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p-Numerical Semigroups of Triples from the Three-Term Recurrence Relations

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Abstract: Many people, including Horadam, have studied the numbers W_n, satisfying the recurrence relation $W_n = uW_{n-1} + vW_{n-2}$ ($n \ge 2$) with $W_0 = 0$ and $W_1 = 1$. In this paper, we study the p-numerical semigroups of the triple (W_i, W_{i+2}, W_{i+k}) for integers $i, k (\ge 3)$. For a nonnegative integer p, the p-numerical semigroup S_p is defined as the set of integers whose nonnegative integral linear combinations of given positive integers $a_1, a_2, \dots, a_\kappa$ with $\gcd(a_1, a_2, \dots, a_\kappa) = 1$ are expressed in more than p ways. When $p = 0$, $S = S_0$ is the original numerical semigroup. The largest element and the cardinality of $\mathbb{N}_0 \backslash S_p$ are called the p-Frobenius number and the p-genus, respectively.

Keywords: Frobenius problem; Frobenius numbers; Horadam numbers; Apéry set

MSC: 11D07; 20M14; 05A17; 05A19; 11D04; 11B68; 11P81

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1. Introduction

We consider the sequence $\{W_n\}_{n=0}^{\infty}$, satisfying

$$W_n = uW_{n-1} + vW_{n-2} \quad (n \ge 2) \quad W_0 = 0,\, W_1 = 1\,, \tag{1}$$

where u and v are positive integers with $\gcd(u, v) = 1$. The values of $W_n = W_n(u, v)$ depend on the values of u and v. If $u = v = 1$, $F_n = W_n(1, 1)$ is the n-th Fibonacci number [1]. If $u = 1$ and $v = 2$, $J_n = W_n(1, 2)$ is the n-th Jacobsthal number [2,3]. If $u = 2$ and $v = 1$, $P_n = W_n(2, 1)$ is the n-th Pell number [4]. However, for simplicity, if we do not specify the values of u or v, we will simply write W_n for $W_n(u, v)$.

This type of number sequence has been well known to many people by Horadam's series of studies ([5–9]) in the 1960s. Because of this fact, this sequence is sometimes called the *Horadam sequence*. Horadam himself used the recurrence relation $W_n = uW_{n-1} - vW_{n-2}$. But recently more people (see, e.g., [10,11]) have used the recurrence relation $W_n = uW_{n-1} + vW_{n-2}$ and such works are still due to Horadam. In general, the initial values are arbitrary, but because of some simplifications, we set $W_0 = 0$ and $W_1 = 1$. According to [6], this sequence has long exercised interest, as seen in, for instance, Bessel-Hagen [12], Lucas [13], and Tagiuri [14], and, for historical details, Dickson [15]. However, it is deplorable that quite a few papers are publishing results that have already been obtained by these authors as new results, either because they are unaware of their or the following important results, or even if they are ignoring them.

Given the set of positive integers $A := \{a_1, a_2, \dots, a_\kappa\}$ ($\kappa \ge 2$), for a nonnegative integer p, let S_p be the set of integers whose nonnegative integral linear combinations of given positive integers $a_1, a_2, \dots, a_\kappa$ are expressed in more than p ways. For a set of nonnegative integers $\mathbb{N}_0$, the set $\mathbb{N}_0 \backslash S_p$ is finite if and only if $\gcd(a_1, a_2, \dots, a_\kappa) = 1$. Then there exists the largest integer $g_p(A) := g(S_p)$ in $\mathbb{N}_0 \backslash S_p$, which is called the *$p$-Frobenius number*. The cardinality of $\mathbb{N}_0 \backslash S_p$ is called the *p-genus* and is denoted by $n_p(A) := n(S_p)$. The sum of the elements in $\mathbb{N}_0 \backslash S_p$ is called the *p-Sylvester sum* and is denoted by $s_p(A) := s(S_p)$. This kind of concept is a generalization of the famous Diophantine problem of Frobenius since $p = 0$ is the case when the original Frobenius number $g(A) = g_0(A)$, the

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genus $n(A) = n_0(A)$ and the Sylvester sum $s(A) = s_0(A)$ are recovered. We can call S_p the *p-numerical semigroup*. Strictly speaking, when $p \ge 1$, S_p does not include 0 since the integer 0 has only one representation, so it satisfies simply additivity, and the set $S_p \cup \{0\}$ becomes a numerical semigroup. For numerical semigroups, we refer to [16–18]. For the p-numerical semigroup, we refer to [19].

We are interested in finding any closed or explicit form of the p-Frobenius number, which is even more difficult when $p > 0$. For three or more variables, no concrete example had been found. Most recently, we have finally succeeded in giving the p-Frobenius number as closed-form expressions for the triangular number triplet ([20]), for repunits ([21,22]).

In this paper, we study the p-numerical semigroups of the triple (W_i, W_{i+2}, W_{i+k}) for integers $i, k (\ge 3)$. We give explicit closed formulas of the p-Frobenius numbers and p-genus of this triple. Note that the special cases for Fibonacci [1], Pell [4], and Jacobsthal triples [2,3] have already been studied.

2. Preliminaries

We introduce the Apéry set (see [23]) below in order to obtain the formulas for $g_p(A)$, $n_p(A)$, and $s_p(A)$ technically. Without loss of generality, we assume that $a_1 = \min(A)$.

Definition 1. *Let p be a nonnegative integer. For a set of positive integers $A = \{a_1, a_2, \dots, a_\kappa\}$ with $\gcd(A) = 1$ and $a_1 = \min(A)$ we denote by*

$$\mathrm{Ap}_p(A) = \mathrm{Ap}_p(a_1, a_2, \dots, a_\kappa) = \{m_0^{(p)}, m_1^{(p)}, \dots, m_{a_1-1}^{(p)}\},$$

the p-Apéry set of A, where each positive integer $m_i^{(p)}$ $(0 \le i \le a_1 - 1)$ satisfies the conditions:

$$\text{(i) } m_i^{(p)} \equiv i \pmod{a_1}, \quad \text{(ii) } m_i^{(p)} \in S_p(A), \quad \text{(iii) } m_i^{(p)} - a_1 \notin S_p(A).$$

Note that $m_0^{(0)}$ is defined to be 0.

It follows that for each p,
$$\mathrm{Ap}_p(A) \equiv \{0, 1, \dots, a_1 - 1\} \pmod{a_1}.$$

Even though it is hard to find any explicit form of $g_p(A)$ as well as $n_p(A)$ and $s_p(A)$ $k \ge 3$, by using convenient formulas established in [24,25], we can obtain such values for some special sequences $(a_1, a_2, \dots, a_\kappa)$ after finding any regular structure of $m_j^{(p)}$. One convenient formula is on the power sum

$$s_p^{(\mu)}(A) := \sum_{n \in \mathbb{N}_0 \setminus S_p(A)} n^\mu$$

by using Bernoulli numbers B_n defined by the generating function

$$\frac{x}{e^x - 1} = \sum_{n=0}^\infty B_n \frac{x^n}{n!},$$

and another convenient formula is on the weighted power sum ([26,27])

$$s_{\lambda,p}^{(\mu)}(A) := \sum_{n \in \mathbb{N}_0 \setminus S_p(A)} \lambda^n n^\mu$$

by using Eulerian numbers $\left\langle {n \atop m} \right\rangle$ appearing in the generating function

$$\sum_{k=0}^\infty k^n x^k = \frac{1}{(1-x)^{n+1}} \sum_{m=0}^{n-1} \left\langle {n \atop m} \right\rangle x^{m+1} \quad (n \ge 1)$$

with $0^0 = 1$ and $\left\langle {0 \atop 0} \right\rangle = 1$. Here, μ is a nonnegative integer and $\lambda \ne 1$. From these convenient formulas, many useful expressions are yielded as special cases. Some useful ones are given as follows. The formulas (3) and (4) are entailed from $s_{\lambda,p}^{(0)}(A)$ and $s_{\lambda,p}^{(1)}(A)$, respectively.

Lemma 1. *Let κ, p and μ be integers with $\kappa \ge 2$ and $p \ge 0$. Assume that $\gcd(a_1, a_2, \dots, a_\kappa) = 1$. We have*

$$g_p(a_1, a_2, \dots, a_\kappa) = \left(\max_{0 \le j \le a_1 - 1} m_j^{(p)} \right) - a_1 \,, \tag{2}$$

$$n_p(a_1, a_2, \dots, a_\kappa) = \frac{1}{a_1} \sum_{j=0}^{a_1 - 1} m_j^{(p)} - \frac{a_1 - 1}{2} \,, \tag{3}$$

$$s_p(a_1, a_2, \dots, a_\kappa) = \frac{1}{2a_1} \sum_{j=0}^{a_1 - 1} \left(m_j^{(p)}\right)^2 - \frac{1}{2} \sum_{j=0}^{a_1 - 1} m_j^{(p)} + \frac{a_1^2 - 1}{12} \,. \tag{4}$$

Remark 1. *When $p = 0$, the formulas (2), (3) and (4) reduce to the formulas by Brauer and Shockley [28] [Lemma 3], Selmer [29] [Theorem], and Tripathi [30] [Lemma 1][1], respectively:*

$$g(a_1, a_2, \dots, a_\kappa) = \left(\max_{0 \le j \le a_1 - 1} m_j \right) - a_1 \,,$$

$$n(a_1, a_2, \dots, a_\kappa) = \frac{1}{a_1} \sum_{j=0}^{a_1 - 1} m_j - \frac{a_1 - 1}{2} \,,$$

$$s(a_1, a_2, \dots, a_\kappa) = \frac{1}{2a_1} \sum_{j=0}^{a_1 - 1} (m_j)^2 - \frac{1}{2} \sum_{j=0}^{a_1 - 1} m_j + \frac{a_1^2 - 1}{12} \,,$$

where $m_j = m_j^{(0)}$ $(1 \le j \le a_1 - 1)$ with $m_0 = m_0^{(0)} = 0$.

