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Review

Magnetic Hopfions: A Review

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Abstract: Recent advances in the research area of 3D magnetic topological solitons (hopfions) in restricted geometries are reviewed. The description of the magnetic solitons is based on macroscopic micromagnetic approach and the Landau-Lifshitz equation of the magnetization motion. The concepts of the gauge emergent vector potential and emergent magnetic field are widely used to calculate 3D topological charge (the Hopf index) of magnetic textures. The relation of the magnetic hopfions with classical field theory is demonstrated and a special role of the curvilinear toroidal coordinates in description of the hopfions is underlined. The hopfion stability and dynamics in ferromagnetic films and dots are considered. Some critical discussion of calculations of the magnetization emergent magnetic field and the Hopf index of the toroidal magnetic hopfions in restricted geometries is presented.

Keywords: 3D magnetization textures; topological solitons; Hopf index

1. Introduction

Non-linear configurations of continuous classical field (solitons) play an important role in the different fields of modern physics such as classical field theory, optics, condensed matter physics, etc. The solitons are localized, particle-like objects, and, in many cases, the solitons can be classified by integer numbers related to their topology (topological charges) [1].

Let us assume that a physical system can be described by a classical 3D vector field $\mathbf{n}(\mathbf{r})$ of the unit length $\mathbf{n}^2 = 1$ (the order-parameter space is a unit sphere S_2 in 3D space) depending on the three spatial coordinates, represented by the vector $\mathbf{r}$. For instance, in the area of ferromagnetism such field is the unit magnetization field $\mathbf{m}(\mathbf{r})$ [2], in optics such field can be the Stokes vector field [3,4], in liquid crystals the role of the field plays a director field [5,6], etc. Topologically non-trivial configurations of the vector field describing an order parameter can be classified by using maps from the coordinate space ($\mathbf{r}$) to the order parameter space $\mathbf{n}(\mathbf{r})$ [7].

The macroscopic equation to describe a magnetization equilibrium configuration and magnetization dynamics in ordered magnetic media (the Landau-Lifshitz equation) is essentially nonlinear and allow several soliton-type solutions including topological magnetic solitons [2]. Topologically non-trivial magnetization configurations in ferromagnets, ferri- and antiferromagnets, such as domain walls, vortices, skyrmions and hopfions, etc. are currently in the focus of activity of researchers working in the area of solid-state magnetism. However, stability of different 3D magnetization configurations $\mathbf{m}(\mathbf{r})$ and the role of 3D (Hopf index), 2D (skyrmion number) topological charges and gyrovectors in their dynamics are still far from complete understanding. Nowadays complicated 3D magnetization configurations in ferromagnetic materials can be observed experimentally using electron holography or X-ray magnetic circular dichroism [8,9].

The vector order parameter for a ferromagnetic media is its net magnetization $\mathbf{M}(\mathbf{r})$. The magnetization field $\mathbf{M}(\mathbf{r})$ in 3D space represented by the unit field vector $\mathbf{m}(\mathbf{r}) = \mathbf{M}(\mathbf{r})/|\mathbf{M}(\mathbf{r})|$

depends, in general, on three spatial coordinates r_α , $\alpha = 1, 2, 3$. There are particular cases when the magnetization configurations depend only on one spatial coordinate (domain walls) or two spatial coordinates (magnetic vortices and skyrmions in thin magnetic films or in flat magnetic dots). For stable magnetization configurations $\mathbf{m}(\mathbf{r})$ it is possible to calculate topological charges, which describe the degrees of mappings (homotopy invariants) of 1D- (R^1) or 2D coordinate space (R^2) to the unit sphere $\mathbf{m}^2 = 1$ in the magnetization space $S_2(\mathbf{m})$, i.e., $R^1 \rightarrow S_2(\mathbf{m})$, $R^2 \rightarrow S_2(\mathbf{m})$. Mapping of the 3D coordinate space $R^3(\mathbf{r})$ to the magnetization unit sphere $\mathbf{m}^2 = 1$ (S_2) is more complicated and is related to the Hopf index. The corresponding magnetization configurations with a non-zero Hopf index are called "magnetic hopfions". The 3D magnetization textures are topologically equivalent if they have the same degree of mapping (the Hopf index). The magnetic energy of the given field configuration $E[\mathbf{m}(\mathbf{r})]$ can be calculated as a functional of the magnetization field $\mathbf{m}(r_\alpha)$ and spatial derivatives of the vector $\mathbf{m}(r_\alpha)$. Such energy includes the exchange, magnetostatic, magnetic anisotropy, etc., energy contributions. The minimization of the energy functional $E[\mathbf{m}(\mathbf{r})]$ yields some equilibrium (stable or metastable) magnetization field configurations $\mathbf{m}(\mathbf{r})$. Then, the magnetization dynamics of these equilibrium configurations can be considered on the base of the Landau-Lifshits equation of the magnetization motion and any parameters, including the topological charges, can be calculated.

In this review, I consider 3D magnetic topological solitons – hopfions and Bloch points. In particular, toroidal hopfions, which were introduced in the field theory, will be analyzed in detail. Such magnetic solitons possess intriguing and novel properties due to their topologically non-trivial magnetization configurations $\mathbf{m}(\mathbf{r})$. The article is organized as follows. The basic properties of the hopfions in the classical field theory are considered in Sec. 2. The concept of the emergent electromagnetic field and definition of 3D topological charges of the magnetic textures are presented in Sec. 3. Static and slow dynamics behavior of the magnetic hopfions are considered in Sec. 4. The fast linear and nonlinear hopfion dynamics are considered in Sec. 5. The review is concluded by a Summary in Sec. 6.

