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Dugundgji's Extension Theorem and Fixed Point Theorem in p-normed Spaces

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Abstract: The goal of this paper is to establish Dugundji's extension theorem in p-normed spaces, and then as applications, fixed theorems in p-normed spaces are given for $p \in (0,1]$. The results in this paper would provide a general fundamental basis for the development of fixed point theory, in supporting for the study of Schauder's conjecture and related nonlinear analysis in p-vector spaces which are either finite or infinite-dimensional. In particularly, how important the Dugundgji type extension and fixed-point theorem in p-normed spaces can be found from Yuan's recent work in [42,43] and related discussion wherein.

Keywords: p-normed space; p-convex set; dugundji type extension theorem in p-normed spaces; fixed point theorem; schauder's conjecture; minkowski p-functional (p-gauge); homeomorphism

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1. Introduction

It is known that the class of p-seminorm spaces for $p \in (0,1]$ is an important generalization of usual normed spaces with rich topological and geometrical structures, and related study has received a lot of attention (e.g., see Agarwal et al.[2], Balachandran [3], Bayoumi [4], Chang et al.[11], Du [12], Ennassik and Taoudi [16], Ennassik et al.[15], Goebel and Kirk [18], Granas and Dugundji [19], Jarchow [20], Kalton [21,22], Kalton et al.[23], Kirk [24], Park [30,31], O'Regan [29], Takahashi [36], Xiao and Zhu [40], Yuan [42,43], Zeidler [44], and many others). However, to the best of our knowledge, the corresponding basic tools and associated results in the category of nonlinear functional analysis have not been well developed.

The goal of this paper is first to establish Dugundji type extension theorem in p-normed spaces, and then as applications, fixed theorems in p-normed spaces are given for $p \in (0,1]$. The results in this paper would provide a general fundamental basis for the development of fixed point theory, in supporting for the study of Schauder's conjecture and related nonlinear analysis in p-vector spaces which are either finite or infinite-dimensional. These results not only provide fundamental tools for the study of nonlinear analysis under the setting of p-vector or locally p-convex spaces for either single-valued or set-valued mappings, but also show the existence of fixed points for continuous mappings in p-normed spaces which unify or improve corresponding results in the existence literature (see Granas and Dugundji [19], Djebali [13], Mauldin [27], McLennan [28], O'Regan [29], Park [30,31], Yuan [42,43], Zeidler [44] and related references there for more in details). The results given by this paper are also fundamental tools for the study of Schauder's conjecture and related nonlinear problems in p-vector spaces for $p \in (0,1]$. In particularly, how important the Dugundgji type extension and fixed point theorem in p-normed spaces established here for $p \in (0,1]$, can be found from Yuan's recent work in [42,43] and related discussion wherein.

Throughout this paper, for the convenience of our discussion, if (X, T_X) and (Y, T_Y) be two topological spaces with topology structures T_X and T_Y, respectively, in short, denoted by X and Y

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without specified unless confusion. All p-convex vector spaces are assumed to be Hausdorff, and p satisfying the condition for $p \in (0,1]$ unless specified. We denote by $\mathbb{N}$ the set of all positive integers, i.e., $\mathbb{N} = \{1, 2, \cdots, \}$. For a given set X, the 2^X denotes the family of all subsets of X. We denote by $\mathbb{R}^n$ for finite n-dimensional Euclidean spaces, where n is a positive integer. The letter $\mathbb{K}$ is denoted for the field of either real numbers or complex numbers unless specified. For the purpose in this book, we also use the symbols $\mathbb{R}$ and $\mathbb{C}$ to represent the real field, and the complex field, respectively.

For a vector space X over a field $\mathbb{K}$ (which is an either real or a complex field), the span of a set A of vectors (not necessarily finite) is defined to be the intersection W of all subspaces of X that contain A. W is referred to as the subspace spanned by W, or by the vectors in A. Conversely, A is called a spanning set of W, and we say that A spans W, and also denoted by $span(A)$ for W.

For $p \in (0,1]$, we denote the interior, the closure, the boundary of a given subset A in a p-normed space X by A^0, $\overline{A}$ and $\partial(A)$, respectively; and the origin (zero) element of vector space X is denoted by θ. By $B(x,r)$, we mean the open ball of X with center $x \in X$ and radius $r > 0$, and B_p for the closed unit ball with center θ in $l^p(n)$ for $n \in \mathbb{N}$, i.e., $B_p : \{x \in l^p(n), \|x\|_p \leq 1\}$. If there is $r > 0$ such that $B(x,r) \cap span(A) \subset A$, then x is said to be an interior point with respect to its (i.e., $span(A)$) relative topology and is denoted by $x \in ri(A)$.

This paper consists of six sections. The Section 1 is for the brief introduction; Section 2 describes general concepts for the p-convex subsets in p-vector spaces for $p \in (0,1]$, and related some basic facts; in Section 3, Dugundji's extension theorem is established in p-normed spaces for $p \in (0,1]$; in Section 4, a fixed point theorem for continuous mappings is proved by the existence of homeomorphisms for s-convex subsets in p-normed spaces, where $s, p \in (0,1]$; Section 5 is the conclusion for the summary of results established in this paper; and Section 6 is for the acknowledgement.

2. Preliminaries

We now recall some notion and definitions on p-convexity, p-vector and locally p-convex spaces and related some fundamental facts (see Jarchow [20], Kalton [21], Bayoumi [4], or Ennassik and Taoudi [16]), which will be used in this paper below.

Definition 2.1. Let $p \in (0,1]$. A set A in a vector space X is said to be p-convex if for any $x, y \in A$, we have $sx + ty \in A$, whenever $0 \leq s, t \leq 1$ with $s^p + t^p = 1$. The set A is said to be absolutely p-convex if for any $x, y \in A$, we have $sx + ty \in A$, whenever $|s|^p + |t|^p \leq 1$. In the case $p = 1$, the concept of the (absolutely) 1-convexity is simply the usually (absolutely) convexity defined in vector spaces.

Definition 2.2. Let $p \in (0,1]$. If A is a subset of a topological vector space X, the the p-convex hull of A and its closed p-convex hull denoted by $co_p(A)$, and $\overline{co}_p(A)$, respectively, which is the smallest p-convex set containing A, and the smallest closed p-convex set containing A, respectively.

Definition 2.3. Let $p \in (0,1]$ and A be p-convex and $x_1, \cdots, x_n \in A$, and $t_i \geq 0, \sum_1^n t_i^{\mathrm{p}} = 1$. Then $\sum_1^n t_i x_i$ is called a p-convex combination of $\{x_i\}$ for $i = 1, 2, \cdots, n$. If $\sum_1^n |t_i|^{\mathrm{p}} \leq 1$, then $\sum_1^n t_i x_i$ is called an absolutely p-convex combination. It is easy to see that $\sum_1^n t_i x_i \in A$ for a p-convex set A.

Definition 2.4. A subset A of a vector space X is called balanced (or circled) if $\lambda A \subset A$ holds for all scalars λ satisfying $|\lambda| \leq 1$. We say that A is absorbing if for each $x \in X$, there is a real number $\rho_x > 0$ such that $\lambda x \in A$ for all $\lambda > 0$ with $|\lambda| \leq \rho_x$.

Definition 2.5. Let X is a vector space and $\mathbb{R}^+$ is a non-negative part of a real line $\mathbb{R}$. Then a mapping $P : X \longrightarrow \mathbb{R}^+$ is said to be a p-seminorm if it satisfies the following requirements for $p \in (0,1]$:

(i) $P(x) \geq 0$ for all $x \in X$;
(ii) $P(\lambda x) = |\lambda|^p P(x)$ for all $x \in X$ and $\lambda \in R$;
(iii) $P(x + y) \leq P(x) + P(y)$ for all $x, y \in X$.

We recall that a p-seminorm P is called an p-norm if $x = 0$ whenever $P(x) = 0$. A topological vector space with a specific p-norm is called a p-normed space. Of course if $p = 1$, X is a usual normed space. By Lemma 3.2.5 of Balachandra [3], the following proposition gives a necessary and sufficient condition for a p-seminorm to be continuous.

