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Article

IS THE PROBABILITY OF TOSSING A COIN REALLY 50-50%?

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Abstract: Considering that a fair coin has two sides and a cylindrical edge, the probability that it would fall on its edge is calculated, yielding the probability of heads or tails less than 50%. Theoretical models for a static case and for five dynamic cases, without and with rebounds, show that there is a small probability that the coin does not fall on its head or tail, depending on initial toss conditions, the coin geometry and conditions of the coin and landing surfaces. For the dynamic model with rebounds, it is found that the probability that a 50 Eurocent coin thrown from a normal height with common initial velocity conditions and appropriate surface conditions will end up on its edge is in the order of one against several thousand.

Keywords: coin toss, probability, impact

1. Introduction

To toss a coin is the simplest of experiments in probability and statistics. Students usually learn that there is a 50-50 % chance that it falls on either side. Repeat this simple experiment with a fair coin a sufficient number of times and the result should show that on average half of the tosses yield heads and half yield tails. But is it really so, even with a fair coin? A coin is a flat cylinder made of some metallic alloy. It has of course two sides, but it has also a small cylindrical surface, the edge.

In this paper, we calculate the probability that a coin would fall on its edge, yielding the probability of heads or tails less than 50%. Several models are considered: a simple static model and five dynamic models, with and without rebounds. Depending on initial toss conditions, the coin geometry and conditions of the coin and landing surfaces, there is a small probability that the coin does not fall on its head or tail.

2. Methods

2.1. Static Model

Let's consider a flat cylindrical coin, perfectly circular and homogeneous with a basis diameter d much larger than its height h. From the centre of mass at the coin mid-height, the height is seen under an angle α. The coin is placed on a perfectly smooth horizontal surface, with the coin edge making an angle θ with the horizontal surface (see Fig. 1). Clearly, the coin will fall on its side if

$$\frac{\alpha}{2} < \theta \leq \frac{\pi}{2} \qquad (1)$$

As $tan(\frac{\alpha}{2}) = \frac{h}{d}$, and if $\frac{h}{d} << 1$, $\frac{\alpha}{2} \approx \frac{h}{d}$, one has

$$\frac{h}{d} < \theta \leq \frac{\pi}{2} \qquad (2)$$

Therefore, the probability that the coin will fall on a side can be estimated as

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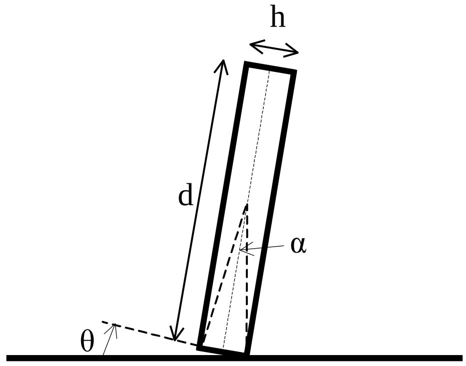

Figure 1: Static model of a coin on its edge.

$$P_{\text{side}} = \left(\frac{\frac{\pi}{2}-\frac{\alpha}{2}}{\frac{\pi}{2}}\right) \approx \left(\frac{\frac{\pi}{2}-\frac{h}{d}}{\frac{\pi}{2}}\right) = 1 - \frac{2h}{\pi d} \quad (3)$$

and the probability that it will stay on its edge is

$$P_{\text{edge}} = 1 - P_{\text{side}} = \left(\frac{\frac{\alpha}{2}}{\frac{\pi}{2}}\right) \approx \frac{2h}{\pi d} \quad (4)$$

Another way to see it is to calculate the probability P_{edge} as the ratio of all values of θ yielding the coin to stay on its edge in the range θ_{min} to θ_{max} to all possible values of θ in the range $-\pi/2$ to $\pi/2$, yielding

$$P_{edge} = \left(\frac{\theta_{max}-\theta_{min}}{\frac{\pi}{2}-\left(-\frac{\pi}{2}\right)}\right) = \left(\frac{\frac{\alpha}{2}-\left(-\frac{\alpha}{2}\right)}{\pi}\right) = \frac{\alpha}{\pi} \approx \frac{2h}{\pi d} \quad (5)$$

For a 50 Eurocent coin, $d \approx 23.5$ mm, $h \approx 2$ mm, yielding $(h/d) \approx 0.085$, which gives a probability P_{edge} in the order of 5%. This means that approximately one out of twenty tosses should end with the coin on its edge, which is clearly too much.

This static model is obviously not adequate to describe a real situation and must be refined. It shows however that the probability of a coin falling on its edge is not nil.

2.2. Dynamic Model

2.2.1 General model and equations

In addition to the hypothesis of a fair coin, i.e., a homogeneous flat circular cylinder of mass m, one considers the following hypotheses (with bold characters denoting vectors):

- the coin is thrown manually from an initial height H with a velocity $\boldsymbol{v_0}$ under an angle β on the horizontal, and an initial angular velocity $\boldsymbol{\omega_0}$; for a manual throw, minimum and maximum possible values are considered to be:
  $$d < H < 2\text{ m}\,;\; 0 < v_0 < 5\text{m/s};\; -\frac{\pi}{2} < \beta < \pi/2\,;\; 0 < \omega_0 < 10\pi \text{ rad/s} \quad (6)$$
  where v_0 and ω_0 are the norms of the vectors $\boldsymbol{v_0}$ and $\boldsymbol{\omega_0}$;
- the coin rotation axis is horizontal and passes through the coin centre of mass at all times during the fall until impacting the landing surface;
- the coin angular velocity after impact is along a (yet) undefined instantaneous axis of rotation that stays horizontal at all times;
- the atmosphere is windless, without any disturbance and the air friction is negligible;
- the landing surface is a perfectly horizontal, plane, solid and immovable surface;

- in a first approach, one considers that there is no rebound of the coin; the rebound case is then addressed at the end.

One considers further a referential frame with its origin at the impact point on the landing surface, its Z-axis perpendicular to the landing surface and directed downward, its X and Y axes in the horizontal plane of the landing surface with X pointing in the direction of the throw (see Fig. 2).

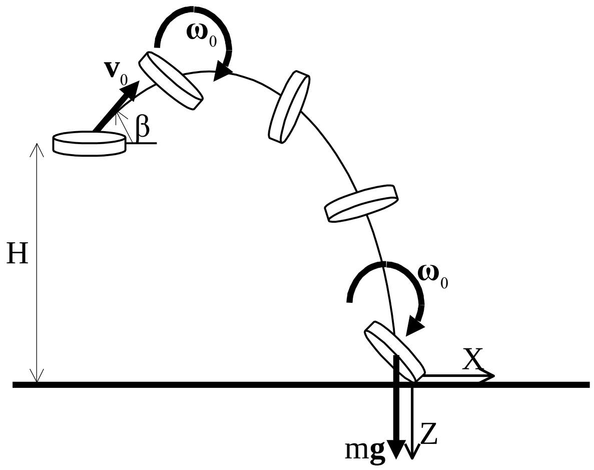

Figure 2: Dynamic model of a coin throw with a clockwise initial rotation.

The velocity vectors of the coin centre of mass before and after impact are noted respectively $\boldsymbol{v}$ and $\boldsymbol{u}$, and the coin angular velocity vectors before and after impact are noted respectively $\boldsymbol{\omega_b}$ and $\boldsymbol{\omega_a}$. Since the landing surface is immovable and with $\otimes$ denoting the vector product, the equations of impact are simply [1] :

$$m\,\boldsymbol{u} \;=\; m\,\boldsymbol{v} \,+\, \boldsymbol{R} \tag{7}$$

$$I_a\,\boldsymbol{\omega_a} \;= I_b\,\boldsymbol{\omega_b} \,+\, \boldsymbol{\rho} \otimes \boldsymbol{R} \tag{8}$$

where $\boldsymbol{R}$ is the impulse of the impact, having components N and T respectively normal to and along the surface at the point of contact; I_a and I_b are the moments of inertia of m at the coin centre of mass with respect to the instantaneous axis of rotation immediately after and before impact; and $\boldsymbol{\rho}$ is the distance vector from the impact point to the coin centre of mass.