2. Hopfions in the Classical Field Theory

Topological three-dimensional solitons with non-zero Hopf index, named hopfions, were first considered in classical field theories in connection with the non-linear σ -model [7,10–12]. The explicit form of the mapping of 3D space (R^3) to the surface of the unit sphere $\mathbf{n}^2 = 1$ of the classical field $\mathbf{n}(\mathbf{r})$, $R^3 \rightarrow S_2$, was introduced by Hopf in 1931 [13]. Later it was shown that the Hopf index (a degree of the mapping $R^3 \rightarrow S_2(\mathbf{n})$) can be represented as some integral of the expression composed by a continuous classical field $\mathbf{n}$ (for instance, the magnetization field $\mathbf{m}(\mathbf{r})$) and its spatial derivatives [14]. For a given magnetization texture $\mathbf{m}(\mathbf{r})$ it is possible to plot pre-images, the 3D curves in real space, $\mathbf{r}(\mathbf{m}_1)$ and $\mathbf{r}(\mathbf{m}_2)$ of two points $\mathbf{m}_1$ and $\mathbf{m}_2$ on the magnetization unit sphere $\mathbf{m}^2 = 1$ and visually check the number of their crossings (a linking number of the pre-images). The linking number introduced solely for the toroidal hopfions is an integer number by definition and is equal to the Hopf index of a magnetization texture $\mathbf{m}(\mathbf{r})$. However, except the toroidal Hopfions, there are other kinds of hopfions, and the Hopf index is, in general, not integer, and cannot be calculated as the linking number of pre-images.

Faddeev [15] suggested a new Lagrangian (sometime referred to as the Faddeev-Skyrme's Lagrangian), which has stable soliton solutions for the three-component classical field $\mathbf{n}(\mathbf{r})$ in 3D coordinate space with conserving a topological charge (the Hopf charge). The Faddeev-Skyrme's Lagrangian is a linear combination of two invariants of the rotation group $O(3)$ represented by the spatial derivatives of the vector field $\mathbf{n}(\mathbf{r})$. Then, de Vega [16] found explicitly the field $\mathbf{n}(\mathbf{r})$ components (calling the field configurations as "closed vortices") for the simplest non-trivial unit Hopf charge in the toroidal coordinates. Now such hopfions are called toroidal hopfions. Nicole [17] suggested a natural form of the Hopf mapping of 3D-coordinate space R^3 to the surface of the unit sphere $\mathbf{n}^2 = 1$, $R^3 \rightarrow S_2(\mathbf{n})$, which allowed to introduce explicitly analytic equations for the soliton field configuration with the unit Hopf index. Later, motivated by the seminal paper by Faddeev and Niemi [7], several papers on the toroidal hopfions in the classical field theory were published [10–

12,18]. Gladikowski et al. [10] introduced the field configurations $\mathbf{n}(\mathbf{r})$ of the toroidal hopfions with arbitrary Hopf index and presented explicit expressions for the emergent magnetic field and vector potential components in the toroidal coordinates. The toroidal coordinates play a special role in the theory of the hopfions because the analytical equations are essentially simplified in these curvilinear coordinates. The toroidal hopfions attracted considerable interest of researchers because such topological solitons are stable solutions of the Faddeev-Skyrme's Lagrangian [15]. The recent review can be found in Ref. [1].

For calculations of the soliton topological charges, it is important to distinguish localized and non-localized solitons. The field $\mathbf{n}(\mathbf{r})$ approaches some constant value $\mathbf{n}_0$ at infinity $|\mathbf{r}| \rightarrow \infty$ for the localized solitons. The field is inhomogeneous at infinity $|\mathbf{r}| \rightarrow \infty$ for non-localized solitons. The condition $\mathbf{n}(\mathbf{r}) \rightarrow \mathbf{n}_0$ implies that the Hopf index, which distinguishes the different homotopy classes for the mapping $R^3 \rightarrow S_2(\mathbf{n})$, is an integer number $\pi_3(S_2) = \mathbb{Z}$ in infinite samples [7]. Therefore, there is a class of the toroidal hopfions in ordered media, which are described by an integer Hopf index $Q_H = 0, \pm 1, \dots$. The Hopf index of the toroidal hopfions is a product of two winding numbers [10], the planar winding (the azimuthal vorticity) and the twisting of the field configuration along the hopfion tube (the poloidal vorticity), respectively. Namely, due to the property that the vector field is asymptotically trivial, $\mathbf{n}(\mathbf{r}) \rightarrow \mathbf{n}_0$ at $|\mathbf{r}| \rightarrow \infty$, it is possible to compactify the three-dimensional coordinate space to the 4-dimensional unit sphere surface, $R^3 \rightarrow S_3$. The toroidal hopfions as localized solitons of a classical three-dimensional field resemble particle-like objects.

Nowadays, different kinds of the hopfions are investigated in the field of condensed matter physics (magnetic media [19], liquid crystals and colloids [5], ferroelectrics [20]) as well as photonics [3], optics [4], electromagnetism and gravitation [21], etc. There is a considerable interest in 3D inhomogeneous magnetization configurations classified by a linking number of the preimages of two distinct points in $S_2(\mathbf{m})$ in the 3D coordinate space (R^3), i.e., by the non-zero Hopf index. Although, 3D localized topological magnetic solitons were introduced by Dzyaloshinskii et al. [22] long time ago, the reincarnation of interest to such magnetization field textures started after the papers by Sutcliffe [23,24] relatively recently.