A vector (or saying, linear) space X on which there is a p-norm is called a p-normed space and is denoted by $(X, \|\cdot\|_p)$. If $p = 1$, then it is a usual normed space. A p-normed space is also a metric linear space with a translation invariant metric $d_p(x, y) := \|x - y\|_p$ for $x, y \in X$. It is a basic fact that the topology of every Hausdorff locally bounded topological vector space is given by some p-norm. The space $L^p(\mu)$ is a p-normed space based on the complete measure space $(\Omega, \mathcal{M}, \mu)$ with the p-norm given by

$$\|f(t)\|_p = \int_\Omega |f(t)|^p d\mu, \text{ for } f \in L^p(\mu).$$

where Ω is a nonempty set, $\mathcal{M}$ is a σ-algebra in Ω, and $\mu : \mathcal{M} \to [\theta, \infty)$ is a positive measure, and

$$L^p(\mu) = \{f | f : \Omega \to \mathbb{K} \text{ is measurable, and } \int_\Omega |f(t)|^p d\mu < \infty\}.$$

If μ is the Lebesgue measure on $[0, 1]$, then it is customary to write $L^p[0, 1]$ instead of $L^p(\mu)$. If μ is the counting measure on $\Omega = \{1, \cdots, n\}$ or $\Omega = \{1, 2, \cdots, \}$, then the corresponding spaces are denoted by $l^p(n)$ and l^p, respectively. Here we note that $l^p(n)$ is an n-dimensional space and l^p is a complete, infinite-dimensional, and separable space. The class of p-normed spaces for $p \in (0, 1]$ is an important generalization of the class of usual normed spaces of p-normed spaces (with $p = 1$).

It is well known that most of the fixed point theorems are concerned with some convex sets. But the unit ball with center θ in a p-normed space for $p \in (0, 1)$ is not a convex set. We know that that every open ball in l^p for $p \in (0, 1)$ does not contain any open convex subset, and there is no open convex subset in $L^P[0, 1]$ for $p \in (0, 1)$, except the case $L^p[0, 1]$ itself (see Kalton et al.[23], Rudin [34] and related references). Here we also point out that the existence of fixed points for non-convex sets is very useful both in theory and in applications, and even in usual normed spaces, the existence of fixed points for operators on non-convex sets is often considered, and much attention attached to these problems. For example, Klee [25] established some fixed point theorems in Hausdorff topological vector space without local convexity under some symmetry conditions. Bayoumi [4] also established generalized Brouwer fixed point theorem and generalized Kakutani fixed point theorem for p-convex subset in p-convex Fréchet spaces; and Zeidler [44] gives a very comprehensive discussion on the development of fixed point theory in functional analysis with applications in mathematics and related various fields; See also more discussion given by Djebali [13], McLennan [28] and references wherein.

Proposition 2.6. Let X be a topological vector space, P is a p-seminorm on X and $V := \{x \in X : P(x) < 1\}$. Then P is continuous if and only if $0 \in V^0$, where V^0 is the interior of V in space X.

Now given a p-seminorm P, the p-seminorm topology determined by P (in short, the p-topology) is the class of unions of open balls $B(x, \epsilon) := \{y \in X : P(y - x) < \epsilon\}$ for $x \in X$ and $\epsilon > 0$.

We also need following notion for the so-called p-gauge (see Balachandra [3]).

Definition 2.7. Let A be an absorbing subset of a vector space X. For $x \in X$ and $p \in (0, 1]$, set $P_A = \inf\{\alpha > 0 : x \in \alpha^{\frac{1}{p}} A\}$, then the non-negative real-valued function P_A is called the p-gauge (gauge if $p = 1$). The p-gauge of A is also known as the Minkowski p-functional.

By Proposition 4.1.10 of Balachandra [3], we have the following proposition.

Proposition 2.8. Let A be an absorbing subset of X and $p \in (0, 1]$. Then p-gauge P_A has the following properties:

(i) $P_A(0) = 0$;
(ii) $P_A(\lambda x) = |\lambda|^p P_A(x)$ if $\lambda \geq 0$;

(iii) $P_A(\lambda x) = |\lambda|^p P_A(x)$ for all $\lambda \in R$ provided A is balanced (circled);
(iv) $P_A(x+y) \leq P_A(x) + P_A(y)$ for all $x, y \in A$ provided A is p-convex.
In particular, P_A is a p-seminorm if A is absolutely p-convex (and also absorbing).

Remark 2.9. It is worthwhile noting that a zero-neighborhood in a topological vector space being an absolutely 0-neighborhood is also absorbing (by Lemma 2.1.16 of Balachandran [3], or Proposition 2.2.3 of Jarchow [20]), this leads us to have the following definition for a topological vector space E being a topological p-vector space (in short, p-vector space) for $p \in (0,1]$ by using the concept of the Minkowski p-functional below.

Definition 2.10. A topological vector space X is said to be a topological p-vector space (in short, p-vector space) if the base of the origin in X is generated by a family of Minkowski p-functionals (p-gauges) (see the Definition 2.7 above), where $p \in (0,1]$.

By incorporating Proposition 2.8, it seems that the following is a nature way for the definition of locally p-convex spaces, where $p \in (0,1]$.

Definition 2.11. A topological vector space X is said to be locally p-convex if the origin in X has a fundamental set of absolutely p-convex 0-neighborhoods. This topology can be determined by p-seminorms which are defined in the obvious way (see pp.52 of Bayoumi [4], Jarchow [20]). When $p = 1$, a locally p-convex space X is reduced to be a usual locally convex space.

By Proposition 4.1.12 of Balachandra [3], we have the following proposition.

Proposition 2.12. Let A be a subset of a vector space X, which is absolutely p-convex for $p \in (0,1]$ and absorbing. Then, we have that
(i) The p-gauge P_A is a p-seminorm such that if $B_1 := \{x \in X : P_A(x) < 1\}$, and $\overline{B_1} = \{x \in X : P_A(x) \leq 1\}$. then $B_1 \subset A \subset \overline{B_1}$; in particular, $ker P_A \subset A$, where $ker P_A := \{x \in X : P_A(x) = 0\}$.
(ii) $A = B_1$ or $\overline{B_1}$ according as A is open or closed in the P_A-topology.

Remark 2.13. Let X be a topological vector space and let U be an open absolutely p-convex neighborhood of the origin, and let ϵ be any given positive number. If $y \in \epsilon^{\frac{1}{p}} U$, then $y = \epsilon^{\frac{1}{p}} u$ for some $u \in U$ and $P_U(y) = P_U(\epsilon^{\frac{1}{p}} u) = \epsilon P_U(u) \leq \epsilon$ (as $u \in U$ implies that $P_U(u) \leq 1$). Thus, P_U is continuous at *zero*, and therefore, P_U is continuous everywhere. Moreover, we have $U = \{x \in X : P_U(x) < 1\}$.
Indeed, since U is open and the scalar multiplication is continuous, we have that for any $x \in U$, there exists $0 < t < 1$ such that $x \in t^{\frac{1}{p}} U$ and so $P_U(x) \leq t < 1$. This shows that $U \subset \{x \in X : P_U(x) < 1\}$. The conclusion follows by Proposition 2.12 above.
The following result is a very important and useful result which allows use to make the approximation for convex subsets containing zero in topological vector spaces by p-convex subsets in locally p-convex spaces (see Lemma 2.1 of Ennassik and Taoudi [16], Remark 2.1 of Qiu and Rolewicz [32]), thus we omit their proof.

Lemma 2.14. Let A be a subset of a vector space X, then we have
(i) If A is r-convex, with $r \in (0,1)$, then $\alpha x \in A$ for any $x \in A$ and any $\alpha \in (0,1]$;
(ii) If A is convex and $0 \in A$, then A is s-convex for any $s \in (0,1]$; and
(iii) If A is r-convex for some $r \in (0,1)$, then A is s-convex for any $s \in (0,r]$.

Remark 2.15. We like to point out that the results (i) and (iii) of Lemma 2.14 do not hold for $r = 1$. Indeed, any singleton $\{x\} \subset X$ is convex in topological vector spaces; but if $x \neq 0$, then it is not p-convex for any $p \in (0,1)$.