The components of the coin velocities before impact are obviously

$$\boldsymbol{v} = \left(v_0 \cos\beta, 0, \sqrt{v_0^2 \sin^2\beta + 2gH}\right) \tag{9}$$

$$\boldsymbol{\omega_b} \;=\; (0\,,(\pm\omega_0), 0) \tag{10}$$

since the initial rotation can be in either direction. The coin moment of inertia before impact is

$$I_b \;=\; \frac{m}{4}\left(\frac{h^2}{3} + \frac{d^2}{4}\right) \tag{11}$$

with respect to the instantaneous horizontal axis of rotation passing through the coin centre of mass.

The components of the distance vector are (see Fig. 3)

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$$\boldsymbol{\rho} = \left(\rho \sin(\theta - \tfrac{\alpha}{2}), 0, \rho \cos(\theta - \tfrac{\alpha}{2})\right) \tag{12}$$

with $\rho = \frac{\sqrt{h^2+d^2}}{2}$ and where θ is the angle from the downward vertical to the mid-plane parallel to the coin sides counted positively in the counter-clockwise direction.

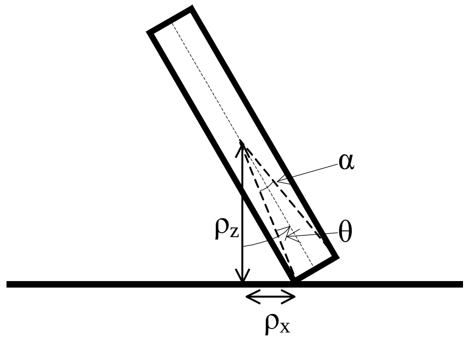

Figure 3: Angles and distances at the moment of impact.

Note that to be exact, the height H from which the coin falls should be less, as the coin's centre of mass never touches the landing surface. The height H should be replaced by $(H - \rho\cos(\theta - \frac{\alpha}{2}))$, but as $\rho\cos(\theta - \frac{\alpha}{2})$ is much smaller than H for any value of θ at impact, one can simplify the problem by considering H only.

With the hypothesis that the angular velocity after impact is along a horizontal instantaneous axis of rotation, one has also

$$\boldsymbol{\omega_a} = (0\,, (\pm\omega_a), 0) \tag{13}$$

The equations (7) and (8) reduce then to:

along X: $m\,u_x = m\,v_0 \cos\beta - T$ (14a)

along Y: $I_a\,(\pm\omega_a) = I_b\,(\pm\omega_0) - N\,\rho_x - T\,\rho_z$ (14b)

along Z: $m\,u_z = m\sqrt{v_0^2 \sin^2\beta + 2gH} + N$ (14c)

An additional condition is given by the nature of the impacting bodies and their surfaces. Four cases can be classically considered:

1. both bodies are inelastic and perfectly rough,
2. both bodies are inelastic and perfectly smooth,
3. both bodies are elastic and perfectly smooth,
4. both bodies are elastic and partially rough.

2.2.2 Inelastic and perfectly rough bodies

In the first case, if the bodies are inelastic and perfectly rough, the particles of the coin and of the landing surface do not separate after collision and their relative velocity is nil, giving the condition

$$\boldsymbol{u} + \boldsymbol{\omega_a} \otimes \boldsymbol{\rho} = \boldsymbol{0} \tag{15}$$

yielding

along X: $u_x = -(\pm\omega_a)\,\rho_z$ (16a)

along Z: $u_z = (\pm\omega_a)\,\rho_x$ (16b) which, with (14a) and (14c), allow to find the tangential and normal components of the impulse at impact

$$T = m\,(v_0\cos\beta + (\pm\omega_a)\,\rho_z) \quad (17a)$$

$$N = m\,((\pm\omega_a)\,\rho_x - \sqrt{v_0^2\sin^2\beta + 2gH}) \quad (17b)$$

Replacing in (14b) yields the angular velocity after impact

$$(\pm\omega_a) = \frac{\kappa(\pm\omega_0) - v_0\cos\beta\cos(\theta - \frac{\alpha}{2}) + \sqrt{v_0^2\sin^2\beta + 2gH}\,\sin(\theta - \frac{\alpha}{2})}{\kappa_a + \rho} \quad (18)$$

where

$$\kappa = \kappa_b = \frac{I_b}{\rho m} = \frac{\left(\frac{h^2}{3} + \frac{d^2}{4}\right)}{2\sqrt{h^2 + d^2}} \quad \text{and} \quad \kappa_a = \frac{I_a}{\rho m} \quad (19)$$

For the coin to stay on its edge, the angular velocity after impact ω_a must become nil while $|\theta| < \frac{\alpha}{2}$. Solving the equation (18), $\omega_a = 0$ for the variable θ yields

$$\theta = \frac{\alpha}{2} - 2\arctan\left(\frac{\sqrt{v_0^2\sin^2\beta + 2gH} \pm \sqrt{v_0^2 + 2gH - \kappa^2\omega_0^2}}{v_0\cos\beta + \kappa(\pm\omega_0)}\right) \quad (20a)$$

or

$$\theta = \frac{\alpha}{2} - 2\arctan\left(\frac{1 \pm \sqrt{1 - \left(\frac{\kappa(\pm\omega_0)}{\sqrt{v_0^2\sin^2\beta + 2gH}}\right)^2 + \left(\frac{v_0\cos\beta}{\sqrt{v_0^2\sin^2\beta + 2gH}}\right)^2}}{\frac{\kappa(\pm\omega_0) + v_0\cos\beta}{\sqrt{v_0^2\sin^2\beta + 2gH}}}\right) \quad (20b)$$

with the resolution condition

$$v_0^2 + 2gH \geq \kappa^2\omega_0^2 \quad (21)$$

The condition $|\theta| < \frac{\alpha}{2}$ yields successively

$$-\frac{\alpha}{2} < \theta < \frac{\alpha}{2} \quad (22a)$$

$$0 < \left(\frac{\sqrt{v_0^2\sin^2\beta + 2gH} \pm \sqrt{v_0^2 + 2gH - \kappa^2\omega_0^2}}{v_0\cos\beta + \kappa(\pm\omega_0)}\right) < \tan\left(\frac{\alpha}{2}\right) = \frac{h}{d} \quad (22b)$$

meaning that the argument of the arctan function in (20) must be positive and smaller than h/d.

For the positive condition, the numerator and the denominator must be of the same sign. Considering the negative sign in front of the second root in the numerator of (20a):

- for both numerator and denominator to be positive, the first root must be greater than the second, yielding

  $$v_0^2\cos^2\beta < \kappa^2\omega_0^2 \quad (23)$$

  and the denominator must be such that

  $$v_0\cos\beta + \kappa(\pm\omega_0) > 0 \quad (24)$$

  which is true only for a positive sign in front of ω_0, i.e., an initial coin counterclockwise rotation, as for a negative sign in front of ω_0, i.e., an initial coin clockwise rotation, one must have in addition

  $$v_0\cos\beta > \kappa\omega_0 \quad (25)$$

  which contradicts (23);
- for both numerator and denominator to be negative, the first root must be smaller than the second, yielding

  $$v_0^2\cos^2\beta > \kappa^2\omega_0^2 \quad (26)$$

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and the denominator must be such that

$$v_0 \cos\beta + \kappa(\pm\omega_0) < 0 \tag{27}$$

which is only possible for a negative sign in front of ω_0, i.e., an initial coin clockwise rotation, and for

$$v_0 \cos\beta < \kappa\omega_0 \tag{28}$$

which contradicts (26).

Considering the positive sign in front of the second root in the numerator of (20a), the denominator must be positive which is true either for a positive sign in front of ω_0, i.e., an initial coin counterclockwise rotation, or for a negative sign in front of ω_0, i.e., an initial coin clockwise rotation, in addition to

$$v_0 \cos\beta > \kappa\omega_0 \tag{29}$$

However, the solution, in this case, yields a too large value of θ.

In conclusion for this first case, the only possible solution is given by an initial coin counterclockwise rotation with the condition (23), which coupled with (21), yields

$$0 < \frac{\sqrt{\kappa^2\omega_0^2 - v_0^2\cos^2\beta}}{\sqrt{v_0^2\sin^2\beta + 2gH}} \leq 1 \tag{30}$$

For the right part of the condition in (19b), considering a positive sign in front of ω_0, it yields

$$\left|\frac{\kappa\omega_0 - \left(\frac{d^2-h^2}{d^2+h^2}\right)v_0\cos\beta}{\sqrt{v_0^2\sin^2\beta + 2gH}}\right| < \frac{2hd}{h^2+d^2} \tag{31}$$

where vertical bars denote the absolute value. These two conditions constrain the vertical velocity $\sqrt{v_0^2\sin^2\beta + 2gH}$ of the falling coin.

