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Exact Similarity Solutions of Unsteady Laminar Boundary Layers

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Abstract: The studies of laminar unsteady boundary layer flows is crucial for understanding laminar-turbulence transition and origins of turbulence. However, the task of finding its solutions poses a significant challenge. In this paper, we propose a novel approach by introducing a similar transformation to convert the 2D unsteady laminar boundary layer equations into a single partial differential equation. By applying this transformation, we are able to obtain the "exact" similarity solutions for the 2D unsteady laminar boundary layer equations, specifically for the case of flat plate boundary flow. Notably, this is the first time that such an exact solution has been obtained. Application of this exact solution is being used to solve Stokes' first problem in 2D boundary layer. Perspectives on the transition fron Rayleigh solution to Blasius solution is provided.

Keywords: Navier-Stokes equations; 2D unsteady laminar boundary layers; similar transformation; exact solutions

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1. Introduction

The study of unsteady flows is crucial for understanding the dynamics of laminar-turbulence transition [1–17], and even for origin of turbulence phenomenon [18–39].

The Navier-Stokes equations, which govern fluid flow, are highly complex and pose significant challenges in obtaining exact solutions. Only a few one-dimensional non-steady problems have been able to achieve exact solutions [1,2], such as Stokes' problem and the flow due to an oscillating infinite plane. The latter was the first exact solution in fluid dynamics [1]. The second example of an exact solution is being the well-known Rayleigh solution for an infinite flat plate [2], which involves an infinite flat plate impulsively started into motion in its own plane with a velocity of U_0. The solution to this problem is given by the equation $u(y,t) = U_0[1 - \mathrm{erf}(\frac{y}{2\sqrt{\nu t}})]$, where erfc is the complementary error function.

If x is the distance along the plate measured from the leading edge and $\tau = Ut/x$, the motion for small τ is described by the solution of Rayleigh (1911) [1], while as $\tau \to \infty$ the ultimate steady-state solution is that given by Blasius (1908) [18]. The problem of describing the transition from the one regim̃e to the other has proved to be a difficult one. It was firstly studied by Stewartson (1951) [4], who obtained approximate solutions by making simplifications of the boundary-layer equations. In particular, Stewartson investigated the nature of the solution near $\tau = 1$ and as $\tau \to \infty$. He found an essential singularity at $\tau = 1$ such that the solution for $\tau \gg 1$ is not an analytic continuation of the solution for $\tau \ll 1$, although all derivatives with respect to x are continuous at $\tau = 1$. As $\tau \to \infty$, the decay to the steady-state Blasius solution was found to be exponential in type, with the departure from the steady solution ultimately concentrated near the plate [10].

A question is how the initial Rayleigh solution [2] having no x-dependence could possibly settle down to the x-dependent but time-independent Blasius solution [18]. Stewartson [4]) was the first to attempt a plausible explanation for it. From the physical requirement, the Rayleigh flow must shift to an x-dependent Blasius-type flow in a smooth manner. Stewartson argued then that, in the light of no analytical development from Rayleigh- to Blasius-type flows, the smooth transition can only be

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possible through an essential singularity. Rather intuitively Stewartson considers $\tau = 1$ as the location of this singularity. It is argued that, for $\tau \ll 1$, since a disturbance at the leading edge has not arrived yet, the flow remains of Rayleigh type; however, at $\tau = 1$, the leading-edge effect is felt suddenly and the flow starts to have x-dependence by means of an essential singularity.? The attempt to construct the mathematical evidence for the existence of such a singularity was made by Stewartson for a flat plate. Takuda [8] proposed a small-time solution in powers of the time, shown a smooth transition from the initial Rayleigh flow to the final Blasius flow without an essential singularity with .

In spite of the these achievements on the subject in the past, the problem still remains intriguing and confusing. The essential difficulty is to describe the transition from the Rayleigh state to the Blasius state owing to no complete solution to the problem has yet been obtained [4]. Unfortunately, to date, no exact solutions have been obtained for the unsteady boundary layer that we are considering. As a result, we are left with no choice but to confront this challenge and strive to find new ideas and approaches to explore new frontiers in understanding unsteady flows.

