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Article

A Novel Prescribed-Time Convergence Acceleration Algorithm with Time Rescaling

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Abstract: In this paper, a novel prescribed-time convergence acceleration algorithm with time rescaling is presented. Two prescribed-time acceleration algorithms are constructed by introducing time rescaling, and the acceleration algorithms are used to solve unconstrained optimization problems and optimization problems containing equation constraints, respectively. Some important theorems are given and the convergence of the acceleration algorithms is proved using the Lyapunov functions method. Finally, we provide numerical simulations to verify the effectiveness and rationality of our theoretical results.

Keywords: exponential prescribed-time convergence; acceleration algorithm; time rescaling; optimization problem; Lyapunov function

MSC: 65K10

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1. Introduction

Machine learning stands as a predominant approach to addressing numerous artificial intelligence challenges in the present day. It has a wide range of applications across various domains, including but not limited to computer vision, natural language processing, secure communication, and search technologies. With the development of the Internet, how to deal with huge data sets has become a topic of concern for scholars. As a result, many requirements are placed on the convergence speed of machine learning, and gradient descent-based algorithms naturally attract attentions.

The origins of the earliest accelerated algorithms[1] can be traced to the heavy-ball method proposed by Polyak in 1964. This method achieved a local linear convergence rate through spectral analysis. Since it was difficult to guarantee global convergence for the heavy ball method, Nesterov later proposed an accelerated gradient (NAG) method[2] using estimated sequences in 1983, which reduced the complexity of the classical gradient descent method and showed the worst-case convergence speed of $O(1/k^2)$ at minimising smooth convex functions. To further advance the development of the acceleration algorithm, Nesterov proposed methods with convergence speed of $O(1/k^2)$ for a class of unconditionally minimizing smooth convex functions in [3]. A universal method for developing optimal algorithms aimed at minimizing smooth convex functions was presented in [4]. Thereafter, the heavy-ball method and the accelerated gradient method had attracted the attention of numerous scholars. In 1994, Pierro et al.[5] proposed a method to speed up iterative algorithms for solving symmetric linear complementarity problems. At the same time, Arihiro et al.[6] proposed an enhancement to the error backpropagation algorithm, which was widely used in multilayer neural networks, by incorporating prediction to improve its speed. However, these methods did not garner significant interest within the machine learning community. It wasn't until Beck and Teboulle[7] introduced the accelerated proximity gradient (APG) method in 2009, aimed at solving composite optimization problems, including sparse and low-rank models. This method was an extension of [4] and simpler compared to [8]. As it happens, the sparse and low-rank models are common in machine learning. The accelerated proximity gradient (APG) method had been widely noticed in the field of machine learning.

However, one of the drawbacks of the methods based on the Nesterov's method was exhibited an oscillatory behavior, which could seriously slow down its convergence speed. In order to avoid this

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phenomenon, many scholars had made a lot of attempts. B.O'Donoghue and E.Candès[9] introduced a function restart strategy and a gradient restart strategy to enhance the convergence speed of Nesterov's method. Further, Nguyen et al.[10] proposed the accelerated residual method, which could be regarded as a finite-difference approximation of the second-order ODE system. The method was shown to be superior to Nesterov's method and was extended to a large class of accelerated residual methods. These strategies successfully mitigated the oscillatory convergence associated with Nesterov's method. The convergence of the above accelerated algorithms is mainly asymptotic, i.e., the solution of the optimization problem or the ODE problem is obtained when the time tends to infinity.

It is well known that finite-time convergence has made great progress in dynamical systems, where the convergence time is linked to the system's initial conditions. In addition, there is another type of convergence which is fixed time convergence. Fixed time convergence that is independent of the initial value have yielded. It is possible to estimate the upper limit of the settling time without relying on any data regarding the initial conditions. While fixed time convergence offers numerous benefits for estimating when a process will stop, there's a deficiency in establishing a straightforward and lucid link between the control parameters and the target maximum stopping time. This frequently results in an overestimation of the stopping time, which in turn misrepresents the system's performance. Additionally, the stopping time isn't a parameter that can be directly adjusted in finite time or fixed time convergence scenarios, since it's also influenced by the design parameters of other control systems. In order to address the challenge of excessive estimations regarding the stopping time, as well as to alleviate the dependence of the stopping time on the design parameters, a prescribed time convergence has been generated[11]. In other words, the system is capable of achieving stability within a predetermined timeframe, irrespective of the initial conditions. It should be highlighted that the integration of prescribed time convergence with optimization problems is likewise a fascinating area of study [12–14].

In practice, many dynamical systems are related to time. The time rescaling is a concept that involves time transformation. Within the realm of non-autonomous dissipative dynamic systems, adjusting the time parameter is a simple yet effective approach to expedite the convergence of system trajectories. As noted in the unconstrained minimization problem in [15–17] and the linear constrained minimization problem in [18,19], the time rescaling parameter has the effect of further increasing the rate of convergence of the objective function values along the trajectory. Balhag et al.[20] developed fast methods for convex unconstrained optimization by leveraging inertial dynamics that combine viscous and Hessian-driven damping with time rescaling. Hulett et al. in their work [21], introduced the time rescaling function that resulted in improved convergence rates. This achievement can also be considered as a further development of the time rescaling approach that was presented for constrained scenarios in [15,16].

Based on the above facts, we propose a novel prescribed-time convergence acceleration algorithm with time rescaling. A distinctive aspect of this paper is the utilization of time rescaling to integrate the concept of prescribed time with second-order systems for tackling optimization problems. This enables the optimization problem to achieve convergence to the optimal value within a prescribed time. What's more, several second-order systems with respect to t for unconstrained optimization problems and optimization problems containing equational constraints was in [22–25]. Under these second-order systems, the above optimization problems yielded asymptotic time convergence. In contrast to [22–25], the contributes of this paper as follows:

(1) We obtain prescribed time convergence rate $e^{-a \cdot M(t)}$, where $0 < a < +\infty$, and $d(s)$ is a positive function, T is a prescribed time, as $t \to \mathrm{T}$, there is

$$M(t) = \int_0^t d(s)\mathrm{d}s \to +\infty.$$

(2) In some cases, the use of time rescaling improves the convergence of the optimization algorithm. So we use $\alpha(s)$ transform the time t into an integral form, i.e. $t = t(\delta) = \int_0^\delta \alpha(s)\mathrm{d}s$, where $\alpha(s)$ is a

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continuous positive function. In this way, the second-order system we construct becomes more flexible and the rate of convergence is improved to certain extent. We give different $\alpha(s)$ and verify the validity of the results with numerical simulations.

The organization of the subsequent sections in this paper is as follows. Section 2 provides a concise overview of the fundamental concepts utilized throughout this paper. In section 3 and 4, we design new algorithms that allow unconstrained optimization problems and optimization problems with equational constraints to converge to the optimum within a prescribed time. We give corresponding examples for different time rescalings in section 5. Numerical simulations are given in section 6. Conclusions and future work done are given in section 7.

2. Preliminaries

Consider the Hilbert space V, and $f : V \to R$ is a properly μ-strongly convex differentiable smooth function. Furthermore, the space V is furnished with an inner product $(\cdot, \cdot)$ and the resulting norm $\|\cdot\|$. The notation $\langle \cdot, \cdot \rangle$ represents the duality pairing between V^* and V, where V^* is the continuous dual space of V, equipped with the standard dual norm $\|\cdot\|_*$. For the sake of simplicity, we consider the real numbers space R^n, whose Euclidean norm is defined as $\|x\|$ for $x \in R^n$. For a given function f, let $x^\star$ is its optimal value point for the optimization problem and the corresponding optimal value is $f(x^\star)$. In addition, the lemmas used in this paper are as follows

Lemma 1 ([23]). *For any $u = u(t), v = v(t), w = w(t) \in R^n$, we have*

$$(u - w, w - v) = \frac{1}{2}(\|u - v\|^2 - \|u - w\|^2 - \|w - v\|^2).$$

Lemma 2 ([23]). *If f is a μ-strongly convex differentiable function, then for any $x = x(t), y = y(t) \in \Omega$, where Ω is the domain of definition of f, we have*

$$f(x) - f(y) \geq \langle \bigtriangledown f(y), x - y \rangle + \frac{\mu}{2}\|x - y\|^2.$$

Assumption 1. *The function $\alpha(s)$ is a continuous positive function, i.e. $\alpha(s) > 0$.*

Assumption 2. *When $\alpha(s) > 0$ holds, let the relationship between t and δ have*

$$t = t(\delta) = \int_0^\delta \alpha(s)ds, \tag{1}$$

i.e. (1) is a time rescaling. And the above equation satisfies that when $\delta \to +\infty$, $t \to T$ holds, where T is positive number. One has

$$t'(\delta) = \alpha(\delta). \tag{2}$$

Also, the inverse function

$$\delta = \delta(t) = t^{-1}(t(\delta)) \tag{3}$$

obtained from the above equation satisfies that when $t \to T$, there is $\delta \to +\infty$.

Assumption 3. *When $\alpha(s) > 0$ holds, let the function*

$$m(\delta) = \int_0^\delta \frac{1}{\alpha(s)}ds \tag{4}$$

and the above equation satisfies when $\delta \to +\infty$, there is $m(\delta) \to +\infty$.

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Assumption 4. *In the case of* $\alpha(\delta)>0$ *and (1),*

$$\alpha(\delta) \cdot d(t) = \frac{1}{\alpha(\delta)} \tag{5}$$

holds.