The simplest magnetic toroidal hopfions with the Hopf index $|Q_H| = 1$ were considered in infinite ferromagnetic films [25,26] and cylindrical dots [24,27–29] using the unit-vector field hopfion ansatz [10,11]. It was shown numerically that the toroidal hopfions with $|Q_H| = 1$ can be the ground state of circular chiral nanodots [28] assuming a strong surface magnetic anisotropy. Basing on the theoretical predictions [24], the first experimental observation of the magnetic hopfions was carried out in the Ir/Co/Pt multilayer films [30]. It was shown that except the toroidal magnetic hopfions there is in soft magnetic materials another class of the magnetization configurations with non-zero Hopf index – Bloch points [31].

3. Emergent Electromagnetic Field and the Hopf Index

Inhomogeneous and moving magnetization texture $\mathbf{m}(\mathbf{r}, t)$ results in appearance of effective magnetic and electric fields, which act on other subsystems. These, so called emergent magnetic and electric fields, are related to the spatial and time derivatives of the magnetization field $\mathbf{m}(\mathbf{r}, t)$. Initially, this effect was calculated by Korenman et al. [32] and Volovik [33] as result of the exchange interaction of the local spins with the spins of itinerant electrons in ferromagnetic metals. Then, it was shown [34] that this is a more general, pure geometrical effect related to choice of the local moving coordinate frame with Oz axis parallel to the local instant magnetization $\mathbf{m}(\mathbf{r}, t)$.

The emergent electromagnetic field tensor (in the units of $\hbar/2e$) can be written as

$$F_{\mu\nu}(\mathbf{m}) = \mathbf{m} \cdot (\partial_\mu \mathbf{m} \times \partial_\nu \mathbf{m}), \quad (1)$$

where $\partial_\mu = \partial/\partial x_\mu$ denote spatial and time derivatives. The indices $\mu, \nu = 0, 1, 2, 3$, where $x_0 = ct$ and $\mathbf{r} = (x_1, x_2, x_3)$ corresponds to the components of 3D radius-vector $\mathbf{r}$ in an orthogonal coordinate system. The field tensor is related to the emergent field vector potential $\mathbf{A}$ as $F_{\mu\nu}(\mathbf{m}) = \partial_\mu A_\nu(\mathbf{m}) - \partial_\nu A_\mu(\mathbf{m})$. The emergent magnetic field $\mathbf{B} = \nabla \times \mathbf{A}$ (gyrocoupling density) can be defined as in the standard electrodynamics $B_\lambda(\mathbf{m}) = \frac{1}{2} \varepsilon_{\lambda\mu\nu} F_{\mu\nu}(\mathbf{m})$ [34]. It is important that the emergent field $\mathbf{B}$ is a

divergence-free vector field, $\nabla \cdot \mathbf{B} = 0$. The flux of the emergent magnetic field through a closed surface defines 2D topological charge (skyrmion number).

Although, this emergent magnetic field differs from real magnetic field defined in the classical electrodynamics and is “fictitious” in some sense, it leads to some experimentally measurable effects. We mention here the topological Hall effect (influence of the emergent magnetic field on the conductivity electrons scattering) and skyrmion Hall effect (deflection of the magnetic soliton motion from the driving force direction due to the gyroforce). A theoretical approach how to use the topological Hall resistance to electrically detect the magnetic hopfion 3D spin textures was recently suggested by Göbel et. al. [29]. The flux of the emergent magnetic field through the plane $z = \text{const}$ in 2D ferromagnets determines the gyrovector, important parameter to describe motion of 2D topological solitons, magnetic vortices and skyrmions. The consequences of the non-zero gyrovector in 2D nanostructures were many times measured experimentally. The prominent example is the vortex/skyrmion gyrotropic excitation mode immediately related to the gyrovector (see the recent review [35]).

The dot product $\mathbf{A} \cdot \mathbf{B}$ of the emergent field vector potential and the emergent magnetic field defines 3D topological charge or the Hopf index

$$Q_H = \frac{1}{(4\pi)^2} \int d^3\mathbf{r} \mathbf{A}(\mathbf{m}) \cdot \mathbf{B}(\mathbf{m}) \quad (2)$$

of a 3D magnetization texture $\mathbf{m}(\mathbf{r})$. To have a physical sense the Hopf index should be gauge invariant. The necessary condition for that in finite samples is either nullification of the emergent magnetic field at the sample borders S , $\mathbf{B}_S = 0$, or the field $\mathbf{B}$ is tangential to the sample surface, $\mathbf{B} \cdot \mathbf{n} = 0$, where $\mathbf{n}$ is the external normal to the surface. The condition $\mathbf{B}_S = 0$ is satisfied for the localized solitons (toroidal hopfions, for instance) in infinite samples. The condition $\mathbf{B} \cdot \mathbf{n} = 0$, in general, is not satisfied. It was shown in Ref. [36] for the toroidal hopfions that if even the Hopf index is gauge invariant there is no freedom in choosing the vector potential $\mathbf{A}$. Only some particular choice of $\mathbf{A}$ related to the hopfion helicity leads to the integer values of the Hopf indices Q_H , allowing to consider them as the degrees of mapping (the linking numbers or integer numbers of crossings of the magnetization configuration $\mathbf{m}(\mathbf{r})$ preimages). In the case of the finite samples and magnetization textures different from the toroidal hopfions, the situation is more complicated. It is difficult to choose a proper form of the vector potential and prove the gauge invariance of the Hopf index. The important question about integer Hopf index and its gauge invariance should be carefully investigated for each magnetization texture in restricted geometry.