In next section, Dugundji's extension theorem in p-normed spaces is first establish, which will be used to prove fixed point theorems in p-normed spaces which are either finite, or infinite-dimensional for $p \in (0,1]$.

3. Dugundgjit Type Extension Theorem in p-Normed Spaces

In this part, we establish Dugundgji type extension theorem under the framework of p-normed spaces for $0 < s \leq p \leq 1$, which includes Theorem 6.1 of Dugundgji [14] as a special case for $s = p = 1$, and we expect that this new extension theorem would be a useful tool for the study of p-normed spaces which are either infinite-dimensional or finite-dimensional in functional analysis and topology.

In topology and functional analysis, the well known Tietze extension theorem says that if a topological space X is normal and $f : A \to [0,1]$ is continuous, where $A \subset X$ is closed, then f has a continuous extension to all of X. The closed subset A in X maps into a finite dimensional Euclidean space is continuous if its component functions are each continuous, so Tietze's theorem is adequate for the study under the framework of finite dimensional spaces and related applications. However, in mathematics, we normally work and study with p-normed spaces $(X, \|\cdot\|_p)$ for $p \in (0,1]$, which are potentially infinite dimensional. Thus we need to establish the extension theorem for closed subset in p-normed spaces as a useful tool, to include Dugundji's extension theorem (of Dugundji [14]) as a special class for the study in functional analysis and topology.

In this section, our goal is to establish the following extension theorem for closed subsets under the framework of p-normed spaces $(X, \|\cdot\|_p)$ for $p \in (0,1]$, which includes the classic Theorem 6.1 of Dugundgji [14] as a special case for $p = 1$. As mentioned above, we expect this new extension theorem in p-normed spaces would be a useful and powerful tool for the development of new results and theory in p-normed spaces which are either infinite-dimensional or finite-dimensional.

For the convenience of our discussion below, by following Dugundji [14] (see also McLennan [28] and Djebali [13], here we first introduce some mathematical notations or symbols to use in this section. For any $\epsilon > 0$ and a point $x \in (X,d)$, where, X is a metric space with metric d, we denote by $U_\epsilon(x)$ the open ball of radius ϵ at a point x in the metric space (X,d). For a subset $A \subset X$, we denote by $d(x,A)$ for the distance of x and A defined by $d(x,A) := \inf\{d(x,a) : a \in A\}$.

Theorem 3.1 (Dugundji type extension theorem in p-normed spaces). Let E be a metric space (E,d) with the metric d and $A \subset E$ a non-empty closed subset, and $(Y, \|\cdot\|_p)$ being a p-normed space for $p \in (0,1]$. Then every continuous mapping $f : A \to Y$ has a continuous extension $F : E \to Y$ such that $F(E) \subset co_s(f(A))$, and $F(x) = f(x)$ for each $x \in A$, where $co_s(f(A))$ is the s-convex hull of $f(A)$ in Y, and $0 < s \leq p \leq 1$.

Proof. Let the family $\{U_{d(x,A)/2}(x)\}_{x \in X \setminus A}$ be an open cover of the space $X \setminus A$. As $X \setminus A$ is paracompact, it implies the existence of an open locally finite refinement $\{W_\alpha\}_{\alpha \in I}$ (see also Dugundji [14]). Then we know that there is an existence of a partition of unity $\{\psi_\alpha\}_{\alpha \in I}$ subordinate to $\{W_\alpha\}_{\alpha \in I}$ (e.g., see Theorem 4.2 in Chapter VIII of Dugundji [14] on the discussion on partitions). For each $\alpha \in I$, choose $a_\alpha \in A$ with $d(a_\alpha, W_\alpha) < 2d(A, W_\alpha)$, and define the extension mapping for f by $F : X \to co_s(f(A)) \subset Y$ as below:

$$F(x) := \begin{cases} f(x), & \text{if } x \in A; \\ \sum_{\alpha \in I} (\psi_\alpha)^{\frac{1}{s}}(x) \cdot f(a_\alpha), & \text{if } x \in X \setminus A. \end{cases} \tag{1}$$

Clearly F is continuous at all points of $X \setminus A$ and at every interior point of A. We only let to prove F is continuous in the boundary of A.

Here we note that for any $s > 0$ with $0 < s \leq p \in (0,1]$, the p-normed space $(Y, \|\cdot\|_p)$ is also a s-normed space with the s-norm $\|\cdot\|_s$, which could be defined by $\|\cdot\|_s = \|\cdot\|_p^{\frac{s}{p}}$.

Let a be a point in the boundary of the closed subset A in X(and may assume $f(a) = 0$; otherwise let $f_1(x) := f(x) - f(a)$ for each $x \in E$, then we use f_1 to replace f to prove its continuity at $f_1(a) = 0$), thus for any open ball $U_\epsilon(a)$, the set $U_\epsilon(a) \cap (X \setminus A) \neq \emptyset$. Now let U be a p-convex neighborhood of the point $F(a)$ (actually $F(a) = f(a)$ as a is a boundary point of a closed subset A) in the space $(Y, \|\cdot\|_p)$, as $f(a)$ is continuous in A, choose $\delta > 0$ small enough that $f(a') \in U$ whenever $a' \in U_\delta(a) \cap A$. Then we will prove that for any $x \in U_{\delta/7}(a) \cap (X \setminus A)$, $F(x) \in U$. This would help us to claim that F is continuous at each boundary point of A, then we can complete the proof to show that F is continuous

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in X, and with the property in the statement of our Theorem 3.1. Here, we note that the p-convex U is also s-convex since the p-normed space $(Y, \|\cdot\|_p)$ is also a s-normed space $(Y, \|\cdot\|_s)$ as mentioned above when $0 < s \leq p \leq 1$.

Indeed, for any $x \in U_{\delta/7}(x) \cap (X \setminus A)$, suppose $\alpha \in I$ such that $x \in W_\alpha$ and any $x' \in X$ such that $W_\alpha \subset U_{d(x,A)/2}(x')$, we first observe that

$$d(a_\alpha, W_\alpha) \geq d(a_\alpha, x') - d(x, A)/2 \geq d(a_\alpha, x') - d(x', a_\alpha)/2 = d(a_\alpha, x')/2,$$

and

$$d(x', x) \leq d(x', A)/2 \leq d(W_\alpha, A) \leq d(W_\alpha, a_\alpha),$$

then it implies that

$$d(a_\alpha, x) \leq d(a_\alpha, x') + d(x, x') \leq 3d(a_\alpha, W_\alpha) \leq 6d(A, W_\alpha) \leq 6d(a, x).$$

Therefore we show that whenever $x \in W_\alpha$, the following is true:

$$d(a_\alpha, a) \leq d(a_\alpha, x) + d(x, a) \leq 7d(x, a) < \delta.$$

This means $a_\alpha \in U_\delta(a) \cap A$, thus $f(a_\alpha) \in U$ by the continuity of f at the point $a \in A$ with associated neighborhood U with the condition and statement given above.

Now by a fact that $\sum_{\alpha \in I}((\psi_\alpha)^{\frac{1}{s}})^s(x) = \sum_{\alpha \in I}(\psi_\alpha)(x) = 1$ for any $x \in X$ as it is a partition of unity subordinate to $\{W_\alpha\}_{\alpha \in I}$ given above, and U is s-convex, it follows $F(x) = \sum_{\alpha \in I}(\psi_\alpha)^{\frac{1}{s}}(x) \cdot f(a_\alpha)$ is in the s-convex hull $co_s(f(a_\alpha)(x) : \alpha \in I)$ of elements $\{f(a_\alpha)(x) : \alpha \in I\}$, thus $F(x) \in U$. This means F is continuous at $a \in A$ (which is a boundary point of A). Therefore we have proved that F is continuous in X. Moreover, by the definition (3.1) above for the mapping F, we do also have that $F(E) \subset co_s(f(A)) \subset Y$, where $co_s(f(A))$ is the s-convex hull of $f(A)$ in Y, for $0 < s \leq p \leq 1$. This completes the proof. □

Remark 3.2. It is clear that Dugundgji's extension theorem under the framework of p-normed spaces for $0 < s \leq p \leq 1$ above includes Theorem 6.1 of Dugundgji [14] as a special case for $s = p = 1$. In addition, we expect that this new extension theorem would be a useful tool for the study of p-normed spaces which are either infinite-dimensional or finite-dimensional for in functional analysis and topology.