Assumption 5. *In the case of satisfying* $d(s)>0$*, let the function*

$$M(t) = \int_0^t d(s)ds \tag{6}$$

and the above equation satisfies that when $t \to T$*, there is* $M(t) \to +\infty$*.*

In [23], for the unconstrained optimization problem

$$\min_{x(t)\in V} f(x(t))$$

of a smooth function f on the entire space V, the second-order ODE constructed to the optimization problem is

$$\gamma(t) \cdot x^{''}(t) + (\mu + \gamma(t)) \cdot x^{'}(t) + \bigtriangledown f(x(t)) = 0.$$

The aforementioned system can be transformed into the following system of first-order ordinary differential equations (ODEs)

$$\begin{cases} x^{'}(t) = v(t) - x(t), \\ \gamma(t) \cdot v^{'}(t) = \mu(x(t) - v(t)) - \bigtriangledown f(x(t)), \end{cases} \tag{7}$$

meantime,

$$\gamma^{'}(t) = \mu - \gamma(t).$$

In [25], the author used primal-dual methods to study the convex optimization problem constrained by linearity

$$\begin{gathered} min f(x(t)) \\ s.t. Ax(t) = b. \end{gathered}$$

Specially, when f is a smooth function, the system constructed to the optimization problem containing the equation constraints is

$$\begin{cases} \theta(t) \cdot \lambda^{'}(t) = \bigtriangledown_{\lambda} L_{\beta}(v(t), \lambda(t)), \\ x^{'}(t) = v(t) - x(t), \\ \gamma(t) \cdot v^{'}(t) = \mu_{\beta}(x(t) - v(t)) - \bigtriangledown_{x} L_{\beta}(x(t), \lambda(t)), \end{cases} \tag{8}$$

meanwhile,

$$\theta^{'}(t) = -\theta(t),$$

$$\gamma^{'}(t) = \mu_{\beta} - \gamma(t).$$

Different second-order systems are constructed for the above two types of optimization problems. Then by introducing different Lyapunov functions, both optimization problems reach asymptotic convergence, i.e. when $t \to +\infty$, there is $f(x) \to f(x^{\star})$.

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Remark 1. *Under certain conditions, the auther[26] used the generalized finite time gain function concerning the variable τ getting when $\tau \to T$ there is $t \to +\infty$. The result obtained is also asymptotically convergent. Inspired by the literature [26], we modify the above two systems to get when $t \to T$, there is $f(x) \to f(x^\star)$, where T is a prescribed time.*

Remark 2. *Instead of giving the second-order system about the variable t directly, we use the time rescaling in [23] to construct the system about the variable δ firstly. By setting up the coefficient relationship between the two systems and applying this relationship, we give the second-order system about the variable t indirectly.*

3. For Unconstrained Optimization Problems

In this subsection, we construct a category of second-order systems designed to address the unconstrained optimization issue (A_1), ensuring that the solution converges to the optimal outcome within a prescribed time under the influence of these second-order systems.

We consider the unconstrained optimization problem

$$\min_{x \in R^n} f(x), \tag{A_1}$$

where $x = x(t) \in R^n$, $f : R^n \to R$, and f is a μ-strongly convex differentiable smooth function. Let its optimal solution is $x^\star$.

Based on the ODE theory, which can provide deeper insights into optimization, we aim to design a second-order system of the following form, that is,

$$\begin{cases} x'(t) = a \cdot d(t) \cdot (-x(t) + v(t)) \\ v'(t) = a \cdot d(t) \cdot \left(\frac{\mu}{\gamma(t)} \cdot (x(t) - v(t)) - \frac{1}{\gamma(t)} \cdot \bigtriangledown f(x(t))\right) \end{cases} \tag{9a}$$

where $t \in [0, T)$, $\gamma(t)$ is a positive function, $T > 0$, $0 < a < +\infty$, $\gamma(0) = \gamma_0 > 0$ and

$$\gamma^{'}(t) = a \cdot d(t) \cdot (\mu - \gamma(t)). \tag{9b}$$

Additionally, our objective is to identify an appropriate $d(t)$ such that (A_1) achieves convergence to the optimum solution within a prescribed time under the influence of equation (9), that is as $t \to T$, there is $f(x(t)) \to f(x^\star)$. In order to find the right $d(t)$, we need to do the following.

Firstly, the variable transformation is used to change the optimization problem (A_1) into the following equivalent optimization problem (A_2). Details are as follows.

Using the relationship between t and δ, we can get

$$x(t) = x(t(\delta)) = y(\delta). \tag{10}$$

Substituting (10) into (A_1), we get the optimization problem

$$\min_{y \in R^n} f(y), \tag{A_2}$$

equivalent to (A_1), where y=y(δ), $f : R^n \to R$, and f is a μ-strongly convex differentiable smooth function. The optimum solution of the optimization problem (A_2) is also $f(x^\star)$.

Secondly, under certain conditions, a second-order system is constructed to solve the optimization problem (A_2), so that (A_2) converges asymptotically to the optimal solution under the action of the system. Details are as follows.

Inspired by [23], a second-order system is constructed for the optimization problem (A_2), that is,

$$\begin{cases} y^{'}(\delta) = a \cdot h(\delta) \cdot (-y(\delta) + w(\delta)) \\ w^{'}(\delta) = a \cdot h(\delta) \cdot \left(\frac{\mu}{p(\delta)} \cdot (y(\delta) - w(\delta)) - \frac{1}{p(\delta)} \cdot \bigtriangledown f(y(\delta))\right), \end{cases} \tag{11a}$$

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$\delta \in [0, +\infty)$, $p(\delta)$ and $h(\delta)$ are positive functions, $0 < a < +\infty$, $p(0) = p_0 = \gamma_0 > 0$ and

$$p'(\delta) = a \cdot h(\delta) \cdot (\mu - p(\delta)). \tag{11b}$$

Again applying the variable transformation

$$\begin{cases} w(\delta) = v(t(\delta)) \\ p(\delta) = \gamma(t(\delta)) \end{cases} \tag{12}$$

along with (2), (9) and (11), we get

$$\begin{cases} y'(\delta) = x'(t) \cdot t'(\delta) = \alpha(\delta) \cdot x'(t) \\ w'(\delta) = v'(t) \cdot t'(\delta) = \alpha(\delta) \cdot v'(t) \\ p'(\delta) = \gamma'(t) \cdot t'(\delta) = \alpha(\delta) \cdot \gamma'(t). \end{cases} \tag{13}$$

Then (9), (11) are analysed, we found that there is a result of

$$h(\delta) = \alpha(\delta) \cdot d(t)$$

Substituting Assumptiom 4 into the above equation, we find that there is

$$h(\delta) = \frac{1}{\alpha(\delta)}$$

holding. Therefore, for the optimization problem (A_2), we use $\alpha(\delta)$ to construct a second-order system (14),

$$\begin{cases} y'(\delta) = a \cdot \frac{1}{\alpha(\delta)} \cdot (-y(\delta) + w(\delta)) \\ w'(\delta) = a \cdot \frac{1}{\alpha(\delta)} \cdot \left(\frac{\mu}{p(\delta)} \cdot (y(\delta) - w(\delta)) - \frac{1}{p(\delta)} \cdot \nabla f(y(\delta)) \right) \end{cases} \tag{14a}$$

$\delta \in [0, +\infty)$, $p(\delta)$ is a positive function, $0 < a < +\infty$, $p(0) = p_0 = \gamma_0 > 0$ and

$$p'(\delta) = a \cdot \frac{1}{\alpha(\delta)} \cdot (\mu - p(\delta)). \tag{14b}$$

Next, we show that the optimization problem (A_2) converges asymptotically to the optimal value under system (14), where $\alpha(\delta)$ satisfies assumptions 1, 2, 3, and 4.

Theorem 1. *The optimization problem (A_2) converges asymptotically to the optimal value under system (14), where $\alpha(\delta)$ satisfies assumptions 1, 2, 3 and 4.*

Proof. We construct the Lyapunov function as

$$L(\delta) = f(y(\delta)) - f(x^\star) + \frac{p(\delta)}{2} \cdot \|w(\delta) - x^\star\|^2. \tag{15}$$

Deriving the above equation with respect to δ yields

$$L'(\delta) = \left\langle \nabla f(y(\delta)), y'(\delta) \right\rangle + \frac{p'(\delta)}{2} \cdot \|w(\delta) - x^\star\|^2 + p(\delta) \cdot \left(w'(\delta), w(\delta) - x^\star \right).$$

Substituting (14) into the above equation and using Lemma 1 and 2, we get

$$L'(\delta) \leq -a \cdot \frac{1}{\alpha(\delta)} \cdot L(\delta).$$

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Furthermore we have:

$$L(\delta) \leq L(0) \cdot e^{-a \cdot \int_0^\delta \frac{1}{\alpha(s)} ds} = L(0) \cdot e^{-a \cdot m(\delta)}.$$

Applying Assumption 3 to the above equation, $L(\delta) \to 0$ can be obtained. So when $\delta \to +\infty$, we are able to get $f(y(\delta)) \to f(x^\star)$. □

In summary, we can make the optimization problem (A_2) asymptotically converge to the optimal solution by using the second-order system (14) constructed by assumption 1, 2 3 and 4. Although we can obtain the expression of $d(t)$ by Assumption 4, substituting the obtained $d(t)$ into the system (9) does not guarantee that the optimization problem (A_1) converges to the optimum solution within a prescribed time T.

