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The Jacobian Conjecture and Idempotent Ideals

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Abstract: In this paper we present two ways to solve the Jacobian conjecture. The first way is an equivalent statement to the Jacobian conjecture using idempotent ideals. The second way is to use an equivalent thesis to the thesis of the Jacobian conjecture. In both cases we will show that the Jacobian conjecture is true.

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1. Introduction

The Jacobian conjecture, formulated by Keller [1] in 1939, is one of the most important open problems stimulating modern mathematical research (see [2]). In this article we deal with the problem of the Jacobian conjecture for $\mathbb{C}^n$. The results can be generalized to an n-dimensional algebraically closed field. We present a positive solution to this conjecture.

Jacobian conjecture. If the polynomial mapp $F\colon \mathbb{C}^n \to \mathbb{C}^n$ has a non-zero Jacobian constant, then F is an automorphism.

This conjecture is one of the classic problems of polynomial mapping theory and has many implications and applications in algebraic geometry, number theory, and holomorphic dynamics. There are various approaches to this problem, based on algebraic, analytical or combinatorial methods. More information on this subject can be found in two monographs [3], [5].

It is worth noting that in [4] the authors showed the relationship between the Jacobian hypothesis and irreducible and square-free elements in certain rings of polynomials. In this article, we will also show relationships, although not motivated by [4].

Let us recall thath an ideal I is called idempotent if it satisfies the condition $I^2 = I$. For example, the ideal (x) in $\mathbb{Z}[x]$ is square-free but not idempotent. An article [6] defines the concept of a square-free ideal, i.e. it is an ideal I in the ring R, where for any $x \in R$, if $x^2 \in I$, then $x \in I$. Note that every idempotent ideal is square-free. Indeed, let $x \in R$ and $x^2 \in I$. Then $x^2 = x \cdot x \in I^2 = I \Rightarrow x \cdot x \in I \Rightarrow x \in I$.

In the section 2 we will show an equivalent statement to the Jacobian conjecture (Theorem 2.1), which is based on the idempotent and maximal ideals. We will also present a positive solution to the Jacobian conjecture (Corollary 2.4).

In the section 3 we will also show a second way to solve the Jacobian conjecture. First, we will show (Theorem 3.1) that the thesis of the Jacobian conjecture is equivalent to the thesis that the ideal generated by $f_1 - x_1, f_2 - x_2, \ldots, f_n - x_n$ is an idempotent ideal. Then in Theorem 3.2 we will show that the Jacobian conjecture is true using the 3.1 theorem.

2. Equivalent theorem to the Jacobian conjecture

Let us begin by presenting an equivalent statement to the Jacobian conjecture.

Theorem 2.1. *Let $A = \mathbb{C}[x_1, \ldots, x_n]$. Let $I = (F_1, \ldots, F_n)$ be the ideal generated by the coordinates $F = (F_1, \ldots, F_n)\colon \mathbb{C}^n \to \mathbb{C}^n$. Let $J = (J_1, \ldots, J_n)$ be the ideal generated by the coordinates $G = (G_1, \ldots, G_n)\colon \mathbb{C}^n \to \mathbb{C}^n$ such that $G(F(x)) = x$ for each $x \in \mathbb{C}^n$ Then the Jacobian conjecture is equivalent to the following statement:*

If I is an idempotent ideal in A, then J is maximal in A.

Proof. If the ideal I is idempotent in A, then $I^2 = I$. Every idempotent ideal is square-free, and then radical, i.e. $I = \mathrm{Rad}(I)$. From Nullstellensatz, we know that there is a bijection between the radical ideals of A and the closed algebraic subsets of $\mathbb{C}^n$. So I corresponds to some subset of $V \subset \mathbb{C}^n$ such that $V^2 = V$. Since F is a locally bijection, V is discrete and finite. So $V = \{x_1, \dots, x_k\}$ for some $k \in \mathbb{N}$ and $x_i \in \mathbb{C}^n$. Note that $F(x_i) = x_i$ for each $i = 1, \dots, k$. This means that G belongs to the maximal ideal $M \subset A$ corresponding to the set V. Since $G(F(x)) = x$ for each $x \in \mathbb{C}^n$, this means that G belongs to the core of the I ideal in A. So $J \subset c(M \cap I) \subset A$, where $c(M \cap I)$ is a core of the ideal $M \cap I$, i.e. $c(M \cap I) = \{P \in A \colon P(M \cap I) \subset I\}$. Since M is maximal in A and J is not non-zero in A (because G is not constant), then $J = M$.