In summary, just after the establishment of the Navier-Stokes equations in 1951, Stokes himself studied two non-steady fluid mechanics problems, known as Stokes' first problem (later referred to as the Rayleigh problem) and Stokes' 2nd problem. Both of these problems are one-dimensional, and there were no results for two-dimensional problems due to their complexity until Stewartson [4] studied two-dimensional unsteady laminar boundary layers in 1951. Afterwards, Takuda (1968) [8], Dennis (1972), and others attempted to study these problems, but unfortunately, none of them obtained satisfactory results. Dennis (1972) [10] pointed out that the reason was the lack of complete analytical solutions. Due to the absence of analytical solutions, Stewartson [4] believed that a singularity must be encountered when transition from the Rayleigh solution to the Blasius solution. This singularity conjecture remains unsolved to this day. In particular, the similar transformation introduced by Stewartson [4] only applies to planar flows, where u =constant. For flow phenomena such as contracting flow, wedge and stagnation flows, a unified similar transformation has yet to be obtained. This article solves this century-old problem in fluid mechanics entirely. Firstly, we introduce a similarity transformation that can handle different flows, $u = Cx^m$. Secondly, for $u = U_0$, we obtain the exact solution for the first time. Thirdly, we fully explain how the Rayleigh solution transforms into the Blasius solution. Lastly, we prove that Stewartson's conjecture is invalid and that there is no singularity.

After introduction in Section 1, the rest of this paper is organized as follows. In Section 2, we formulate the 2D turbulent boundary layers and introduce a similar transformation. Under special conditions, the partial differential equations of the 2D turbulent boundary layers can be reduced to a single partial differential equation. In Section 3, A semi-infinite flat plate impulsively started with velocity is studied, whose exact solution is obtained for the first time. In Section 4, A semi-infinite flat plate in an uniform flow with velocity is studied, whose exact solution is obtained for the first time. In Section 5, An open problem of from Rayleigh solution to Blasius solution is discussed. Finally, in Section 6, conclusions are drawn.

2. Similar transformations of 2D laminar boundary layers equations

A thin flat plate is immersed at zero incidence in a uniform stream as shown in Figure 1, which flows with speed $U(x)$ and is assumed not to be affected by the presence of the plate, except in the boundary layer. The fluid is supposed unlimited in extent, and the origin of coordinates is taken at the leading edge, with x measured downstream along the plate and y perpendicular to it.

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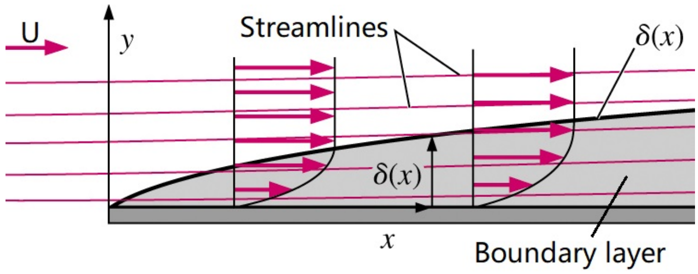

Figure 1. Turbulent boundary layer. μ is the dynamical viscosity, ρ is the flow density, $\delta(x)$ is "boundary layer thickness". Strictly speaking, $\delta(x)$ is not the boundary layer thickness,but rather a scaled measure of the boundary layer thickness which is equal to the boundary layer thickness up to some numerical factor [13].

The unsteady Navier-Stokes equations of the two dimensional boundary layers flow under gradient, $\frac{dp}{dx}$, are reduced to

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \tag{1}$$

$$-\frac{1}{\rho}\frac{\partial p}{\partial y} = 0, \tag{2}$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{dp}{dx} + \nu\frac{d^2 u}{dy^2}, \tag{3}$$

and initial-boundary conditions together with initial conditions at $x = x_0$, $t = 0$. In which, ν is the kinematic viscosity, ρ is flow density, p is pressure, $U(x)$ is outer of boundary layer potential flow velocity. The pressure gradient must be negative, namely $\frac{dp}{dx} < 0$, to maintain the flow motion. For a curved boundary layers, the coordinates (x, y) should be replaced by (s, n), where s is arc length and n is normal to the curve layers.