3. Main Results

We use the following properties repeatedly. The proof is trivial and omitted.

Lemma 2. *For $i, k \ge 1$, we have*

$$W_k | W_i \Leftrightarrow k | i \,, \tag{5}$$

$$\gcd(W_i, W_{i+2}) = \begin{cases} u & \text{if } i \text{ is even;} \\ 1 & \text{if } i \text{ is odd,} \end{cases} \tag{6}$$

$$W_{i+k} = W_{i+1} W_k + v W_i W_{k-1} \,, \tag{7}$$

$$W_n \equiv \begin{cases} 0 \pmod u & \text{if } n \text{ is even;} \\ v^{\frac{n-1}{2}} \pmod u & \text{if } n \text{ is odd}\,. \end{cases} \tag{8}$$

First of all, if i is odd and $3 \le i \le k - 1$ then by (1) and (7),

$$\begin{aligned} W_{i+k} - g_0(W_i, W_{i+2}) &\ge W_{2i+1} - W_i W_{i+2} + W_i + W_{i+2} \\ &= W_{i+1} W_{i-1} + W_{i+2} + W_i > 0 \,. \end{aligned}$$

1 There was a typo, but it was corrected in [31].

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Hence, $g_0(W_i, W_{i+2}, W_{i+k}) = g_0(W_i, W_{i+2})$. Therefore, from now on, we consider the case only when i is even and k is odd, or when i is odd, with $i \ge k \ge 3$.

3.1. The Case k Is Odd

When k is odd, we choose non-negative integers $\mathfrak{q}$ and $\mathfrak{r}$ as

$$W_i = \mathfrak{q} W_k + \mathfrak{r} u, \quad 0 \le \mathfrak{r} < W_k\,, \tag{9}$$

where $\mathfrak{q} = W_i/W_k$ if $k|i$ due to (5), otherwise $\mathfrak{q}$ is the largest integer, satisfying

$$\mathfrak{q} \le \frac{W_i}{W_k} \quad \text{and} \quad \mathfrak{q} \equiv \begin{cases} 0 \pmod u & \text{if } i \text{ is even;} \\ v^{\frac{i-k}{2}} \pmod u & \text{if } i \text{ is odd.} \end{cases} \tag{10}$$

More directly, when i is even (and k is odd),

$$\mathfrak{q} = u \left\lfloor \frac{1}{u} \left\lfloor \frac{W_i}{W_k} \right\rfloor \right\rfloor. \tag{11}$$

When i is odd (and k is odd),

$$\mathfrak{q} = u \left\lfloor \frac{1}{u} \left(\left\lfloor \frac{W_i}{W_k} \right\rfloor - v^{\frac{i-k}{2}} \right) \right\rfloor + v^{\frac{i-k}{2}}. \tag{12}$$

Note that if $u = 1$ ([2]), then always $\mathfrak{q} = \lfloor W_i/W_k \rfloor$.

In particular, if i is even and

$$u > \frac{W_i}{W_k}, \quad \text{then} \quad \mathfrak{q} = 0, \quad \text{so} \quad \mathfrak{r} = W_i/u\,.$$

If $k|i$, then by (5) $W_k|W_i$. So, when i is even, by (8) $u|W_i$ (8). Thus, we get

$$\mathfrak{q} = \frac{W_i}{W_k}, \quad \text{so} \quad \mathfrak{r} = 0\,.$$

When $k|i$ and i is odd, by $W_i \equiv v^{\frac{i-1}{2}}$ and $W_k \equiv v^{\frac{k-1}{2}}$, there exists an integer h such that $v^{\frac{i-1}{2}} \equiv h v^{\frac{k-1}{2}} \pmod u$. By $\gcd(u, v) = 1$, $h \equiv v^{\frac{i-k}{2}} \pmod u$. Thus,

$$u \left| \left(\frac{W_i}{W_k} - v^{\frac{i-k}{2}} \right) \right.$$

Thus, we get

$$\mathfrak{q} = \frac{W_i}{W_k}, \quad \textit{so} \quad \mathfrak{r} = 0\,.$$

We use the following identity.

Lemma 3. *For $i, v \ge 3$, we have*

$$\mathfrak{r} W_{i+2} + \mathfrak{q} W_{i+k} = \big(W_{i+1} + v(\mathfrak{q} W_{k-1} + \mathfrak{r})\big) W_i\,.$$

Proof. By (1) and (7) together with (9), we get

$$\text{LHS} - \text{RHS} = \mathfrak{r}(u^2 + v) W_i + \mathfrak{r} u v W_{i-1} + \mathfrak{q}(W_{i+1} W_k + v W_i W_{k-1})$$

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$$
\begin{aligned}
&\quad - (uW_i + vW_{i-1})W_i - rvW_i - \mathfrak{q}vW_iW_{k-1} \\
&= 0\,.
\end{aligned}
$$

□

Assume that $k \nmid i$ (The case $k \mid i$ is discussed later). Then the elements of the (0-)Apéry set are given in Table 1. Here, we consider the expression

$$t_{y,z} := yW_{i+2} + zW_{i+k} \quad (y, z \geq 0)$$

or simply the position (y, z).

Table 1. $\mathrm{Ap}_0(W_i, W_{i+2}, W_{i+k})$ for odd k

$(0,0)$	$(1,0)$	$\cdots$	$\cdots$	$(W_k-1,0)$
$(0,1)$	$(1,1)$	$\cdots$	$\cdots$	$(W_k-1,1)$
$\vdots$	$\vdots$			$\vdots$
$(0,\mathfrak{q}-1)$	$(1,\mathfrak{q}-1)$	$\cdots$	$\cdots$	$(W_k-1,\mathfrak{q}-1)$
$(0,\mathfrak{q})$	$\cdots$	$(\mathfrak{r}-1,\mathfrak{q})$		
$\vdots$		$\vdots$		
$(0,\mathfrak{q}+u-1)$	$\cdots$	$(\mathfrak{r}-1,\mathfrak{q}+u-1)$		

We shall show that all the elements in Table 1 constitute the sequence $\{\ell W_{i+2} \pmod{W_i}\}_{\ell=0}^{W_i-1}$ in the vertical y direction. However, if i is odd and i is even, the situation of this sequence is different. In short, if i is odd, the sequence appears continuously, but if i is even, the sequence is divided into u subsequences.

First, let i be odd. Then by $\gcd(W_i, W_{i+2}) = 1$, we have

$$\{\ell W_{i+2} \pmod{W_i}\}_{\ell=0}^{W_i-1} = \{\ell \pmod{W_i}\}_{\ell=0}^{W_i-1}\,.$$

By (7), we get

$$W_{i+2}W_k - uW_{i+k} = v^2 W_i W_{k-2} \tag{13}$$

Hence,

$$W_{i+2}W_k \equiv uW_{i+k} \pmod{W_i} \quad \text{and} \quad W_{i+2}W_k > uW_{i+k}. \tag{14}$$

Thus, the element at (W_k, j) $(0 \leq j \leq \mathfrak{q}-1)$ cannot be an element of $\mathrm{Ap}_0(A)$ but $(0, u+j)$ as the same residue modulo W_i, where $A = \{W_i, W_{i+2}, W_{i+k}\}$. Next, by Lemma 3, we have

$$\mathfrak{r}W_{i+2} + \mathfrak{q}W_{i+k} \equiv 0 \pmod{W_i} \quad \text{and} \quad \mathfrak{r}W_{i+2} + \mathfrak{q}W_{i+k} > 0\,.$$

Thus, the element at $(\mathfrak{r}, \mathfrak{q}+j)$ $(0 \leq j \leq u-1)$ cannot be an element of $\mathrm{Ap}_0(A)$ but $(0, j)$.

