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On the Unitarity of the Quantum Telegraph Equation and Measurement as Bayesian Update from Maximum Entropy Prior Distribution

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Abstract

The quantum telegraph equation is solved for unitary solutions, which links the eigenvalues of the Hamiltonian directly to the oscillation frequency. As it has been showed previously that this PDE is the formal square of a Dirac operator, and on the other hand it is a linearized Hamilton-Jacobi-Bellman PDE, from which the Schrödinger equation can be deduced in a nonrelativistic limit, it is clear that it is the key equation in relativistic quantum mechanics. We give a stationary solution for the quantum telegraph equation and a Bayesian interpretation for the measurement. The stationary solution is understood as a maximum entropy prior distribution and measurement is understood as Bayesian update.

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1. Introduction

In [1] it was shown, that the Stueckelberg wave equation, invented in 1941-1942, can be derived as a linearized Hamilton-Jacobi-Bellman equation for a relativistic and stochastic optimal control problem on a Minkowski spacetime, as a linear limit. The ontology of the theory is such that the spacetime is randomly oscillating at Planck scales, and the test particle is affected by Brownian noise and some effective potential. It is plausible, that these properties link to random metrics of a spacetime. Consider a square-integrable wave function $\phi \in L^2$ on some domain in a complex Hilbert space of functions. The Stueckelberg wave equation is

$$i\hbar\frac{\partial\phi}{\partial\tau} = \frac{\hbar^2}{2m}\Box\phi - V(\mathbf{x})\phi. \tag{1}$$

Where $\phi(\tau, t, x, y, z)$ is the square-integrable wave function on Hilbert space and $\Box = -\frac{1}{c^2}\frac{\partial^2}{\partial t^2} + \Delta$ is the normal d'Alembertian and Δ is the normal Laplacian operator. $V(x, y, z)$ is some potential function, for example quadratic or other well-known real valued potential. In [2], it was shown that the square of the Dirac operator yields exactly a telegrapher's equation, where the first order term is analytically continued, that is, the derivative operator first order in time contains the imaginary unit. It is therefore clear, that this type of PDE has an immanent role in relativistic quantum mechanics. It is remarkable, that the same equation can be obtained from the Stueckelberg PDE as follows. As the Hamilton-Jacobi-Bellman PDE is backwards, we reverse the proper time $\tau \longrightarrow -\tau$ to obtain

$$i\hbar\frac{\partial\phi}{\partial\tau} = -\frac{\hbar^2}{2m}\Box\phi + V(\mathbf{x})\phi. \tag{2}$$

As on a Minkowski spacetime, proper time τ can be related to coordinate time t in a frame by:

$$d\tau = \sqrt{1 - \frac{v^2}{c^2}}dt, \tag{3}$$

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where v is the velocity of the particle in that frame, we obtain immediately a Telegrapher's equation

$$\frac{i\hbar}{\sqrt{1-\frac{v^2}{c^2}}}\frac{\partial\phi}{\partial t} = -\frac{\hbar^2}{2m}\Box\phi + V(\mathbf{x})\phi. \tag{4}$$

or more clearly written

$$\frac{i\hbar}{\sqrt{1-\frac{v^2}{c^2}}}\frac{\partial\phi}{\partial t} - \frac{\hbar^2}{2mc^2}\frac{\partial^2\phi}{\partial t^2} = -\frac{\hbar^2}{2m}\Delta\phi + V(\mathbf{x})\phi. \tag{5}$$

This can be recast using the self-adjoint, unbouded and linear Hamiltonian operator $\mathcal{H}$:

$$\frac{i\hbar}{\sqrt{1-\frac{v^2}{c^2}}}\frac{\partial\phi}{\partial t} - \frac{\hbar^2}{2mc^2}\frac{\partial^2\phi}{\partial t^2} = \mathcal{H}\phi. \tag{6}$$

The benefit is now that we can study the temporal part by using the separation of variables method. The eigenvalues for the Hamiltonian are real, and the eigenfunctions give a complete orthonormal basis on this Hilbert space, which is convenient, as we express the solution as an infinite series, when the system is confined on a compact interval.

