tag{2}$$

The integral in (1) can be evaluated explicitly,

$$\int_1^N \ln x \, dx = N \ln N - N + 1, \quad \text{and} \quad \sum_{k=1}^{N-1} \frac{\ln(k+1) - \ln k}{2} = \frac{\ln N}{2}$$

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 18 September 2025 doi:10.20944/preprints202509.1468.v1

is a telescoping sum, as the terms cancel pairwise. In this way,

$$\begin{aligned}\ln N! =& N\ln N - N + 1 + \frac{\ln N}{2} - \sum_{k=1}^{N-1} D_k = \\ &= \ln\left(N^N\right) - N + 1 + \ln\left(N^{1/2}\right) - \sum_{k=1}^{N-1} D_k,\end{aligned}$$

which leads to

$$N! = N^{N+1/2}\, \mathrm{e}^{1-N} \exp\left(-\sum_{k=1}^{N-1} D_k\right), \tag{3}$$

where D_k is given in (2).

It is well known, from the Mercator series, that

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots = \sum_{i=1}^{+\infty} (-1)^{i+1}\frac{x^i}{i}, \quad -1 < x \le 1.$$

Substituting $x = 1/k$, where $k = 1, 2, 3, \ldots$, yields

$$\ln\left(1+\frac{1}{k}\right) = \frac{1}{k} - \frac{1}{2k^2} + \frac{1}{3k^3} - \frac{1}{4k^4} + \ldots = \sum_{i=1}^{+\infty} (-1)^{i+1}\frac{1}{ik^i}.$$

Thus, D_k in (2) can be rewritten as

$$\begin{aligned}D_k &= \left(k+\frac{1}{2}\right)\ln\left(1+\frac{1}{k}\right) - 1 = \\ &= \left(1 - \frac{1}{2k} + \frac{1}{3k^2} - \frac{1}{4k^3} + \ldots\right) + \frac{1}{2}\left(\frac{1}{k} - \frac{1}{2k^2} + \frac{1}{3k^3} - \ldots\right) - 1 = \\ &= \left(\frac{1}{3} - \frac{1}{2.2}\right)\frac{1}{k^2} - \left(\frac{1}{4} - \frac{1}{2.3}\right)\frac{1}{k^3} + \left(\frac{1}{5} - \frac{1}{2.4}\right)\frac{1}{k^4} - \ldots = \\ &= \sum_{i=2}^{+\infty} (-1)^i \left(\frac{1}{i+1} - \frac{1}{2i}\right)\frac{1}{k^i} = \sum_{i=2}^{+\infty} (-1)^i \frac{a_i}{k^i},\end{aligned}$$

where $a_i = 1/(i+1) - 1/(2i)$, for $i = 2, 3, 4, \ldots$, is a positive decreasing sequence of real numbers that converges to zero.

Thus,

$$\sum_{k=1}^{N-1} D_k = \sum_{k=1}^{N-1}\sum_{i=2}^{+\infty} (-1)^i \frac{a_i}{k^i} = \sum_{i=2}^{+\infty} (-1)^i a_i \left(\sum_{k=1}^{N-1} \frac{1}{k^i}\right).$$

From the Riemann Zeta function (see [2] for details),

$$\zeta(x) = \sum_{j=1}^{+\infty} \frac{1}{j^x},$$

which is convergent for $x \in \mathbf{R}$ with $x > 1$, it follows that

$$\sum_{k=1}^{N-1} \frac{1}{k^i} = \zeta(i) - \sum_{k=N}^{+\infty} \frac{1}{k^i},$$

and, therefore,

$$\sum_{k=1}^{N-1} D_k = \sum_{i=2}^{+\infty} (-1)^i a_i \, \zeta(i) - \sum_{i=2}^{+\infty} (-1)^i a_i \left(\sum_{k=N}^{+\infty} \frac{1}{k^i} \right).$$

From [2,3],

$$\sum_{i=2}^{+\infty} (-1)^i \frac{\zeta(i)}{2i} = \frac{1}{2} \left(\frac{\zeta(2)}{2} - \frac{\zeta(3)}{3} + \frac{\zeta(4)}{4} - \frac{\zeta(5)}{5} + \ldots \right) = \frac{\gamma}{2},$$

where γ is the Euler-Mascheroni constant.