Therefore, the sequence $\{\ell W_{i+2} \pmod{W_i}\}_{\ell=0}^{W_i-1}$ is divided into the longer parts with length W_k and the shorter parts with length $\mathfrak{r}$: Namely, the longer part is of the subsequence

$$(0, j), (1, j), \ldots, (W_k - 1, j) \quad (j = 0, 1, \ldots, \mathfrak{q}-1)$$

with the next element at $(0, u+j)$. The shorter part is of the subsequence

$$(0, \mathfrak{q}+j), (1, \mathfrak{q}+j), \ldots, (\mathfrak{r}-1, \mathfrak{q}+j) \quad (j = 0, 1, \ldots, u-1)$$

with the next element at $(0, j)$. Since $\gcd(W_{i+2}, W_{i+k}) = 1$, all elements in $\{\ell W_{i+2} \pmod{W_i}\}_{\ell=0}^{W_i-1}$ are different modulo W_i.

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Next, let i be even. Then by $\gcd(W_i, W_{i+2}) = u$, we have

$$\{\ell W_{i+2} \pmod{W_i}\}_{\ell=0}^{W_i/u-1} = \{\ell \pmod{W_i/u}\}_{\ell=0}^{W_i/u-1}.$$

Hence,

$$\{\ell \pmod{W_i}\}_{\ell=0}^{W_i-1} = \cup_{\kappa=0}^{u-1}\{\ell W_{i+2} + \kappa W_{i+k} \pmod{W_i}\}_{\ell=0}^{W_i/u-1}$$

with $\{\ell W_{i+2} + \kappa_1 W_{i+k} \pmod{W_i}\}_{\ell=0}^{W_i/u-1} \cap \{\ell W_{i+2} + \kappa_2 W_{i+k} \pmod{W_i}\}_{\ell=0}^{W_i/u-1} = \emptyset$ $(\kappa_1 \neq \kappa_2)$. By the determination of $\mathfrak{q}$ in (11), we see that $u|\mathfrak{q}$. So, by using the relations (14). Thus, each subsequence is given as the following points. For $z = 0, 1, \ldots, u-1$

$$\begin{aligned}
&(0,z),(1,z),\ldots,(W_k-1,z),\ (0,u+z),(1,u+z),\ldots,(W_k-1,u+z),\\
&(0,2u+z),(1,2u+z),\ldots,(W_k-1,2u+z),\ldots\ldots,\\
&(0,\mathfrak{q}-u+z),(1,\mathfrak{q}-u+z),\ldots,(W_k-1,\mathfrak{q}-u+z),\\
&(0,\mathfrak{q}+z),(1,\mathfrak{q}+z),\ldots,(\mathfrak{r}-1,\mathfrak{q}+z)
\end{aligned}$$

with next element is at $(0,z)$, coming back to the first one, because of Lemma 3. In addition, by (8), all terms of the above subsequence are

$$yW_{i+2} + zW_{i+k} \equiv zv^{\frac{i+k-1}{2}} \pmod{u}.$$

Since $\gcd(u,v) = 1$, this is equivalent to $z \pmod{u}$ $(z = 0, 1, \ldots, u-1)$. Therefore, there is no overlapped element among all subsequences. By (9), the total number of terms in each subsequence is

$$\frac{\mathfrak{q}}{u}W_k + \mathfrak{r} = \frac{W_i}{u}$$

as expected.

By Table 1, the candidates of the largest element of $\mathrm{Ap}_0(A)$ are at $(\mathfrak{r}-1, \mathfrak{q}+u-1)$ or at $(W_k - 1, \mathfrak{q}-1)$. Since $(\mathfrak{r}-1)W_{i+2} + (\mathfrak{q}+u-1)W_{i+k} > (W_k-1)W_{i+2} + (\mathfrak{q}-1)W_{i+k}$ is equivalent to $\mathfrak{r}W_{i+2} > v^2 W_i W_{k-2}$, by Lemma 1 (2), if $\mathfrak{r}W_{i+2} \geq v^2 W_i W_{k-2}$, then

$$g_0(W_i, W_{i+2}, W_{i+k}) = (\mathfrak{r}-1)W_{i+2} + (\mathfrak{q}+u-1)W_{i+k} - W_i.$$

If $\mathfrak{r}W_{i+2} \leq v^2 W_i W_{k-2}$, then

$$g_0(W_i, W_{i+2}, W_{i+k}) = (W_k-1)W_{i+2} + (\mathfrak{q}-1)W_{i+k} - W_i.$$

3.1.1. The Case k Is Odd with $k|i$

When k is odd and $k|i$, we get $\mathfrak{q} = W_i/W_k$ and $\mathfrak{r} = 0$. Hence, the elements of the (0-)Apéry set are given in Table 2.

Table 2. $\mathrm{Ap}_0(W_i, W_{i+2}, W_{i+k})$ when $k|i$

$(0,0)$	$(1,0)$	$\cdots$	$\cdots$	$(W_k-1,0)$
$(0,1)$	$(1,1)$	$\cdots$	$\cdots$	$(W_k-1,1)$
$\vdots$	$\vdots$	$\cdots$	$\cdots$	$\vdots$
$(0,W_i/W_k-1)$	$(1,W_i/W_k-1)$	$\cdots$	$\cdots$	$(W_k-1,W_i/W_k-1)$

Similarly to the case $k \nmid i$, when i is odd, so $uW_k \nmid W_i$, the sequence $\{\ell W_{i+2} \pmod{W_i}\}_{\ell=0}^{W_i-1}$ simply becomes one sequence by combining all the subsequences with length W_k and with length $\mathfrak{r}$. When i is even, so $uW_k \mid W_i$, the sequence $\{\ell W_{i+2} \pmod{W_i}\}_{\ell=0}^{W_i-1}$ consists of u subsequences with the same length W_i/u.

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By Table 2, the largest element of $\mathrm{Ap}_0(A)$ is at $(W_k - 1, W_i/W_k - 1)$. Hence,

$$g_0(W_i, W_{i+2}, W_{i+k}) = (W_k - 1)W_{i+2} + \left(\frac{W_i}{W_k} - 1\right)W_{i+k} - W_i\,.$$

In fact, this is included in the case where $k \nmid i$ and $rW_{i+2} \le v^2 W_i W_{i-2}$.

3.2. The Case k Is Even

When k is even (so i is odd), we choose non-negative integers q and r as

$$W_i = q\frac{W_k}{u} + r, \quad 0 \le r < \frac{W_k}{u}\,, \tag{15}$$

where $q = \lfloor uW_i/W_k \rfloor$. Note that W_k/u is an integer for even k.

Note that $k \nmid i$ because otherwise i is also even. Then the elements of the (0-)Apéry set are given in Table 3.

Table 3. $\mathrm{Ap}_0(P_{2i+1}(u), P_{2i+3}(u), P_{2i+k+1}(u))$ for even k

$(0,0)$	$(1,0)$	$\cdots$	$\cdots$	$(W_k/u-1,0)$
$(0,1)$	$(1,1)$	$\cdots$	$\cdots$	$(W_k/u-1,1)$
$\vdots$	$\vdots$			$\vdots$
$(0,q-1)$	$(1,q-1)$	$\cdots$	$\cdots$	$(W_k/u-1,q-1)$
$(0,q)$	$\cdots$	$(r-1,q)$		

Similarly to the case where k is odd in (14), we have

$$W_{i+2}\frac{W_k}{u} \equiv W_{i+k} \pmod{W_i} \quad \text{and} \quad W_{i+2}\frac{W_k}{u} > W_{i+k}.$$

Thus, the element at $(W_k/u, j)$ $(0 \le j \le q-1)$ cannot be an element of $\mathrm{Ap}_0(A)$ but $(0, j+1)$ as the same residue modulo W_i. The sequence $\{\ell W_{i+2} \pmod{W_i}\}_{\ell=0}^{W_i-1}$ is divided into the longer parts with length W_k/u and one shorter part with length r: Namely, the longer part is of the subsequence

$$(0,j), (1,j), \dots, (W_k/u - 1, j) \quad (j = 0, 1, \dots, q-1)$$

with the next element at $(0, j+1)$. One shorter part is of the subsequence

$$(0,q), (1,q), \dots, (r-1, q)$$

with the next element at $(0,0)$. Notice that similarly to Lemma 3, we have

$$rW_{i+2} + qW_{i+k} \equiv 0 \pmod{W_i}\,.$$

Since $\gcd(W_{i+2}, W_{i+k}) = 1$, all elements in $\{\ell W_{i+2} \pmod{W_i}\}_{\ell=0}^{W_i-1}$ are different modulo W_i. Then by $\gcd(W_i, W_{i+2}) = 1$, we have

$$\{\ell W_{i+2} \pmod{W_i}\}_{\ell=0}^{W_i-1} = \{\ell \pmod{W_i}\}_{\ell=0}^{W_i-1}\,.$$

By Table 3, the candidates of the largest element of $\mathrm{Ap}_0(A)$ are at $(r-1, q)$ or at $(W_k/u - 1, q-1)$. Since $(r-1)W_{i+2} + qW_{i+k} > (W_k/u - 1)W_{i+2} + (q-1)W_{i+k}$ is equivalent to $ruW_{i+2} > v^2 W_i W_{k-2}$, by Lemma 1 (2), if $ruW_{i+2} \ge v^2 W_i W_{k-2}$, then

$$g_0(W_i, W_{i+2}, W_{i+k}) = (r-1)W_{i+2} + qW_{i+k} - W_i\,.$$

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If $ruW_{i+2} \le v^2 W_i W_{k-2}$, then

$$g_0(W_i, W_{i+2}, W_{i+k}) = \left(\frac{W_k}{u} - 1\right) W_{i+2} + (q-1)W_{i+k} - W_i\,.$$

Notice that $ruW_{i+2} = v^2 W_i W_{k-2}$ may occur in some cases. For example, $(i,k,u,v) = (9,2,6,133)$. In this case, both of the two formulas are valid, yielding the Frobenius number $g_0(A) = 5949962315313983$.