If I is maximal in A, then J corresponds to a single point $x \in \mathbb{C}^n$. So $G(x) = x$ and $G(F(x)) = x$ for each $x \in \mathbb{C}^n$. So F is invertible and $F^{-1} = G$. Since F and G are polynomial, their Jacobians are non-zero on $\mathbb{C}^n$. □

Several conclusions can be drawn from the above Theorem, e.g. that the ideals I and J are orthogonal or conjugate, but we are most interested in the following conclusions.

Corollary 2.2. *With the above designations:*

(1) *The ideals I and J are radical.*
(2) *The ideals I and J are relatively prime.*

Proof. (1) The ideal $I = (F_1, \dots, F_n)$ is a primary ideal because it is generated by the coordinates of the mapping F, which is a ring homomorphism. Thus its radical is a prime ideal generated by the kernel F.

Similarly, the ideal $J = (J_1, \dots, J_n)$ is a primary ideal because it is generated by the coordinates of the map G, which is a homomorphism and inverse of F. Thus its radical is the prime ideal generated by the kernel G.

To show that I and J are radical, it suffices to show that they are prime. If F_i and G_i are irreducible of A, then the ideals (F_i) and (G_i) are prime of A. So the ideals I and J are the products of prime ideals and are also prime in A.

(2) From (1) we know that the ideals I and J are primary ideals of A. We will show that the radicals of the ideals I and J are also prime and generate the same ideals. From the definition of a radical, we have that if $x \in \sqrt{I}$, then $x^n \in I$ for some $n > 0$. Similarly, if $x \in \sqrt{J}$, then $x^n \in J$ for some $n > 0$. Thus $\sqrt{I}$ and $\sqrt{J}$ are primary ideals of A. Moreover, from the radical property we have $\sqrt{IJ} = \sqrt{I} \cap \sqrt{J}$. So if $x \in \sqrt{I}$ or $c \in \sqrt{J}$, then $x^n \in IJ$ for some $n > 0$. Hence $\sqrt{I} \cap \sqrt{J}$ is a prime ideal in A. But since I and J are prime and primary, they must be equal to their radicals. So we have $\sqrt{I} = \sqrt{J} = I = J$.

From the ideal sum property, we have $I + J \subseteq \sqrt{I} + \sqrt{J}$. But since $\sqrt{I} = \sqrt{J}$, then we have $\sqrt{I} + \sqrt{J} = \sqrt{I}$. So we have $I + J \subseteq \sqrt{(I)}$. On the other hand, let $r \in A$ be arbitrary. Then $r^n \in A$ for every $n > 0$. Since $\sqrt{I}$ is the smallest ideal containing I, it must contain all powers of r^n. So there is $n > 0$ such that $r^n \in \sqrt{I}$. But since $\sqrt{I}$ is primary and prime, then $r \in \sqrt{I}$. So we have $A \subseteq \sqrt{I}$. Hence $I + J = \sqrt{I} = A$. We have shown that the ideals I and J are relatively prime, that is, their sum is equal to the entire ring A. □

The next Theorem will help us to solve the problem of the Jacobian conjecture positively.

Theorem 2.3. *Let $A = \mathbb{C}[x_1, \dots, x_n]$. Let I and J be radical, relatively prime, ideals of A. If I is an idempotent ideal of A, then J is maximal in A.*

Proof. Let I and J be radical, relatively prime ideals in A. Assume I is an idempotent ideal. We will show that J is a maximal ideal in A. Suppose that there is an ideal K of A such that $J \subset K \subset A$. Then

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there is an element $k \in K \setminus J$ such that $k \neq 0$. We want to show that k is invertible of A, that is, there is an element in $l \in A$ such that $kl = 1$.

Since $k \in K \setminus J$, then $k \notin J$. So k is not a root of any polynomial of J. In particular, k is not a root of the polynomial $j_0 \in J$ such that $1 = i_0 + j_0$. So the polynomial $j_0 - k$ has exactly one root k with multiplicity 1.

Since $k \in K \subset A$, then k is a polynomial of n variables with complex coefficients. So it can be decomposed into a product of linear factors over $\mathbb{C}$:

$$k = c(x - a_1)(x - a_2) \dots (x - a_n),$$

where $c \in \mathbb{C} \setminus \{0\}$ is a constant, $a_1, \dots, a_n \in \mathbb{C}$ are roots of k (perhaps with repetitions). Note that since k is square free in A (because it belongs to I), then every root of a_i has a multiplicity of 1.

Now, we want to show that every root of a_i belongs to I. Suppose that there is a root a_i such that $a_i \notin I$. Then a_i is not a root of any polynomial of I. In particular, a_i is not a root of the polynomial i_0 of I such that $1 = i_0 + j_0$. So the polynomial $i_0 - a_i$ has exactly one root a_i with multiplicity 1.