2. Unitary Solutions

Consider the following separation of variables: $\phi = T(t)f(\mathbf{x})$. The spatial part is the familiar eigenvalue equation for the Hamiltonian operator:

$$\mathcal{H}f = \lambda f. \tag{7}$$

The obtained equation can be solved by separating variables, we have for the temporal part:

$$\beta T' + \alpha T'' = \lambda T \tag{8}$$

Try $T(t) = Ae^{rt}$ we have

$$\beta rT + \alpha r^2 T - \lambda T = 0 \tag{9}$$

Thus we have the quadratic

$$\alpha r^2 + \beta r - \lambda = 0 \tag{10}$$

The quadratic formula gives:

$$r = \frac{-\beta \pm \sqrt{\beta^2 + 4\alpha\lambda}}{2\alpha} \tag{11}$$

Therefore for the unitary solution such that r is purely imaginary, we demand that the discriminant vanishes

$$\beta^2 + 4\alpha\lambda = 0 \tag{12}$$

Which in turn relates the eigenvalues of the Hamiltonian operator as follows

$$\lambda = -\frac{\beta^2}{4\alpha}. \tag{13}$$

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As we have $\alpha = -\frac{\hbar^2}{2mc^2}$, $\beta = \frac{i\hbar}{\sqrt{1-\frac{v^2}{c^2}}}$ We have thus a set of negative eigenvalues, which correspond to mass. The oscillation frequency for the wave function is thus given by:

$$r = -\frac{\beta}{2\alpha} = \frac{imc^2}{\sqrt{1-\frac{v^2}{c^2}}\hbar} = \frac{iE_j}{\hbar}, \tag{14}$$

where we have the relativistic energy $E_j = \frac{mc^2}{\sqrt{1-\frac{v^2}{c^2}}}$. For the limiting case of a nonrelativistic particle, we have $r = \frac{imc^2}{\hbar}$. Note that also $\lambda = \frac{1}{2}\beta r$. So that the ultra-high frequency of the oscillation of the wave function is essentially given by the eigenvalues of the Hamiltonian. The unitarity now ensures that norms are conserved, and the probability is conserved.

3. Relativistic Mass as Frequency

In [2], it was shown that the telegrapher's equation is obtained from the Dirac equation. We can consider a general solution for the quantum telegraph equation by considering the eigenfunction expansion. As the spectral theorem provides that the Hamiltonian eigenfunctions form an orthonormal basis in our complex Hilbert space, and that the eigenvalues are real, the general solution can be expressed as

$$\phi = \sum_{j=1}^{\infty} < \phi, f_j > f_j e^{\frac{imc^2}{\sqrt{1-\frac{v^2}{c^2}}\hbar} t}, \tag{15}$$

given that for the orthonormal eigenfunctions f_j we have $< f_j, f_j >= 1 \; \forall j$. Given Parseval's theorem, we have

$$\sum_{j=1}^{\infty} < \phi, f_j >^2 = 1, \tag{16}$$

so that we may interpret the squares of the coefficients as the probabability of measuring an eigenvalue λ_j with the corresponding eigenfunction f_j. From the physical point of view, the discrete energy eigenvalues represent discrete relativistic masses, and the probability of measuring such eigenstate is proportional to the inner product $< \phi, f_j >$. Operating in the stationary solution with the Hamiltonian operator, we have

$$\mathcal{H}\phi = \sum_{j=1}^{\infty} < \phi, f_j > \lambda_j f_j e^{\frac{imc^2}{\sqrt{1-\frac{v^2}{c^2}}\hbar} t}, \tag{17}$$

Taking an inner product $< \phi, \mathcal{H}\phi >$ we obtain the expected energy for the quantum system. Therefore energy is a discrete random variable, with probability given as $< \phi, f_j >^2$.

4. Bayesian Update

The measurement and collapse to an eigenstate can be seen through Bayesian inference, see [3]. The stationary or separable solution corresponds to a stationary distribution, where the test particle under Brownian motion reaches a thermal equilibrium, see [4]. According to Jaynes, this maximum entropy distribution should be used as the prior in Bayesian inference and update, [5]. The stationary distribution is a Gibbs distribution or a maximum entropy distribtution. Consider the following interpretation for quantum state measurement: we assume that the general solution or the stationary solution for the quantum telegraph equation is the prior distribution, so that we update the probability according to Bayes' rule:

$$P(\phi|f_j)P(f_j) = P(f_j|\phi)P(\phi) \tag{18}$$

$$< \phi, f_j >< f_j, f_j >=< f_j, \phi >< \phi, \phi > \tag{19}$$

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Giving

$$< f_j < f_j, \phi >, f_j >=< f_j, \phi >< \phi, \phi >, \tag{20}$$

from which we can see that the Bayesian update gives us essentially the individual coefficients of the eigenfunction expansion, when we treat the stationary general solution as the prior distribution. The wave function general solution ϕ can be interpreted as a maximum entropy prior distribution in Bayesian terms, where the updated probability is given by the measurement, collapsing on the eigenstate f_j.

5. Conclusions

The quantum telegraph equation is deduced from the well-known and manifestly covariant Stueckelberg wave equation. The telegraph equation separates, and gives unitary solutions when the real eigenvalues of the Hamiltonian operator are tied to the relativistic mass of the particle. Finally, the measurement is shown to match mathematically to Bayesian update, when the prior distribution is taken as the general solution, expressed as an infinite series of Hamiltonian eigenfunctions.

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