4. The Case where $p > 0$

It is important to see that the elements of $\mathrm{Ap}_p(A)$ are determined from those of $\mathrm{Ap}_{p-1}(A)$.

4.1. When k Is Odd

4.1.1. When $p = 1$

The corresponding relations from $\mathrm{Ap}_0(A)$ to $\mathrm{Ap}_1(A)$ are as follows. See Table 4.
[The first u rows]

$$\begin{aligned}
&(y,z) \to (y+\mathfrak{r}, z+\mathfrak{q}) \quad (0 \le y \le W_k - \mathfrak{r} - 1,\ 0 \le z \le u-1)\,,\\
&(y,z) \to (y - W_k + \mathfrak{r}, z + \mathfrak{q} + u) \quad (W_k - \mathfrak{r} \le y \le W_k - 1,\ 0 \le z \le u-1)
\end{aligned}$$

by Lemma 3 and

$$(-W_k + \mathfrak{r})W_{i+2} + (\mathfrak{q}+u)W_{i+k} = (W_{i+1} + v(\mathfrak{q}W_{k-1} + \mathfrak{r}) - v^2 W_{k-2})W_i \qquad (\text{Lemma 3 and (13)})\,,$$

respectively. Note that when $\mathfrak{r} = 0$, the second corresponding relation does not exist. This also implies that all the elements at $(y+\mathfrak{r}, z+\mathfrak{q})$ and $(y - W_k + \mathfrak{r}, z+\mathfrak{q}+u)$ can be expressed in terms of (W_i, W_{i+2}, W_{i+k}) in at least two ways.
[Others]

$$\begin{aligned}
(y,z) \to (y+W_k, z-u) \quad &(0 \le y \le W_k - 1,\ u \le z \le \mathfrak{q}-1;\\
&\ 0 \le y \le \mathfrak{r} - 1,\ \mathfrak{q} \le z \le \mathfrak{q}+u-1)
\end{aligned}$$

by the identity (13). This also implies that all the elements at $(y+W_k, z-u)$ can be expressed in at least two ways.

By Table 4, there are four candidates to take the largest value of $\mathrm{Ap}_1(A)$. Namely, the values at

$$\begin{aligned}
&(\mathfrak{r}-1, \mathfrak{q}+2u-1), \quad (W_k - 1, \mathfrak{q}+u-1),\\
&(W_k + \mathfrak{r} - 1, \mathfrak{q}-1), \quad (2W_k - 1, \mathfrak{q}-u-1).
\end{aligned}$$

If $2uW_{i+k} > W_k W_{i+2}$, one of the elements at $(\mathfrak{r}-1, \mathfrak{q}+2u-1)$ and at $(W_k - 1, \mathfrak{q}+u-1)$ is the largest. In this case, if $\mathfrak{r}W_{i+2} \ge v^2 W_i W_{k-2}$, then

$$g_1(W_i, W_{i+2}, W_{i+k}) = (\mathfrak{r}-1)W_{i+2} + (\mathfrak{q}+2u-1)W_{i+k} - W_i\,.$$

If $\mathfrak{r}W_{i+2} \le v^2 W_i W_{k-2}$, then

$$g_1(W_i, W_{i+2}, W_{i+k}) = (W_k - 1)W_{i+2} + (\mathfrak{q}+u-1)W_{i+k} - W_i\,.$$

If $2uW_{i+k} < W_k W_{i+2}$, one of the elements at $(W_k + \mathfrak{r} - 1, \mathfrak{q} - 1)$ and at $(2W_k - 1, \mathfrak{q} - u - 1)$ is the largest. In this case, if $\mathfrak{r}W_{i+2} \ge v^2 W_i W_{k-2}$, then

$$g_1(W_i, W_{i+2}, W_{i+k}) = (W_k + \mathfrak{r} - 1)W_{i+2} + (\mathfrak{q} - 1)W_{i+k} - W_i\,.$$

If $\mathfrak{r}W_{i+2} \le v^2 W_i W_{k-2}$, then

$$g_1(W_i, W_{i+2}, W_{i+k}) = (2W_k - 1)W_{i+2} + (\mathfrak{q} - u - 1)W_{i+k} - W_i\,.$$

Table 4. $\mathrm{Ap}_p(W_i, W_{i+2}, W_{i+k})$ $(p = 0, 1)$ for odd k

$(0,0)$	$(1,0)$	$\cdots$	$\cdots$	$W_k-1,0)$	$(W_k,0)$	$(W_k+1,0)$	$\cdots$	$\cdots$	$(2W_k-1,0)$
$(0,1)$	$(1,1)$	$\cdots$	$\cdots$	$(W_k-1,1)$	$(W_k,1)$	$(W_k+1,1)$	$\cdots$	$\cdots$	$(2W_k-1,1)$
$\vdots$	$\vdots$			$\vdots$	$\vdots$	$\vdots$			$\vdots$
$(0,\mathfrak{q}-u-1)$	$(1,\mathfrak{q}-u-1)$	$\cdots$	$\cdots$	$(W_k-1,\mathfrak{q}-u-1)$	$(W_k,\mathfrak{q}-u-1)$	$(W_k+1,\mathfrak{q}-u-1)$	$\cdots$	$\cdots$	$(2W_k-1,\mathfrak{q}-u-1)$
$(0,\mathfrak{q}-u)$	$(1,\mathfrak{q}-u)$	$\cdots$	$\cdots$	$(W_k-1,\mathfrak{q}-u)$	$(W_k,\mathfrak{q}-u)$	$\cdots$	$(W_k+\mathfrak{r}-1,\mathfrak{q}-u)$		
$\vdots$	$\vdots$			$\vdots$	$\vdots$		$\vdots$		
$(0,\mathfrak{q}-1)$	$(1,\mathfrak{q}-1)$	$\cdots$	$\cdots$	$(W_k-1,\mathfrak{q}-1)$	$(W_k,\mathfrak{q}-1)$	$\cdots$	$(W_k+\mathfrak{r}-1,\mathfrak{q}-1)$		
$(0,\mathfrak{q})$	$\cdots$	$(\mathfrak{r}-1,\mathfrak{q})$	$\cdots$	$(W_k-1,\mathfrak{q})$					
$\vdots$		$\vdots$		$\vdots$					
$(0,q+\mathfrak{u}-1)$	$\cdots$	$(\mathfrak{r}-1,\mathfrak{q}+u-1)$	$\cdots$	$(W_k-1,\mathfrak{q}+u-1)$					
$(0,\mathfrak{q}+u)$	$\cdots$	$(\mathfrak{r}-1,\mathfrak{q}+u)$							
$\vdots$		$\vdots$							
$(0,\mathfrak{q}+2u-1)$	$\cdots$	$(\mathfrak{r}-1,\mathfrak{q}+2u-1)$							

Examples.
When $(i, k, u, v) = (5, 3, 4, 3)$, the first identity is applied:

$$\begin{aligned} g_1(W_5, W_7, W_8) &= g_1(409, 8827, 41008) \\ &= 11W_7 + 26W_8 - W_5 = 1162896\,. \end{aligned}$$

Indeed, there are two representations in terms of W_5, W_7, W_8 as

$$11W_7 + 26W_8 = 2155W_5 + 18W_7 + 3W_8\,,$$

which is the largest element of $\mathrm{Ap}_1(W_5, W_7, W_8)$. In fact, the second, the third and the fourth identities yield the smaller values

$$\begin{aligned} 1060653 &= 18W_7 + 22W_8 - W_5 (= 2164W_5 + 6W_7 + 3W_8 - W_5)\,, \\ 1002545 &= 30W_7 + 18W_8 - W_5 (= 9W_5 + 11W_7 + 22W_8 - W_5)\,, \\ 900302 &= 37W_7 + 14W_8 - W_5 (= 9W_5 + 18W_7 + 18W_8 - W_5)\,, \end{aligned}$$

respectively.

When $(i, k, u, v) = (5, 3, 2, 7)$, the second identity is applied:

$$\begin{aligned} &g_1(W_5, W_7, W_8) = g_1(149, 2143, 8136) \\ &= 10W_7 + 14W_8 - W_5 (= 753W_5 + 7W_7 + W_8 - W_5) = 135185\,. \end{aligned}$$

In fact, the first, the third and the fourth identities yield the smaller values

$$134313, \quad 125342, \quad 126214\,,$$

respectively.