Next by applying Dugundgji type extension theorem above, we are able to prove fixed point theorems in p-normed spaces for $p \in (0, 1]$, which unify the corresponding fixed point theorems in normed spaces as special case.

4. Fixed Point Theorems in p-Normed Space

The goal of this section is to establish fixed theorems in p-normed spaces by applying Dugundji's extension theorem in p-normed spaces for $p \in (0, 1]$ as a tool, which would support the study of Schauder's conjecture and related nonlinear analysis in p-vector spaces which are either finite or infinite-dimensional (see Yuan [43] for more in details). In order to establish the general fixed point theorems in p-normed spaces for $p \in (0, 1]$, the starting point is the following classical Brouwer fixed point theorem without proof as it available in almost any book on (nonlinear) functional analysis (see, Granas and Dugundji [19], Djebali [13], McLennan [28], Rudin [34], or Zeidler [44] and reference wherein).

Brouwer Fixed Point Theorem. Every continuous function from a closed ball of an n-dimensional Euclidean space $\mathbb{R}^n$ into itself has a fixed point, where n is a positive integer. In addition, if the space $\mathbb{R}^n$ is replaced by a normed space X, the conclusion still holds when the closed ball is replaced by a compact convex of X.

We now go to give some fundamental results in p-vector spaces which will be used in supporting our study for the development of fixed point theory in p-vector spaces. The first one is the following Lemma 4.1.

Lemma 4.1. Let $(X, \|\cdot\|_p)$ be a p-normed space for $0 < s \leq p \leq 1$. Then we have:
(a) The ball $B(\theta, r)$ is s-convex, where $r > 0$.
(b) If $C \subset X$ is s-convex and $\alpha \in K$, then αC is s-convex.
(c) If $C_1, C_2 \subset X$ are s-convex, then $C_1 + C_2$ is also s-convex.
(d) If $C_i \subset X, i = 1, 2, \cdots$ are all s-convex, then $\cap_{i=1}^{\infty} C_i$ is s-convex.
(e) If $A \subset X$ and $\theta \in A$, then $co_s A \subset coA$, where coA is the convex hull of A in $(X, \|\cdot\|_p)$.
(f) If C is a closed s-convex set and $0 < k < s$, then C is a closed k-convex set.

Proof. They are Lemma 2.1 of Ennassik and Taoudi [16], and also Lemma 1.4 by Xiao and Zhu [40]. The proof is complete. □

Lemma 4.2. Let $(X, \|\cdot\|_p)$ be a p-normed space for $p \in (0, 1]$ and $0 < s \leq p$, and C be an s-convex subset of X with $\theta \in C$. Let q_c be the Minkowski s-functional of C. Then we have
(a) $q_c(\theta) = 0$;
(b) q_c is positively s-homogeneous, i.e.,, $q_c(tx) = t^s q_c(x)$, for each $x \in X$ and $t > 0$;
(c) q_c is sub-additive, i.e., $q_c(x + y) \leq q_c(x) + q_c(y)$, for each $x, y \in X$;
(d) if C is bounded, then $q_c(x) > 0$ for each non-zero point $x \in X$;
(e) if C is closed, then $q_c(x)$ is lower semi-continuous and $C := \{x \in X : q_c(x) \leq 1\}$;
(f) if C is absorbing, then $q_c(x) < +\infty$ for each $x \in X$;
(g) if $\theta \in C^0$, then q_c is continuous, and $C^0 = \{x \in X : q_c(x) < 1\}$ and $\overline{C} = \{x \in X : q_c(x) \leq 1\}$.

Proof. See Jarchow [20] (or see Lemma 1.5 of Xiao and Zhu [40]). This completes the proof. □

Lemma 4.3. Let $(X, \|\cdot\|_p)$ be a p-normed space for $p \in (0, 1]$ and $0 < s \leq p$, and C be a bounded closed s-convex subset of X with $\theta \in C^0$. For each $x \in X$, we define an operator by

$$r(x) = \frac{x}{max\{1, (q_c(x))^{\frac{1}{s}}\}},$$

where q_c is the Minkowski s-functional of C in X. Then C is a retract of X and $r : X \to C$ is a continuous operator such that
(a) if $x \in C$, then $r(x) = x$;
(b) if $x \notin C$, then $r(x) \in \partial C$.

Proof. Since C is a closed and s-convex subset and $\theta \in C^0$, by Lemma 4.2, the Minkowski s-functional q_c of C is a positively s-homogeneous, sub-additive and continuous functional with $C = \{x \in X : q_c(x) \leq 1\}$ and $C^0 = \{x \in X : q_c(x) < 1\}$. It is also clear r is continuous. If $x \in C$, then $q_c(x) \leq 1$, so we have $r(x) = x \in C$; if $x \notin C$, then $q_c(x) > 1$, also we have

$$r(x) = (\frac{1}{q_c(x)})^{\frac{1}{s}} x + (1 - \frac{1}{q_c(x)})^{\frac{1}{s}} \theta \in C.$$

Hence, r is a continuous operator from X into C, C is a retract of X and (a) is valid.

If $x \notin C$, then by $q_c(x) > 1$ we have $q_c(r(x)) = q_c(\frac{x}{(q_c(x))^{\frac{1}{s}}}) = \frac{q_c(x)}{q_c(x)} = 1$, which means that $r(x) \in \{x \in X : q_c(x) = 1\} = C \setminus C^0 = \partial C$, so (b) is also true. This completes the proof. □

Now we have the following result in p-normed spaces for $p \in (0, 1]$, which includes the corresponding classic results in normed spaces (when $p = 1$) as a special case, e.g., see Theorem 7.1.21 of Djebali [13] and references wherein.

Lemma 4.4. Let $(X, \|\cdot\|_p)$ be a p-normed space for $p \in (0, 1]$ and D a non-empty closed s-convex subset of X, where $0 < s \leq p$. Then D is a retract of X.

Proof. Let $Id_D : D \to D$ be the restriction of the continuous identity mapping on D. Now by Theorem 3.1, the mapping Id_D admits a continuous extension $\overline{Id_D} : X \to X$ such that $\overline{Id_D}(X) \subset co_s(Id_D(D)) = D$ as D is closed and s-convex. Now let $F(x) = \overline{Id_D}(x)$ for each $x \in X$. Then F is an extension of Id_D and also continuous, thus F is a retraction of D. This completes the proof. □

Now let (X, T_X) and (Y, T_Y) be two topological spaces with topology structures T_X and T_Y. We first recall that a function $f : X \to Y$ is said to be bijective if and only if it is invertible; that is, a function $f : X \to Y$ is bijective if and only if there is a function $g : Y \to X$, the inverse of f, such that each of the two ways for composing the two functions produces an identity function: $g(f(x)) = x$ for each $x \in X$, and $f(g(y)) = y$ for each $y \in Y$. We now recall the following definition (see also Rudin [34] in pp.8 and pp.17).

Definition 4.5. Let (X, T_X) and (Y, T_Y) be two topological spaces and the mapping $f : X \to Y$ be a bijection. Then f is said to be homeomorphism if the bijection $f : X \to Y$ is continuous, and its inverse $f^{-1} : Y \to X$ is also continuous, with respect to the given topologies. In addition, two spaces X and Y are said to be homeomorphic.

Definition 4.6. Let (X, T_X) and (Y, T_Y) be two topological vector spaces. A linear mapping $T : X \to Y$ is said to be an isomorphism if T is one to one, and onto; and two vector spaces X and Y is said to be isomorphic.

By the definition, we note that in mathematics languages, an isomorphism is a structure-preserving mapping between two structures of the same type that can be reversed by an inverse mapping. Two mathematical structures are isomorphic if an isomorphism exists between them.