Recently Zheng et al. [37] reported on a direct observation of the magnetic hopfions forming coupled states with skyrmion strings in the submicron FeGe plates. They also provided a theoretical interpretation of the observed hopfions (hopfion rings). Zheng et al. [37] used the hopfion ring magnetization to calculate the Hopf invariant (Hopf index) applying the concept of the emergent magnetic field. The hopfion topological charge (Q_H) was calculated for confined samples. However, the large values of Q_H are very strange from the point of view of the theory of 3D magnetization configurations (including the magnetic hopfions [7,26,36]). The calculation method resulted in such large integer values of Q_H and gauge invariance of the calculated values of the Hopf index Q_H should be analyzed.

Obviously, the magnetization configurations considered by Zheng et al. are not so called “toroidal” hopfions (torus like vortex rings) introduced in the field theory for infinite media, see Refs. [26,36] and references therein. It was proved that the Hopf index is an integer only for the toroidal hopfions in infinite space (see Sec. 2). In this article [37] Zheng et al. considered neither toroidal magnetic hopfions nor infinite samples. Therefore, there is no ground for the speculations about the integer Hopf index and its equivalence to linking number of their 3D magnetization textures. The Hopf index can be calculated by the equation (2) and is, in general, non-integer for arbitrary 3D magnetization configurations in a finite domain like ones considered in the paper by Zheng et al. [37].

The strong statement about the integer Hopf index of an arbitrary magnetization configuration in a finite sample was done by Zheng et al. without proof or a proper reference. In the modern hopfion theory such statement is only valid for the toroidal hopfions in an infinite media. The simulations of the equilibrium magnetization configurations $\mathbf{m}(\mathbf{r})$ and the corresponding Hopf indices were

conducted by Zheng et al. for finite samples: 0.5- μm diameter disk with a thickness of 180 nm, the square plate 180 x 1000 x 1000 nm or "bulk" sample 700 x 350 x 350 nm. None of the samples used for the simulations in Ref. [37] is an infinite sample.

The Hopf index can be calculated only if an explicit expression for the vector potential $\mathbf{A}$ of the emergent magnetic field is known. The authors of Ref. [37] suggested heuristic equation to calculate the vector potential $\mathbf{A}$. However, they did not discuss the important problem of the vector potential gauge and the gauge invariance of the Hopf index. It is easy to show that to make the Hopf index (2) gauge invariant [26] the emergent field $\mathbf{B}(\mathbf{r})$ components should decay fast enough approaching zero increasing the absolute value of the radius vector $\mathbf{r}$ up to infinity $|\mathbf{r}| \rightarrow \infty$. This is possible to implement only in an infinite sample. In other words, the Hopf index is invariant and does not depend on the vector potential gauge in infinite samples. If even one introduces the artificial boundary condition for the magnetization $\mathbf{m}$ in a finite sample at the sample borders ($\mathbf{m} = \mathbf{m}_0 = \text{const}$) as Zheng et al. [37] suggested, such boundary conditions do not guarantee that the emergent magnetic field $\mathbf{B}$ goes to zero at the borders because this field is defined via the magnetization space derivatives, not via the magnetization $\mathbf{m}$ itself. There is a specific case of the axially symmetric magnetization configuration $\mathbf{m}(\mathbf{r}) = \mathbf{m}(\rho, z)$ in a cylindrical sample [26,36] ($\mathbf{r} = (\rho, \varphi, z)$ are the cylindrical coordinates), for which $\mathbf{B} = 0$ at the sample surface if $\mathbf{m} = \mathbf{m}_0 = \mathbf{z}$ ($\mathbf{z}$ is unit vector in the out-of-plane direction) at the borders. None of the magnetization configurations simulated by Zheng et al. is axially symmetric. Moreover, in many cases, the square plates or "bulk" rectangular samples were used for the simulations. The boundary condition $\mathbf{m} = \mathbf{z}$ used in Ref. [37] means a strong surface uniaxial magnetic anisotropy. It has no physical sense for such magnetic material as FeGe with a cubical crystal structure [38]. Therefore, we can exclude this particular case of axial symmetry from the consideration. It is important that if $\mathbf{B} \neq 0$ at the sample borders, then the Hopf index is not gauge invariant [26] and cannot be used to characterize a magnetization texture $\mathbf{m}(\mathbf{r})$. Even if one can suggest an artificial expression for the vector potential like equation used by Zheng et al. to satisfy the equation $\mathbf{B} = \nabla \times \mathbf{A}$, this does not mean that the introduced vector potential is correct.