Thirdly, under certain conditions, a second-order system is constructed to solve the optimization problem (A_1), so that (A_1) converges to the optimum solution within a prescribed time T.

Theorem 2. *When $\alpha(\delta)$ and $d(t)$ satisfy Assumptions 1, 2, 3, 4 and 5, then the optimization problem (A_1) can converge to the optimum solution within a prescribed time under the action of system (9).*

Proof. From Assumptions 1, 2, 3 and 4, we are able to get $d(t)$. In addition, it is known from Assumption 2 that there is $t \to T$, when $\delta \to +\infty$, and vice versa.

For (9), the adaptive Lyapunov function is

$$L(t) = f(x(t)) - f(x^\star) + \frac{\gamma(t)}{2} \cdot \|v(t) - x^\star\|^2 \tag{16}$$

and (16) uses the feature that $\gamma(t)$ is a positive function. Deriving the above equation with respect to t yields

$$L^{'}(t) = \left\langle \bigtriangledown f(x(t)), x^{'}(t) \right\rangle + \frac{\gamma^{'}(t)}{2} \cdot \|v(t) - x^\star\|^2 + \gamma(t) \cdot \left(v^{'}(t), v(t) - x^\star \right).$$

Substituting (9) into the above equation and using Lemma 1 and 2, we get

$$L^{'}(t) \leq -a \cdot d(t) \cdot L(t).$$

Further we have

$$L(t) \leq L(0) \cdot e^{-a \cdot \int_0^t d(s) ds} = L(0) \cdot e^{-a \cdot M(t)}.$$

Applying Assumption 5 to the above equation, $L(t) \to 0$ can be obtained. So we are able to get $f(x(t)) \to f(x^\star)$. □

Thus, under the assumptions of 1, 2, 3, 4, and 5, we turn the optimization problem (A_1) of the strongly convex objective function into an equivalent optimization problem (A_2). The latter converges asymptotically to the optimal solution under the action of system (14), while the former converges to the optimum solution within a prescribed time under the action of system (9).

Remark 3. *When $a = 1$ and $\alpha(\delta) = 1$, (9) and (14) become asymptotic systems that solve unconstrained optimization problems in [23].*

For the unconstrained optimization problem (A_1), our algorithm is summarised as follow.

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Algorithm 1:

Input: $a > 0$, $\gamma_0 > 0$, $\mu > 0$, $T > 0$, $x_0 \in R^n$, $v_0 \in R^n$
1.for k=1,2,...,K.
2.$x_{k+1} = h \cdot (a \cdot d_k \cdot (-x_k + v_k)) + x_k$.
3.$v_{k+1} = h \cdot \left(a \cdot d_k \cdot \left(\frac{\mu}{\gamma_k} \cdot (x_k - v_k) - \frac{1}{\gamma_k} \cdot \bigtriangledown f(x_k)\right)\right) + x_k$
4.$\gamma_{k+1} = h \cdot (a \cdot d_k \cdot (\mu - \gamma_k)) + \gamma_k$
end for

4. For Optimization Problems with Equational Constraints

In this subsection we consider optimization problems with equational constraints

$$\begin{aligned} &minf(x) \\ &s.t.Ax = b, \end{aligned} \tag{B_1}$$

where $x = x(t) \in R^n$, $f : R^n \to R$, and f is a $\mu-$strongly convex differentiable smooth function, $A \in R^{m\times n}$, $b \in R^m$. The Lagrangian function for problem (B_1) is

$$L(x(t), \lambda(t)) = f(x(t)) + (\lambda(t), Ax(t) - b).$$

Let (x^*, λ^*) is the saddle point of $L(x(t), \lambda(t))$, thus

$$L(x^*, \lambda(t)) \le L(x^*, \lambda^*) \le L(x(t), \lambda^*).$$

There is x^* as the optimal value point of the problem (B_1), that is,

$$x^* = x^\star.$$

Based on the ODE theory, we want to design a second-order system that has the following form,

$$\begin{cases} x^{'}(t) = a \cdot d(t)(v(t) - x(t)) \\ v^{'}(t) = a \cdot d(t)\left(\frac{\mu}{\gamma(t)} \cdot (x(t) - v(t)) - \frac{1}{\gamma(t)} \cdot \bigtriangledown_x L(x(t), \lambda(t))\right) \\ \lambda^{'}(t) = a \cdot d(t) \cdot \frac{1}{\beta(t)} \cdot \bigtriangledown_\lambda L(v(t), \lambda(t)), \end{cases} \tag{17a}$$

where $t \in [0, T)$, $\gamma(t)$ and $\beta(t)$ are positive functions, $T > 0$, $0 < a < +\infty$, $\gamma(0) = \gamma_0 > 0$, $0 < \beta(0) = \beta_0 < +\infty$ and

$$\begin{cases} \beta(t) = \beta_0 \cdot e^{-a\cdot\int_0^t d(s)ds} \\ \gamma^{'}(t) = a \cdot d(t) \cdot (\mu - \gamma(t)). \end{cases} \tag{17b}$$

Clearly, from the first equation of (17b) we get

$$\beta^{'}(t) = -a \cdot d(t) \cdot \beta(t).$$

Further, our aim is also to select a suitable $d(t)$ so that (B_1) can converge to the optimum solution within a prescribed time under the action of (17), that is, there is $f(x(t)) \to f(x^\star)$ when $t \to T$. To determine the appropriate $d(t)$, our work is divided into the following steps.

Firstly, the variable transformation (10) is used to change the optimization problem (B_1) into the following equivalent optimization problem (B_2). Details are as follows.

By substituting (10) into (B_1), we get the optimization problem

$$\begin{aligned} &minf(y) \\ &s.t.Ay = b, \end{aligned} \tag{B_2}$$

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where $y = y(\delta)$, $f : R^n \to R$, and f is a $\mu-$strongly convex differentiable smooth function, $A \in R^{m\times n}$, $b \in R^m$. The optimal solution of the optimization problem (B_2) is also $x^\star$. The Lagrangian function of problem (B_2) is

$$L(y(\delta), l(\delta)) = f(y(\delta)) + (l(\delta), Ay(\delta) - b),$$

where (x^*, λ^*) is the saddle point of $L(y(\delta), l(\delta))$.

Secondly, under certain conditions, a second-order system is constructed to solve the optimization problem (B_2), so that (B_2) converges asymptotically to the optimal solution under the action of the system. Details are as follows.

Inspired by [25], a second-order system is constructed for the optimization problem (B_2),

$$\begin{cases} y^{'}(\delta) = a \cdot h(\delta) \cdot (w(\delta) - y(\delta)) \\ w^{'}(\delta) = a \cdot h(\delta) \cdot \left(\frac{\mu}{p(\delta)} \cdot (y(\delta) - w(\delta)) - \frac{1}{p(\delta)} \cdot \nabla_y L(y(\delta), l(\delta)) \right) \\ l^{'}(\delta) = a \cdot h(\delta) \cdot \frac{1}{u(\delta)} \cdot \nabla_l L(w(\delta), l(\delta)), \end{cases} \tag{18a}$$

$\delta \in [0, +\infty)$, $p(\delta)$ and $u(\delta)$ are positive functions, $0 < a < +\infty$, $p(0) = p_0 = \gamma_0 > 0$, $u(0) = u_0 = \beta_0$ and

$$\begin{cases} u(\delta) = u_0 \cdot e^{-a \cdot \int_0^\delta \frac{1}{\alpha(s)} ds} \\ p^{'}(\delta) = a \cdot h(\delta) \cdot (\mu - p(\delta)). \end{cases} \tag{18b}$$

Clearly, from the first equation of (18b) we get

$$u^{'}(\delta) = -a \cdot \frac{1}{\alpha(\delta)} \cdot u(\delta).$$

In addition to (10) and (12), we apply the variable transformation

$$\begin{cases} u(\delta) = \beta(t(\delta)) \\ l(\delta) = \lambda(t(\delta)) \end{cases} \tag{19}$$

to get

$$\begin{cases} y'(\delta) = x'(t) \cdot t'(\delta) = \alpha(\delta) \cdot x'(t) \\ w'(\delta) = v'(t) \cdot t'(\delta) = \alpha(\delta) \cdot v'(t) \\ l'(\delta) = \lambda'(t) \cdot t'(\delta) = \alpha(\delta) \cdot \lambda'(t) \\ u'(\delta) = \beta'(t) \cdot t'(\delta) = \alpha(\delta) \cdot \beta'(t) \\ p'(\delta) = \gamma'(t) \cdot t'(\delta) = \alpha(\delta) \cdot \gamma'(t). \end{cases}$$

Then (17), (18) are analysed, we also found that there is a result of

$$h(\delta) = \alpha(\delta) \cdot d(t).$$

Substituting Assumptiom 4 into the above equation, we find

$$h(\delta) = \frac{1}{\alpha(\delta)}$$

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holding. Therefore, for the optimization problem (B_2), we use $\alpha(\delta)$ to construct a second-order system,

$$\begin{cases} y^{'}(\delta) = a \cdot \frac{1}{\alpha(\delta)} \cdot (w(\delta) - y(\delta)) \\ w^{'}(\delta) = a \cdot \frac{1}{\alpha(\delta)} \cdot \left(\frac{\mu}{p(\delta)} \cdot (y(\delta) - w(\delta)) - \frac{1}{p(\delta)} \cdot \bigtriangledown_y L(y(\delta), l(\delta)) \right) \\ l^{'}(\delta) = a \cdot \frac{1}{\alpha(\delta)} \cdot \frac{1}{u(\delta)} \cdot \bigtriangledown_l L(w(\delta), l(\delta)), \end{cases} \tag{20a}$$

$\delta \in [0, +\infty)$, $p(\delta)$ and $u(\delta)$ are positive functions, $0 < a < +\infty$, $p(0) = p_0 = \gamma_0 > 0$, $u(0) = u_0 = \beta_0$ and

$$\begin{cases} u(\delta) = u_0 \cdot e^{-a \cdot \int_0^\delta \frac{1}{\alpha(s)} ds} \\ p^{'}(\delta) = a \cdot \frac{1}{\alpha(\delta)} \cdot (\mu - p(\delta)). \end{cases} \tag{20b}$$

Next, we show that the optimization problem (B_2) converges asymptotically to the optimal value under system (20), where $\alpha(\delta)$ satisfies Assumptions 1, 2, 3, and 4.