Integration of Eq.2 yields $\frac{p}{\rho} = \frac{p_e}{\rho}$, where p_e is a function of x only [13], then $\frac{\partial p}{\partial x} \approx \frac{dp_0}{dx}$. From Bernoulli equation, we have relation: $p_e + \frac{1}{2}\rho U^2 = constant.$, leads to $\frac{dp_e}{dx} = \rho U\frac{dU}{dx}$. The boundary equations are reduced to following:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \tag{4}$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = U\frac{dU}{dx} + \nu\frac{d^2 u}{dy^2}, \tag{5}$$

Introducing a stream function $\psi(x, y, t)$ and express the velocity components as follows

$$u = \frac{\partial \psi}{\partial y}, \quad v = -\frac{\partial \psi}{\partial x}, \tag{6}$$

with the relation in Eq.6, the mass conservation Eq.4 is satisfied, and the momentum conservation Eq.(5) becomes

$$\frac{\partial^2 \psi}{\partial t \partial y} + \frac{\partial \psi}{\partial y}\frac{\partial \psi}{\partial x \partial y} - \frac{\partial \psi}{\partial x}\frac{\partial \psi}{\partial y^2} = U\frac{dU}{dx} + \nu\frac{\partial^3 \psi}{\partial y^3}, \tag{7}$$

and corresponding boundary conditions.

We can assume that the velocity profiles at different distances from the leading edge are affine or similarity to one another, i.e. that the velocity profile u at different distances x can be mapped onto each other by suitable choice of scaling factors for u and y. A suitable scaling factor for u could be the free stream velocity $U(x)$, while for y, "boundary-layer thickness" $\delta(x)$, which increases with distance

x, could be used. The similarity law of the velocity profile can thus be written as $u/[\delta(x)U] = f(\eta)$ with $\eta = y/\delta(x)$.

Regarding the similarity of time t, it is known that the time for any bulk property of the fluid, such as vorticity or momentum, to diffuse through a distance δ is of the order of δ^2/ν (the "diffusion time" proposed by Stuart [7]), hence, we can introduce a dimensionless time as $\tau = t/(\delta^2/\nu)$.

Based on the above understanding, introducing following transformations

$$\psi = U(x)\delta(x)f(\eta,\tau), \tag{8}$$
$$\eta = \frac{y}{\delta(x)}, \tag{9}$$
$$\tau = \frac{\nu}{\delta^2}t, \tag{10}$$

where the dimensionless time transformation in Eq.10 is the key of success to solve the 2D unsteady laminar boundary layer flows and is introduced firstly in this paper.

To formulate the Eq.7 in terms of $f(\eta,\tau)$, we need to calculate some derivatives of function ψ respect to both x and y, for simplification, we denote $\psi_{,x} = \frac{\partial\psi}{\partial x}$ and $f_{,\eta} = \frac{\partial f}{\partial\eta}$ and so on. Noting the $\eta_{,x} = -\eta\delta^{-1}\delta_{,x}$ and $\tau_{,x} = -2\tau\delta^{-1}\delta_{,x}$, and by the chain rule for derivatives, we can obtain some useful relations as follows:

$$\psi_{,x} = (U\delta)_{,x}f - U\eta\delta_{,x}f_{,\eta} - 2U\tau\delta_{,x}f_{,\tau}, \tag{11}$$
$$\psi_{,y} = Uf_{,\eta}, \tag{12}$$
$$\psi_{xy} = U_{,x}f_{,\eta} - \eta U\delta^{-1}\delta_{,x}f_{,\eta\eta} - 2U\tau\delta^{-1}\delta_{,x}f_{,\eta\tau}, \tag{13}$$
$$\psi_{,yy} = U\delta^{-1}f_{,\eta\eta}, \tag{14}$$
$$\psi_{,yyy} = U\delta^{-2}f_{,\eta\eta\eta}, \tag{15}$$
$$\psi_{ty} = U\nu\delta^{-2}f_{,\tau\eta}, \tag{16}$$

Thus the velocity components become

$$u = Uf_{,\eta}, \tag{17}$$
$$v = -\left[(U\delta)_{,x}f - U\eta\delta_{,x}f_{,\eta} - 2U\tau\delta_{,x}f_{,\tau}\right]. \tag{18}$$

Substituting Eqs.13-18 into Eq.(7), we have a single partial differential equation as follows

$$f_{,\eta\eta\eta} + \alpha f f_{,\eta\eta} + \beta[1 - (f_{,\eta})^2] = f_{,\tau\eta} + \gamma\tau(f_{,\tau}f_{,\eta\eta} - f_{,\eta}f_{\tau\eta}), \tag{19}$$

where the coefficients are $\alpha = \frac{\delta}{\nu}\frac{dU\delta}{dx}$, $\beta = \frac{\delta^2}{\nu}\frac{dU}{dx}$, and $\gamma = \frac{U}{\nu}\frac{d\delta^2}{dx}$.