When $(i, k, u, v) = (5, 3, 1, 4)$, the third identity is applied:

$$\begin{aligned} &g_1(W_5, W_7, W_8) = g_1(29, 181, 441) \\ &= 8W_7 + 4W_8 - W_5 (= 16W_5 + 3W_7 + 5W_8 - W_5) = 3183\,. \end{aligned}$$

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In fact, the first, the second and the fourth identities yield the smaller values

$$3160, \quad 2900, \quad 2923\,,$$

respectively.

When $(i,k,u,v) = (5,3,3,35)$, the fourth identity is applied:

$$\begin{aligned} & g_1(W_5, W_7, W_8) = g_1(2251, 123929, 898467) \\ & = 87W_7 + 46W_8 - W_5 (= 1225W_5 + 43W_7 + 49W_8 - W_5) = 521090543\,. \end{aligned}$$

In fact, the first, the second and the third identities yield the smaller values

$$51396298, \quad 52046980, \quad 51458372\,,$$

respectively.

4.1.2. When $p \ge 2$

The similar corresponding relations to the case $p = 1$ are also applied for $p \ge 2$. When $p = 2$, the elements of the first u rows of the main area (the second block from the left) correspond to fill the gap below the left-most block:

$$\begin{aligned} (y,z) &\to (y - W_k + \mathfrak{r}, z + \mathfrak{q} + u) \quad (W_k \le y \le 2W_k - \mathfrak{r} - 1,\ 0 \le z \le u - 1)\,, \\ (y,z) &\to (y - 2W_k + \mathfrak{r}, z + \mathfrak{q} + 2u) \quad (2W_k - \mathfrak{r} \le y \le 2W_k - 1,\ 0 \le z \le u - 1) \end{aligned}$$

The other elements of the main area correspond to those in the block immediately to the right to go up the u row:

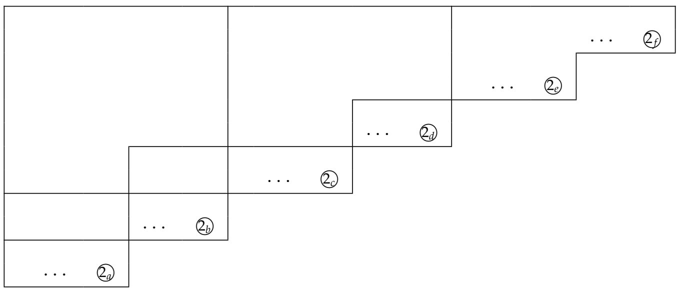
$$\begin{aligned} (y,z) \to (y + W_k, z - u) \quad & (W_k \le y \le 2W_k - 1,\ u \le z \le \mathfrak{q} - u - 1; \\ & W_k \le y \le W_k + \mathfrak{r} - 1,\ \mathfrak{q} - u \le z \le \mathfrak{q} - 1)\,. \end{aligned}$$

The elements of the stair areas correspond to those in the block immediately to the right in the form as it is to go up the $2u$ row:

$$\begin{aligned} (y,z) \to (y + W_k, z - 2u) \quad & (\mathfrak{r} \le y \le W_k - 1,\ \mathfrak{q} + u \le z \le \mathfrak{q} + 2u - 1; \\ & 0 \le y \le \mathfrak{r} - 1,\ \mathfrak{q} + 2u \le z \le \mathfrak{q} + 3u - 1)\,. \end{aligned}$$

See Table 5. We can also show that all these elements have at least three distinct representations in terms of W_i, W_{i+2}, W_{i+k}.

Table 5. $\mathrm{Ap}_p(W_i, W_{i+2}, W_{i+k})$ ($p = 0,1,2$) for odd k



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From Table 5, there are six candidates to take the largest element of $\mathrm{Ap}_2(A)$. These elements are indicated as follows.

$$
\begin{aligned}
&②_a : (\mathfrak{r}-1, \mathfrak{q}+3u-1) && ②_b : (W_k-1, \mathfrak{q}+2u-1)\\
&②_c : (W_k+\mathfrak{r}-1, \mathfrak{q}+u-1) && ②_d : (2W_k-1, \mathfrak{q}-1)\\
&②_e : (2W_k+\mathfrak{r}-1, \mathfrak{q}-u-1) && ②_f : (3W_k-1, \mathfrak{q}-2u-1)\,.
\end{aligned}
$$

If $uW_{i+k} > (W_k-\mathfrak{r})W_{i+2}$ (or $\mathfrak{r}W_{i+2} \ge v^2 W_i W_{k-2}$), one of those at $②_a$, $②_c$ and $②_e$ is the largest. Otherwise, one of those at $②_b$, $②_d$ and $②_f$ is the largest. However, it is clear that one of the values at $②_a$ or $②_e$ (respectively, $②_b$ or $②_f$) is larger than at $②_c$ (respectively, $②_d$). Hence, if $2uW_{i+k} > W_k W_{i+2}$, then the element at $②_a$ (respectively, $②_b$) is the largest. Otherwise, the element at $②_e$ (respectively, $②_f$) is the largest.

In conclusion, if $2uW_{i+k} > W_k W_{i+2}$ and $\mathfrak{r}W_{i+2} \ge v^2 W_i W_{k-2}$, then

$$
g_2(W_i, W_{i+2}, W_{i+k}) = (\mathfrak{r}-1)W_{i+2} + (\mathfrak{q}+3u-1)W_{i+k} - W_i\,.
$$

If $2uW_{i+k} > W_k W_{i+2}$ and $\mathfrak{r}W_{i+2} \le v^2 W_i W_{k-2}$, then

$$
g_2(W_i, W_{i+2}, W_{i+k}) = (W_k-1)W_{i+2} + (\mathfrak{q}+2u-1)W_{i+k} - W_i\,.
$$

If $2uW_{i+k} < W_k W_{i+2}$ and $\mathfrak{r}W_{i+2} \ge v^2 W_i W_{k-2}$, then

$$
g_2(W_i, W_{i+2}, W_{i+k}) = (2W_k+\mathfrak{r}-1)W_{i+2} + (\mathfrak{q}-u-1)W_{i+k} - W_i\,.
$$

If $2uW_{i+k} < W_k W_{i+2}$ and $\mathfrak{r}W_{i+2} \le v^2 W_i W_{k-2}$, then

$$
g_2(W_i, W_{i+2}, W_{i+k}) = (3W_k-1)W_{i+2} + (\mathfrak{q}-2u-1)W_{i+k} - W_i\,.
$$

In general, for an integer $p > 0$, it is sufficient to compare two elements at both ends. See Table 6. If $2uW_{i+k} > W_k W_{i+2}$ and $\mathfrak{r}W_{i+2} \ge v^2 W_i W_{k-2}$, then

$$
g_p(W_i, W_{i+2}, W_{i+k}) = (\mathfrak{r}-1)W_{i+2} + \big(\mathfrak{q}+(p+1)u-1\big)W_{i+k} - W_i\,.
$$

If $2uW_{i+k} > W_k W_{i+2}$ and $\mathfrak{r}W_{i+2} \le v^2 W_i W_{k-2}$, then

$$
g_p(W_i, W_{i+2}, W_{i+k}) = (W_k-1)W_{i+2} + (\mathfrak{q}+pu-1)W_{i+k} - W_i\,.
$$

If $2uW_{i+k} < W_k W_{i+2}$ and $\mathfrak{r}W_{i+2} \ge v^2 W_i W_{k-2}$, then

$$
g_p(W_i, W_{i+2}, W_{i+k}) = (pW_k+\mathfrak{r}-1)W_{i+2} + \big(\mathfrak{q}-(p-1)u-1\big)W_{i+k} - W_i\,.
$$

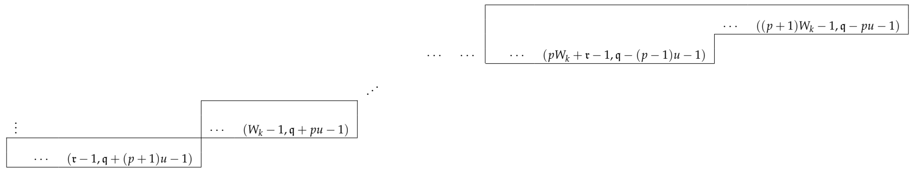
If $2uW_{i+k} < W_k W_{i+2}$ and $\mathfrak{r}W_{i+2} \le v^2 W_i W_{k-2}$, then

$$
g_p(W_i, W_{i+2}, W_{i+k}) = \big((p+1)W_k-1\big)W_{i+2} + (\mathfrak{q}-pu-1)W_{i+k} - W_i\,.
$$

The positions of the elements of $\mathrm{Ap}_p(A)$ below the left-most block and the positions of $\mathrm{Ap}_p(A)$ in the right-most block are arranged as shown in Table 6.

This situation is continued as long as $z = \mathfrak{q} - pu \ge 0$. However, when $p > q/u - 1$, the shape of the block on the right side collapses. Thus, the regularity of taking the maximum value of $\mathrm{Ap}_p(A)$ is broken. Hence, the fourth case holds until $p \le \lfloor q/u \rfloor - 1$ and other cases hold for $p \le \lfloor q/u \rfloor$.