Theorem 3. *The optimization problem (B_2) converges asymptotically to the optimal value under system (20), where $\alpha(\delta)$ satisfies Assumptions 1, 2, 3 and 4.*

Proof. We construct the Lyapunov function as

$$G(\delta) = L(y(\delta), \lambda^*) - L(x^*, l(\delta)) + \frac{p(\delta)}{2} \cdot \|w(\delta) - x^*\|^2 + \frac{u(\delta)}{2} \cdot \|l(\delta) - \lambda^*\|^2. \tag{21}$$

Deriving the above equation with respect to δ yields

$$\begin{aligned} G^{'}(\delta) =& \left\langle \bigtriangledown_y L(y(\delta), \lambda^*), y^{'}(\delta) \right\rangle + \frac{p^{'}(\delta)}{2} \cdot \|w(\delta) - x^*\|^2 + \left(p(\delta) \cdot w^{'}(\delta), w(\delta) - x^* \right) \\ &+ \frac{u^{'}(\delta)}{2} \cdot \|l(\delta) - \lambda^*\|^2 + \left(u(\delta) \cdot l^{'}(\delta), l(\delta) - \lambda^* \right), \end{aligned}$$

and

$$\bigtriangledown_l L(x^*, l(\delta)) = 0$$

is used for the above equation to hold. Substituting (20) into the above equation, we get

$$G^{'}(\delta) = I_1 + I_2, \tag{22}$$

where

$$\begin{aligned} I_1 =& a \cdot \frac{1}{\alpha(\delta)} \cdot \langle \bigtriangledown_y L(y(\delta), \lambda^*), w(\delta) - y(\delta) \rangle - a \cdot \frac{1}{\alpha(\delta)} \cdot \langle \bigtriangledown_y L(y(\delta), l(\delta)), w(\delta) - x^* \rangle, \\ I_2 =& a \cdot \frac{1}{\alpha(\delta)} \cdot \frac{\mu}{2} \cdot \|w(\delta) - x^*\|^2 - a \cdot \frac{1}{\alpha(\delta)} \cdot \frac{p(\delta)}{2} \cdot \|w(\delta) - x^*\|^2 \\ &+ a \cdot \frac{1}{\alpha(\delta)} \cdot (\mu \cdot (y(\delta) - w(\delta)), w(\delta) - x^*) - a \cdot \frac{1}{\alpha(\delta)} \cdot \frac{u(\delta)}{2} \cdot \|l(\delta) - \lambda^*\|^2 \\ &+ a \cdot \frac{1}{\alpha(\delta)} \cdot (Aw(\delta) - b, l(\delta) - \lambda^*). \end{aligned}$$

From

$$\bigtriangledown_y L(y(\delta), l(\delta)) = \bigtriangledown_y L(y(\delta), \lambda^*) + A(l(\delta) - \lambda^*)$$

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and when f is a strong convex function with

$$\langle \nabla_y L(y(\delta),\lambda^*), -y(\delta)+x^* \rangle \le L(x^*,\lambda^*) - L(y(\delta),\lambda^*) - \frac{\mu}{2}\cdot\|y(\delta)-x^*\|^2$$

holding, we get

$$\begin{aligned} I_1 \le & a\cdot\frac{1}{\alpha(\delta)}\cdot\left(L(x^*,\lambda^*) - L(y(\delta),\lambda^*) - \frac{\mu}{2}\cdot\|y(\delta)-x^*\|^2\right) \\ & - a\cdot\frac{1}{\alpha(\delta)}\cdot(Aw(\delta)-b, l(\delta)-\lambda^*). \end{aligned}$$

Substituting the above equation into (22), we have

$$G^{'}(\delta) \le -a\cdot\frac{1}{\alpha(\delta)}\cdot G(\delta).$$

Further, one has

$$G(\delta) \le G(0)\cdot e^{-a\cdot\int_0^\delta \frac{1}{\alpha(s)}ds} = G(0)\cdot e^{-a\cdot m(\delta)}. \tag{23}$$

Assuming that the initial point is not the optimal point, it is clear that $G(0) > 0$ is true. Applying Assumption 3 to the above equation, $G(\delta)\to 0$ can be obtained. However we are not able to get the convergence speed and this requires further work. Details are as follows.

Substituting (21) into (23), we get

$$\begin{aligned} L(y(\delta),\lambda^*) - L(x^*,l(\delta)) = & f(y(\delta)) - f(x^*) + (\lambda^*, Ay(\delta)-b) \\ \le & G(0)\cdot e^{-a\cdot m(\delta)}, \end{aligned} \tag{24}$$

$$\|l(\delta)-\lambda^*\| \le \sqrt{\frac{2G(0)\cdot e^{-a\cdot m(\delta)}}{u(\delta)}}. \tag{25}$$

Let

$$H(\delta) = l(\delta) - \frac{1}{u(\delta)}\cdot(Ay(\delta)-b). \tag{26}$$

By the expression of $u(\delta)$ in (20b), we obtain

$$\left(\frac{1}{u(\delta)}\right)^{'} = a\cdot\frac{1}{\alpha(\delta)}\cdot\frac{1}{u(\delta)}.$$

Deriving (26) with respect to δ and substituting into the first and third equations of (20a) yields $H^{'}(\delta) = 0$. So there is

$$H(\delta) = H(0) = l_0 - \frac{1}{u_0}(Ay_0 - b).$$

From (26), we have $Ay(\delta) - b = u(\delta)\cdot(l(\delta) - H(\delta))$, thus

$$\begin{aligned} \|Ay(\delta)-b\| = & |u(\delta)|\cdot\|l(\delta)-H(\delta)\| = u(\delta)\cdot\|l(\delta)-H(0)\| \\ = & u(\delta)\cdot\left\|l(\delta) - l_0 + \frac{1}{u_0}(Ay_0-b)\right\| \\ = & u(\delta)\cdot\left\|l(\delta) - \lambda^* + \lambda^* - l_0 + \frac{1}{u_0}(Ay_0-b)\right\| \\ \le & C_1\cdot e^{-a\cdot m(\delta)}, \end{aligned}$$

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where $C_1 = \sqrt{2G(0)u_0} + u_0\|\lambda^* - l_0\| + \|Ay_0 - b\|$. From the above equation and (25), we achieve

$$f(y(\delta)) - f(x^*) \leq -(\lambda^*, Ay(\delta) - b) + G(0) \cdot e^{-a \cdot m(\delta)}.$$

Further, we have

$$\begin{aligned} \|f(y(\delta)) - f(x^*)\| &= |f(y(\delta)) - f(x^*)| \\ &= f(y(\delta)) - f(x^*) = f(y(\delta)) - f(x^\star) \\ &\leq C_2 \cdot e^{-a \cdot m(\delta)}, \end{aligned}$$

where $C_2 = \|\lambda^*\| \cdot C_1 + G(0)$, so

$$f(y(\delta)) - f(x^\star) \leq C_2 \cdot e^{-a \cdot m(\delta)}.$$

Since $G(0) > 0$, substituting Assumption 3 into the above equation yields $f(y(\delta)) \to f(x^\star)$. □

In summary, we can make the optimization problem (B_2) asymptotically converge to the optimal solution by using the second-order system (20) constructed by Assumption 1, 2, 3 and 4. Although we can get the expression $d(t)$ by substituting $\alpha(\delta)$ into Assumption 4, substituting the obtained $d(t)$ into the system (17), it does not guarantee that the optimization problem (B_1) converges to the optimum solution within a prescribed time T.

Thirdly, under certain conditions, a second-order system is constructed to solve the optimization problem (B_1), so that (B_1) converges to the optimum solution within a prescribed time T.

Theorem 4. *When $\alpha(\delta)$ and $d(t)$ satisfy Assumptions 1, 2, 3, 4 and 5, then the optimization problem (B_1) can converge to the optimum solution within a prescribed time under the action of system (17).*

Proof. From Assumptions 1, 2, 3 and 4, we are able to get $d(t)$. In addition, it is known from Assumption 2 that when $\delta \to +\infty$, there is $t \to T$, and vice versa.