Within the above conditions, the maximum and minimum values of θ are

$$\theta_{max} = \frac{\alpha}{2} - \arctan\left(\frac{\sqrt{v_0^2\sin^2\beta + 2gH} - \sqrt{v_0^2 + 2gH - \kappa^2\omega_0^2}}{v_0\cos\beta + \kappa\omega_0}\right)_{min} \tag{32a}$$

$$\theta_{min} = \frac{\alpha}{2} - \arctan\left(\frac{\sqrt{v_0^2\sin^2\beta + 2gH} - \sqrt{v_0^2 + 2gH - \kappa^2\omega_0^2}}{v_0\cos\beta + \kappa\omega_0}\right)_{max} \tag{32b}$$

The probability that the coin will end on its edge is then

$$P_{edge} = \left(\frac{\theta_{max} - \theta_{min}}{\pi}\right) = \frac{2}{\pi}\left(\arctan\left(\frac{\sqrt{v_0^2\sin^2\beta + 2gH} - \sqrt{v_0^2 + 2gH - \kappa^2\omega_0^2}}{v_0\cos\beta + \kappa\omega_0}\right)_{max} - \arctan\left(\frac{\sqrt{v_0^2\sin^2\beta + 2gH} - \sqrt{v_0^2 + 2gH - \kappa^2\omega_0^2}}{v_0\cos\beta + \kappa\omega_0}\right)_{min}\right) \tag{33}$$

The maximum value of this probability will occur for the largest value of θ_{max} and the smallest value of θ_{min}. Considering that θ is bound by $\alpha/2$ and $-\alpha/2$ (22a), $\theta_{max} = \alpha/2$ and $\theta_{min} = -\alpha/2$, which would yield the same value of the maximum probability as for the static case (5). However, one has still to verify that the values of θ_{max} and θ_{min} can be achieved within the ranges (6) of the four initial parameters H, v_0, β and ω_0.

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For $\theta_{max} = \alpha/2$ to occur, the argument of the arctan function in (32a) must be nil, i.e., for the limiting case where $\omega_{0\,min} = 0$ and either $v_{0\,min} = 0$ and $-\pi/2 \le \beta \le \pi/2$, or $\beta = \pm\pi/2$ and v_0 taking any value, with H taking any value.

For $\theta_{min} = -\alpha/2$, the arctan function in (32b) must be equal to or larger than $-\alpha/2$, which in most cases is not possible within the conditions (30) and (31). In fact, the argument of the arctan function in θ_{min} (32b) is maximum for $\omega_{0\,max}$, $v_{0\,min}$ and H_{min}, with β taking any value between $-\pi/2$ and $\pi/2$, yielding

$$\theta_{min} = \frac{\alpha}{2} - \arctan\left(\frac{\sqrt{v_{0\,min}^2 \sin^2\beta + 2gH_{min}} - \sqrt{v_{0\,min}^2 + 2gH_{min} - \kappa^2\omega_{0\,max}^2}}{v_{0\,min}\cos\beta + \kappa\omega_{0\,max}}\right)_{max} \qquad (33)$$

yielding for the ranges of values (6) with $v_{0\,min} = 0$ and $H_{min} = d$,

$$\theta_{min} = \frac{\alpha}{2} - \arctan\left(\frac{\sqrt{2gd}}{\kappa\omega_{0max}}\left(1 - \sqrt{1 - \frac{\kappa^2\omega_{0max}^2}{2gd}}\right)\right) \qquad (34)$$

The maximum probability that the falling coin stays on its edge at impact becomes then slightly less than in the static case, namely

$$P_{edge\,max} = \left(\frac{\theta_{max} - \theta_{min}}{\pi}\right)_{max} = \frac{2}{\pi}\arctan\left(\frac{\sqrt{2gd}}{\kappa\omega_{0max}}\left(1 - \sqrt{1 - \frac{\kappa^2\omega_{0max}^2}{2gd}}\right)\right) \qquad (35)$$

which, for the values given in (6) for a 50 Eurocent coin, yields a probability $P_{edge\,max}$ = 0.04366 or nearly one throw every 23 that ends with the coin on its edge.

For more common and practical values, let's consider a series of vertical throws up or down (i.e., $\beta = \pm\pi/2$) from a height $H = 1.5$ m with the coin initial velocities ranging from 1 to 5 m/s and initial rotation ranging from 0.5 to 5 turns/s. The probability that the coin will end on its edge is then $P_{edge} = 4.96\ 10^{-3}$ or approximately one throw every 202. For another series of throws with $\beta = \pi/4$, $H = 1.5$ m and the coin initial velocities ranging from 0.01 to 0.1 m/s and initial rotation ranging from 0.5 to 5 turns/s, $P_{edge} = 1.17\ 10^{-3}$ or approximately one throw every 856.

2.2.3 Inelastic and perfectly smooth bodies

In the second case, if the bodies are inelastic and perfectly smooth, at the moment of impact, the X and Y components of the relative velocity of the coin contact point with respect to the landing surface are unaltered, while the Z component is reduced to zero [1], giving the condition (with • denoting the scalar product):

$$(\boldsymbol{u} - \boldsymbol{v}) - (\boldsymbol{u} - \boldsymbol{v}) \bullet \boldsymbol{z} = 0 \qquad (36)$$

$$(\boldsymbol{u} + \boldsymbol{\omega_a} \otimes \boldsymbol{\rho}) \bullet \boldsymbol{z} = 0 \qquad (37)$$

where $\boldsymbol{z}$ is the unit vector following the Z-axis, $\boldsymbol{z} = (0, 0, 1)$, yielding

along X: $u_x = v_x = v_0\cos\beta$ (38a)

along Z: $u_z = (\pm\omega_a)\,\rho_x = (\pm\omega_a)\rho\sin(\theta - \frac{\alpha}{2})$ (38b)

Replacing these last two relations in (14a) and (14c) yield

$$T = 0 \qquad (39a)$$

$$N = m\,((\pm\omega_a)\rho\sin(\theta - \tfrac{\alpha}{2}) - \sqrt{v_0^2\sin^2\beta + 2gH}) \qquad (39b)$$

indicating that the impulse lies only in the normal direction to the surface. Replacing in (14b) yields

$$(\pm\omega_a) = \frac{\kappa(\pm\omega_0)+\sqrt{v_0^2 \sin^2\beta+2gH}\sin(\theta-\frac{\alpha}{2})}{\kappa_a+\rho\sin^2(\theta-\frac{\alpha}{2})} \tag{40}$$

For the coin to stay on its edge, the angular velocity after impact ω_a must become nil while $|\theta| < \alpha/2$. Solving the equation (40) $\omega_a = 0$ for the variable θ yields

$$\theta = \frac{\alpha}{2} - \arcsin\left(\frac{\kappa(\pm\omega_0)}{\sqrt{v_0^2\sin^2\beta+2gH}}\right) \tag{41}$$

under the condition that the denominator in (40) is different from zero, which is always the case as all terms are positive. For the condition $|\theta| < \frac{\alpha}{2}$ to hold, the argument of the arcsin function in (41) must be first, positive which is the case only for a positive sign in front of ω_0, i.e., an initial coin counterclockwise rotation, and second, smaller than $sin\ \alpha$, yielding for a positive sign in front of ω_0

$$\frac{\kappa\omega_0}{\sqrt{v_0^2\sin^2\beta+2gH}} < \left(\frac{2hd}{h^2+d^2}\right) \tag{42}$$

If this condition holds, the probability that the coin will end on its edge is then

$$P_{edge} = \left(\frac{\theta_{max}-\theta_{min}}{\pi}\right) =$$

$$\frac{1}{\pi}\left(\arcsin\left(\frac{\kappa\omega_0}{\sqrt{v_0^2\, sin^2\beta+2gH}}\right)_{max} - \arcsin\left(\frac{\kappa\omega_0}{\sqrt{v_0^2\, sin^2\beta+2gH}}\right)_{min}\right) \tag{43}$$

The maximum probability will occur for $\theta_{max} = \alpha/2$ for the limiting case where $\omega_{0\,min}$ = 0 and v_0, H and β taking any value $(-\pi/2 \le \beta \le \pi/2)$, and for the argument of the arcsin function in θ_{min} being maximum, i.e., for $\omega_{0\,max}$, H_{min} and either $v_{0\,min}$ and $-\pi/2 \le \beta \le \pi/2$, or $\beta = 0$ and v_0 taking any value, yielding

$$P_{edge\,max} = \left(\frac{\theta_{max}-\theta_{min}}{\pi}\right)_{max} = \frac{1}{\pi} arcsin\left(\frac{\kappa\omega_{0\,max}}{\sqrt{2gH_{min}}}\right) \tag{44}$$

which, for the values given in (6) for a 50 Eurocent coin, yields a probability $P_{edge\,max}$ = 0.04366 or nearly one throw every 23 that ends with the coin on its edge, like in the previous case.