Table 6. $\mathrm{Ap}_p(W_i, W_{i+2}, W_{i+k})$ for odd k



Theorem 1. *Let i be an integer and k be odd with $3 \le k \le i$. Let $\mathfrak{q}$ and $\mathfrak{r}$ be determined as (9) and (10). For $0 \le p \le \mathfrak{q}/u$, if $2uW_{i+k} > W_k W_{i+2}$ and $\mathfrak{r}W_{i+2} \ge v^2 W_i W_{k-2}$, then*

$$g_p(W_i, W_{i+2}, W_{i+k}) = (\mathfrak{r} - 1)W_{i+2} + \big(\mathfrak{q} + (p+1)u - 1\big)W_{i+k} - W_i\,.$$

If $2uW_{i+k} > W_k W_{i+2}$ and $\mathfrak{r}W_{i+2} \le v^2 W_i W_{k-2}$, then

$$g_p(W_i, W_{i+2}, W_{i+k}) = (W_k - 1)W_{i+2} + (\mathfrak{q} + pu - 1)W_{i+k} - W_i\,.$$

If $2uW_{i+k} < W_k W_{i+2}$ and $\mathfrak{r}W_{i+2} \ge v^2 W_i W_{k-2}$, then

$$g_p(W_i, W_{i+2}, W_{i+k}) = (pW_k + \mathfrak{r} - 1)W_{i+2} + \big(\mathfrak{q} - (p-1)u - 1\big)W_{i+k} - W_i\,.$$

If $2uW_{i+k} < W_k W_{i+2}$ and $\mathfrak{r}W_{i+2} \le v^2 W_i W_{k-2}$, then for $p \le q/u - 1$

$$g_p(W_i, W_{i+2}, W_{i+k}) = \big((p+1)W_k - 1\big)W_{i+2} + (\mathfrak{q} - pu - 1)W_{i+k} - W_i\,.$$

Examples.
When $(i, k, u, v) = (5, 3, 3, 7)$, the first identity is applied. Since $\mathfrak{q} = 19$ and $\mathfrak{r} = 5$, for $0 \le p \le \lfloor 19/3 \rfloor = 6$ we have

$$\begin{aligned}\{g_p(W_5, W_7, W_8)\}_{p=0}^{6} &= \{g_p(319, 6553, 29739)\}_{p=0}^{6}\\ &= 650412, 739629, 828846, 918063, 1007280, 1096497, 1185714.\end{aligned}$$

Namely, the corresponding element for each integer is at $(4, 3p + 21)$ $(p = 0, 1, \dots, 6)$. However, for $p \ge 7$, the p-Frobenius numbers can be computed neither by the above formula nor by any other closed formulas. Namely, the real value is $g_7(A) = 1218479$, corresponding to $(9, 39)$, though the formula gives 1274931, corresponding to $(4, 42)$.

4.2. When k Is Even

4.2.1. When $p = 1$

Similarly to the odd case where k is odd, the elements of $\mathrm{Ap}_p(A)$ can be determined from those of $\mathrm{Ap}_{p-1}(A)$. When $p = 1$, there are corresponding relations as follows.
[The first row $z = 0$]

$$\begin{aligned}&(y, 0) \to (y + r, z + \mathfrak{q}) \quad (0 \le y \le W_k/u - r - 1)\,,\\ &(y, 0) \to (y - W_k/u + r, z + q + 1) \quad (W_k/u - r \le y \le W_k/u - 1)\end{aligned}$$

with

$$rW_{i+2} + qW_{i+k} = (W_{i+1} + v(qW_{k-1} + r))W_i$$

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due to (15). Note that when $r = 0$ the second corresponding relation does not exist. This also implies that all the elements at $(y + r, z + q)$ and $(y - W_k/u + r, z + q + 1)$ can be expressed in terms of (W_i, W_{i+2}, W_{i+k}) in at least two ways.
[Others]

$$(y, z) \to (y + W_k/u, z - 1) \quad (0 \le y \le W_k/u - 1,\ 1 \le z \le \mathfrak{q} - 1;\ 0 \le y \le r - 1,\ z = q)$$

by the identity (13). This also implies that all the elements at $(y + W_k/u, z - 1)$ can be expressed in at least two ways.

By Table 7, there are four candidates to take the largest value of $\mathrm{Ap}_1(A)$. Namely, the values at

$$(r - 1, q + 1), \quad (W_k/u - 1, \mathfrak{q}),$$
$$(W_k/u + r - 1, q - 1), \quad (2W_k/u - 1, q - 2).$$

If $2uW_{i+k} > W_k W_{i+2}$, one of the elements at $(r - 1, q + 1)$ and at $(W_k - 1, q)$ is the largest. In this case, if $ruW_{i+2} \ge v^2 W_i W_{k-2}$, then

$$g_1(W_i, W_{i+2}, W_{i+k}) = (\mathfrak{r} - 1)W_{i+2} + (q + 1)W_{i+k} - W_i\,.$$

If $ruW_{i+2} \le v^2 W_i W_{k-2}$, then

$$g_1(W_i, W_{i+2}, W_{i+k}) = \left(\frac{W_k}{u} - 1\right)W_{i+2} + qW_{i+k} - W_i\,.$$

If $2uW_{i+k} < W_k W_{i+2}$, one of the elements at $(W_k/u + r - 1, q - 1)$ and at $(2W_k/u - 1, q - 2)$ is the largest. In this case, if $ruW_{i+2} \ge v^2 W_i W_{k-2}$, then

$$g_1(W_i, W_{i+2}, W_{i+k}) = \left(\frac{W_k}{u} + r - 1\right)W_{i+2} + (q - 1)W_{i+k} - W_i\,.$$

If $ruW_{i+2} \le v^2 W_i W_{k-2}$, then

$$g_1(W_i, W_{i+2}, W_{i+k}) = \left(\frac{2W_k}{u} - 1\right)W_{i+2} + (q - 2)W_{i+k} - W_i\,.$$

Table 7. $\mathrm{Ap}_p(W_i, W_{i+2}, W_{i+k})$ $(p = 0, 1)$ for even k

$(0,0)$	$(1,0)$	$\cdots$	$\cdots$	$(W_k/u-1,0)$	$(W_k/u,0)$	$(W_k/u+1,0)$	$\cdots$	$\cdots$	$(2W_k/u-1,0)$
$(0,1)$	$(1,1)$	$\cdots$	$\cdots$	$(W_k/u-1,1)$	$(W_k/u,1)$	$(W_k/u+1,1)$	$\cdots$	$\cdots$	$(2W_k/u-1,1)$
$\vdots$	$\vdots$			$\vdots$	$\vdots$	$\vdots$			$\vdots$
$(0,q-2)$	$(1,q-2)$	$\cdots$	$\cdots$	$(W_k/u-1,q-2)$	$(W_k/u,q-2)$	$(W_k/u+1,q-2)$	$\cdots$	$\cdots$	$(2W_k/u-1,q-2)$
$(0,q-1)$	$(1,q-1)$	$\cdots$	$\cdots$	$(W_k/u-1,q-1)$	$(W_k/u,q-1)$	$\cdots$	$(W_k/u+r-1,q-1)$		
$(0,q)$	$\cdots$	$(r-1,q)$	$\cdots$	$(W_k/u-1,q)$					
$(0,q+1)$	$\cdots$	$(r-1,q+1)$							

4.2.2. When $p \ge 2$

The situation is similar for $p \ge 2$. From Table 8, there are six candidates to take the largest element of $\mathrm{Ap}_2(A)$. These elements are indicated as follows.

$$②_a : (r - 1, q + 2) \qquad ②_b : (W_k/u - 1, q + 1)$$
$$②_c : (W_k/u + r - 1, q) \qquad ②_d : (2W_k/u - 1, q - 1)$$
$$②_e : (2W_k/u + r - 1, q - 2) \qquad ②_f : (3W_k/u - 1, q - 3)\,.$$

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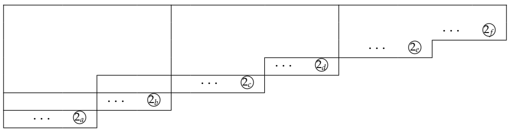
Similarly to the case where k is odd, middle element at $②_c$ and at $②_d$ cannot take the largest value. Hence, if $2uW_{i+k} > W_kW_{i+2}$, then the element at $②_a$ (respectively, $②_b$) is the largest. Otherwise, the element at $②_e$ (respectively, $②_f$) is the largest.