For a positive integer $n \in \mathbb{N}$, the real field $\mathbb{R}$ and complex field $\mathbb{C}$, maybe the simplest finite-dimensional Banach spaces are $\mathbb{R}^n$ and $\mathbb{C}^n$, the standard n-dimensional vector spaces over $\mathbb{R}$ and $\mathbb{C}$, respectively, normed by the mean of the usual Euclidean metric below:

If, for example, for each $z_i \in \mathbb{C}$, $i = 1, 2, \cdots, n$, $z = (z_1, \cdots, z_n)$ is a vector in $\mathbb{C}^n$, then

$$\|z\| = (|z_1|^2 + \cdots + |z_n|^2)^{\frac{1}{2}}.$$

Other norms can be defined on $\mathbb{C}^n$, for example,

$$\|z\| := |z_1| + \cdots + |z_n|, \text{ or } \|z\| := \max\{|z_i| : 1 \le i \le n\}.$$

These norms correspond, of course, to different metrics on $\mathbb{C}^n$ (when $n > 1$), but one can see very easily that they all induce the same topology on $\mathbb{C}^n$. Actually, more is true as shown by result below.

If X is a topological vector space over $\mathbb{C}$ (the complex field) with n dimensions (denoted by, $dimX = n$), then every basis of X induces an isomorphism of X onto $\mathbb{C}$. The following result shows that this isomorphism must be a homeomorphism. In other words, this says that the topology of $\mathbb{C}^n$ is the only vector topology that an n-dimensional complex ($\mathbb{C}$) topological vector space can have. We shall also see that finite-dimensional subspaces are always closed. We note that everything below discussed remain true with real scalars ($\mathbb{R}$) in place of complex ones ($\mathbb{C}$). We now have the following fundamental result (which is actually Theorem 2.21 of Rudin [34] given in pp.16-17).

Theorem 4.7. If n is a positive integer and Y is an n-dimensional subsapces of a complex (or real) topological vector space X, then we have: (a) every isomorphism of Euclidean space $\mathbb{C}^n$ (or $\mathbb{R}^n$) onto Y is a homeomorphism, and (b) Y is closed.

Proof. It is actually Theorem 1.21 of Rudin [34] and thus we omit its proof here. The proof is completes. □

Theorem 4.7 says that each finite n-dimensional $(X, \|\cdot\|_p)$ p-normed space ($p \in (0, 1]$) is (linear) homeomorphic to n-dimensional Euclidean space $\mathbb{R}^n$, here the integer $n > 0$, and $p \in (0, 1]$.

It is known that every Hausdorff topological vector spaces of finite dimension n over $\mathbb{K}$(either a real field, or a complec field) is linear homeomorphic with an Euclidean space $\mathbb{K}^n$ by Theorem 4.7 above, and so is $(X, \|\cdot\|_p)$. By following the argument used by Rudin [34], or Xiao and Wang [39], we have the following result (which is actually Lemma 2.5 of Xiao and Zhu [40]) and thus omit its proof in details.

Lemma 4.8. Let $(X, \|\cdot\|_p)$ for $p \in (0,1]$ be an n-dimensional p-normed space. Then there is a linear homeomorphism L of X into $l^1(n)$ with positive constants m, M such that

$$m\|x\|_p \leq \|Lx\|^* \leq M\|x\|_p$$

for all $x \in X$, where, $\|\cdot\|^*$ denotes the norm in terms of Euclidean metric discussed above. In addition, let L be a linear homeomorphism of $(X, \|\cdot\|_p)$ for $p \in (0,1]$ into $l^1(n)$. It is easy to see that if C is a closed s-convex set, then so is $L(C)$ and vice versa.

Proof. By following the argument used by Rudin [34] (see also proof given by Xiao and Wang [39] in pp.149-150), then the proof is complete. □

Lemma 4.9. Let $(X, \|\cdot\|_p)$ be a p-normed space for $p \in (0,1]$ and C a compact s-convex subset of X, where $0 < s \leq p$. Let $\theta \neq x_0 \in C$ and $D = C - \overline{co_s}\{x_0\}$. Then D is a compact s-convex set, $\theta \in D$, and $C \subset D$. Let $x \in D$, and $C_x = \{y \in C : x = y - tx_0\}$, for some $t \in [0,1]\}$ and

$$\alpha(x) = \min_{y \in C_x}[\|x - y\|_p / \|x_0\|_p].$$

Then $\alpha(x) \in [0,1]$ is continuous on D, and each $x \in D$ has a unique decomposition:

$$x = y(x) - \alpha x_0 \text{ for some } y(x) \in C.$$

For a given operator $T : C \to C$, the operator $T_0 : D \to D$ is defined by

$$T_0(x) = T[x + \alpha(x)x_0] - \alpha(x)x_0$$

for each $x \in D$. Then the operator T_0 has the following properties:
(a) T is continuous if and only if T_0 is continuous.
(b) T has a fixed point if and only if T_0 has a fixed point.

Proof. Indeed, it is Lemma 2.3 of Xiao and Zhu [40]. This completes the proof. □

Lemma 4.10. Let B_p and B_1 be the unit closed balls with center θ in $l^p(n)$ for $p \in (0,1]$, and $l^1(n)$, respectively. Then there is a homeomorphism H of $l^p(n)$ into $l^1(n)$ with $H(B_p) = B_1$.

Proof. For $\alpha \in K$, we denote by $sgn\alpha$ for $(sgn\alpha)|\alpha| = \alpha$. Define $H : l^p(n) \to l^1(n)$ by

$$H(\alpha_1, \cdots, \alpha_n) := (\beta_1, \cdots .\beta_n)$$

where $\beta_i = (sgn\alpha_i)|\alpha_i|^p$ for $i = 1, \cdots, n$. It is easy to verify that H is a bijective mapping. As $\Sigma_{i=1}^n |\alpha_i|^p \leq 1$ if and only if $\Sigma_{i=1}^n |\beta_i| \leq 1$, it follows that $H(B_p) = B_1$. Since the functionals $f_i(\alpha_i) = (sgn\alpha_i)|\alpha_i|^p$ and $g_i(\beta_i) = (sgn\beta_i)|\beta_i|^{\frac{1}{p}}$ are continuous in K for $i = 1, \cdots, n$, we conclude that H and H^{-1} are continuous, so H is a homeomorphism. This completes the proof. □

Lemma 4.11. Let B_p be the unit closed balls with center θ in $l^p(n)$ for $p \in (0,1]$ and $T : B_p \to B_p$ a continuous operator. Then there exists $u \in B_p$ such that $T(u) = u$.

Proof. Let H be the homeomorphism which is defined as the proof of Lemma 4.10 above and B_1 the closed unit ball with center θ in $l^1(n)$. Then $HTH^{-1} : B_1 \to B_1$ is continuous. By the classical Brouwer fixed point theorem in a finite n-dimensional space, there exists one $x \in B_1$ such that $HTH^{-1}(x) = x$. Now let $H^{-1}(x) = u$. Then $u \in B_p$, which is a fixed point of T, i.e., $T(u) = u$. The proof is complete. □

Theorem 4.12 (Brouwer fixed point in finite-dimensional p-normed spaces). Let $(X, \|\cdot\|_p)$ be a finite-dimensional p-normed space and C a bounded closed s-convex subset of X, where $0 < s \le p \le 1$. If $T : C \to C$ is continuous, then T has a fixed point in C, i.e., there exists $z \in C$ such that $T(z) = z$.