Throwing the coin vertically, upward or downward, i.e., $\beta = \pm\pi/2$, from an initial height H = 1.5 m with the coin initial velocities ranging from 1 to 5 m/s and initial rotation ranging from 0.5 to 5 turns/s yields the same result as in the first case above. For a series of throws with initial conditions $\beta = \pi/4$, H = 1.5 m and v_0 ranging from 0.1 to 5 m/s and ω_0 varying between 0.5 and 5 turn/s, P_{edge} = 4.99 10^{-3} or approximately one throw every 200 will deliver the coin on its edge.

2.2.4 Elastic and perfectly smooth bodies

In the third case, if the bodies are elastic and perfectly smooth, at the moment of greatest compression at impact, the Z component of the relative velocity of the coin contact point with respect to the landing surface is reduced to zero, while the X and Y

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components are unaltered. The magnitude of the normal impulse can be computed like in the second case, multiplying it by $(1+e)$ where e is the coefficient of restitution.

This yields from (39b)

$$N = m(1+e)\left((\pm\omega_a)\rho\sin\left(\theta-\frac{\alpha}{2}\right) - \sqrt{v_0^2\sin^2\beta+2gH}\right) \quad (45)$$

with T still being nil. Replacing in (14b) yields

$$(\pm\omega_a) = \frac{\kappa(\pm\omega_0)+(1+e)\sqrt{v_0^2\sin^2\beta+2gH}\sin\left(\theta-\frac{\alpha}{2}\right)}{\kappa_a+(1+e)\rho\sin^2\left(\theta-\frac{\alpha}{2}\right)} \quad (46)$$

Like above, for the coin to stay on its edge, the angular velocity after impact ω_a must become nil while $|\theta| < \frac{\alpha}{2}$. Solving the equation (46) $\omega_a = 0$ for the variable θ yields

$$\theta = \frac{\alpha}{2} - \arcsin\left(\frac{\kappa\omega_0}{(1+e)\sqrt{v_0^2\sin^2\beta+2gH}}\right) \quad (47)$$

where the positive sign in front of ω_0, i.e., an initial coin counterclockwise rotation, was chosen to guarantee that the first part of the condition $|\theta| < \frac{\alpha}{2}$ holds, while the second part yields

$$\frac{\kappa\omega_0}{\sqrt{v_0^2\sin^2\beta+2gH}} < (1+e)\left(\frac{2hd}{h^2+d^2}\right) \quad (48)$$

Under this condition, the probability that the coin will end on its edge is then

$$P_{edge} = \frac{1}{\pi}\left(\arcsin\left(\frac{\kappa\omega_0}{(1+e)\sqrt{v_0^2\sin^2\beta+2gH}}\right)_{max} - \arcsin\left(\frac{\kappa\omega_0}{(1+e)\sqrt{v_0^2\sin^2\beta+2gH}}\right)_{min}\right) \quad (49)$$

Like in the previous case, the maximum probability will occur for $\theta_{max} = \alpha/2$ and for the maximum value of the argument of the arcsin function in θ_{min}, yielding

$$P_{edge\ max} = \left(\frac{\theta_{max}-\theta_{min}}{\pi}\right)_{max} = \frac{1}{\pi}\arcsin\left(\frac{\kappa\omega_{0\ max}}{(1+e)\sqrt{2gH_{min}}}\right) \quad (50)$$

The coefficient of restitution e depends obviously on the nature of the surface of both bodies and can take values [2] from 0.2 for lead on lead to 0.95 for glass on glass, with approximately 0.55 for steel on steel, yielding values of $(1+e)$ ranging approximately from 1.2 to slightly less than 2.

Assuming a coefficient of restitution $e = 0.5$, i.e., both coin and landing surface being made of steel alloy, for a 50 Eurocent coin and the values given in (6), (50) yields a probability $P_{edge\ max} = 0.02905$ or nearly one throw every 34 that ends with the coin on its edge.

Throwing the coin vertically, upward or downward, i.e., $\beta = \pm\pi/2$, from an initial height $H = 1.5$ m with the coin initial velocities ranging from 1 to 5 m/s and initial rotation ranging from 0.5 to 5 turns/s yields $P_{edge} = 3.304\ 10^{-3}$ or approximately one throw every 303 will end with the coin on its edge, i.e., less than in the first two cases.

For a series of throws of a 50 Eurocent coin with initial conditions $\beta = \pi/4$, $H = 1.5$ m and v_0 ranging from 0.1 to 5 m/s and ω_0 varying between 0.5 and 5 turns/s, $P_{edge} = 3.33\ 10^{-3}$ or approximately one throw every 300 will deliver the coin on its edge.

2.2.5 Elastic and partially rough bodies

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Contrary to the first three idealised cases, this fourth case depicts reality from closer as the partially rough character of both surfaces in contact will cause slippage to occur. One assumes for simplicity that first, the slip is always in the same direction, second, the frictional impulse has a magnitude μN, where μ is the coefficient of friction, and a direction opposite to the relative motion of the point of contact on the landing surface. The normal impulse is still calculated as in the third case.

This yields from (45)

$$N = m\,(1+e)\left((\pm\omega_a)\rho\sin\left(\theta-\frac{\alpha}{2}\right) - \sqrt{v_0^2\sin^2\beta + 2gH}\right) \tag{51a}$$

$$T = m\,\mu\,(1+e)\left((\pm\omega_a)\rho\sin\left(\theta-\frac{\alpha}{2}\right) - \sqrt{v_0^2\sin^2\beta + 2gH}\right) \tag{51b}$$

Replacing in (14b) yields

$$(\pm\omega_a) = \frac{\kappa(\pm\omega_0)+(1+e)\sqrt{v_0^2\sin^2\beta+2gH}\left(\mu\cos\left(\theta-\frac{\alpha}{2}\right)+\sin\left(\theta-\frac{\alpha}{2}\right)\right)}{\kappa_a+(1+e)\rho\sin\left(\theta-\frac{\alpha}{2}\right)\left(\mu\cos\left(\theta-\frac{\alpha}{2}\right)+\sin\left(\theta-\frac{\alpha}{2}\right)\right)} \tag{52}$$

Like above, for the coin to stay on its edge, the angular velocity after impact ω_a must be nil while $|\theta| < \frac{\alpha}{2}$. Solving the equation (52) $\omega_a = 0$ for the variable θ yields

$$\theta = \frac{\alpha}{2} - 2\arctan\left(\frac{1\pm\sqrt{1+\mu^2-\left(\frac{\kappa(\pm\omega_0)}{(1+e)\sqrt{v_0^2\sin^2\beta+2gH}}\right)^2}}{\frac{\kappa(\pm\omega_0)}{(1+e)\sqrt{v_0^2\sin^2\beta+2gH}}-\mu}\right) \tag{53}$$

under the condition that the denominator of (52) is different from zero, i.e., for

$$\theta \neq \frac{\alpha}{2} - \arctan\left(\frac{\mu(1+e)\pm\sqrt{\mu^2(1+e)^2-4\frac{\kappa_a}{\rho}\left(\frac{\kappa_a}{\rho}+1\right)}}{2\left((1+e)+\frac{\kappa_a}{\rho}\right)}\right) \tag{54}$$

To verify condition (54), one must evaluate in κ_a (19) the coin moment of inertia I_a after impact, which depends on the position of the instantaneous rotation axis after impact. Under the hypothesis that the coin rotation axis after impact stays horizontal along the Y direction, its position can be assumed between two extreme cases. The first corresponds to the coin rotation axis after impact being identical to the one before impact, namely along the horizontal axis passing through the coin centre of mass, i.e., $I_a = I_b$ and $\kappa_a = \kappa$. The second case corresponds to the coin rotation horizontal axis after impact passing through the coin contact point with the landing surface, yielding

$$\kappa_a = \frac{I_a}{\rho m} = \frac{\left(\frac{4h^2}{3}+\frac{5d^2}{4}\right)}{2\sqrt{h^2+d^2}} \tag{55}$$

These two extreme cases can be extended to other positions of the instantaneous axis of rotation, e.g., outside the coin at a distance from the coin centre several times the diameter of the coin, which would increase the values of the parameters k_1 and k_2. For the sake of the argument, we limit ourselves to the two positions indicated.