Additionally, from [3,4], based on the Riemann Zeta function and Rational Zeta series, it follows that

$$\sum_{i=2}^{+\infty} (-1)^i \frac{\zeta(i) - 1}{i+1} = \frac{1}{2} \left(\gamma + 3 - \ln \sqrt{2\pi} \right) - \ln 2.$$

Moreover, using the Mercator series evaluated at $x = 1$,

$$\sum_{i=2}^{+\infty} (-1)^i \frac{1}{i+1} = \ln 2 - \frac{1}{2},$$

we arrive at the Suryanarayana formula (see [3] for example),

$$\sum_{i=2}^{+\infty} (-1)^i \frac{\zeta(i)}{i+1} = \frac{1}{2} \left(\gamma + 3 - \ln \sqrt{2\pi} \right) - \ln 2 + \left(\ln 2 - \frac{1}{2} \right) = \frac{\gamma}{2} + 1 - \ln \sqrt{2\pi},$$

and, consequently,

$$\sum_{i=2}^{+\infty} (-1)^i a_i \, \zeta(i) = \frac{\gamma}{2} + 1 - \ln \sqrt{2\pi} - \frac{\gamma}{2} = 1 - \ln \sqrt{2\pi}.$$

In this way,

$$\sum_{k=1}^{N-1} D_k = 1 - \ln \sqrt{2\pi} - \sum_{i=2}^{+\infty} \sum_{k=N}^{+\infty} (-1)^i \frac{a_i}{k^i}.$$

Therefore,

$$\exp\left(- \sum_{k=1}^{N-1} D_k \right) = \mathrm{e}^{-1} \sqrt{2\pi} \, \mathrm{e}^{r_N},$$

where r_N is given by

$$r_N = \sum_{i=2}^{+\infty} \sum_{k=N}^{+\infty} (-1)^i \frac{a_i}{k^i}, \tag{4}$$

and, as before, $a_i = 1/(i+1) - 1/(2i)$, for $i = 2, 3, 4, \ldots$, is a positive decreasing sequence of real numbers that converges to zero.

Finally, Eq. (3), known in the literature as Stirling's formula, can be expressed as

$$N! = \sqrt{2\pi} \, N^{N+1/2} \, \mathrm{e}^{-N} \, \mathrm{e}^{r_N}, \tag{5}$$

where the remainder r_N is given in (4).

 doi:10.20944/preprints202509.1468.v1

Eq (5) was first presented in [5], where it was also established that $11/2 < r_N + \ln\sqrt{2\pi} < 1$. Since then, the term r_N has been the subject of detailed study (see [6–9] for historical notes), particularly through works such as [10], who first provided sharp two-sided bounds for it,

$$\frac{1}{12N+1} < r_N < \frac{1}{12N}.$$

Several years later, [11] improved the estimation of r_N,

$$\frac{1}{12N} - \frac{1}{360N^3} < r_N < \frac{1}{12N}.$$

More recently, [12] obtained the following result,

$$\frac{1}{12N} - \frac{1}{360N^3} < r_N < \frac{1}{12N} - \frac{1}{360N(N+1)(N+2)},$$

[8] proved that

$$\frac{1}{12N+c_1} \le r_N < \frac{1}{12N+c_2},$$

with the best possible constants $c_1 = -12 + 1/(1 - \ln\sqrt{2\pi}) \approx 0.336317$ and $c_2 = 0$, and [9] derived the following inequality:

$$\frac{1}{12N} - \frac{7}{360N(7N^2+2)} - \frac{13}{35280N^7} < r_N < \frac{1}{12N} - \frac{7}{360N(7N^2+2)}.$$

It is also important to highlight the works [1,7,13–15] who focused on obtaining explicit formulas for the coefficients of the asymptotic expansion of r_N.

The main goal of this work is to present an algorithm, relying only on elementary mathematical methods techniques, for efficiently calculating many terms of r_N exactly. To the best of our knowledge, we understand that our algorithm represents an alternative to the procedures presented herein.