Proof. Without loss of generality we assume that $span(C) = X$, which is a finite dimensional space with $dimX = n$. Since X is linear homeomorphic with $l^p(n)$ by Lemma 4.8, we also assume that $X = l^p(n)$, which is a finite-dimensional space. Without loss of generality, we assume that the interior C^0 of the subset C is not empty, i.e., $C^0 \neq \emptyset$. By Lemma 4.9, we may assume that $\theta \in C^0$. Then we have that the Minkowski s-functional q_c of C is a positively s-homogeneous, sub-additive and continuous functional with $C = \{x \in l^p(n) : q_c(x) \le 1\}$. Since C is bounded, for $\theta \neq x \in l^p(n)$, we have $p_c > 0$, i.e., q_c is positive definite. Now define a mapping $S : B_p \to C$ for each $x \in B_p$ by

$$y = S(x) = \begin{cases} (\frac{\|x\|_p}{q_c(x)})^{\frac{1}{s}} x, & \text{if } x \neq \theta, \\ \theta, & \text{if } x = \theta; \end{cases} \tag{2}$$

where B_p is the unit closed ball with center θ in $l^p(n)$. Since $q_c(y) = (\|x\|_p / q_c(x)) q_c(x) = \|x\|_p$, we have

$$(\frac{q_c(y)}{\|y\|_p})^{\frac{1}{p}} = (\frac{\|x\|_p}{(\frac{\|x\|_p}{q_c(x)})^{\frac{p}{s}} \|x\|_p})^{\frac{1}{p}} = (\frac{q_c(x)}{\|x\|_p})^{\frac{1}{s}}, \text{ for all } \theta \neq y \in C.$$

Hence there exists the inverse $S^{-1} : C \to B_p$ given by for each $y \in C$

$$x = S^{-1}(y) = \begin{cases} (\frac{q_c(y)}{\|y\|_p})^{\frac{1}{p}} y, & \text{if } y \neq \theta, \\ \theta, & \text{if } y = \theta. \end{cases} \tag{3}$$

Clearly, by the continuity of q_c and $\|\cdot\|_p$, from (4.1) and (4.2) we see that S is continuous at $x \neq \theta$ and S^{-1} is continuous at $y \neq \theta$. Suppose that $x_n \to \theta$. By (4.1) we have $q_c(S(x_n)) = \|x_n\|_p \to \theta$. Since C is bounded, there is $M > 0$ such that $\|x\|_p \le M$ for all $x \in C$. For each $\epsilon > 0$, there is a positive integer N such that $q_c(S(x_n)) < \epsilon$ for all $n \ge N$. By the definition of q_c, there is t_n such that $0 < t_n < \epsilon$ and $S(x_n) \in t_n^{\frac{1}{s}} C$. So we have $\|S(x_n)\|_p \le t_n^{\frac{p}{s}} M < \epsilon^{\frac{p}{s}} M$, for all $n \ge N$. This shows that $S(x_n) \to \theta$. Suppose that $y_n \to \theta$. By (4.2) and the continuity of q_c, we have $\|S^{-1}(y_n)\|_p = q_c(y_n) \to \theta$. Therefore, S and S^{-1} are all continuous, and so $S : B_P \to C$ is a homeomorphism. Since $S^{-1}TS : B_p \to B_p$ is continuous, by Lemma 4.11, there exists $u \in B_p$ such that $S^{-1}TS(u) = u$. Let $z := S(u)$. Then, we have $z \in C$ and $T(z) = z$, which is the fixed point of T. The proof is complete. □

The rest part of this section is to establish fixed point theorems for s-convex subsets in p-normed spaces by applying the existence of homeomorphisms for s-convex subsets in p-normed spaces, with a different proof by comparing those in the existing literature, where $s, p \in (0, 1]$.

By Proposition 2 of Shapiro [35], for a space l^p, where $p \in (0, 1]$, we have the following result which is often called the universal property of the space l^p.

Lemma 4.13. Every complete, separable p-normed space for $p \in (0, 1]$ is a continuous linear image of l^p.

By Lemma 4.13 and the space decomposition approach, we have the following result (see also Corollary of Theorem 4.3.1 by Wang [37]; and also see close related Corollary 2.11 in pp.26 given by Kalton et al.[23]). For its completeness and convenience of readers, we give its proof here below.

Lemma 4.14. Let $(X, \|\cdot\|_p)$ be a complete and separable p-normed space for $p \in (0, 1]$. Then there exists a closed subspace l_X^p of l^p and a linear operator $T : l^p \to X$ such that $T : l_X^p \to X$ is a homeomorphism.

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Proof. By Lemma 4.13, there exists a continuous linear operator $T : l^p \to X$. We denote the kernel (null) space of T by $kerT := \{x \in l^p : Tx = 0\}$, then $kerT$ is closed as T is continuous. Assume E is the closed subspace of l^p such that the (topological) direct sum (for the decomposition) of l^p is given by $l^p := kerT + E$. Then T is a linear one to one continuous mapping from E to X. Now let $l^p_X := \{x \in E : T(x) \in X\}$. Then l^p_X is a closed subspace of l^p. We still denoted by T as the restriction of the mapping T on l^p_X, and T is a linear and one to one continuous mapping from l^p_X onto X, thus it is a homeomorphism. This completes proof. □

Now we have the following result for the existence of a homeomorphism for a given non-empty s-convex compact subset in p-normed space $(X, \|\cdot\|_p)$ with $0 < s \leq p \leq 1$.

Theorem 4.15. Let $(X, \|\cdot\|_p)$ be a complete p-normed space and D a s-convex compact subset of X, where $0 < s \leq p \leq 1$. Then there exists a linear operator $F : D \to l^p$ such that $F(D) \subset C_s$ and $F : D \to F(D)$ is a homeomorphism, where

$$C_s := \{x = (\alpha_1, \cdots, \alpha_n, \cdots) \in l^p : \ |\alpha_n|^s \leq 1/n^2\}.$$

Proof. Let $Y := \overline{\text{span}D}$. Since D is compact, thus Y is separable. As Y is a closed subspace of X, Y is also complete. By Lemma 4.14, there exists a closed subspace l^p_Y of l^p and a linear operator $T : l^p \to Y$ such that $T : l^p_Y \to Y$ is a homeomorphism. Let $F = T^{-1}$. Then $F : D(\subset l^p) \to l^p$ is a linear operator, and continuous such that $F : D \to F(D)$ is a homeomorphism, thus $F(D)$ is also s-convex, and compact by the continuity of the mapping F from $(l^p, \|\cdot\|_p)$ to $(l^p, \|\cdot\|_p)$. In addition, it is easy to verify that $F(D) \subset C_s$ is true. This completes the proof. □

Remark 4.16. Here, we first would like to point out one very important fact that when $0 < s < p \leq 1$, there is no linear continuous mappings from l^p to l^s (resp., $L^p[0,1]$ to $L^s[0,1]$), but we do have the existence of a homeomorphism for a s-convex compact in the p-normed space $(X, \|\cdot\|_p)$ due to the universal property of l^p for $0 < s \leq p \in (0,1]$. Secondly, we know the dual space $(L^p[0,1])^*$ of $L^p[0,1]$ is with only zero element when $p \in (0,1)$, which means that the linear topological space $L^p[0,1]$ has no separability property, so Theorem 2.1 of Bessaga and Pełczyński [6] is not applicable to claim the non-existence of homeomorphisms by using Roberts' example in [33] for a compact convex set with no extreme points in (not separable) space $L^p[0,1]$ to derive her/his Corollary 2.5 by Yu in [41]!

We now have the continuous extension result in p-normed spaces for $p \in (0,1]$ below.

Lemma 4.17. Let $(X, \|\cdot\|_p)$ be a complete separable p-normed space for $p \in (0,1]$, and D a bounded closed subset of X. Let $T : D \to X$ be a continuous bounded operator. Then T has a continuous extension $S : X \to X$ such that $S(X) \subset \overline{co_s}(T(D))$.

Proof. By Theorem 3.1, the conclusion follows. This completes the proof. □

By Theorem 4.12, we have a fixed point theorem in l^p space for $p \in (0,1]$ below.

Lemma 4.18. Let C_s be the closed cuboid in l^p defined by

$$C_s := \{x = (\alpha_1, \cdots, \alpha_n, \cdots) \in l^p : \ |\alpha_n|^s \leq 1/n^2\}$$

and $T : C_s \to C_s$ a continuous operator, where $0 < s \leq p \leq 1$. Then C_s is s-convex and compact, and there exists $z \in C_s$ such that $T(z) = z$.