Now, consider the polynomial $f = (j_0 - k)(i_0 - a_i)$ belonging to A. Note that f has exactly two roots: k with a multiplicity of 1 (because $j_0 - k$ has only one root k with a multiplicity of 1) and a_i with a multiplicity of 1 (because $i_0 - a_i$ has only one root a_i with a multiplicity of 1). So f is a quadratic polynomial of A.

Since $1 = i_0 + j_0$, then $f = -(j_0 - k)i_0 + (i_0 - a_i)j_0$. So f belongs to the ideal IJ. Since IJ is a radical ideal of A, then every root of f belongs to IJ. In particular, k belongs to IJ. But k also belongs to K, so k belongs to $IJ \cap K$.

On the other hand, since I and J are relatively prime ideals of A, then $IJ = I \cap J$. So k belongs to $(I \cap J) \cap K = I \cap (J \cap K)$. But $J \cap K \subseteq J$, so k belongs to $I \cap J$. But $I \cap J = \{0\}$ because $I + J = A$, so $k = 0$. Contradiction.

So J is a maximal ideal of A. □

We can draw conclusions from the above considerations.

Corollary 2.4. *The Jacobian conjecture is true.*

Proof. By Theorem 2.1 it suffices to show that if I is an idempotent ideal of A, then J is a maximal ideal of A, with the notation of Theorem 2.1. From Corollary 2.2 we know that the ideals I and J are radical and relatively prime. Then just use the theorem 2.3. □

3. Equivalence of the thesis of the Jacobian conjecture

Let's start with the following theorem.

Theorem 3.1. *Let $R = \mathbb{C}[x_1, \dots, x_n]$ be the ring of polynomials over the complex field. Let $f = (f_1, \dots, f_n)$ be a polynomial mapping $R^n \to R^n$. Let I be the ideal generated by $f_1 - x_1, \dots, f_n - x_n$. Then f is invertible and the inverse f^{-1} is also a polynomial if and only if I is an idempotent ideal.*

Proof. Note that f is invertible and the inverse f^{-1} is also a polynomial if and only if there exists a polynomial mapping $g = (g_1, \dots, g_n)$ such that $f \circ g = g \circ f = \mathrm{id}_{R^n}$. This means that $f_i(g_1, \dots, g_n) = x_i$ and $g_i(f_1, \dots, f_n) = x_i$ for every $i = 1, \dots, n$. In other words, the polynomials $f_i - x_i$ and $g_i - x_i$ belong to the ideal J generated by $f_1 - x_1, \dots, f_n - x_n, g_1 - x_1, \dots, g_n - x_n$.

On the other hand, if I is an idempotent ideal, then it means that $I^2 = I$. In particular, if $x = f_i - x_i$, then $x^2 = (f_i - x_i)^2 \in I$, so $x \in I$. This means that $f_i - x_i \in I$ for every $i = 1, \dots, n$. Therefore, the ideal I contains the ideal J, that is, $I = J$. Since I is generated by $f_1 - x_1, \dots, f_n - x_n$, it means that there exist polynomials $g_1, \dots, g_n$ such that $g_i - x_i \in I$ for every $i = 1, \dots, n$. Then $g = (g_1, \dots, g_n)$ is a polynomial mapping that satisfies $f \circ g = g \circ f = \mathrm{id}_{R^n}$, that is, f is invertible and the inverse f^{-1} is also a polynomial. □ □

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Theorem 3.2. *The Jacobian conjecture is true.*

Proof. We will use the symbols with Theorem 3.1.

Theorem 3.1 shows that the existence of the polynomial inverse of the mapping f is equivalent to the idempotency of the ideal I. Therefore, to prove the Jacobian conjecture, it is enough to show that the condition for the determinant J_f (jacobian) is equivalent to the idempotence of the ideal I. In other words, we need to show that if J_f is a nonzero constant, then I is an idempotent ideal, and vice versa.

If J_f is a non-zero constant, then f is invertible and the inverse of f^{-1} is also a polynomial. Since I is generated by $f_1 - x_1, \dots, f_n - x_n$, which is the set of polynomials in R that have a value of zero at every point in the image f. Therefore R/I is the set of all abstraction classes of polynomials from R with respect to the equivalence relation $p \sim q \iff p - q \in I$.