2. The Algorithm

Observe that r_N in (4) can be rewritten as

$$r_N = \sum_{k=N}^{+\infty}\sum_{i=2}^{+\infty}(-1)^i\frac{a_i}{k^i} = \sum_{k=N}^{+\infty}\left(\frac{a_2}{k^2} - \frac{a_3}{k^3} + \frac{a_4}{k^4} - \frac{a_5}{k^5} + \dots\right),$$

a numerical series whose general term forms a convergent alternating series for each N. Define

$$S_j(k) = \sum_{i=2}^{j}(-1)^i\frac{a_i}{k^i},$$

the sequence of partial sums of this series.

Step 1: Consider the polynomial p_1 defined by the following equation:

$$S_4(k) - \alpha_1\left(\frac{1}{k} - \frac{1}{k+1}\right) = \frac{p_1(k)}{k^4(k+1)},$$

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 18 September 2025 doi:10.20944/preprints202509.1468.v1

where α_1 is a real parameter to be determined later. This leads to:

$$\begin{aligned} p_1(k) =& k^4(k+1)\left(S_4(k) - \alpha_1\left(\frac{1}{k} - \frac{1}{k+1}\right)\right) = \\ =& k^4(k+1)\left(\frac{a_2}{k^2} - \frac{a_3}{k^3} + \frac{a_4}{k^4} - \alpha_1\left(\frac{1}{k} - \frac{1}{k+1}\right)\right) = \\ =& (a_2 - \alpha_1)k^3 + (a_2 - a_3)k^2 + (a_4 - a_3)k + a_4. \end{aligned}$$

Now, α_1 is chosen in order to reduce the degree of p_1 to the lowest possible value. In this case, $\alpha_1 = a_2 = 1/12$ and, after some calculations, we obtain

$$p_1(k) = (a_4 - a_3)k + a_4 = \frac{-1}{120}k + \frac{3}{40}$$

which is a polynomial of degree 1, that is, $\deg(p_1) = 1$. We denote by $\eta(p_1) = -1/120$ the coefficient associated with the highest degree term of the polynomial p_1.

In this way, the partial sum S_4 can be expressed as

$$S_4(k) = \frac{a_2}{k^2} - \frac{a_3}{k^3} + \frac{a_4}{k^4} = \frac{1}{12}\left(\frac{1}{k} - \frac{1}{k+1}\right) + \frac{p_1(k)}{k^4(k+1)}.$$

As an important remark, r_N in (4) can then be approximated by

$$\begin{aligned} r_N \approx& \sum_{k=N}^{+\infty} S_4(k) = \frac{1}{12}\sum_{k=N}^{+\infty}\left(\frac{1}{k} - \frac{1}{k+1}\right) + \sum_{k=N}^{+\infty}\frac{p_1(k)}{k^4(k+1)} = \\ =& \frac{1}{12N} + \sum_{k=N}^{+\infty}\frac{p_1(k)}{k^4(k+1)}. \end{aligned}$$

The sum above can be estimated using the integral test, comparing it to an improper integral.

Step 2: Similarly, we seek a polynomial p_3 that satisfies

$$\frac{p_1(k)}{k^4(k+1)} - \frac{a_5}{k^5} + \frac{a_6}{k^6} - \alpha_3\left(\frac{1}{k^3} - \frac{1}{(k+1)^3}\right) = \frac{p_3(k)}{k^6(k+1)^3},$$

where α_3 is a real parameter to be determined.

Thus,

$$p_3(k) = k^2(k+1)^2 p_1(k) - a_5 k(k+1)^3 + a_6(k+1)^3 - \alpha_3 k^3\left((k+1)^3 - k^3\right).$$

Again, α_3 is chosen with the aim of reducing the degree of p_3 to the lowest possible value. In this case, $\alpha_3 = \eta(p_1)/3 = -1/360$.