For (17), the adaptive Lyapunov function is

$$G(t) = L(x(t), \lambda^*) - L(x^*, \lambda(t)) + \frac{\gamma(t)}{2} \cdot \|v(t) - x^*\|^2 + \frac{\beta(t)}{2} \cdot \|\lambda(t) - \lambda^*\|^2. \tag{27}$$

Deriving the above equation with respect to t, it gives

$$\begin{aligned} G'(t) =& \left\langle \nabla_x L(x(t), \lambda^*), x'(t) \right\rangle + \frac{\gamma'(t)}{2} \cdot \|v(t) - x^*\|^2 + \left(\gamma(t) \cdot v'(t), v(t) - x^*\right) \\ &+ \frac{\beta'(t)}{2} \cdot \|\lambda(t) - \lambda^*\|^2 + \left(\beta(t) \cdot \lambda'(t), \lambda(t) - \lambda^*\right), \end{aligned}$$

where

$$\nabla_\lambda L(x^*, \lambda(t)) = 0$$

is used for the above equation to hold. Substituting (17) into the above equation, we get

$$G'(t) = I_3 + I_4, \tag{28}$$

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where

$$\begin{aligned} I_3 =& a \cdot d(t) \cdot \langle \bigtriangledown_x L(x(t), \lambda^*), v(t) - x(t) \rangle - a \cdot d(t) \cdot \langle \bigtriangledown_x L(x(t), \lambda(t)), v(t) - x^* \rangle, \\ I_4 =& a \cdot d(t) \cdot \frac{\mu}{2} \cdot \|v(t) - x^*\|^2 - a \cdot d(t) \cdot \frac{\gamma(t)}{2} \cdot \|v(t) - x^*\|^2 \\ &+ a \cdot d(t) \cdot (\mu \cdot (x(t) - v(t)), v(t) - x^*) - a \cdot d(t) \cdot \frac{\beta(t)}{2} \cdot \|\lambda(t) - \lambda^*\|^2 \\ &+ a \cdot d(t) \cdot (Av(t) - b, \lambda(t) - \lambda^*). \end{aligned}$$

From

$$\bigtriangledown_x L(x(t), \lambda(t)) = \bigtriangledown_x L(x(t), \lambda^*) + A(\lambda(t) - \lambda^*)$$

and when f is a strong convex function with

$$\langle \bigtriangledown_x L(x(t), \lambda^*), -x(t) + x^* \rangle \le L(x^*, \lambda^*) - L(x(t), \lambda^*) - \frac{\mu}{2} \cdot \|x(t) - x^*\|^2$$

holding, we get

$$\begin{aligned} I_3 \le& a \cdot d(t) \cdot \left(L(x^*, \lambda^*) - L(x(t), \lambda^*) - \frac{\mu}{2} \cdot \|x(t) - x^*\|^2 \right) \\ &- a \cdot d(t) \cdot (Av(t) - b, \lambda(t) - \lambda^*). \end{aligned}$$

Substituting the above equation into (28), we have

$$G'(t) \le -a \cdot d(t) \cdot G(t).$$

Further, we get

$$G(t) \le G(0) \cdot e^{-a \cdot \int_0^t d(s) ds} = G(0) \cdot e^{-a \cdot M(t)}. \tag{29}$$

Assuming that the initial point is not the optimal point, it is clear that $G(0) > 0$. Applying Assumption 5 to the above equation, $G(t) \to 0$ can be obtained. However we are not able to get the convergence speed and this requires further work.

Substituting (27) into (29), we get

$$\begin{aligned} L(x(t), \lambda^*) - L(x^*, \lambda(t)) =& f(x(t)) - f(x^*) + (\lambda^*, Ax(t) - b) \\ \le& G(0) \cdot e^{-a \cdot M(t)}, \end{aligned} \tag{30}$$

$$\|\lambda(t) - \lambda^*\| \le \sqrt{\frac{2G(0) \cdot e^{-a \cdot M(t)}}{\beta(t)}}. \tag{31}$$

Let

$$H(t) = \lambda(t) - \frac{1}{\beta(t)} \cdot (Ax(t) - b). \tag{32}$$

By the expression of $\beta(t)$, we obtain

$$\left(\frac{1}{\beta(t)} \right)' = a \cdot d(t) \cdot \frac{1}{\beta(t)}.$$

Deriving (32) with respect to t and substituting into (17a), it yields $H'(t) = 0$. So, there is

$$H(t) = H(0) = \lambda_0 - \frac{1}{\beta_0}(Ax_0 - b).$$

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From (32), we have $Ax(t) - b = \beta(t) \cdot (\lambda(t) - H(t))$, thus

$$\begin{aligned}\|Ax(t) - b\| =& |\beta(t)| \cdot \|\lambda(t) - H(t)\| = \beta(t) \cdot \|\lambda(t) - H(0)\| \\ =& \beta(t) \cdot \left\| \lambda(t) - \lambda_0 + \frac{1}{\beta_0}(Ax_0 - b) \right\| \\ =& \beta(t) \cdot \left\| \lambda(t) - \lambda^* + \lambda^* - \lambda_0 + \frac{1}{\beta_0}(Ax_0 - b) \right\| \\ \leq& C_1 \cdot e^{-a \cdot M(t)},\end{aligned}$$

where $C_1 = \sqrt{2G(0)\beta_0} + \beta_0\|\lambda^* - \lambda_0\| + \|Ax_0 - b\|$. And from the above equation and (28), we get

$$f(x(t)) - f(x^*) \leq -(\lambda^*, Ax(t) - b) + G(0) \cdot e^{-a \cdot M(t)}.$$

Further, we have

$$\begin{aligned}\|f(x(t)) - f(x^*)\| &= |f(x(t)) - f(x^*)| \\ &= f(x(t)) - f(x^*) = f(x(t)) - f(x^\star) \\ &\leq C_2 \cdot e^{-a \cdot M(t)},\end{aligned}$$

where $C_2 = \|\lambda^*\| \cdot C_1 + G(0)$, so

$$f(x(t)) - f(x^\star) \leq C_2 \cdot e^{-a \cdot M(t)}.$$

Since $G(0) > 0$ satisfies, substituting Assumption 5 into the above equation yields $f(x(t)) \to f(x^\star)$. □

Thus, under the Assumptions of 1, 2, 3, 4, and 5, we turn the optimization problem (B_1) of the strongly convex objective function into an equivalent optimization problem (B_2). The latter converges asymptotically to the optimal solution under the action of system (20), while the former converges to the optimum solution within a prescribed time under the action of system (17).

Remark 4. *When $a = 1$ and $\alpha(\delta) = 1$, (17) and (20) become asymptotic systems that solve optimization problems with equality constraints in [25].*

For the optimization problem (B_1), our algorithm is summarised as follow.

Algorithm 2:

Input: $a > 0, \gamma_0 > 0, \beta_0 > 0, \mu > 0, T > 0, A \in R^{m\times n}, b \in R^m, x_0 \in R^n, v_0 \in R^n, \lambda_0 \in R^m$.
1.for k=1,2,...,K.
2.$x_{k+1} = h \cdot (a \cdot d_k \cdot (-x_k + v_k)) + x_k$.
3.$v_{k+1} = h \cdot \left(a \cdot d_k \cdot \left(\frac{\mu}{\gamma_k} \cdot (x_k - v_k) - \frac{1}{\gamma_k} \cdot (\bigtriangledown f(x_k) + A\lambda_k)\right)\right) + x_k$
4.$\lambda_{k+1} = h \cdot \left(a \cdot d_k \cdot \frac{1}{\beta_k} \cdot (Av_k - b)\right) + \lambda_k$
5.$\beta_{k+1} = \beta_0 \cdot e^{-a \cdot \int_0^{t_k} d(s)ds} = \beta_0 \cdot e^{-a \cdot M_k}$
6.$\gamma_{k+1} = h \cdot (a \cdot d_k \cdot (\mu - \gamma_k)) + \gamma_k$
end for

5. Examples

Furthermore, we construct the following functions $\alpha_i(s)$, for $i = 1, 2, 3, 4$, based on linear functions, exponential function and Pearl function, and correspondingly the functions $d_i(t), i = 1, 2, 3, 4$. We prove that these functions we construct satisfy Assumptions 1, 2, 3, 4 and 5. Then, we prove that substituting the corresponding $d_i(t)$, $i = 1, 2, 3, 4$. into system (9) or system (17) can make the unconstrained

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optimization problem (A_1) and the optimization problem with equality constraints (B_1) converge to the optimum solution within a prescribed time T, respectively. Details are as follows.

Example 1. We construct

$$\alpha_1(s) = \frac{(2\beta-1)T}{(1+s)^{2\beta}}$$

based on the linear functions, where $\beta > \frac{1}{2}$ and

$$d_1(t) = \frac{\left(\frac{T}{T-t}\right)^{\frac{4\beta}{2\beta-1}}}{(2\beta-1)^2 \cdot T^2}.$$

Obviously, $\alpha_1(s) > 0$, so Assumption 1 holds and

$$\alpha_1(\delta) = \frac{(2\beta-1)T}{(1+\delta)^{2\beta}}.$$

From $t = t(\delta) = \int_0^\delta \alpha_1(s)ds$, we know

$$t = T\left(1 - \frac{1}{(1+\delta)^{2\beta-1}}\right).$$

Since $\beta > \frac{1}{2}$, $\delta \in [0, +\infty)$, it follows that $t \geq 0$. Clearly, when $\delta \to +\infty$, there is $t = t(\delta) \to T$ and the reverse is also true, so Assumption 2 holds. Next,

$$m_1(\delta) = \int_0^\delta \frac{1}{\alpha_1(s)}ds = \frac{(1+\delta)^{2\beta+1} - 1}{(4\beta^2-1)T}.$$