Proof. It is easy to verify that the set C_s is a compact subset of l^p, and C_s is s-convex. For each n, we define an operator P_n on l^p by

$$P_n(\alpha_1, \cdots, \alpha_n, \cdots) = (\alpha_1, \cdots, \alpha_n, 0, \cdots)$$

for each $(\alpha_1, \cdots, \alpha_n, \cdots) \in l^p$. Then P_n is linear and continuous, and $P_n(C_s) \subset C_s$. It follows from s-convexity and compactness of C_s that $P_n(C_s)$ is an s-convex and compact subset of $l^p(n)$. Since P_nT is continuous and $P_nT(P_n(C_s)) \subset P_nT(C_s) \subset P_n(C_s)$, by Theorem 4.12 (which is Brouwer fixed point theorem in a finite dimensional space $l^p(n)$), there is a point $x_n \in P_n(C_s)$ such that $P_nT(x_n) = x_n$. Let $T(x_n) = \{\alpha_i^{(n)}\}_{i=1}^{\infty}$. Then by the definition of P_n, we have

$$\|x_n - T(x_n)\|_p = \|P_nT(x_n) - T(x_n)\|_p = \Sigma_{i=n+1}^{\infty}|\alpha_i^{(n)}|^p \leq \Sigma_{i=n+1}^{\infty}|\alpha_i^{(n)}|^s \leq \Sigma_{i=n+1}^{\infty}\frac{1}{i^2}.$$

Now $\{T(x_n)\}$ is a sequence in the compact set C_s, so there is a point $z \in C_s$ and a subsequence $\{T(x_{n_j})\}$ such that $T(x_{n_j}) \to z$. By above inequality, it follows that $x_{n_j} \to z$. Since T is continuous, we have $T(z) = \lim_{j\to\infty} T(x_{n_j}) = z$, which is the desired conclusion. The proof is complete. □

Now we have the following fixed point theorem in complete p-normed spaces which are either infinite-dimensional or finite-dimensional for $s, p \in (0, 1]$.

Theorem 4.19. Let $(X, \|\cdot\|_p)$ be a complete p-normed space and C a compact s-convex subset of X, where $s, p \in (0, 1]$. If $T : C \to C$ is continuous, then there exists $z \in C$ such that $Tz = z$.

Proof. We prove this result by following two steps below.

The first step: we prove the conclusion for $p \in (0, 1]$ and $0 < s \leq p \leq 1$. By Theorem 4.15, there exists a linear homeomorphism $F : C \to F(C)$ such that $F(C) \subset C_s$, where C_s is an s-convex compact subset of l^p defined by Theorem 4.15 (see also Lemma 4.18), then $F(C)$ is s-convex compact. Since T is continuous, the mapping $FTF^{-1} : F(C) \to F(C)$ is also continuous. By Lemma 4.17, FTF^{-1} has a continuous extension $S : C_s \to C_s$ such that $S(C_s) \subset \overline{co_s}F(C)$. By Lemma 4.18, which is a fixed point theorem for a compact s-convex subset C_s in l^p spaces (where, $0 < s \leq p \leq 1$), there is $u \in C_s$ such that $u = S(u) \in F(C)$, and so $u = S(u) = FTF^{-1}(u)$. Let $z = F^{-1}(u)$. It follows that $z \in C$ and $T(z) = z$, which is a fixed point of the mapping T.

The second step: we prove the case $0 < p < s < 1$ and $0 < p < s = 1$. For the case $0 < p < s < 1$, by Lemma 2.1, C is also p-convex. By applying the conclusion given in the first step above, it follows that there exists $x \in C$ such that $T(x) = x$.

Now the only case left to prove is for $0 < p < s = 1$, which means C is convex. We choose an arbitrary $x_0 \in C$ and let $C_0 := \{x - x_0 : x \in C\}$. Then it is clear that C_0 is a compact convex subset of X which contains the zero element. By Lemma 2.1 again, we conclude that C_0 is s-convex for any $s \in (0, 1)$. Now, define the mapping $S : C_0 \to C_0$ by $S(x - x_0) = T(x) - x_0$ for each $x - x_0 \in C_0$. Clearly, S is continuous, and applying the result of the first step to S, we conclude that there exists $x \in C$ such that $S(x - x_0) = x - x_0$, it implies that $x = T(x)$, which is a fixed point of T. This completes the proof. □

In order to prove Theorem 4.19 also holds for a p-normed space which may not be complete for $s, p \in (0, 1]$, we need to recall the following result (see also Theorem 2.2 of Ennassik and Taoudi [16]).

Lemma 4.20. Let $(X, \|\cdot\|_p)$ be a p-normed space, where $p \in (0, 1]$. Then there is a complete p-normed space $\hat{X}$ and a linear isometry mapping i from X onto the subspace $W := i(X)$ which is the dense in $\hat{X}$.

Proof. It is Theorem 2.2 of Ennassik and Taoudi [16]. Its proof completely follows the way for the proof of Theorem 2.3.2 by Kreyszig [26].This completes the proof. □

As applications of Theorem 4.19 and Lemma 4.20, we have the following general fixed point theorem in p-normed spaces which may not be complete.

Theorem 4.21. Let $(X, \|\cdot\|_p)$ be a p-normed space and C a compact s-convex subset of X, where $s, p \in (0, 1]$. If $T : C \to C$ is continuous, then there exists $z \in C$ such that $Tz = z$.

Proof. Let $\hat{X}$ be the completion of X. By Lemma 4.20, there exists a linear isometric embedding $i : X \to \hat{X}$ with $i(X)$ dense in X. We define $\hat{T} : i(C) \to i(C)$ by $\hat{T}(i(x)) = i(T(x))$ for each $x \in C$. Then

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this mapping is easily checked to be well defined, and it is continuous since i is a linear isometry and T is continuous on C. Furthermore, the set $i(C)$ is compact, being the image of a compact set under a continuous mapping. It is also s-convex as it is the image of an s-convex set under a linear mapping.

Now by Theorem 4.19, there exists $x \in C$ such that $\hat{T}(i(x)) = i(x)$. Thus, $i(T(x)) = i(x)$ so $T(x) = x$, which means T has a fixed point in C. This completes the proof. □.

Let $p = 1$ in Theorem 4.21, we have the following fixed point theorem in normed spaces for non-empty compact s-convex subset, where $0 < s \leq 1$.

Theorem 4.22. Let $(X, \|\cdot\|)$ be a normed space and C a non-empty compact s-convex subset of X, where, $0 < s \leq 1$. If $T : C \to C$ is continuous, then there exists $z \in C$ such that $Tz = z$.

Remark 4.23. By applying Theorem 4.19 and Lemma 4.20, we establish general fixed point theorems for s-convex subsets in p-normed spaces, which is Theorem 4.21 above for $s, p \in (0, 1]$ by using Dugundgji's extension theorem in p-normed spaces for $p \in (0, 1]$ given in Appendix A below as a powerful tool, with the application of the existence of homeomorphisms for s-convex subsets in p-normed spaces. Here we also like to share with readers that the proof for the existence of homeomorphisms for s-convex subsets in p-normed spaces is new, and different from those existing literature (e.g., see Xiao and Zhu [40] and references wherein). By a fact that each p-normed space includes normed spaces as a special class (with $p = 1$), the main result Theorem 4.21 in this paper unifies corresponding results in the existing literature, for more in details, see Agarwal et al.[1], Agarwal et al.[2], Ben-El-Mechaiekh [5], Browder [7], Cauty [8–10], Chang et al.[11], Djebali [13], Du [12], Ennassik et al.[15], Fan [17], Granas and Dugundji [19], Mauldin [27], McLennan [28], O'Regan [29], Park [30,31], Xiao and Zhu [40], Zeidler [44], and references wherein. In addition, for the fixed point theorem in p-vector and locally p-convex spaces, we refer to Ennassik et al.[15], Ennassik and Taoudi [16], and Yuan [42,43]'s recent works for more in details.

5. Conclusion

In this paper, by establishing Dugundgji's extension theorem in p-normed spaces for $p \in (0, 1]$ a powerful tool, with the applications for the existence of homeomorphisms for s-convex subsets in p-normed spaces, where $s, p \in (0, 1]$, we establish the general fixed point theorems for p-normed spaces, which include corresponding Schauder fixed point theorem in normed spaces as a special class; in addition, these now fixed point theorems in p-normed spaces seem interesting and would be important tools for the nonlinear analysis under the framework of p-vector spaces for $p \in (0, 1]$.