The condition (54) then yields

$$\theta \neq \frac{\alpha}{2} - \arctan\left(\frac{\mu(1+e)\pm\sqrt{\mu^2(1+e)^2-4\left(\frac{k_1h^2+k_2d^2}{h^2+d^2}\right)\left(\frac{(1+k_1)h^2+(1+k_2)d^2}{h^2+d^2}\right)}}{2\left((1+e)+\frac{k_1h^2+k_2d^2}{h^2+d^2}\right)}\right) \tag{56}$$

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where $k_1$and k_2 take values respectively k_1= 1/3 and k_2= 1/4 in the first case ($I_a = I_b$) and k_1= 4/3 and k_2= 5/4 in the second case (55).

For θ to be real in (53), the condition for the radical in the numerator reads

$$\frac{\kappa(\pm\omega_0)}{\sqrt{v_0^2 \sin^2\beta + 2gH}} \le (1+e)\sqrt{1+\mu^2} \tag{57}$$

For the condition $|\theta| < \frac{\alpha}{2}$ to hold, the argument of the arctan function in (53) must be positive and smaller than h/d. For the positive condition, the numerator and the denominator must be of the same sign. Considering the negative sign in front of the root in the numerator of (53):

- both numerator and denominator are positive if

$$(1+e)\mu < \frac{\kappa(\pm\omega_0)}{\sqrt{v_0^2 \sin^2\beta + 2gH}} \tag{58}$$

that combined with (57) yields

$$(1+e)\mu < \frac{\kappa(\pm\omega_0)}{\sqrt{v_0^2 \sin^2\beta + 2gH}} \le (1+e)\sqrt{1+\mu^2} \tag{59}$$

which is true only for a positive sign in front of ω_0, i.e., an initial coin counterclockwise rotation;

- both numerator and denominator are negative if

$$\frac{\kappa(\pm\omega_0)}{\sqrt{v_0^2 \sin^2\beta + 2gH}} < (1+e)\mu \tag{60}$$

which includes condition (57) and is true either for a positive sign in front of ω_0, i.e., an initial coin counterclockwise rotation, and with values of ω_0 complying with (60), or for all values of ω_0 for a negative sign, i.e., an initial coin clockwise rotation.

For the positive sign in front of the root in the numerator of (53), the denominator must be positive which is true if condition (58) holds, meaning that the positive sign in front of ω_0 must be chosen, i.e., an initial coin counterclockwise rotation. However, the solution, in this case, yields a too large value of θ.

The other part of the condition, i.e., the argument of the arctan in (53) smaller than h/d like in (22b), yields

- if $\mu < \frac{h}{d}$: $$(1+e)\mu < \frac{\kappa\omega_0}{\sqrt{v_0^2 \sin^2\beta + 2gH}} < (1+e)\left(\frac{2hd-\mu(d^2-h^2)}{h^2+d^2}\right) \tag{61a}$$

- if $\mu = \frac{h}{d}$: $$\frac{\kappa\omega_0}{\sqrt{v_0^2 \sin^2\beta + 2gH}} = (1+e)\frac{h}{d} \tag{61b}$$

- if $\mu > \frac{h}{d}$: $$(1+e)\left(\frac{2hd-\mu(d^2-h^2)}{h^2+d^2}\right) < \frac{\kappa\omega_0}{\sqrt{v_0^2 \sin^2\beta + 2gH}} < (1+e)\mu \tag{61c}$$

These three conditions include condition (57) as the right part of (57) is always greater than the right parts of (61a) to (61c).

Summarizing the conditions for this case:

- if $\mu < \frac{h}{d}$, then (61a) includes (59) and ω_0 must be positive, i.e., an initial coin counterclockwise rotation;

- if $\mu = \frac{h}{d}$, then (61b) is the limiting case of (58) and (60), and ω_0 is positive, i.e., an initial coin counterclockwise rotation;

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- if $\frac{h}{d} < \mu < \frac{2hd}{d^2-h^2}$, then (61c) includes (60) and ω_0 is positive, i.e., an initial coin counterclockwise rotation;

- if $\mu > \frac{2hd}{d^2-h^2}$, then (61c) includes (60) and ω_0 can be either positive or negative, i.e., an initial coin counterclockwise rotation or clockwise rotation.

Discussing the probabilities like in the first case and depending on the above conditions, the probability that the coin will end on its edge is

$$P_{\text{edge}} = \frac{2}{\pi}\left(\arctan\left(\frac{1-\sqrt{1+\mu^2-\left(\frac{\kappa(\pm\omega_0)}{(1+e)\sqrt{v_0^2\sin^2\beta+2gH}}\right)^2}}{\frac{\kappa(\pm\omega_0)}{(1+e)\sqrt{v_0^2\sin^2\beta+2gH}}-\mu}\right)_{max} - \arctan\left(\frac{1-\sqrt{1+\mu^2-\left(\frac{\kappa(\pm\omega_0)}{(1+e)\sqrt{v_0^2\sin^2\beta+2gH}}\right)^2}}{\frac{\kappa(\pm\omega_0)}{(1+e)\sqrt{v_0^2\sin^2\beta+2gH}}-\mu}\right)_{min}\right) \quad (62)$$

The maximum probability will occur first, for $\theta_{max} = \alpha/2$ for the limiting case where

$$\frac{\kappa(\pm\omega_0)}{\sqrt{v_0^2\sin^2\beta+2gH}} \approx (1+e)\mu \quad (63)$$

where the approximate equal sign is taken to recall that the exact equality is not allowed by conditions (58) or (60), and second, for the argument of the arctan function in θ_{min} being maximum, i.e., for $\omega_{0\,max}$, H_{min} and either $v_{0\,min}$ and $-\pi/2 \le \beta \le \pi/2$, or $\beta = 0$ and v_0 taking any value, yielding

$$P_{edge\;max} = \left(\frac{\theta_{max}-\theta_{min}}{\pi}\right)_{max} = \frac{2}{\pi}\arctan\left(\frac{1-\sqrt{1+\mu^2-\left(\frac{\kappa(\pm\omega_{0\,max})}{(1+e)\sqrt{2gH_{min}}}\right)^2}}{\frac{\kappa(\pm\omega_{0\,max})}{(1+e)\sqrt{2gH_{min}}}-\mu}\right) \quad (64)$$

The coefficient of friction μ depends on the nature of the surface of both bodies and can take values from 0.05 to 0.2 for steel on steel depending on whether the surfaces are wet or dry, or up to 0.6 for steel on wood.

Assuming, for a 50 Eurocent coin, coefficients of restitution e = 0.5 and of friction μ = 0.05, i.e., both coin and landing surface being made of steel alloy with wet surfaces, the case $\mu < \frac{h}{d}$ applies and with the values (6), (64) yields a probability $P_{edge\;max}$ = 0.04492 or nearly one throw every 22 that ends with the coin on its edge.

For more common values, the condition (61a) restricts drastically the choice of values of the coin's initial rotation velocities with respect to the coin's vertical velocity at impact. For the above values of coefficients e and μ, the condition (61a) yields

$$0.075 < \left(\frac{\kappa\omega_0}{\sqrt{v_0^2\sin^2\beta+2gH}}\right) < 0.1796 \quad (65)$$

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with κ = 2.955 10^{-3}m, which leaves a relatively limited range of values for the four initial parameters. This shows that a coin ending on its edge is a very rare event in reality.