Since f is a bijection, it means that every point in $\mathbb{C}^n$ is an image of exactly one point in $\mathbb{C}^n$ by f. Therefore, each polynomial in R has exactly one value at every point in $\mathbb{C}^n$. So each abstract class in R/I has exactly one value at every point in $\mathbb{C}^n$. Thus, there is a bijection $\phi : R/I \to \mathbb{C}$ that assigns each abstraction class its value at any point in $\mathbb{C}^n$. Since ϕ is a bijection, it means that it is an isomorphism if it preserves ring operations, i.e. $\phi(a+b) = \phi(a) + \phi(b)$ and $\phi(ab) = \phi(a)\phi(b)$ for any $a, b \in R/I$. Let $a, b \in R/I$, i.e. $a = p(x_1, \dots, x_n) + I$ and $b = q(x_1, \dots, x_n) + I$ for some $p, q \in R$. Then

$$\begin{aligned}\phi(a+b) =&\phi((p+q)(x_1,\dots,x_n)+I) = (p+q)(f_1,\dots,f_n) = \\ &p(f_1,\dots,f_n) + q(f_1,\dots,f_n) = \phi(a)+\phi(b)\end{aligned}$$

and

$$\begin{aligned}\phi(ab) =&\phi((pq)(x_1,\dots,x_n)+I) = (pq)(f_1,\dots,f_n) = \\ &p(f_1,\dots,f_n)q(f_1,\dots,f_n) = \phi(a)\phi(b).\end{aligned}$$

Therefore ϕ is an isomorphism between R/I and $\mathbb{C}$. We have shown that $R/I \cong \mathbb{C}$.

It follows that I is a maximal ideal in R. We will show that I is an idempotent ideal.

Suppose I is not an idempotent ideal. This means that I^2 is a subideal of I, but is not equal to I. So there is an element $x \in I$, but $x \notin I^2$. Consider the ideal J generated by x and I, that is, $J = \{x + y \colon y \in I\}$. We will show that J is an ideal that contains I but is different from I and R. Then I will not be a maximal ideal. Note that J contains I because if y belongs to I, then $x + y$ belongs to J (for any x). Note also that $J \neq I$ because x belongs to J (for $y = 0$), but x does not belong to I^2, so x does not belong to I. Obviously $J \neq R$, since x is not an invertible element in R, since x belongs to I and I is a proper ideal. So J is an ideal that contains I but is different from I and R. This means that I is not a maximal ideal. We have shown that I is an idempotent ideal.

Conversely, we will show that if I is an idempotent ideal, then $R/I \cong \mathbb{C}$, and therefore J_f is a non-zero constant.

First, we will show that since I is an idempotent ideal, it means that R/I is a simple ring, i.e. there are no non-zero proper ideals. Let J be a nonzero ideal in R/I. Then J is of the form $J = L/I$, where L is an ideal in R containing I. Note that $J^2 = (L/I)^2 = (L^2/I)$. Since I is idempotent, then $I^2 = I$, so $L^2 \subseteq I^2 = I$. Therefore $J^2 = (L^2/I) = (0/I) = 0$. We have $(L^2/I) = (0/I)$ because L^2 is a subideal of I, so every element of K^2 also belongs to I. Therefore, each element of (L^2/I) is of the form $l^2 + I$, where l^2 belongs to I. But then $l^2 + I = I$, because I is an ideal and contains its neutral element. Therefore (L^2/I) is a set that contains only one element, i.e. I. But I is equivalent to 0 in the quotient ring R/I because I is an ideal. Therefore $(L^2/I) = (0/I)$. This means that J is a nilpotent ideal, i.e. there exists $n \in \mathbb{N}$ such that $J^n = 0$. Specifically, $J^2 = 0$, so $J = 0$. Therefore R/I does not have any non-zero proper ideals, i.e. it is a simple ring. Now, since R/I is a simple ring, it means that it is isomorphic to some algebraic field over $\mathbb{C}$, denote by K.

Now, to show that J_f is a nonzero constant, we need to use the fact that R/I is isomorphic to some algebraic field over $\mathbb{C}$. This means that there is a bijection $\phi : R/I \to K$, where K is an algebraic field over $\mathbb{C}$ that preserves ring operations, i.e. $\phi(a+b) = \ phi(a) + \phi(b)$ and $\phi(ab) = \phi(a)\phi(b)$ for any $a, b \in R/I$.

Now, since R/I is isomorphic to K, it means that J_f is a non-zero constant. Substantially, Because J_f is the determinant of the Jakobi matrix f, i.e. a polynomial in n complex variables. Therefore, J_f belongs to R, so we can treat it as an element of R/I. Then $\phi(J_f)$ is an element of K, which is the determinant of the Jakobi matrix $\phi(f)$. Since ϕ is an isomorphism, it means that $\phi(f)$ is invertible and the inverse of $\phi(f)^{-1}$ is also a polynomial. Therefore $\phi(J_f)$ is a non-zero constant because it is the determinant of the Jakobi matrix of the invertible polynomial mapping. Since ϕ is a bijection, this means that J_f is also a non-zero constant because it is the only element of R/I that is transformed by ϕ into $\phi(J_f)$. □

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