An expression, alternative to the equation (2) for the Hopf index, was suggested in Refs. [39,40]. This expression is based on the double volume integral from the emergent magnetic field $\mathbf{B}(\mathbf{r})$ components and can be derived as a result of application of the Helmholtz theorem [41] (the fundamental theorem of vector calculus) to the field $\mathbf{B}(\mathbf{r})$. Important consequence of the Helmholtz theorem used in Refs. [39,40] is the integral representation of the vector potential $\mathbf{A}$ via the emergent field $\mathbf{B}$, namely

$$\mathbf{A}(\mathbf{r}) = -\frac{1}{4\pi} \int d^3\mathbf{r}' \frac{\mathbf{R} \times \mathbf{B}(\mathbf{r}')}{R^3}, \quad \mathbf{R} = \mathbf{r} - \mathbf{r}'. \quad (3)$$

This expression for $\mathbf{A}(\mathbf{r})$ (3) is different from the expression for the emergent field vector potential given by Zheng et al. [37]. Therefore, the heuristic vector potential introduced by Zheng et al. contradicts to the Helmholtz theorem [41] and the integral expressions for the Hopf index used in Refs. [39,40]. The similar wrong heuristic expressions for the vector potential (represented as one-fold integral from the emergent magnetic field $\mathbf{B}$) were used in Refs. [19] and [42] without any justification.

An expression for the vector potential was used by Liu, Jiang et al. [27] for calculations of the Hopf index of the circular magnetic dots in the momentum representation. Liu et al. calculated the Hopf index to be 0.96 for their dot parameters. However, Liu et al. [25] used the Coulomb gauge for the vector potential, $\nabla \cdot \mathbf{A} = 0$, which is not compatible with the equation (3) for the potential $\mathbf{A}(\mathbf{r})$ given above. The gauge $\nabla \cdot \mathbf{A} = 0$ is wrong for the particular case of the toroidal hopfions. This can be checked if one uses the toroidal vector potential components calculated explicitly in Ref. [36].

The value of the Hopf index represented by Eq. (2) depends essentially on the choice of the vector potential $\mathbf{A}$ of the emergent magnetic field. Recently it was shown [36] that some definite gauge of the vector potential $\mathbf{A}$ should be chosen for the toroidal hopfions in infinite samples to guarantee the integer values of the Hopf index Q_H . The heuristic vector potential given by Zheng et al. [37] can guarantee neither the integer Hopf indices nor their stability with respect to the vector potential gauge change in the finite samples. It seems that the integer values of Q_H simulated in the paper by Zheng et al. [37] are some arbitrary and ungrounded numbers resulting in a confusion of the readers

who are not specialists in the theory of 3D magnetic textures (hopfions, in particular). Therefore, the methods and calculations used in Ref. [37] should be reconsidered to bring them in accordance with the hopfion theory.

4. Static Properties of Magnetic Hopfions

Recently, stability of the hopfions was studied numerically in chiral ferromagnets FeGe [27,28] and in confined ferroelectric nanoparticles (nanospheres of PbZrTi oxide) [20]. Basing on the theoretical predictions [27,28], the first experimental observation of magnetic hopfions was carried out in the Ir/Co/Pt multilayer systems [30] with ultrathin Co layers. Restricted sample cylindrical geometry was used in for the chiral ferromagnets FeGe and Ir/Co/Pt with a strong Dzyaloshinsky-Moriya interaction and out-of-plane uniaxial magnetic anisotropy on the sample faces. The calculated Hopf index Q_H for chiral ferromagnets was equal to 1 [28,30] or close to 1 [27]. It was simulated in Ref. [27] that the toroidal hopfions are metastable states of a chiral nanodisk existing at large enough disk thickness and radius. Their magnetic energy is higher than the energy of the monopole-antimonopole pair 3D magnetization configuration.

The hopfions were considered also in ferroelectrics. The Hopf index of the polarization field $\mathbf{P}$ was defined using a non-standard equation in Ref. [20]. If one assumes that the polarization field $\mathbf{P}(\mathbf{r})$ for ferroelectrics is similar to the magnetization field $\mathbf{M}(\mathbf{r})$ for ferromagnets, then in the definition of the Hopf index should be the emergent magnetic field $\mathbf{B}(\mathbf{r})$, not the field $\mathbf{M}(\mathbf{r})$ (or $\mathbf{P}(\mathbf{r})$). Therefore, the parameter H defined in Ref. [20] as a volume integral from the dot product $\mathbf{P} \cdot \mathbf{A}$ is not a Hopf index, although the spatial configuration $\mathbf{P}(\mathbf{r})$ is similar to one for the toroidal hopfion of the magnetization field $\mathbf{M}(\mathbf{r})$. It is not surprise that the Hopf index H is not integer for the considered spherical samples. The calculated values of the Hopf index are not listed in Ref. [20].

The magnetization configurations of the toroidal magnetic hopfions in infinite media with the integer Hopf index $Q_H = mn$ and the arbitrary poloidal and azimuthal vorticities m, n were explicitly calculated in Ref. [36] using the toroidal and cylindrical coordinates. The calculation method was based on the Hopf mapping definition and the concept of the emergent magnetic field which is expressed via spatial derivatives of the magnetization field $\mathbf{m}(\mathbf{r})$. It was shown that the Hopf charge density can be represented as a Jacobian of the transformation from the toroidal to the cylindrical coordinates [36]. The calculated components of the hopfion emergent magnetic field and vector potential can be used, in particular, for calculations of the topological Hall effect and skyrmion Hall effect of the toroidal magnetic hopfions, respectively.