Therefore, we can present $S_6(k)$ as

$$S_6(k) = \frac{1}{12}\left(\frac{1}{k} - \frac{1}{k+1}\right) - \frac{1}{360}\left(\frac{1}{k^3} - \frac{1}{(k+1)^3}\right) + \frac{p_3(k)}{k^6(k+1)^3},$$

where

$$p_3(k) = \frac{1}{252}k^3 + \frac{3}{56}k^2 + \frac{47}{420}k + \frac{5}{84}.$$

Note that $\deg(p_3) = 3$ and $\eta(p_3) = 1/252$.

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 18 September 2025 doi:10.20944/preprints202509.1468.v1

Again, as a remark, r_N in (4) is better approximated by

$$r_N \approx \sum_{k=N}^{+\infty} S_6(k) = \frac{1}{12}\sum_{k=N}^{+\infty}\left(\frac{1}{k} - \frac{1}{k+1}\right) - \frac{1}{360}\sum_{k=N}^{+\infty}\left(\frac{1}{k^3} - \frac{1}{(k+1)^3}\right) + \\ + \sum_{k=N}^{+\infty}\frac{p_3(k)}{k^6(k+1)^3} = \frac{1}{12N} - \frac{1}{360N^3} + \sum_{k=N}^{+\infty}\frac{p_3(k)}{k^6(k+1)^3}.$$

Step 3: Similarly, aiming for a better approximation of r_N, we write

$$\frac{p_3(k)}{k^6(k+1)^3} - \frac{a_7}{k^7} + \frac{a_8}{k^8} - \alpha_5\left(\frac{1}{k^5} - \frac{1}{(k+1)^5}\right) = \frac{p_5(k)}{k^8(k+1)^5},$$

where α_5 is, as before, a real parameter to be determined.

The polynomial p_5 can be rewritten as

$$p_5(k) = k^2(k+1)^2\, p_3(k) - a_7 k(k+1)^5 + a_8(k+1)^5 - \alpha_5 k^3\Big((k+1)^5 - k^5\Big).$$

The parameter α_5 is chosen with the aim of reducing the degree of p_5 to the lowest possible value. Thus, $\alpha_5 = \eta(p_3)/5 = 1/1260$.

Then, $S_8(k)$ is given by

$$S_8(k) = \frac{1}{12}\left(\frac{1}{k} - \frac{1}{k+1}\right) - \frac{1}{360}\left(\frac{1}{k^3} - \frac{1}{(k+1)^3}\right) + \\ + \frac{1}{1260}\left(\frac{1}{k^5} - \frac{1}{(k+1)^5}\right) + \frac{p_5(k)}{k^8(k+1)^5},$$

where

$$p_5(k) = -\frac{1}{240}k^5 + \frac{29}{720}k^4 + \frac{13}{72}k^3 + \frac{5}{18}k^2 + \frac{191}{1008}k + \frac{7}{144}.$$

Note that $\deg(p_5) = 5$ and $\eta(p_5) = -1/240$.

Therefore, a better approximation for r_N in (4) is obtained:

$$r_N \approx \sum_{k=N}^{+\infty} S_8(k) = \frac{1}{12}\sum_{k=N}^{+\infty}\left(\frac{1}{k} - \frac{1}{k+1}\right) - \frac{1}{360}\sum_{k=N}^{+\infty}\left(\frac{1}{k^3} - \frac{1}{(k+1)^3}\right) + \\ + \frac{1}{1260}\sum_{k=N}^{+\infty}\left(\frac{1}{k^5} - \frac{1}{(k+1)^5}\right) + \sum_{k=N}^{+\infty}\frac{p_5(k)}{k^8(k+1)^5} = \\ = \frac{1}{12N} - \frac{1}{360N^3} + \frac{1}{1260N^5} + \sum_{k=N}^{+\infty}\frac{p_5(k)}{k^8(k+1)^5}.$$

Step 4: Likewise, the polynomial p_7 is defined as

$$p_7(k) = k^2(k+1)^2\, p_5(k) - a_9 k(k+1)^7 + a_{10}(k+1)^7 - \alpha_7 k^3\Big((k+1)^7 - k^7\Big).$$

Taking $\alpha_7 = \eta(p_5)/7 = -1/1680$, we obtain

$$p_7(k) = \frac{1}{132}k^7 + \frac{7}{132}k^6 + \frac{3349}{13860}k^5 + \frac{8119}{13860}k^4 + \frac{10891}{13860}k^3 + \\ + \frac{105}{176}k^2 + \frac{479}{1980}k + \frac{9}{220},$$

where $\deg(p_7) = 7$ and $\eta(p_7) = 1/132$.