Nevertheless, if one extends the ranges (6) of allowed initial conditions, throwing either the coin horizontally ($\beta = 0$) with any initial velocity v_0 or vertically ($\beta = -\pi/2$) with a nil initial velocity, from initial heights between 5 and 10 cm and with initial rotation velocities between 6 and 12 turns/s yield a probability P_{edge} = 9.916 10^{-3} or a throw every approximately 100 that would end with the coin on its edge.

From a normal initial height H = 1.5 m, tossing the coin upward at 45° ($\beta = \pi/4$) with an initial rotation of 33 turns/s with initial velocities ranging from 0.1 to 1 m/s leads to a probability P_{edge} = 1.993 10^{-4} or a throw every 5017 ending with the coin on its edge.

For larger values of the friction coefficient, i.e., $\mu \approx 0.1$, the friction becomes quickly too important and the coin cannot come to a vertical position after impact: it falls immediately on one of its sides.

For all the above values, the term under the root sign in condition (56) is negative yielding imaginary values for the argument of the arctan function, which shows that condition (56) is complied with for all practical values of θ.

2.2.6 The case of rebounds

The four cases above have been analysed assuming that the coin is not rebounding after impact. This is obviously the case by definition for the first two hypotheses, as for inelastic surfaces, the velocity component normal to the landing surface becomes nil and the coin does not rebound.

For the last two cases, where elastic surfaces are involved, one has to be more cautious. A simple condition of no-rebound can be found by writing that the normal impulse of the landing surface at impact must be smaller than the integral over time of the coin weight considered as an impulsive force during the short period of impact, i.e.,

$$|N| = \left|\int_{t_0}^{t_f} mg\, dt\right| = mg\, \Delta t \tag{66}$$

For this condition to be respected in the last two hypotheses of elastic surfaces, the duration of impact Δt must be such as

$$\Delta t > \left|\left(\frac{1+e}{g}\right)\left((\pm\omega_a)\rho \sin(\theta - \frac{\alpha}{2}) - \sqrt{v_0^2 \sin^2 \beta + 2gH}\right)\right| \tag{67}$$

which yields respectively for the cases of elastic and perfectly smooth bodies and of elastic and partially rough bodies

$$\Delta t > \left|\frac{\kappa(\pm\omega_0)\sin(\theta-\frac{\alpha}{2}) - \frac{\kappa a}{\rho}\sqrt{v_0^2 \sin^2\beta + 2gH}}{g\left(\frac{\kappa a}{(1+e)\rho} + \sin^2(\theta-\frac{\alpha}{2})\right)}\right| \tag{68}$$

$$\Delta t > \left|\frac{\kappa(\pm\omega_0)\sin(\theta-\frac{\alpha}{2}) - \frac{\kappa a}{\rho}\sqrt{v_0^2 \sin^2\beta + 2gH}}{g\left(\frac{\kappa a}{(1+e)\rho} + \sin^2(\theta-\frac{\alpha}{2}) + \frac{\mu}{2}\sin(2\theta-\alpha)\right)}\right| \tag{69}$$

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For the numerical values previously considered, (68) and (69) yield impact duration Δt longer than several seconds, which is practically impossible for usual surfaces. This shows that the condition (66) cannot be practically fulfilled and the coin will rebound.

In this case, one can still calculate the probability that the coin will stay on its edge at the second impact by considering that the first impact will be such as to deliver the favourable initial conditions (Event 1) for the second impact to result in the coin staying on its edge (Event 2). The probability of the first event E_1 is obviously independent of the probability of the second event E_2. The opposite is of course not true and the probability of the second event E_2 depends on the first event E_1 having occurred. Therefore, the probability that the coin will stay on its edge at the second impact knowing that the first impact has delivered the favourable conditions for the second event to occur is

$$P_{edge\ 2} = P(E_1)\, P(E_2\ E_1) = P_{Fav.Impact\ 1}\, P_{edge\ 2\ 1} \tag{70}$$

where indexes 1 and 2 refer respectively to the first and second impacts.

With the following hypotheses:

- between the first and second impacts, the coin attains a maximum height

$$H_{12} = e^2 H \tag{71}$$

- the rotation velocities and moments of inertia of the coin respectively before the second impact and after the first impact are the same, yielding

$$\pm\omega_{b2} = \pm\omega_{a1} \tag{72}$$

$$I_{b2} = I_{a1} \tag{73}$$

$$\kappa_1 = \kappa_{b1} = \frac{I_{b1}}{\rho m}, \qquad \kappa_2 = \kappa_{b2} = \kappa_{a1} = \frac{I_{a1}}{\rho m} = \frac{I_{b2}}{\rho m} \tag{74}$$

- the vertical components of the coin velocity before and after the second impact are respectively

$$v_{z2} = e\sqrt{2gH} \tag{75}$$

$$u_{z2} = (\pm\omega_{a2})\, \rho_x = (\pm\omega_{a2})\rho \sin(\theta_2 - \tfrac{\alpha}{2}) \tag{76}$$

- the coin and landing surface are elastic and partially rough bodies (fourth hypothesis above), such as the normal and tangential impulses read

$$N_2 = m\,(1+e)\left((\pm\omega_{a2})\,\rho \sin(\theta_2 - \tfrac{\alpha}{2}) - e\sqrt{2gH}\right) \tag{77a}$$

$$T_2 = m\,\mu\,(1+e)\left((\pm\omega_{a2})\,\rho \sin(\theta_2 - \tfrac{\alpha}{2}) - e\sqrt{2gH}\right) \tag{77b}$$

and the impact equation reduces to

$$I_{a2}\,(\pm\omega_{a2}) = I_{a1}\,(\pm\omega_{a1}) - N_2\,\rho_x - T_2\,\rho_z \tag{78}$$

yielding

$$(\pm\omega_{a2}) = \frac{\kappa_2(\pm\omega_{a1}) + e(1+e)\sqrt{2gH}\left(\mu\cos(\theta_2-\frac{\alpha}{2}) + \sin(\theta_2-\frac{\alpha}{2})\right)}{\kappa_{a2} + (1+e)\rho\sin(\theta_2-\frac{\alpha}{2})\left(\mu\cos(\theta_2-\frac{\alpha}{2}) + \sin(\theta_2-\frac{\alpha}{2})\right)} \tag{79}$$

where

$$\kappa_{a2} = \frac{I_{a2}}{\rho m} \tag{80}$$

Like before, the coin will stay on its edge if the angular velocity after the second impact ω_{a2} is nil while $|\theta_2| < \frac{\alpha}{2}$. Solving the equation (79) $\omega_{a2} = 0$ for θ_2 yields

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$$\theta_2 = \frac{\alpha}{2} - 2\arctan\left(\frac{1 \pm \sqrt{1+\mu^2 - \left(\frac{\kappa_2(\pm\omega_{a1})}{e(1+e)\sqrt{2gH}}\right)^2}}{\frac{\kappa_2(\pm\omega_{a1})}{e(1+e)\sqrt{2gH}} - \mu}\right) \tag{81}$$

under the condition that the denominator of (79) is different from zero, i.e., for

$$\theta_2 \neq \frac{\alpha}{2} - \arctan\left(\frac{\mu(1+e) \pm \sqrt{\mu^2(1+e)^2 - 4\frac{\kappa_{a2}}{\rho}\left(\frac{\kappa_{a2}}{\rho}+1\right)}}{2\left((1+e)+\frac{\kappa_{a2}}{\rho}\right)}\right) \tag{82}$$

that can be modified like (56).

For θ_2 to be real, the condition on the root in the numerator of (81) reads

$$\frac{\kappa_2(\pm\omega_{a1})}{\sqrt{2gH}} \leq e(1+e)\sqrt{1+\mu^2} \tag{83}$$

For the condition $|\theta_2| < \frac{\alpha}{2}$ to hold, the argument of the arctan function in (81) must be positive and smaller than h/d.