Table 8. $\mathrm{Ap}_p(W_i, W_{i+2}, W_{i+k})$ $(p = 0, 1, 2)$ for even k



In conclusion, if $2uW_{i+k} > W_kW_{i+2}$ and $ruW_{i+2} \ge v^2W_iW_{k-2}$, then

$$g_2(W_i, W_{i+2}, W_{i+k}) = (r-1)W_{i+2} + (q+2)W_{i+k} - W_i\,.$$

If $2uW_{i+k} > W_kW_{i+2}$ and $ruW_{i+2} \le v^2W_iW_{k-2}$, then

$$g_2(W_i, W_{i+2}, W_{i+k}) = \left(\frac{W_k}{u} - 1\right)W_{i+2} + (q+1)W_{i+k} - W_i\,.$$

If $2uW_{i+k} < W_kW_{i+2}$ and $ruW_{i+2} \ge v^2W_iW_{k-2}$, then

$$g_2(W_i, W_{i+2}, W_{i+k}) = \left(\frac{2W_k}{u} + r - 1\right)W_{i+2} + (q-2)W_{i+k} - W_i\,.$$

If $2uW_{i+k} < W_kW_{i+2}$ and $ruW_{i+2} \le v^2W_iW_{k-2}$, then

$$g_2(W_i, W_{i+2}, W_{i+k}) = \left(\frac{3W_k}{u} - 1\right)W_{i+2} + (q-3)W_{i+k} - W_i\,.$$

In general, for an integer $p > 0$, it is sufficient to compare two elements at both ends. See Table 9. If $2uW_{i+k} > W_kW_{i+2}$ and $ruW_{i+2} \ge v^2W_iW_{k-2}$, then

$$g_p(W_i, W_{i+2}, W_{i+k}) = (r-1)W_{i+2} + (q+p)W_{i+k} - W_i\,.$$

If $2uW_{i+k} > W_kW_{i+2}$ and $ruW_{i+2} \le v^2W_iW_{k-2}$, then

$$g_p(W_i, W_{i+2}, W_{i+k}) = \left(\frac{W_k}{u} - 1\right)W_{i+2} + (q+p-1)W_{i+k} - W_i\,.$$

If $2uW_{i+k} < W_kW_{i+2}$ and $ruW_{i+2} \ge v^2W_iW_{k-2}$, then

$$g_p(W_i, W_{i+2}, W_{i+k}) = \left(\frac{pW_k}{u} + r - 1\right)W_{i+2} + (q-p)W_{i+k} - W_i\,.$$

If $2uW_{i+k} < W_kW_{i+2}$ and $ruW_{i+2} \le v^2W_iW_{k-2}$, then

$$g_p(W_i, W_{i+2}, W_{i+k}) = \left(\frac{(p+1)W_k}{u} - 1\right)W_{i+2} + (q-p-1)W_{i+k} - W_i\,.$$

The positions of the elements of $\mathrm{Ap}_p(A)$ below the left-most block and the positions of $\mathrm{Ap}_p(A)$ in the right-most block are arranged as shown in Table 6.

This situation is continued as long as $z = q - p - 1 \ge 0$. However, when $p = q$, the shape of the block on the right side collapses. Namely, we cannot take the value at $\big((p+1)W_k/u - 1, q - p - 1\big)$. Thus, the regularity of taking the maximum value of $\mathrm{Ap}_p(A)$ is broken. Hence, the fourth case holds until $p \le q - 1$, and other cases hold for $p \le q$.

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Table 9. $\mathrm{Ap}_p(W_i, W_{i+2}, W_{i+k})$ for even k

				$\cdots$ $((p+1)W_k/u-1, q-p-1)$
		$\cdots$ $\cdots$	$\cdots$ $(pW_k/u+r-1, q-p)$	
	$\iddots$			
$\vdots$	$\cdots$ $(W_k/u-1, q+p-1)$			
$\cdots$ $(\mathfrak{r}-1, q+p)$				

Theorem 2. *Let i be an integer and k be even with $3 \le k \le i$. Let q and r be determined as (15). For $0 \le p \le q$, if $2uW_{i+k} > W_k W_{i+2}$ and $ruW_{i+2} \ge v^2 W_i W_{k-2}$, then*

$$g_p(W_i, W_{i+2}, W_{i+k}) = (r-1)W_{i+2} + (q+p)W_{i+k} - W_i\,.$$

If $2uW_{i+k} > W_k W_{i+2}$ and $ruW_{i+2} \le v^2 W_i W_{k-2}$, then

$$g_p(W_i, W_{i+2}, W_{i+k}) = \left(\frac{W_k}{u} - 1\right)W_{i+2} + (q+p-1)W_{i+k} - W_i\,.$$

If $2uW_{i+k} < W_k W_{i+2}$ and $ruW_{i+2} \ge v^2 W_i W_{k-2}$, then

$$g_p(W_i, W_{i+2}, W_{i+k}) = \left(\frac{pW_k}{u} + r - 1\right)W_{i+2} + (q-p)W_{i+k} - W_i\,.$$

If $2uW_{i+k} < W_k W_{i+2}$ and $ruW_{i+2} \le v^2 W_i W_{k-2}$, then for $0 \le p \le q-1$

$$g_p(W_i, W_{i+2}, W_{i+k}) = \left(\frac{(p+1)W_k}{u} - 1\right)W_{i+2} + (q-p-1)W_{i+k} - W_i\,.$$

Example

When $(i, k, u, v) = (5, 4, 2, 3)$, we have $q = 6$ and $r = 1$. So, the elements of $\mathrm{Ap}_6(W_5, W_7, W_9)$, where $(W_5, W_7, W_9) = (61, 547, 4921)$, are given as in Table 10. The largest element is at $(W_k/u - 1, q + p - 1) = (9, 11)$, which comes from the second identity. Thus,

$$g_6(W_5, W_7, W_9) = 9W_7 + 11W_9 - W_5 = 58993\,.$$

Notice that the right-most element is at $(pW_k/u + r - 1, q - p) = (60, 0)$ and the block of the right side is empty. Therefore, the formula does not hold for $p = 7$. In fact, $g_7(A) = 59542$, corresponding to $(19, 10)$, though the formula gives 63914, corresponding to $(9, 12)$.

Table 10. $\mathrm{Ap}_6(W_5, W_7, W_9)$ for $(u, v) = (2, 3)$

												(60, 0)
									(51, 1)	…	(59, 1)	
								(50, 2)				
					(41, 3)	…	(49, 3)					
				(40, 4)								
	(31, 5)	…	(39, 5)									
(30, 6)												
									(21, 7)	…	(29, 7)	
								(20, 8)				
					(11, 9)	…	(19, 9)					
				(10, 10)								
	(1, 11)	…	(9, 11)									
(0, 12)												

5. p-Genus

5.1. The Case Where k Is Odd

Let k be odd. For a non-negative integer p, the areas of the p-Apéry set can be divided into three parts: the stairs part (left), the stairs part (right) and the main part. By referring to Table 6 (with Tables 4 and 5), we can compute

$$
\begin{aligned}
&\sum_{w\in \mathrm{Ap}_p(A)} w\\
&= \sum_{l=0}^{p}\sum_{z=\mathfrak{q}+(p-2l)u}^{\mathfrak{q}+(p-2l+1)u-1}\sum_{y=lW_k}^{lW_k+\mathfrak{r}-1}(yW_{i+2}+zW_{i+k})\\
&\quad+\sum_{l=0}^{p}\sum_{z=\mathfrak{q}+(p-2l-1)u}^{\mathfrak{q}+(p-2l)u-1}\sum_{y=lW_k+\mathfrak{r}}^{(l+1)W_k-1}(yW_{i+2}+zW_{i+k})\\
&\quad+\sum_{z=0}^{\mathfrak{q}-pu-1}\sum_{y=pW_k}^{pW_k+\mathfrak{r}-1}(yW_{i+2}+zW_{i+k})+\sum_{z=0}^{\mathfrak{q}-(p+1)u-1}\sum_{y=pW_k+\mathfrak{r}}^{(p+1)W_k-1}(yW_{i+2}+zW_{i+k})\\
&=\frac{W_i}{2u}\big((W_i-u)W_{i+2}+u(u-1)W_{i+k}-\mathfrak{q}v^2(2W_i-uW_k)W_{k-2}\\
&\qquad+\mathfrak{q}^2v^2W_kW_{k-2}\big)\\
&\quad+\frac{pW_i}{2}W_k(2W_{i+2}-uv^2W_{k-2})-\frac{p^2W_i}{2}uv^2W_kW_{k-2}\,.
\end{aligned}
$$

Here, we used the relation (9) to simplify the expression. In addition, by $qvW_{k-2}\equiv qW_k\equiv W_i \pmod u$, we have

$$
\begin{aligned}
&(W_i-u)W_{i+2}+u(u-1)W_{i+k}-\mathfrak{q}v^2(2W_i-uW_k)W_{k-2}+\mathfrak{q}^2v^2W_kW_{k-2}\\
&\equiv vW_i^2-2vW_i^2+vW_i^2\equiv 0 \pmod u.
\end{aligned}
$$

By Lemma 1 (3), we have

$$
\begin{aligned}
&n_p(W_i,W_{i+2},W_{i+k})\\
&=\frac{1}{2u}\big((W_i-u)W_{i+2}+u(u-1)W_{i+k}-\mathfrak{q}v^2(2W_i-uW_k)W_{k-2}\\
&\qquad+\mathfrak{q}^2v^2W_kW_{k-2}\big)\\
&\quad+\frac{p}{2}W_k(2W_{i+2}-uv^2W_{k-2})-\frac{p^2}{2}uv^2W_kW_{k-2}-\frac{W_i-1}{2}\\
&=\frac{1}{2u}\big((W_i-1)(W_{i+2}-1)+u(u-1)(W_{i+k}-1)-\mathfrak{q}v^2(2W_i-uW_k)W_{k-2}\\
&\qquad+\mathfrak{q}^2v^2W_kW_{k-2}\big)\\
&\quad+\frac{p}{2}W_k(2W_{i+2}-uv^2W_{k-2})-\frac{p^2}{2}uv^2W_kW_{k-2}\,.
\end{aligned}
$$

Since the z value of the right-most side must be non-negative, $\mathfrak{q}-pu-1\ge 0$. Namely, the above formula is valid for $p\le(\mathfrak{q}-1)/u$.