The important question is stability of the different 3D magnetization configurations $\mathbf{m}(\mathbf{r})$. It was established in the field theory [43,44] that any physical system with a squared gradient field term in the Lagrangian has no stable, time-independent, localized solutions in 3D case for any form of the potential. This statement known as the Hobard-Derrick theorem. However, stable localized solutions (localized solitons) can exist if there are any energy contributions linear with respect to spatial derivatives or with higher-order spatial derivatives of the field [45,46]. There are the energy terms with the first-order derivatives, so called Lifshitz invariants, accounting for the Dzyaloshinskii-Moriya interactions (DMI) in magnetic materials with broken inversion symmetry. Other opportunity to get stable 3D field configurations is accounting for the higher-order spatial derivatives in the Lagrangian. The terms quartic in spatial derivatives were suggested by Skyrme [47] and Faddeev [15] within the classical field theory. It was shown that the Faddeev-Skyrme Lagrangian has stable 3D localized soliton solutions in the form of toroidal hopfions [10,11]. The toroidal hopfions are some kinds of the localized topological solitons and are characterized by non-zero values of 3D topological charge (Hopf index) [13,14]. It was shown recently [19] that the classical Heisenberg model with competing long-range exchange interactions can result in the terms quadratic in the second spatial derivatives of magnetization field. Although such model is beyond the standard theory of micromagnetism, it may result in stabilization of the toroidal magnetic hopfions. The question is: whether is it possible to stabilize the magnetic hopfions in a ferromagnet within the standard exchange approach (avoid exotic exchange interactions) due to non-zero DMI terms and/or magnetostatic energy? Such energy contributions are beyond the field theory and, therefore,

applicability of the Hobart-Derrick theory to the evaluation of stability of the magnetic field configurations should be reconsidered. The simple scaling analysis accounting for the DMI terms was conducted in Ref. [48]. However, this analysis ignored the magnetostatic interaction (which is unavoidable in real ferromagnetic samples) and the finite sample sizes. The complicated non-local magnetostatic interaction is usually not considered in the theory of magnetic skyrmions and hopfions or accounted in a simplified form. The skyrmions are considered either in the bulk magnetic crystals without inversion symmetry or in ultrathin films. In both cases the magnetostatic interaction is reduced to a local form of some extra contribution to the magnetic anisotropy energy. Accounting for the magnetostatic interaction in relatively thick magnetic dots [49] allows to stabilize quasi-2D skyrmions without presence of any DMI, if a small out-of-plane magnetic anisotropy is included to the energy functional. The magnetostatic interaction was not included to the energy functional in Refs. [24,28,29] describing the magnetic hopfions in thick cylindrical dots. We note that the magnetostatic interaction can also lead to stabilization of other kinds of magnetization textures: the Bloch point hopfions with non-zero Hopf index or half-hedgehog (3D quasi-skyrmion) magnetization textures even in soft magnetic materials with no DMI [31,50,51].

It was demonstrated that in Ref. [52] that the toroidal hopfion magnetization configuration is a metastable state of a thick cylindrical ferromagnetic dot or a long wire of a finite radius R . Existence of this state is a result of the competition of the exchange and magnetostatic energies. The Dzyaloshinskyi-Moriya exchange interaction and magnetic anisotropy are of the second importance for the hopfion stabilization. Important role of the magnetostatic interaction in the toroidal hopfion stabilization in a cylindrical dot was confirmed by simulations [42].

5. Dynamics of Magnetic Hopfions

The study of the spin excitation spectra of magnetic Hopfions was performed in the recent papers [53,54]. Very recently the theoretical papers on the magnetic hopfion dynamics [25,42,55,56] were published.

The current induced 3D dynamics of the toroidal magnetic hopfions were studied both analytically (the collective coordinates approach) and numerically in Ref. [25] in an infinite frustrated ferromagnet (different signs of the exchange integrals for the nearest neighbor spins). The hopfions exhibit complicated dynamics including a longitudinal motion along the current direction, a transverse motion perpendicular to the current direction, a rotational motion and dilation [25]. The skyrmion Hall effect (nonzero hopfion gyrovector) is clearly seen in the solutions, although the gyrovector was not explicitly defined. The calculated hopfion velocity is quite small, < 10 m/s, for the typical values of the current intensity. The results by Liu et al. [25] are in some contradiction with the hopfion dynamics calculated by Wang et al. in Ref. [26], where the hopfion internal structure dynamics were ignored. Wang et al. did not observe any skyrmion Hall effect in the course of the hopfion motion in magnetic stripes. The hopfions moved under drive by the spin-transfer torque or spin Hall torque along the nanostripe with velocities up to 20 m/s. The origin of this discrepancy is still unclear. The Hopf index was $|Q_H| = 1$ in both cases.

The spin wave excitation spectra of the Bloch and Neel toroidal hopfions in FeGe nanodisks were investigated numerically in Ref. [42]. It was shown that the Bloch hopfions reveal very dense eigenfrequencies in the low frequency region (< 5 GHz). Only few eigenfrequencies were detected for the Neel hopfions. The eigenmode spatial distributions are found for some selected spin modes. The modes are mainly localized at the disk edge for the Bloch hopfions or near the dot center for Neel hopfions [42]. The mode classification according to their symmetry or number of the nodes in different spatial directions were not presented. It seems that the Bloch hopfion frequency spectra are numerical artifacts and further careful analysis is necessary. The Hopf index of the Neel hopfions calculated using the concept of the emergent electromagnetic is above 0.9. It is unclear why the Neel hopfions transform from circularly symmetric shape (azimuthal symmetry) to square shapes increasing the axial magnetic field. There is no reason to break the azimuthal symmetry of the static magnetization configurations.