Clearly when $\delta \to +\infty$, there is $m_1(\delta) \to +\infty$, so Assumption 3 holds. Substituting $t = t(\delta)$ into $d(t)$, then

$$\alpha_1(\delta) \cdot d_1(t) = \frac{1}{\alpha_1(\delta)},$$

and obviously Assumption 4 is true. Let

$$Q_1(t) = \left(\frac{1}{T-t}\right)^{\frac{2\beta+1}{2\beta-1}} - \left(\frac{1}{T}\right)^{\frac{2\beta+1}{2\beta-1}}$$

then

$$M_1(t) = \int_0^t d_1(s)ds = \frac{T^{\frac{2}{2\beta-1}}}{4\beta^2-1} \cdot Q_1(t).$$

Clearly, there is $M_1(t) \to +\infty$, when $t \to T$, so Assumption 5 holds. Then, we will show that $\gamma(t)$ is a positive function. Substitute $d_1(t)$ into (9b) or (17b) to get

$$\begin{aligned}\gamma(t) =& \mu - (\mu - \gamma_0) \cdot e^{-a \cdot \frac{T^{\frac{2}{2\beta-1}}}{4\beta^2-1} \cdot Q_1(t)} \\ =& \mu - (\mu - \gamma_0) \cdot e^{-a \cdot M_1(t)}.\end{aligned}$$

Since the coefficient of $\mu - \gamma(t)$ is positive, when we consider $\mu = \gamma(t)$, $\mu > \gamma(t)$ and $\mu < \gamma(t)$, respectively, there is $min\{\gamma_0, \mu\} \leq \gamma(t) \leq max\{\gamma_0, \mu\}$ according to the image method. When $t \to T$, there is $\gamma(t) \to \mu$. Because $\gamma_0 > 0$, $\gamma(t)$ is a positive function.

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case1: For the unconstrained optimization problem (A_1), when we use (16) as the Lyapunov function, and take (16) for the derivative of the variable t, we can obtain $L^{'}(t) \leq -a \cdot d_1(t) \cdot L(t)$ by further calculation. Thus, when $t \to T$, $f(x(t) \to f(x^{\star})$ holds.

case2: When considering the optimization problem with equation constraints, we still have $\gamma(t)$ and $\beta(t)$ are positive functions. When we use (27) as the Lyapunov function, and take (27) for the derivative of the variable t, we can receive

$$G^{'}(t) \leq -a \cdot d_1(t) \cdot G(t).$$

Further, we have

$$G(t) \leq G(0) \cdot e^{-a \cdot \frac{T^{\frac{2}{2\beta-1}}}{4\beta^2-1} \cdot Q_1(t)} = G(0) \cdot e^{-a \cdot M_1(t)},$$

where $\beta > \frac{1}{2}$. Assuming that the selected initial value point is not the optimal value point, it is clear that there is $G(0) > 0$. So when $t \to T$, there is $G(t) \to 0$. We also get

$$\left(\frac{1}{\beta(t)}\right)^{'} = a \cdot d_1(t) \cdot \frac{1}{\beta(t)}.$$

Further work yields when $t \to T$, there is $f(x(t)) \to f(x^{\star})$.

Example 2. If we choose

$$\alpha_2(s) = \frac{T^2}{(T+s)^2}$$

and

$$d_2(t) = \frac{T^4}{(T-t)^4}$$

the same result can be obtained.

Example 3. We use the exponential function as the basis for constructing the exponential function type

$$\alpha_3(s) = kT \cdot e^{-ks}$$

where $k > 0$. Clearly, $\alpha_3(s) > 0$, and

$$\alpha_3(\delta) = kT \cdot e^{-k\delta}.$$

Besides,

$$d_3(t) = \frac{1}{k^2(T-t)^2}.$$

We are able to show that the problem (A_1) and (B_1) can converge to the optimum solution $f(x^{\star})$ within a prescribed time T under system (9) and (17), respectively.

Example 4. We construct the following function based on the Pearl function

$$\alpha_4(s) = \frac{2T}{ln\frac{1+b}{1-b}} \cdot \frac{1}{\frac{1}{b^2} \cdot e^{bs} - e^{-bs}},$$

where $0 < b < 1$. Clearly, we have

$$\alpha_4(\delta) = \frac{2T}{ln\frac{1+b}{1-b}} \cdot \frac{1}{\frac{1}{b^2} \cdot e^{b\delta} - e^{-b\delta}}. \tag{33}$$

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In addition, we construct

$$d_4(t) = \frac{\left(ln\frac{1+b}{1-b}\right)^2}{4b^2T^2} \cdot \left(\frac{\left(\frac{1+b}{1-b}\right)^{1-\frac{t}{T}} + 1}{\left(\frac{1+b}{1-b}\right)^{1-\frac{t}{T}} - 1} - \frac{\left(\frac{1+b}{1-b}\right)^{1-\frac{t}{T}} - 1}{\left(\frac{1+b}{1-b}\right)^{1-\frac{t}{T}} + 1} \right)^2.$$

Let us first prove that $\alpha_4(s) > 0$. Since $0 < b < 1$, $\frac{1+b}{1-b} > 1$, it follows that

$$ln\frac{1+b}{1-b} > 0,$$

so the first term in (33) is positive. The second term of (33) is decreasing with respect to the variable δ, and when $\delta \rightarrow +\infty$, the second term of (33) tends to 0, so the second term of (33) is always a positive function. Therefore $\alpha_4(s) > 0$ is true, which satisfies Assumption 1. From $t = t(\delta) = \int_0^\delta \alpha_4(s)ds$, we know

$$t = t(\delta) = \int_0^\delta \alpha_4(s)ds = \frac{T}{ln\frac{1+b}{1-b}} \cdot ln\left(\frac{1+b}{1-b} \cdot \frac{\frac{1}{b}e^{b\delta} - 1}{\frac{1}{b}e^{b\delta} + 1}\right).$$

Let

$$g(\delta) = \frac{1+b}{1-b} \cdot \frac{\frac{1}{b}e^{b\delta} - 1}{\frac{1}{b}e^{b\delta} + 1},$$

where $\delta \in [0, +\infty)$, $0 < b < 1$, $\frac{1+b}{1-b} > 0$, so we have

$$g'(\delta) = \frac{1+b}{1-b} \cdot \frac{2e^{b\delta}}{\left(\frac{1}{b}e^{b\delta} + 1\right)^2} > 0,$$

i.e.

$$g'(\delta) > 0.$$

When $\delta = 0$, $g(0) = 1$, so $g(\delta) \geq 1$ and $t = t(\delta) \geq 0$. And when $\delta \rightarrow +\infty$, there is $\frac{\frac{1}{b}e^{b\delta}-1}{\frac{1}{b}e^{b\delta}+1} \rightarrow 1$ and $g(\delta) \rightarrow \frac{1+b}{1-b}$. So there is $t = t(\delta) \rightarrow T$. Further, we get

$$\delta(t) = ln\left(\frac{(H+1)b}{H-1}\right)^{\frac{1}{b}},$$

where

$$H = \left(\frac{1+b}{1-b}\right)^{1-\frac{t}{T}} = \frac{e^{b\delta} + b}{e^{b\delta} - b}.$$

From the above equation, we know

$$\frac{(H+1)b}{H-1} = e^{b\delta}$$

and when $t \rightarrow T$, $H \rightarrow 1$. So when $t \rightarrow T$, there is δ(t) $\rightarrow +\infty$. Then Assumption 2 holds. Next,

$$m_4(\delta) = \int_0^\delta \frac{1}{\alpha_4(s)}ds = \frac{ln\frac{1+b}{1-b}}{2bT}\left(\frac{1}{b^2}e^{b\delta} + e^{-b\delta} - 1 - \frac{1}{b^2}\right).$$

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From the above equation, we get that when $\delta \to +\infty$ is satisfied, there is $m_4(\delta) \to +\infty$. So Assumption 3 holds. Substituting $t = t(\delta)$ into $d_4(t)$,

$$\alpha_4(\delta) \cdot d_4(t) = \frac{1}{\alpha_4(\delta)}$$

holds, and obviously Assumption 4 is satisfied. In addition, there is

$$M_4(t) = \int_0^t d_4(s)ds = \frac{\left(\frac{1+b}{1-b}\right)^2 - \left(\frac{1+b}{1-b}\right)^{2\cdot(1-\frac{t}{T})}}{\left(\left(\frac{1+b}{1-b}\right)^{2\cdot(1-\frac{t}{T})} - 1\right) \cdot \left(\left(\frac{1+b}{1-b}\right)^2 - 1\right)}.$$

So $M_4(t) \to +\infty$, when $t \to T$, thus Assumption 5 holds. Therefore, using a similar process above, we can get that either the unconstrained optimization problem (A_1) or the optimization problem (B_1) with the equality constraint converges to the optimum solution within a prescribed time T.

6. Numerical Results

By selecting different $\alpha(\delta)$, $d(s)$ and changing the size of the parameter a, we consider the system (9) of the variable t and the system (14) of the variable δ for the unconstrained optimization problem (A_1) and (B_1), respectively. We also consider the system (17) of the variable t and the system (20) of the variable δ for the optimization problem (A_2) and (B_2), respectively.

Figures 1–8 show images of unconstrained optimization problems, where Figures 1, 2, 5 and 6 show systems with variable t, and Figures 3, 4, 7 and 8 show systems with variable δ. Figures 9–16 show images of optimisation problems containing equation constraints, where Figures 9, 10, 13 and 14 show systems with variable t, and Figures 11, 12, 15 and 16 show systems with variable δ. The findings indicate that both systems arrive at the identical optimal solution for the strongly convex objective function.