Example

When $(i,k,u,v)=(5,3,3,7)$, by

$$
\mathfrak{q}=3\left\lfloor\frac{1}{3}\left(\left\lfloor\frac{319}{16}\right\rfloor-7^{\frac{5-3}{2}}\right)\right\rfloor+7^{\frac{5-3}{2}}=19\,,
$$

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for $0 \le p \le (\mathfrak{q}-1)/u = 6$ we have for $0 \le p \le \lfloor \mathfrak{q}/u \rfloor = 6$

$$\begin{aligned}\{n_p(W_5, W_7, W_8)\}_{p=0}^{6} &= \{n_p(319, 6553, 29739)_{p=0}^{6} \\ &= 330327, 432823, 532967, 630759, 726199, 819287, 910023.\end{aligned}$$

However, for $p \ge 7$, the p-genus cannot be obtained by the above formula. The real values are given by

$$\{n_p(W_5, W_7, W_8)\}_{p=7}^{9} = 965215, 1021448, 1067956,$$

though the formula gives

$$998407, 1084439, 1168119.$$

5.2. The Case Where k Is Even

Similarly to the case for k is odd, when k is even, by referring to Table 9 (with Tables 7 and 8), we can compute

$$\begin{aligned}&\sum_{w \in \mathrm{Ap}_p(A)} w \\ &= \sum_{l=0}^{p} \sum_{y=lW_k/u}^{lW_k/u+r-1} \big(yW_{i+2} + (q+p-2l)W_{i+k}\big) \\ &\quad + \sum_{l=0}^{p} \sum_{y=lW_k/u+r}^{(l+1)W_k/u-1} \big(yW_{i+2} + (q+p-2l-1)W_{i+k}\big) \\ &\quad + \sum_{z=0}^{q-p-1} \sum_{y=pW_k/u}^{pW_k/u+r-1} \big(yW_{i+2} + zW_{i+k}\big) + \sum_{z=0}^{q-p-2} \sum_{y=pW_k/u+r}^{(p+1)W_k/u-1} \big(yW_{i+2} + zW_{i+k}\big) \\ &\text{(When } p = q-1 \text{, the fourth term is empty, and} \\ &\text{when } p = q \text{, the third and the fourth terms are empty.)} \\ &= \frac{1}{2u^2} W_i \big(u^2 W_{i+2}(W_i - 1) - qv^2 W_{k-2}(2uW_i - W_k) \\ &\qquad + q^2 v^2 W_k W_{k-2}\big) \\ &\quad + \frac{p}{2u^2} W_i W_k (2uW_{i+2} - v^2 W_{k-2}) - \frac{p^2}{2u^2} v^2 W_i W_k W_{k-2} \,.\end{aligned}$$

Here, we used the relation (15) to simplify the expression. In addition,

$$\begin{aligned}\frac{W_{k-2}(2uW_i - W_k)}{u^2} &= \frac{W_{k-2}}{u}\left(2W_i - \frac{W_k}{u}\right), \\ \frac{v^2 W_k W_{k-2}}{u^2} &= v^2 \frac{W_k}{u} \frac{W_{k-2}}{u}, \\ \frac{W_k(2uW_{i+2} - v^2 W_{k-2})}{u^2} &= \frac{W_k}{u}\left(2W_{i+2} - v^2 \frac{W_{k-2}}{u}\right), \\ \frac{v^2 W_i W_k W_{k-2}}{u^2} &= v^2 W_i \frac{W_k}{u} \frac{W_{k-2}}{u}\end{aligned}$$

are all positive integers. By Lemma 1 (3), we have

$$\begin{aligned}&n_p(W_i, W_{i+2}, W_{i+k}) \\ &= \frac{1}{2u^2} \big(u^2 W_{i+2}(W_i - 1) - qv^2 W_{k-2}(2uW_i - W_k) \\ &\qquad + q^2 v^2 W_k W_{k-2}\big)\end{aligned}$$

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$$
\begin{aligned}
&+\frac{p}{2u^2}W_k(2uW_{i+2}-v^2W_{k-2})-\frac{p^2}{2u^2}v^2W_kW_{k-2}-\frac{W_i-1}{2}\\
&=\frac{1}{2u^2}\big(u^2(W_i-1)(W_{i+2}-1)-qv^2W_{k-2}(2uW_i-W_k)\\
&\quad+q^2v^2W_kW_{k-2}\big)\\
&+\frac{p}{2u^2}W_k(2uW_{i+2}-v^2W_{k-2})-\frac{p^2}{2u^2}v^2W_kW_{k-2}\,.
\end{aligned}
$$

Theorem 3. *Let i and k be integers with $\gcd(i,k)=1$ and $i\ge k\ge 3$. When k is odd, for $0\le p\le \mathfrak{q}/u$ we have*

$$
\begin{aligned}
&n_p(W_i,W_{i+2},W_{i+k})\\
&=\frac{1}{2u}\big((W_i-1)(W_{i+2}-1)+u(u-1)(W_{i+k}-1)-\mathfrak{q}v^2(2W_i-uW_k)W_{k-2}\\
&\quad+\mathfrak{q}^2v^2W_kW_{k-2}\big)\\
&+\frac{p}{2}W_k(2W_{i+2}-uv^2W_{k-2})-\frac{p^2}{2}uv^2W_kW_{k-2}\,,
\end{aligned}
$$

where $\mathfrak{q}$ and $\mathfrak{r}$ are given in (9). When k is even (and i is odd), for $0\le p\le q$ we have

$$
\begin{aligned}
&n_p(W_i,W_{i+2},W_{i+k})\\
&=\frac{1}{2u^2}\big(u^2(W_i-1)(W_{i+2}-1)-qv^2W_{k-2}(2uW_i-W_k)\\
&\quad+q^2v^2W_kW_{k-2}\big)\\
&+\frac{p}{2u^2}W_k(2uW_{i+2}-v^2W_{k-2})-\frac{p^2}{2u^2}v^2W_kW_{k-2}\,,
\end{aligned}
$$

where q and r are given in (15).

Example
Let $(i,k,u,v)=(5,4,2,3)$. So, $q=\lfloor 2W_5/W_4\rfloor=\lfloor 2\cdot 61/20\rfloor=6$. Then for $0\le p\le 6$ by the formula we have

$$
\begin{aligned}
\{n_p(W_5,W_7,W_9)\}_{p=0}^6&=\{n_p(61,547,4921)\}_{p=0}^6\\
&=14976,20356,25646,30846,35956,40976,45906\,.
\end{aligned}
$$

However, contrary to the fact that $n_7(W_5,W_7,W_9)=46885$, the formula gives 50746.

6. Final Comments

The original numbers studied by Horadam satisfy the recurrence relation $W_n=uW_{n-1}-vW_{n-2}$. From this point of view, almost all the above identities hold by replacing v by $-v$, though the condition $u>|v|$ is necessary. For example, the identities of (7) and (8) are replaced by

$$
\begin{aligned}
&W_{i+k}=W_{i+1}W_k-vW_iW_{k-1}\,,\\
&W_n\equiv\begin{cases}0\pmod u & \text{if } n \text{ is even;}\\ (-v)^{\frac{n-1}{2}}\pmod u & \text{if } n \text{ is odd}\,.\end{cases}
\end{aligned}
$$

respectively. For example, when $(i,k,u,v)=(8,5,4,-3)$, by $\mathfrak{q}=24$ for $0\le p\le 6=24/4$ by the first identity of Theorem 1, we have

$$
\{g_p(W_5,W_7,W_9)\}_{p=0}^6=24265799,27454443,30643087,
$$

$$33831731, 37020375, 40209019, 43397663\,.$$

When $(i,k,u,v) = (5,4,3,-2)$, by $q = 6$ for $0 \le p \le 6$ by the first identity of Theorem 2, we have

$$\{g_p(W_5, W_7, W_9)\}_{p=0}^{6} = 3035, 3546, 4057, 4568, 5079, 5590, 6101\,.$$

Horadam also studied the number W_n with arbitrary initial values W_0 and W_1. However, with arbitrary initial values many identities (e.g., (7)) do not hold as they are. Hence, the situation becomes too complicated.

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