For the positive condition, the numerator and the denominator must be of the same sign. Considering the negative sign in front of the numerator root in (81):

- both denominator and numerator are positive if

$$\frac{\kappa_2(\pm\omega_{a1})}{\sqrt{2gH}} > e(1+e)\mu \tag{84}$$

that combined with (83) yields

$$e(1+e)\mu < \frac{\kappa_2(\pm\omega_{a1})}{\sqrt{2gH}} \leq e(1+e)\sqrt{1+\mu^2} \tag{85}$$

which is true only for a positive sign in front of ω_{a1}, i.e., a coin counterclockwise rotation;

- both denominator and numerator are negative if

$$\frac{\kappa_2(\pm\omega_{a1})}{\sqrt{2gH}} < e(1+e)\mu \tag{86}$$

which includes condition (83) and is true either for a positive sign in front of ω_{a1}, i.e., a coin counterclockwise rotation after the first impact, as long as values of ω_{a1} comply with (86), or for all negative values of ω_{a1}, i.e., a coin clockwise rotation.

For the positive sign in front of the numerator root of (81), the denominator must be positive which is true if condition (84) holds, meaning that the positive sign in front of ω_{a1} must be chosen, i.e., a coin counterclockwise rotation. However, the solution, in this case, yields a too large value of θ_2.

The other part of the condition, i.e., the argument of the arctan in (81) smaller than h/d like in (22b), yields like in the previous section

- if $\mu < \frac{h}{d}$: $$e(1+e)\mu < \frac{\kappa_2\omega_{a1}}{\sqrt{2gH}} < e(1+e)\left(\frac{2hd-\mu(d^2-h^2)}{h^2+d^2}\right) \tag{87a}$$

- if $\mu = \frac{h}{d}$: $$\frac{\kappa_2\omega_{a1}}{\sqrt{2gH}} = e(1+e)\frac{h}{d} \tag{87b}$$

- if $\mu > \frac{h}{d}$: $$e(1+e)\left(\frac{2hd-\mu(d^2-h^2)}{h^2+d^2}\right) < \frac{\kappa_2\omega_{a1}}{\sqrt{2gH}} < e(1+e)\mu \tag{87c}$$

These three conditions include condition (83) as the right part of (83) is always greater than the right parts of (87a) to (87c).

Summarizing the conditions for this case:

- if $\mu < \frac{h}{d}$, then (87a) includes (85) and ω_{a1} must be positive, i.e., a coin counterclockwise rotation after first impact;

- if $\mu = \frac{h}{d}$, then (87b) is the limiting case of (84) and (86) and ω_{a1} is positive, i.e., a coin counterclockwise rotation after first impact;

- if $\frac{h}{d} < \mu < \frac{2hd}{d^2-h^2}$, then (87c) includes (85) and ω_{a1} is positive, i.e., a coin counterclockwise rotation after first impact;

- if $\mu > \frac{2hd}{d^2-h^2}$, then (87c) includes (86) and ω_{a1} can be either positive or negative, i.e., a coin counterclockwise rotation or clockwise rotation after first impact.

The three conditions (87a) to (87c) constrain the favourable values of ω_{a1}, which can be translated into conditions on θ_1 through (52) (where the index 1 has been added to θ, ω_a and κ).

Assuming that $\mu < \frac{h}{d}$, the left and right parts of (87a) yield respectively

$$\sin^2(\theta_1 - \tfrac{\alpha}{2}) - \sin(\theta_1 - \tfrac{\alpha}{2})\left(B - \mu\cos(\theta_1 - \tfrac{\alpha}{2})\right) - \left(A + B\mu\cos(\theta_1 - \tfrac{\alpha}{2})\right) < 0 \quad (88)$$

$$\sin^2(\theta_1 - \tfrac{\alpha}{2}) - \sin(\theta_1 - \tfrac{\alpha}{2})\left(D - \mu\cos(\theta_1 - \tfrac{\alpha}{2})\right) - \left(C + D\mu\cos(\theta_1 - \tfrac{\alpha}{2})\right) > 0 \quad (89)$$

with

$$A = \kappa_2\left(\frac{\kappa_1(\pm\omega_0) - e(1+e)\mu\sqrt{2gH}}{e(1+e)^2\mu\rho\sqrt{2gH}}\right) \quad (90a)$$

$$B = \frac{\kappa_2\sqrt{v_0^2\sin^2\beta + 2gH}}{e(1+e)\mu\rho\sqrt{2gH}} \quad (90b)$$

$$C = \kappa_2\left(\frac{\kappa_1(\pm\omega_0) - e(1+e)\left(\frac{2hd-\mu(d^2-h^2)}{h^2+d^2}\right)\sqrt{2gH}}{e(1+e)^2\rho\left(\frac{2hd-\mu(d^2-h^2)}{h^2+d^2}\right)\sqrt{2gH}}\right) \quad (91a)$$

$$D = \frac{\kappa_2\sqrt{v_0^2\sin^2\beta + 2gH}}{e(1+e)\rho\left(\frac{2hd-\mu(d^2-h^2)}{h^2+d^2}\right)\sqrt{2gH}} \quad (91b)$$

Analytical solutions of (88) and (89) for $(\theta_1 - \frac{\alpha}{2})$ require solving analytically fourth-degree equations in $sin(\theta_1 - \frac{\alpha}{2})$, which is not an easy task. However, numerical solutions can be found to define the range $[\theta_{1min}, \theta_{1max}]$ of allowed values of θ_1 at first impact to deliver favourable conditions. Relations (88) and (89) yield numerically values of respectively θ_{1min} and θ_{1max}, involving functions Φ_{min} and Φ_{max} of ω_0, H, v_0, β, e, μ, h and d, respectively through A and B and through C and D

$$\theta_{1min} = \frac{\alpha}{2} + \Phi_{min}(A, B, \mu); \qquad \theta_{1max} = \frac{\alpha}{2} + \Phi_{max}(C, D, \mu) \quad (92)$$

The probability that the first impact delivers favourable conditions for the second impact to bring the coin on its edge reads then

$$P_{Fav.impact\ 1} = \left(\frac{\theta_{1\ max}-\theta_{1\ min}}{\pi}\right) = \left(\frac{\Phi_{max}(C,D,\mu)-\Phi_{min}(A,B,\mu)}{\pi}\right) \quad (93)$$

For this case $\mu < \frac{h}{d}$, the condition (87a) constrains the favourable coin rotation velocity ω_{a1} after the first impact between minimum and maximum values ω_{a1min} and ω_{a1max} corresponding to θ_{1min} and θ_{1max} through (52):

$$\omega_{a1\ min} = \frac{\kappa_1(\pm\omega_0)+(1+e)\sqrt{v_0^2\sin^2\beta+2gH(\mu\cos\Phi_{min}+\sin\Phi_{min})}}{\kappa_2+(1+e)\rho\sin\Phi_{min}(\mu\cos\Phi_{min}+\sin\Phi_{min})} \quad (94a)$$

$$\omega_{a1\ max} = \frac{\kappa_1(\pm\omega_0)+(1+e)\sqrt{v_0^2\sin^2\beta+2gH(\mu\cos\Phi_{max}+\sin\Phi_{max})}}{\kappa_2+(1+e)\rho\sin\Phi_{max}(\mu\cos\Phi_{max}+\sin\Phi_{max})} \quad (94b)$$

Replacing these two values ω_{a1min} and ω_{a1max} in (81) yield the two limiting values θ_{2min} and θ_{2max}

$$\theta_{2min} = \frac{\alpha}{2} - \arctan\left(\frac{1-\sqrt{1+\mu^2-v_{min}^2}}{v_{min}-\mu}\right);\ \theta_{2max} = \frac{\alpha}{2} - \arctan\left(\frac{1-\sqrt{1+\mu^2-v_{max}^2}}{v_{max}-\mu}\right) \quad (95)$$

with

$$v_{min} = \frac{\kappa_2\omega_{a1\ min}}{e(1+e)\sqrt{2gH}} = \frac{\frac{\kappa_1(\pm\omega_0)}{\sqrt{2gH}}+(1+e)\sqrt{1+\left(\frac{v_0\sin\beta}{\sqrt{2gH}}\right)^2(\mu\cos\Phi_{min}+\sin\Phi_{min})}}{e(1+e)\left(1+(1+e)\frac{\rho}{\kappa_2}\sin\Phi_{min}(\mu\cos\Phi_{min}+\sin\Phi_{min})\right)} \quad (96)$$

and a similar expression for v_{max} with Φ_{max} replacing Φ_{min}.