The spin excitation modes of the Bloch toroidal hopfions in nanodisks (the radius R is 100 nm and thickness L is 70 nm) were simulated in Ref. [53] in a wide frequency range 0 – 20 GHz. The spin eigenmodes were classified according to their symmetry, number of the nodes in the azimuthal and out-of-plane directions and degree of the mode localization. The dot edge, dot middle and dot center localized modes were identified. No simple rule establishing a connection of the mode localization with its frequency was found. The edge and middle localized modes have large azimuthal indices. Such complicated inhomogeneous mode patterns mean that ferromagnetic resonance intensity (average mode volume magnetization) of the modes is small and it will be difficult to observe these modes by the standard FMR spectroscopy technique. It was found numerically [54] that the toroidal hopfions have distinctly less resonance peaks in comparison with skyrmion tubes. It was also found that the hopfion breathing and rotating spin modes hybridize applying the oscillation magnetic field along the hopfion axis $0z$.

As it is well known, the Hall effect (existence of a gyroforce perpendicular to the soliton velocity) for magnetic topological solitons is attributed to a gyrovector, which, in the case of the magnetic vortices and skyrmions, is not equal to zero and determines their low frequency dynamics both in 3D and 2D case [57,58]. Therefore, the question of the magnitude and direction of gyrovector of the magnetic hopfion is of practical importance. The components of the emergent magnetic field in the cylindrical coordinates, defining the gyrovector $\mathbf{G}$ of the magnetic Hopfion, were presented in the article Ref. [26]. It was proved unambiguously that the gyrovector axial component G_z of an axially symmetric magnetic Hopfion with the Hopf index $|Q_H| = 1$ is equal to zero. As for the zero values of the other two components of the gyrovector, only a plausible assumption was made that they vanish due to the hopfion and system symmetry. Nevertheless, this assumption was accepted on faith in a number of subsequent articles, where the equality to zero of all components of the hopfion gyrovector is mentioned as a proven fact [55,56,59]. Meanwhile, this issue needs to be clarified and proved, since the possible non-zero values of the gyrovector components affect both the three-dimension current-induced hopfion translational motion [60], and the driven by external magnetic field dynamics of spin waves over the hopfion background [55]. Liu et al. [56] excluded the hopfion gyrovector from consideration and considered the hopfion dynamics in terms of the hopfion emergent magnetic field toroidal $\mathbf{T} = 1/2 \int dV(\mathbf{r} \times \mathbf{B})$ and octupole moments. It resulted in a specific hopfion dynamics presented in Ref. [56]. The hopfion ansatz for the particular case $|Q_H| = 1$ in Ref. [60] was initially written incorrectly and then it was corrected in Ref. [59] of the same group.

The calculation approach suggested in Ref. [61] is based on the concept of the emergent magnetic field $\mathbf{B}(\mathbf{r})$ introduced in Sec. 3 and calculation of the field components and their volume averages for the toroidal hopfion spin texture in the appropriate curvilinear coordinate systems. The simplest non-trivial toroidal hopfion with the Hopf index $|Q_H| = 1$ in the cylindrical magnetic dot was considered and the dependencies of the Hopfion gyrovector components on the dot sizes were calculated. It was demonstrated by the analytical calculations that the magnetic Hopfion gyrovector is not equal to zero and does not vanish even in the limit of an infinite sample. Namely, two components of the gyrovector in the curvilinear cylindrical coordinates, G_z and G_φ are non-zero. The out of-plane z -component of the gyrovector (G_z) goes to zero increasing the dot radius, however, the in-plane φ -component (G_φ) remains finite [61]. The calculated components of the Hopfion emergent field and gyrovector can be used for calculations of the topological Hall effect and skyrmion Hall effect of the toroidal magnetic hopfions, respectively. It was recently shown [62] that the toroidal hopfions reveal a Hall motion (skyrmion Hall effect) under the current pulses, while the skyrmionium moves only along the current direction. This is because the gyrovector (2D topological charge) of skyrmionium vanishes, whereas it is not equal to zero for the toroidal hopfions in accordance with Ref. [61]. The fractional Hopf index $Q_H = 0.8$ was calculated in Ref. [62] due to the restricted geometry of FeGe samples. The current pulses were employed to drive the dynamics of the fractional hopfions. An asymmetric Hall motion of the hopfions with respect to the current direction was detected [62].

Space time magnetic hopfions were suggested in Ref. [63]. They are treated as 2D magnetic textures (skyrmions) excited by an oscillating magnetic or electric field. The authors considered the

coupled dynamics of the skyrmion radius and helicity using the model of the exchange interaction quadratic and quartic on the magnetization spatial derivatives. The authors believe that the hopfion topological invariant, the spacetime Hopf index ($Q_H = \pm 1$), can be tuned by the applied electric field. The emergent magnetic field $B_i(\mathbf{m}) = \varepsilon_{ijk} \mathbf{m} \cdot (\partial_j \mathbf{m} \times \partial_k \mathbf{m})$ ($i, j, k = x, y, t$) and gauge vector potential A were introduced using the standard equations. However, these equations are valid only if the indices i, j, k used in their definition mark the spatial coordinates, for instance the Cartesian coordinates (x, y, z) . However, one of the indices i, j, k corresponds to time (t) according to Knapman et al. [63]. If the index $i = t$, then the emergent field component B_t has no physical sense because $\mathbf{B} = (B_x, B_y, B_z)$ is a vector in three-dimensional space (x, y, z) . If the indices $j = t$ or $k = t$, then the vector $\mathbf{B}$ is not the emergent magnetic field, and the equation $\mathbf{B} = \nabla \times \mathbf{A}$ is not valid. In the case $j = t$ or $k = t$, the field $\mathbf{B}$ is an emergent electric field $\mathbf{E}$, proportional to time derivative of the magnetization vector $\mathbf{m}$. The relation of the emergent electric field $\mathbf{E}$ and the vector potential A is essentially different from the equation $\mathbf{B} = \nabla \times \mathbf{A}$ for the emergent magnetic field. Only the space Hopf index defined by Eq. (2) has physical sense. Therefore, the equations defining the space-time Hopf index used by Knapman et al. [63] are wrong and should be corrected.