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 18 September 2025 doi:10.20944/preprints202509.1468.v1

Thus,

$$S_{10}(k) = \frac{1}{12}\left(\frac{1}{k} - \frac{1}{k+1}\right) - \frac{1}{360}\left(\frac{1}{k^3} - \frac{1}{(k+1)^3}\right) + \\ + \frac{1}{1260}\left(\frac{1}{k^5} - \frac{1}{(k+1)^5}\right) - \frac{1}{1680}\left(\frac{1}{k^7} - \frac{1}{(k+1)^7}\right) + \frac{p_7(k)}{k^{10}(k+1)^7},$$

and then

$$r_N \approx \sum_{k=N}^{+\infty} S_{10}(k) = \frac{1}{12N} - \frac{1}{360N^3} + \frac{1}{1260N^5} - \frac{1}{1680N^7} + \sum_{k=N}^{+\infty} \frac{p_7(k)}{k^{10}(k+1)^7},$$

where $\deg(p_7) = 7$.

In summary, the following algorithm is presented.

Step 5: Algorithm

$$\begin{aligned}
&\alpha_1 = 1/12 \\
&p_1(k) = -(1/120)k + 3/40 = \eta(p_1)k + 3/40 \\
&\textbf{For } l = 1, 2, 3, \ldots \\
&\quad p_{2l+1}(k) = k^2(k+1)^2\, p_{2l-1}(k) - \alpha_{2l+3}\, k(k+1)^{2l+1} + \alpha_{2l+4}(k+1)^{2l+1} \\
&\qquad\qquad - \alpha_{2l+1}\, k^3\left((k+1)^{2l+1} - k^{2l+1}\right) \\
&\quad \alpha_{2l+1} = \eta(p_{2l-1})/(2l+1)
\end{aligned}$$

Remark 1. *The sequence* (α_{2l+1}), *for* $l = 0, 1, 2, 3, \ldots$, *defined above, is crucial because it leads to*

$$r_N = \sum_{l=0}^{+\infty} (-1)^l \frac{\alpha_{2l+1}}{N^{2l+1}}.$$

Furthermore, based on computational tests, it was observed that, for $l = 0, 1, 2, \ldots$,

$$\deg(p_{2l+1}) = 2l + 1.$$

The exact calculations of $p_{2l+1}(k)$ and α_{2l+1}, for $l = 1, 2, 3, \ldots$, were carried out using symbolic computations in the Mathematica software [16], using the script that follows:

```
np = 19; (* number of terms to compute *)
nn = 2*np + 4;
ind = Range[1, nn]; (* index range for initial coefficient calculation *)

a = 1/(ind + 1) - 1/(2*ind); (* initial sequence used for recursion *)

p1 = (k^2)*(k + 1)^2; (* base polynomial for recursion *)
q = (a[[4]] - a[[3]]) k + a[[4]]; (* initial q polynomial *)
alpha = {a[[2]]}; (* initialize list of alpha coefficients *)

Timing[
  For[j = 1, j <= np, j++,
    l = 2*j + 1; (* degree index for current step *)
    p2 = (k + 1)^l; (* auxiliary polynomial for current term *)
    p = p1*q + (a[[l + 3]] - a[[l + 2]]*k)*p2; (* compute new polynomial *)
    aux = Reverse[CoefficientList[p, k]]; (* coefficients of p in descending order *)
    q = p - (aux[[1]]/l)*(k^3)*(p2 - k^l);(*update q for next iteration*)
    alpha = Append[alpha, aux[[1]]/l]; (* store current alpha coeff. *)
  ]
]
```

```
l (* last degree index computed *)
Length[CoefficientList[q, k]] - 1  (* degree of final polynomial pl *)
Expand[q];  (* display expanded polynomial pl *)
alpha; (* display list of alpha coefficients *)
```