The new fixed point theorems established in this paper would provide a general foundation for the development of fixed point theory in p-vector spaces for $p \in (0, 1]$. The results given by this paper provide fundamental tools for the study of Schauder's conjecture and related general fixed point theory in p-vector spaces, as shown by Cauty [8–10], Ennassi and Taoudi [16], Granas and Dugundji [19], Mauldin [27], Park [30,31], Yuan [42,43] and related references wherein for more discussion in details. Once again, we like to share with readers that how important the Dugundgji type extension and fixed point theorem in p-normed spaces established in this paper, can be found from Yuan's recent work in [42,43] and related discussion wherein.

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References

1. R.P. Agarwal, M. Meehan and D. O'Regan, *Fixed Point Theory and Applications*, Cambridge Tracts in Mathematics, vol. 141, Cambridge University Press, Cambridge, 2001.
2. R.P. Agarwal, E. Karapinar, D. O'Regan, and A. Roldán-López-de-Hierro, *Fixed Point Theory in Metric Type Spaces*, Springer, Cham., 2015.
3. V.K. Balachandran, *Topological Algebras*, vol. 185, Elsevier, Amsterdam, 2000.

4. A. Bayoumi, *Foundations of complex analysis in non locally convex spaces: Function theory without convexity condition*, North-Holland Mathematics Studies, Vol. 193, Elsevier Science B.V., Amsterdam, 2003.
5. H. Ben-El-Mechaiekh, *Approximation and selection methods for set-valued maps and fixed point theory*, in Fixed Point Theory, Variational Analysis, and Optimization, Saleh A. R. Al-Mezel, Falleh R. M. Al-Solamy and Qamrul H. Ansar (eds.), CRC Press, Boca Raton, 2014, pp.77–136.
6. C. Bessaga and A. Pełczyński, *Selected Topics in Infinite-Dimensional Topology*, Volume 58, PWN-Polish Scientific Publishers, Warsaw, Poland, 1975.
7. F.E. Browder, *Fixed point theory and nonlinear problems*, Bull. Am. Math. Soc.(N.S.) **9(1)**(1983), 1–39.
8. R. Cauty, *Solution du problème de point fixe de Schauder*, (French) [*Solution of Schauder's fixed point problem*], Fund. Math. **170(3)**(2001), 231–246.
9. R. Cauty, *Rétractes absolus de voisinage algébriques*, (French) [*Algebraic absolute neighborhood retracts*], Serdica Math. J. **31(4)**(2005), 309–354.
10. R. Cauty, *Le théoréme de Lefschetz - Hopf pour les applications compactes des espaces ULC*, (French) [*The Lefschetz - Hopf theorem for compact maps of uniformly locally contractible spaces*], J. Fixed Point Theory Appl. **1(1)**(2007), 123–134.
11. S.S. Chang, Y.J. Cho, S. Park and G.X. Yuan, *Fixed point theorems for quasi upper semicontinuous set-valued mappings in p-vector and locally p-convex spaces*, in Advanced Mathematical Ananlysis and its Applications, Pradip Debnath, Delfim F. M. Torres and Yeol Je Cho (eds), CRC Press, Boca Raton, FL., USA, 2023, Chapter 5, pp.57–72. DOI.https://doi.org/10.1201/9781003388678
12. W.-S. Du, *On coincidence point and fixed point theorems for nonlinear multivalued maps*, Topology Appl. **159(1)**(2012), 49–56.
13. S. Djebali, *Algebraic Topology: Constructions, Retractions, and Fixed Point Theory*, De Gruyter, Berlin, 2024.
14. J. Dugundji, *Topology*, Allyn and Bacon Series in Advanced Mathematics, Allyn and Bacon, Inc., Boston, 1978.
15. M. Ennassik, L. Maniar and M.A. Taoudi, *Fixed point theorems in r-normed and locally r-convex spaces and applications*, Fixed Point Theory **22(2)**(2021), 625–644.
16. M. Ennassik and M.A. Taoudi, *On the conjecture of Schauder*, J. Fixed Point Theory Appl. **23(4)** (2021): Paper (article) No.52.
17. K. Fan, *Fixed-point and minimax theorems in locally convex topological linear spaces*, Proc. Nat. Acad. Sci. U.S.A. **38**(1952), 121–126.
18. K. Goebel and W.A. Kirk, *Topics in Metric Fixed Point Theory*, Cambridge Stud. Adv. Math., 28, Cambridge University Press, Cambridge, 1990.
19. A. Granas and J. Dugundji, *Fixed Point Theory*, Springer Monographs in Mathematics (SMM), Springer-Verlag, New York, 2003.
20. H. Jarchow, *Locally Convex Spaces*, Stuttgart: B.G. Teubner, 1981.
21. N.J. Kalton, *Compact p-convex sets*, Q.J. Math. Oxf. Ser. **28(2)**(1977), 301–308.
22. N.J. Kalton, *Universal spaces and universal bases in metric linear spaces*, Studia Math. **61**(1977), 161–191.
23. N.J. Kalton, N.T. Peck and J.W. Roberts, *An F-space Sampler*, London Math. Soc., Lecture Note Series 89, London, UK, 1984.
24. W.A. Kirk, *An abstract fixed point theorem for nonexpansive mappings*, Proc. Amer. Math. Soc. **82(4)**(1981), 640–642.
25. V. Klee, *Leray-Schauder theory without local convexity*, Math. Ann. **141**(1960), 286–296.
26. E. Kreyszig, *Introductory Functional Analysis with Applications*, Wiley Classics Library, Wiley, New York, 1989.
27. R.D. Mauldin, *The Scottish book, Mathematics from the Scottish Café with Selected Problems from the New Scottish Book*, Second Edition, Birkhauser, 2015.
28. A. McLennan, *Advanced Fixed Point Theory for Economics*, Springer, New York, 2018.?
29. D. O'Regan, *Continuous selecting families and collectively fixed point theory*, Aequationes Math. **97(4)**(2023), 871–881.
30. S. Park, *The KKM principle implies many fixed point theorems*, Topology Appl. **135(1-3)**(2004), 197–206.
31. S. Park, *One hundred years of the Brouwer fixed point theorem*, J. Nat. Acad. Sci., ROK, Nat. Sci. Ser. **60(1)**(2021), 1–77.
32. J. Qiu and S. Rolewicz, *Ekeland's variational principle in locally p-convex spaces and related results*, Stud. Math. **186(3)**(2008), 219–235.
33. J.W. Roberts, *A compact convex set with no extreme points*, Studia Math. **60(3)** (1977), 255 - 266.
34. W. Rudin, *Functional Analysis*, 2nd edition, McGraw-Hill, New York, 1991.

 doi:10.20944/preprints202502.2199.v1

35. J.H. Shapiro, *Examples of proper, closed, weakly dense subspaces in non locally convex F-spaces*, Israel J. Math. **7**(1969), 369–380.
36. W. Takahashi, *Nonlinear Functional Analysis*, Yokohama Publishers, Yokohama, Japan, 2000.
37. J. Y. Wang, *An introduction to locally p-convex spaces*, Scientific Press, Beijing, pp.1–81, 2013.
38. J. Y. Wang, *The subrepresentation theorem of the conjugate cone of* $l^p(X)$ $(0 < p < 1)$, Adv. Math.(China). **39(6)**(2010), 709 - 718.
39. J.-Z. Xiao and Z.Y. Wang, *Foundations of Abstract Analysis*, 2nd edition (in Chinese), Tsinghua University Press, Beijing, 2022.
40. J.-Z. Xiao and X.H. Zhu, *Some fixed point theorems for s-convex subsetes in p-nomed spaces*, Nonlinear Anal. **72**(2011), 1738–1748.
41. L. Yu, *Remarks on "Some fixed point theorems for s-convex subsets in p-normed spaces"*, J. Anal. **32(2)** (2024), 1139 - 1143.
42. G.X. Yuan, *Fixed point theorems and applications in p-vector spaces*, Fixed Point Theory Algorithms Sci Eng. **2023**(2023): Paper (Article) No.10. https://doi.org/10.1186/s13663-023-00747-w
43. G.X. Yuan, *Fixed Point Theory in p-Vector Spaces* (in press), World Scientific, London and Singapore, 2025.
44. E. Zeidler, *Nonlinear Functional Analysis and Its Applications I, Fixed-Point Theorems*, Springer, Berlin, 1993.