Case 1: when μ=0.5, a=2, β=1, T=6, $A=\begin{bmatrix} 1 & 0 \\ 0 & 0.5 \end{bmatrix}$, we choose $\alpha_1(\delta)$ and $d_1(s)$, then we consider the minimum of the strongly convex function $f = \frac{1}{2}x^T Ax$.

(a): Figures 1 and 2 show the variation of variables x and $f(x(t))$ with respect to t when the system (9) is applied to solve the problem, respectively.

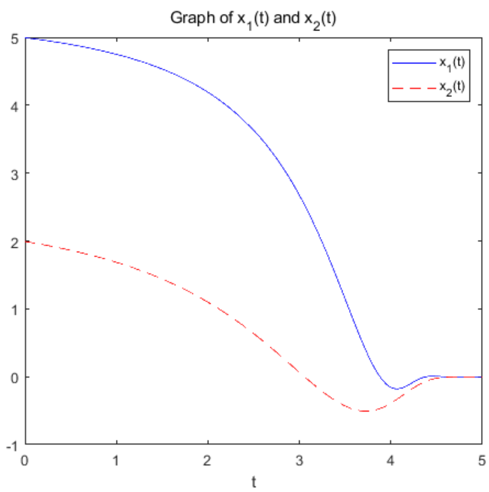

Figure 1. $x(t)$ change with respect to t

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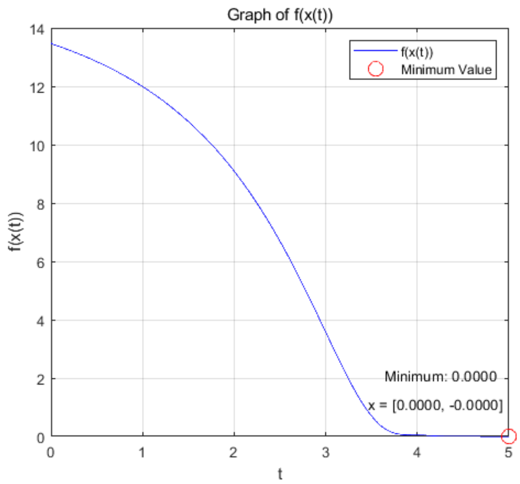

Figure 2. $f(x(t))$ change with respect to t

(b): Figures 3 and 4 show the variation of variables y and f(y(δ)) with respect to δ when the system (14) is applied to solve the problem, respectively.

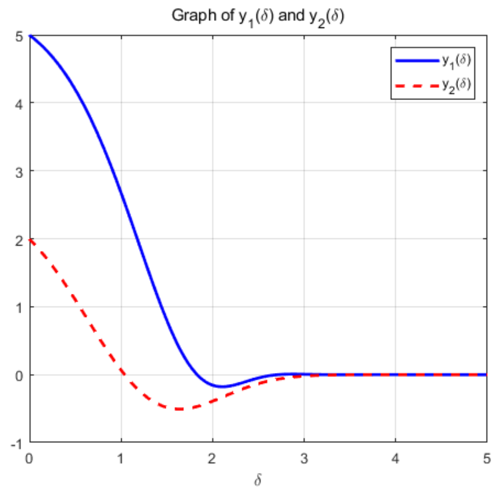

Figure 3. $y(\delta)$ change with respect to δ

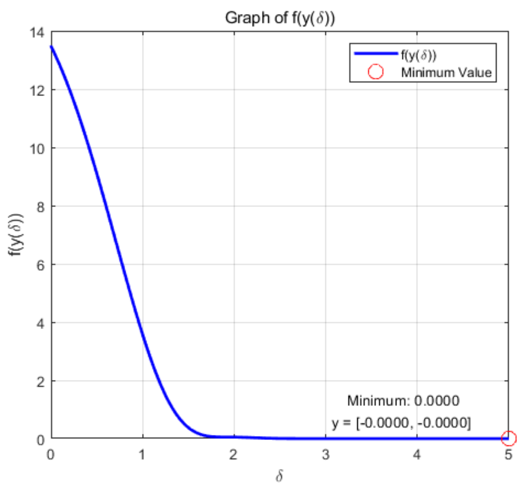

Figure 4. $f(y(\delta))$ change with respect to δ

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Remark 5. *the optimization problem reaches the optimal solution* $x^\star = (-0.0000, 0.0000)$ *within a prescribed time T=6 under the action of system (9), and the optimization problem of its equivalence converges asymptotically to the same optimal solution under the action of system (14).*

Case 2: when μ=1, a=3, T=9.5, $A=\begin{bmatrix} 4 & -1 & 2 \\ -1 & 5 & -3 \\ 2 & -3 & 6 \end{bmatrix}$, $c=\begin{bmatrix} 1 & 1 & 0 \end{bmatrix}$, d=1, b=1/2, we choose $\alpha_4(\delta)$ and $d_4(s)$, then we consider the minimum of the strongly convex function $f = \frac{1}{2}x^T Ax + cx + d$.

(a): Figures 5 and 6 show the variation of variables x and $f(x(t))$ with respect to t when the system (9) is applied to solve the problem, respectively.

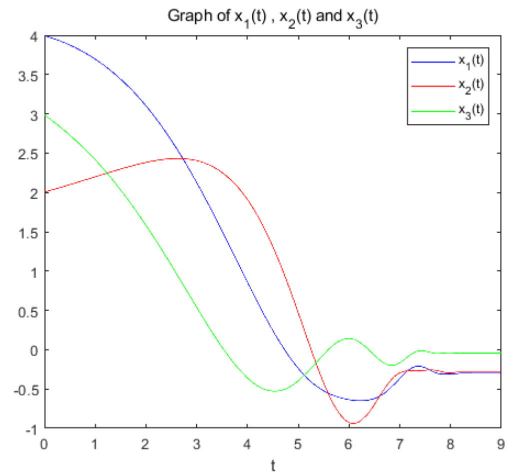

Figure 5. $x(t)$ change with respect to t

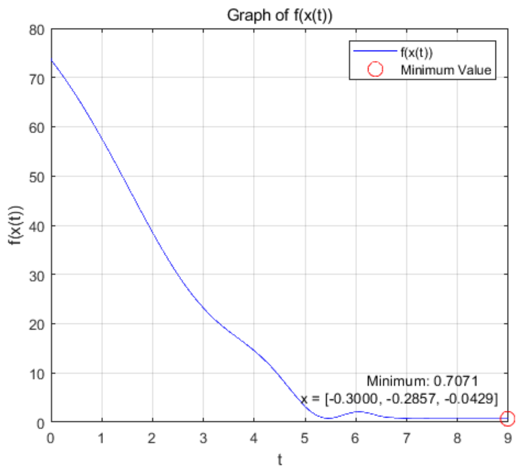

Figure 6. $f(x(t))$ change with respect to t

(b): Figures 7 and 8 show the variation of variables y and $f(y(\delta))$ with respect to δ when the system (14) is applied to solve the problem, respectively.

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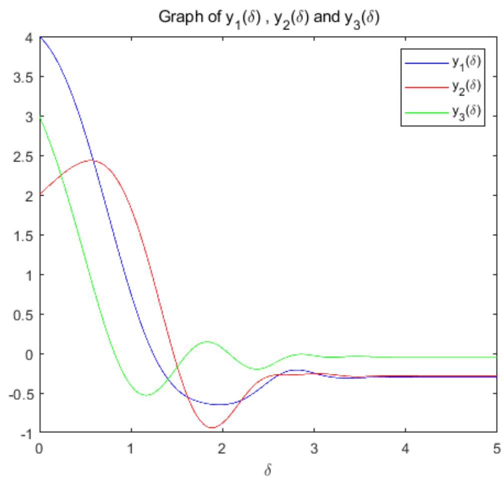

Figure 7. $y(\delta)$ change with respect to δ

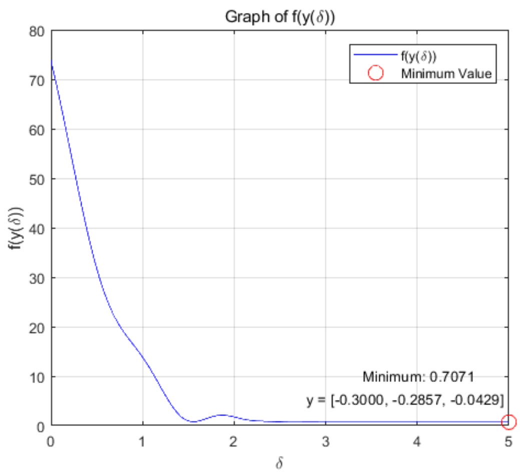

Figure 8. $f(y(\delta))$ change with respect to δ

Remark 6. *the optimization problem reaches the optimal solution* $x^{\star} = (-0.3000, -0.2857, -0.04$ $29)$ *within a prescribed time T=9.5 under the action of system (9), and the optimization problem of its equivalence converges asymptotically to the same optimal solution under the action of system (14).*

Case 3: when μ=1, a=0.5, T=6.5, k=0.9, A=$\begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}$, b=$\begin{bmatrix} 1 \\ 2.5 \end{bmatrix}$, we choose $\alpha_3(\delta)$ and $d_3(s)$, then we consider the minimum of the strongly convex function $f = \frac{1}{2}x^T Ax$.

(a): Figures 9 and 10 show the variation of variables x and $f(x(t))$ with respect to t when the system (17) is applied to solve the problem, respectively.