To illustrate our algorithm, we calculate $\alpha_1, \alpha_3, \alpha_5, ..., \alpha_{39}$ using the Mathematica script provided above (CPU time approximately 0.1 seconds, executed in the trial version of Mathematica Online) and then, r_N in (4) can be approximated by

$$r_N = \sum_{l=0}^{+\infty}(-1)^l \frac{\alpha_{2l+1}}{N^{2l+1}} \approx \frac{1}{12N} - \frac{1}{360N^3} + \frac{1}{1260N^5} - \frac{1}{1680N^7} + \frac{1}{1188N^9} - \\ - \frac{691}{360360N^{11}} + \frac{1}{156N^{13}} - \frac{3617}{122400N^{15}} + \frac{43867}{244188N^{17}} - \frac{174611}{125400N^{19}} + \\ + \frac{77683}{5796N^{21}} - \frac{236364091}{1506960N^{23}} + \frac{657931}{300N^{25}} - \frac{3392780147}{93960N^{27}} + \frac{1723168255201}{2492028N^{29}} - \\ - \frac{7709321041217}{505920N^{31}} + \frac{151628697551}{396N^{33}} - \frac{26315271553053477373}{2418179400N^{35}} + \\ + \frac{154210205991661}{444N^{37}} - \frac{261082718496449122051}{21106800N^{39}}.$$

As a test, the CPU time to calculate the first 300 terms of r_N was approximately 200 seconds. Remark 1 have been verified in this case.

Conflicts of Interest: The authors have no conflicts of interest and there is no financial interest to report.

References

1. Marsaglia, G., Marsaglia, J.C.W.: A new derivation of stirling's approximation to n! Am. Math. Mon. **97:9**, 826–829 (1990)
2. Lagarias, J.C.: Euler's constant: Euler's work and modern developments. Bulletin of the American Mathematical Society **50**(4), 527–628 (2013)
3. Coppo, M.-A.: A note on some alternating series involving zeta and multiple zeta values. Journal of Mathematical Analysis and Applications **475**(2), 1831–1841 (2019)
4. Borwein, J.M., Bradley, D.M., Crandall, R.E.: Computational strategies for the riemann zeta function. Journal of Computational and Applied Mathematics **121**(1), 247–296 (2000)
5. Hummel, P.M.: A note on stirling's formula. Am. Math. Mon. **47**, 97–99 (1940)
6. Beesack, P.R.: Improvement of stirling's formula by elementary methods. Univ. Beograd. Publ. Elektrotehn. Fak. Set. Mat. Fiz. **274-301**, 17–21 (1969)
7. Nemes, G.: On the coefficients of the asymptotic expansion of n! (2010). https://arxiv.org/abs/1003.2907
8. Chen, C.P., Batir, N.: Some inequalities and monotonicity properties associated with the gamma and psi functions. Appl. Math. Comput. **218**, 8217–8225 (2012)
9. Lin, L.: On stirling's formula remainder. Applied Mathematics and Computation **247**, 494–500 (2014)
10. Nanjundiah, T.S.: Note on stirling's formula. Am. Math. Mon. **66**, 701–703 (1959)
11. Robbins, H.: A remark on stirling's formula. Am. Math. Mon. **62**, 26–29 (1955)
12. Shi, X., Liu, F., M., H.: A new asymptotic series for the gamma function. J. Comput. Appl. Math. **195**, 134–154 (2006)
13. Comtet, L.: Advanced Combinatorics: The Art of Finite and Infinite Expansions - Stirling Numbers, pp. 204–229. Springer, Dordrecht (1974). https://doi.org/10.1007/978-94-010-2196-8_5
14. S., B., M´endez, A.: The asymptotic expansion for n! and Lagrange inversion formula (2010). https://arxiv.org/abs/1002.3894
15. Impens, C.: Stirling's series made easy. Am. Math. Mon. 110:8, 730–735 (2003)
16. Inc., W.R.: Mathematica Online, Version 14.2. Champaign, IL, 2024. https://www.wolfram.com/mathematica

doi:10.20944/preprints202509.1468.v1

disclaim responsibility for any injury to people or property resulting from any ideas, methods, instructions or products referred to in the content.