The hopfions considered by Knapman et al. are a particular case of 3D winding hopfions introduced by Kobayashi et al. [64] in the $R^2 \times S_1$ coordinate space, where the third coordinate S_1 (z) is along the sample thickness, L . Knapman et al. substituted the thickness coordinate (z) to time (t). We use the angular parameterization of the magnetization $\mathbf{m}(\mathbf{r})$ components via spherical angles $\theta(\mathbf{r}), \Phi(\mathbf{r})$. It is naturally to choose the hopfion magnetization spherical angles in axially symmetric form, $\theta(\mathbf{r}) = \theta(\rho, z), \Phi(\mathbf{r}) = Q\varphi + \gamma(\rho, z)$ as it was done for the toroidal hopfions in Refs. [36,52]. If following Ref. [64] one introduces the hopfion helicity as $\gamma(\rho, z) = Pg(z)$, where the phase satisfies to the boundary condition $g(L) - g(0) = 2\pi$ and P is integer, then the calculated Hopf index of the winding hopfion is integer. The calculation yields $Q_H = QP$, where Q is the skyrmion number (2D topological charge). However, in the finite samples Q is not integer and the condition $g(L) - g(0) = 2\pi$, in general, is not satisfied. Therefore, the Hopf index of the winding hopfions can be arbitrary number. The winding hopfions are very different from the toroidal magnetic hopfions, which have almost integer Hopf indices even in finite samples.

Very recently the paper by Saji et al. [65] was published where the authors rewrote the toroidal hopfion magnetization in the spherical coordinates regardless the hopfion has a cylindrical (azimuthal) symmetry. Validity of the ansatz derived in Ref. [64] for the Hopf index $Q_H > 1$ should be proven. It also unclear, why the emergent vector potential calculated by Saji et al. [65] differs from the vector potential presented in Ref. [59] calculated on the base of the field theory [10,11]. In the articles [59,60,65] the toroidal hopfion magnetization, emergent magnetic field and vector potential were presented via an arbitrary smooth function $f(r)$ satisfying the boundary conditions $f(0) = 0, f(\infty) = \pi$ to ensure an integer Hopf index Q_H in infinite sample. Therefore, these important local parameters $\mathbf{m}(\mathbf{r}), \mathbf{B}(\mathbf{r}), \mathbf{A}(\mathbf{r})$ are not completely defined and cannot be used for calculations of any effects related to the toroidal magnetic hopfions. Using the function $f(r)$ is not necessary because the exact explicit expressions for $\mathbf{m}(\mathbf{r}), \mathbf{B}(\mathbf{r}), \mathbf{A}(\mathbf{r})$ of the toroidal magnetic hopfions in infinite media were found in Ref. [36] for any values of the 3D topological charge Q_H .

The authors of Ref. [65] demonstrated that the spin waves excited over the hopfion background experience the emergent electromagnetic field generated by the hopfion magnetization texture $\mathbf{m}(\mathbf{r})$. In particular, it was shown that the spin waves propagating along the hopfion symmetry axis ($0z$) are deflected by the hopfion magnetic texture. This effect using analogy with the skyrmion topological effect was named as "magnonic Hall effect".

6. Summary

I briefly reviewed above the current status of research of 3D magnetic topological solitons (hopfions) in restricted and infinite geometries. The hopfions similarly to the skyrmions were initially introduced in the classical field theory. Then, these complicated 3D textures were calculated and measured in ferromagnetic media. It is shown that the concepts of the gauge emergent vector potential and emergent magnetic field are absolutely necessary to describe the magnetic hopfions.

The central place among the magnetic hopfions occupy the toroidal hopfions with the integer Hopf index found in the field theory. One should be careful using the equations developed for infinite media to calculate the hopfions in finite samples. Except the toroidal hopfions, other hopfions with non-zero and non-integer Hopf index are also possible (the Bloch points and winding hopfions, for instance). Note that the Hopf index (3D topological charge) has no direct physical sense in comparison to 2D topological charge (skyrmion number) because the latter defines the soliton gyrovectore resulting, in particular, in the vortex/skyrmion gyrotropic excitation mode.

The main articles on the toroidal hopfion stability and dynamics in ferromagnetic films and dots are reviewed. Both these topics are considered mainly theoretically (numerically) because it is difficult to observe stable hopfions and their dynamics experimentally. No applications of the magnetic hopfions are realized to the moment. I believe that it can be done in the nearest future.

Some critical discussion of the calculations and simulations of the magnetization, emergent magnetic field and the Hopf index of the toroidal magnetic hopfions and other kinds of magnetic hopfions in restricted geometries is presented. The critical point there is using proper equations for the hopfion emergent magnetic field and vector potential and proof of gauge invariance of the calculated Hopf index.

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