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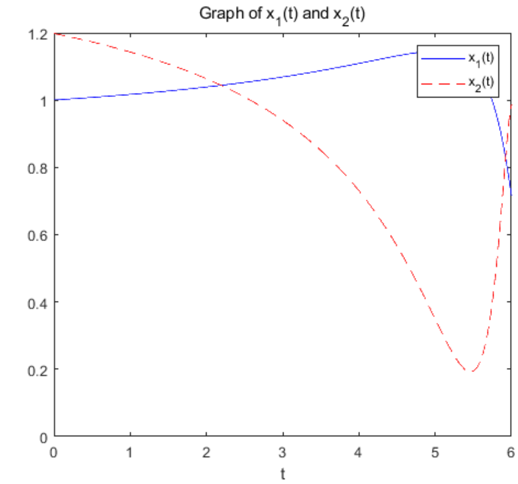

Figure 9. $x(t)$ change with respect to t

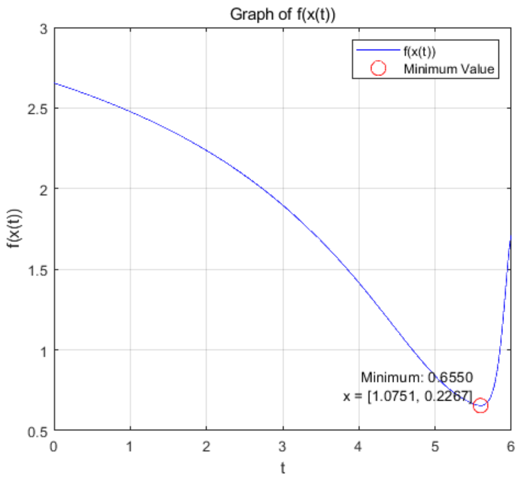

Figure 10. $f(x(t))$ change with respect to t

(b): Figures 11 and 12 show the variation of variables y and $f(y(\delta))$ with respect to δ when the system (20) is applied to solve the problem, respectively.

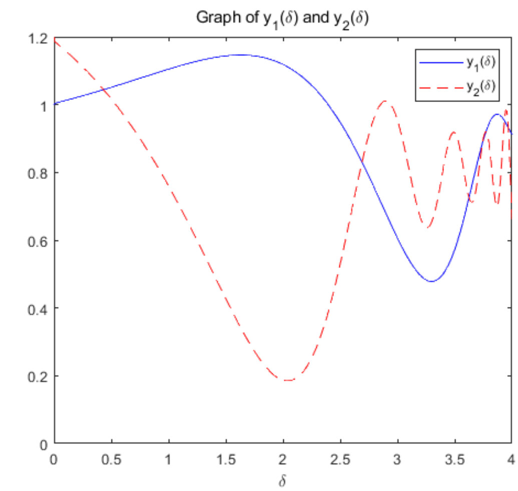

Figure 11. $y(\delta)$ change with respect to δ

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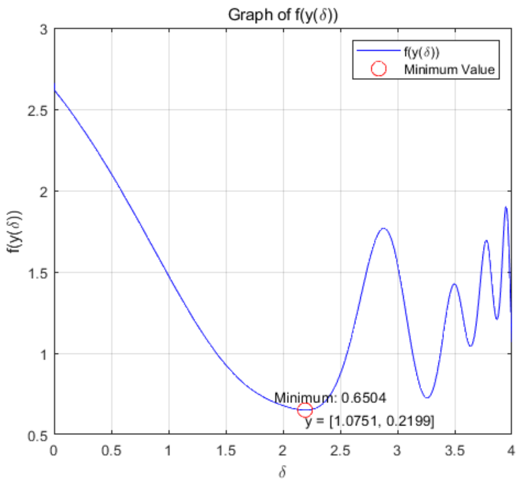

Figure 12. $f(y(\delta))$ change with respect to δ

Remark 7. *Within the allowable range of error, the optimization problem reaches the optimal solution* $x^{\star} = (1.0751, 0.2200)$ *within a prescribed time T=6.5 under the action of system (17), and the optimization problem of its equivalence converges asymptotically to the same optimal solution under the action of system (20).*

Case 4: when μ=0.35, a=0.8, T=8, $A=\begin{bmatrix} 1.2 & 1.1 & 0.3 \\ 0.2 & 0.3 & 0.1 \\ 1.5 & 0.4 & 0.5 \end{bmatrix}$, $b=\begin{bmatrix} 1.1 \\ 3 \\ 2 \end{bmatrix}$, we choose $\alpha_2(\delta)$ and $d_2(s)$, then we consider the minimum of the strongly convex function $f = \frac{1}{2}(x-1)^2$.

(a): Figures 13 and 14 show the variation of variables x and $f(x(t))$ with respect to t when the system (17) is applied to solve the problem, respectively.

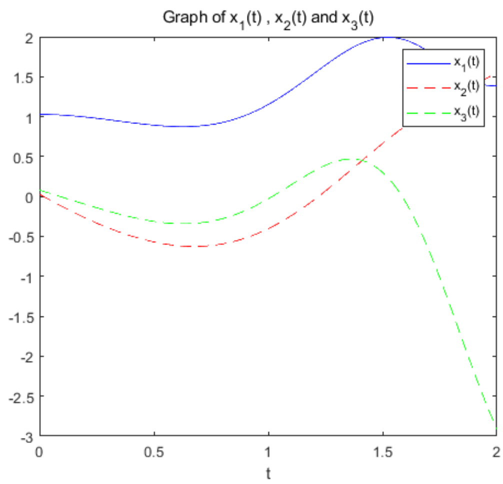

Figure 13. $x(t)$ change with respect to t

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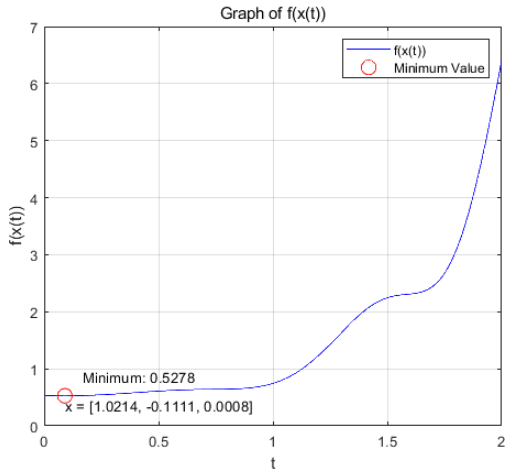

Figure 14. $f(x(t))$ change with respect to t

(b): Figures 15 and 16 show the variation of variables y and $f(y(\delta))$ with respect to δ when the system (20) is applied to solve the problem, respectively.

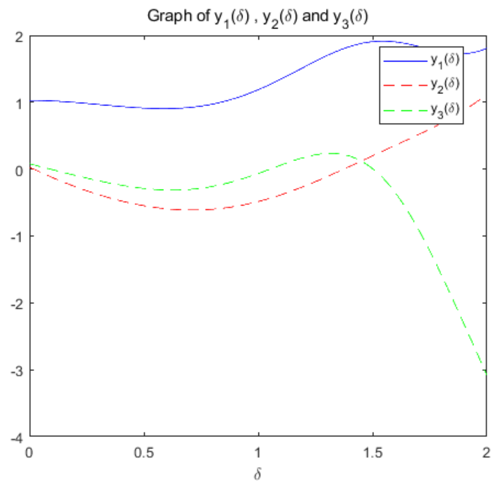

Figure 15. $y(\delta)$ change with respect to δ

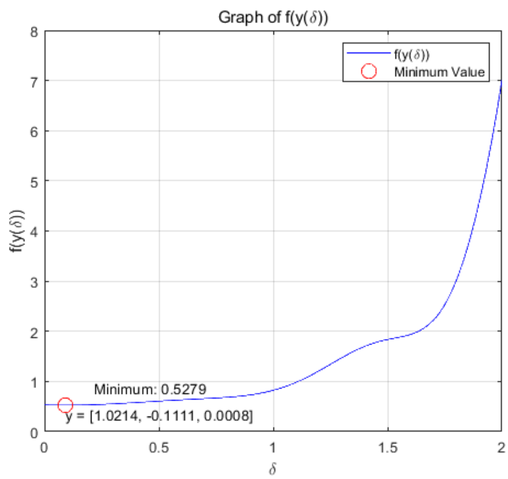

Figure 16. $f(y(\delta))$ change with respect to δ

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Remark 8. *the optimization problem reaches the optimal solution $x^\star = (1.0214, -0.1111, 0.0008)$ within a prescribed time T=8 under the action of system (17), and the optimization problem of its equivalence converges asymptotically to the same optimal solution under the action of system (20).*

7. Conclusions and Future Work

For the unconstrained optimization problem of strongly convex objective function and the optimization problem with equation constraints, we develop a novel prescribed-time convergence acceleration algorithm with time rescaling. Our basic idea is to construct different second-order systems under certain conditions, so that the two types of optimization problems converge to the optimum solution within a prescribed-time T. This is more flexible than the traditional exponential asymptotic time convergence and improves the convergence rate of the algorithm to a large extent. According to this idea, our next work can be focus on

(1) Modify the optimization algorithm with sub-linear convergence rate $O\left(\frac{1}{t^2}\right)$ so that it can also converge to the optimum solution of the optimization problem within a prescribed-time T .

(2) Considering the problem that the strongly convex objective function has a prescribed time T convergence under the constraints of general convex ensembles or inequalities.

(3) In addition to the Euler method, it is possible to discretize a system that converges within a prescribed time T by methods such as Lunger-Kutta.

(4) Consider the problem of acceleration of algorithms that converge in prescribed time T in a distributed manner.

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