The probability that the coin stays on its edge at the second impact knowing that the first impact has delivered the appropriate favourable conditions is then

$$P_{edge\ 2|1} = \left(\frac{\theta_{2\ max} - \theta_{2\ min}}{\pi}\right)$$

$$= \frac{2}{\pi}\left(\arctan\left(\frac{1-\sqrt{1+\mu^2-v_{min}^2}}{v_{min}-\mu}\right) - \arctan\left(\frac{1-\sqrt{1+\mu^2-v_{max}^2}}{v_{max}-\mu}\right)\right)$$

$$= \frac{2}{\pi}\arctan\left(\frac{(v_{max}-\mu)\left(1-\sqrt{1+\mu^2-v_{min}^2}\right)-(v_{min}-\mu)\left(1-\sqrt{1+\mu^2-v_{max}^2}\right)}{(v_{min}-\mu)(v_{max}-\mu)+\left(1-\sqrt{1+\mu^2-v_{min}^2}\right)\left(1-\sqrt{1+\mu^2-v_{max}^2}\right)}\right) \quad (97)$$

where the formula $\arctan(X)$ - $\arctan(Y) = \arctan\left(\frac{X-Y}{1-XY}\right)$ was used [3].

The total probability that the coin stays on its edge at the second impact after the first rebound has delivered the appropriate conditions is given by (70).

As in the previous section, coefficients of restitution and of friction are e = 0.5 and μ = 0.05 for a 50 Eurocent coin, thus $\mu < \frac{h}{d}$. One considers further the two extreme cases of position of the instantaneous rotation axis after the first impact, i.e., respectively along the horizontal axis passing first through the coin centre of mass, and second through the coin contact point with the landing surface (see the previous section), yielding values of κ_2 varying between 2.955 10^{-3} and 1.475 10^{-2}, while κ_1 = 2.955 10^{-3} under the initial hypothesis.

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Like in the previous case, the condition (87a) imposes a quite stringent range of values for the ratio $\frac{\kappa_2\omega_{a1}}{\sqrt{2gH}}$, namely

$$12.689 < \frac{\kappa_2\omega_{a1}}{\sqrt{2gH}} < 30.381 \text{ and } 2.543 < \frac{\kappa_2\omega_{a1}}{\sqrt{2gH}} < 6.088 \quad (98)$$

respectively for κ_{2min} = 2.955 10^{-3}m and κ_{2max} = 1.475 10^{-2}m, yielding respectively for $H = d$ and for $H = 2\,m$

$$8.616 < \omega_{a1} < 20.629 \text{ and } 1.726 < \omega_{a1} < 4.134 \text{ (rad/s)} \quad (99a)$$

$$79.490 < \omega_{a1} < 190.311 \text{ and } 15.928 < \omega_{a1} < 38.135 \text{ (rad/s)} \quad (99b)$$

The allowed range for the coin angular velocities after the first impact thus increases for increasing initial height H.

Within the value ranges (6) of initial parameters, the largest value of the probability $P_{edge2\ max}$ is obviously attained for the largest values of $P_{Fav.Impact1\ max}$ and $P_{edge2\ 1\ max}$. The largest value of $P_{Fav.Impact1}$ is obtained for the largest value of Φ_{max} and the smallest value of Φ_{min}. Φ_{max} is the largest for C_{min} and D_{min}, i.e., for ω_0 = 0 and v_0 = 0, while Φ_{min} is the smallest for A_{max} and B_{max}, i.e., for ω_0= 10π rad/s, v_0 = 5 m/s, $\beta = \pm\pi/2$ and $H = d$.

For the first case of $\kappa_1 = \kappa_{2min}$ = 2.955 10^{-3}m, a series of coin tosses having initial conditions in the ranges (6), one finds successively

Φ_{max} = 0.0101, Φ_{min} = -0.0588 (rad)

θ_{1max} = 0.0950, θ_{1min} = 0.0261 (rad)

ω_{a1max} = 190.311, ω_{a1min} = 8.616 (rad/s)

θ_{2max} = -0.0849, θ_{2min} = -0.0150 (rad)

$P_{Fav.Impact1max}$ = 2.193 10^{-2}, $P_{edge2\ 1\ max}$= 2.225 10^{-2}, $P_{edge2\ max}$= 4.880 10^{-4}

or approximately a throw every 2050.

For the second case of κ_{2max} = 1.475 10^{-2}m, one has similarly

Φ_{max} = 0.0099, Φ_{min} = -0.0588 (rad)

θ_{1max} = 0.0948, θ_{1min} = 0.0261 (rad)

ω_{a1max} = 38.135, ω_{a1min} = 1.727 (rad/s)

θ_{2max} = -0.0849, θ_{2min} = -0.0150 (rad)

$P_{Fav.Impact1max}$ = 2.188 10^{-2}, $P_{edge2\ 1\ max}$ = 2.225 10^{-2}, $P_{edge2\ max}$= 4.868 10^{-4}

or approximately a throw every 2054.

For more common values, a series of throw from an initial height H = 1.5 m with a velocity v_0= 1 m/s under an angle of $\beta = \pi/4$ and an initial rotation velocity ω_0 varying between 0.5 and 5 turns/s yield successively first for κ_{2min} = 2.955 10^{-3}m,

θ_{1max} = 0.0829, θ_{1min} = 0.0585 (rad)

ω_{a1max} = 164.814, ω_{a1min} = 68.840 (rad/s)

θ_{2max} = -0.0849, θ_{2min} = -0.0150 (rad)

$P_{Fav.Impact1}$ = 7.774 10^{-3}, $P_{edge2\ 1}$ = 2.224 10^{-2} and P_{edge2} = 1.729 10^{-4}

or approximately a throw every 5783, and second for κ_{2max} = 1.475 10^{-2}m,

θ_{1max} = 0.0829, θ_{1min} = 0.0586 (rad)

ω_{a1max} = 33.026, ω_{a1min} = 13.794 (rad/s),

θ_{2max} = -0.0849, θ_{2min} = -0.0150 (rad),

$P_{Fav.Impact1}$ = 7.759 10^{-3}, $P_{edge2\ 1}$ = 2.224 10^{-2} and P_{edge2} = 1.726 10^{-4}

or approximately a throw every 5794.

For the sake of completeness, for all the above values, the term under the root sign in condition (82) is negative yielding imaginary values for the argument of the arctan function, which shows that condition (62) is complied with for all practical values of θ_2.

The cases of successive rebounds can be treated similarly, albeit with more and more complexity in the various relations.

3. Discussion

The six theoretical models developed in this paper show that there is a non-nil probability that a falling coin will not end up on one of its sides but on its edge, with decreasing probabilities for models describing reality from closer. It is interesting to note that the probabilities calculated in all the above case models are independent of the coin mass but strongly depend on several other factors, mainly on the coin vertical velocity before impact, which depends on the initial height H and the initial angle β of the throw. It appears that increasing the initial height decreases the probability that the coin will end on its edge while increasing the initial rotation will increase this probability. Depending on the characteristics of the surface, throwing the coin vertically seems to decrease the probability of the coin ending on its edge. The role of friction is obviously important: if the surface conditions are such as to increase the friction coefficient μ above a certain value depending on the models, the coin can no longer stop on its edge and will inevitably fall on one side. Finally, the rebound case model shows that very limited initial conditions and coin and landing surface conditions would deliver the proper conditions for the coin to stop on its edge at the second impact. For a series of throws from an average height with common velocity values and appropriate surface conditions, the probability that a 50 Eurocent coin ends up on its edge is calculated to be in the order of one against several thousand.

The coin geometry is also an important factor. A 25 US Dollar cent coin has a thinner edge (approximately $d \approx 1.5$ mm), which would decrease all above calculated probabilities, although these would stay in the same order of magnitude.

However, these theoretical models are just what they are: theoretical models. All effects are not taken into account, e.g., the coin may spin around its symmetry axis, the coin may not be exactly ‘fair’ (i.e., with a perfectly homogeneous mass distribution), surface conditions of the coin and the landing surface may not be optimal as assumed above, etc. Therefore, to throw a coin in the air and let it fall will usually end up with the coin on one of its sides. Nevertheless, the probability that the coin ends on its edge is not nil, even in reality.

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The author declares no conflict of interest.

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