If the coefficient α, β and γ were constants, the Eq.19 is solvable because it will get rid of the variable x. Following from the relation $\frac{d\delta^2 U}{dx} = (2\alpha - \beta)\nu$, its integration leads to $\delta^2 U = (2\alpha - \beta)\nu x$, canceling out δ^2 by $\beta = \frac{\delta^2}{\nu}\frac{dU}{dx}$, we have $\frac{dU}{U} = \frac{\beta}{(2\alpha-\beta)}\frac{dx}{x}$, hence when $\beta \neq 2\alpha$, the general solution of this equation is of the form:

$$U(x) = Cx^m, \tag{20}$$
$$\delta(x) = \left[\nu\left|\frac{(2\alpha - \beta)x}{U(x)}\right|\right]^{1/2}, \tag{21}$$

where the exponent $m = \frac{\beta}{2\alpha-\beta}$. In the same way, we have $\frac{d\delta^2 U}{dx} = (\beta + \gamma)\nu$, leads to a relation $2\alpha - \beta = \beta + \gamma$, namely $\gamma = 2(\alpha - \beta)$. It implies that γ is constant if both α and β were constants. Regarding the solution of Eq.19, for constants α, β and γ, Eq.19 can be numerically solved.

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It can be clearly seen that the Eq.19 does not explicitly contain the coordinate x, indicating that we have successfully transformed the original partial differential equation Eq.7 with three independent variables (x, y, t) into a partial differential equation with two similar variables (η, τ). This not only facilitates solving the equation but also allows us to express the obtained results as a single profile in terms of the similar variable η. The solution with a single profile at a certain moment t is called as the exact solution according to the arguments of Wang [14]. In particular, although scholars such as Stewaetson [4] and Dennis [10] have also obtained transformed equations by introducing their similar variables, they failed to obtain exact solutions due to inappropriate similar transformations. In other words, an inappropriate similar transformation does not necessarily result in a well-structured transformed equation. Therefore, the similar transformations we introduced in Eq.8 are of essential importance for solving the flat boundary layer problem.

3. A semi-infinite flat plate impulsively started with velocity U_0

As an application of our formulation in previous section. Let's consider the Stokes' first problem [1], Stokes' first problem is a fundamental unsteady fluid problem from which an exact solution has been found for its one dimensional case. Here we discuss the Stokes' first problem of 2D laminar flow.

An semi-infinitely flat plate is immersed in a viscous and stationary fluid. At $t = 0$, the plate suddenly accelerates in its own plane to a velocity U_0, and then continues to move at a constant speed U_0. At the moment of initiation, the fluid surrounding the plate remains stationary due to viscosity. As time passes, the fluid near the plate gradually starts to move layer by layer due to the effect of viscosity. Clearly, this is an unsteady two-dimensional flow. This problem was first proposed by Stokes in 1851, known as Stokes' first problem [1], and its 1D similar solution is called the Rayleigh solution [2].

The problem is to find the motion of a viscous fluid past a semi-infinite flat plate which, at time $t = 0$, is suddenly set in motion with constant velocity U_0 parallel to itself. This problem first attracted the attention of Stewartson [4]. The equations governing the boundary layer flow are Eq.1, Eq.2 and Eq.3 with $U(x) = U_0$ (constant), together with the initial-boundary conditions:

$$\begin{aligned} &u = 0,\ v = 0: \quad \text{when } y = 0,\ x > 0,\ t > 0; \\ &u = U_0: \quad \text{when } y > 0 \\ &\text{and either } t \geq 0,\ x = 0, \text{ or } x \geq 0,\ t = 0; \\ &u = 0: \quad as\ y \to \infty,\ x \geq 0,\ t \geq 0. \end{aligned} \tag{22}$$

In the case of two dimensional plate boundary layers, if $\beta = 0$ then we have $m = 0$, flow velocity $U(x) = U_0$. Since α can be any real number, without loss of generality, we set $\alpha = 1$, hence $\gamma = 2(\alpha - \beta) = 2$. The boundary thickness is $\delta(x) = (\frac{2\nu x}{U_0})^{1/2}$.

The stream function is of the form

$$\psi = (2\nu U_0 x)^{1/2} f(\eta, \tau), \tag{23}$$

where $\eta = y(\frac{U_0}{2\nu x})^{1/2}$, and diffusion time $\tau = \frac{U_0}{2}\frac{t}{x}$.

The Eq.19 can be reduced to the following:

$$f_{,\eta\eta\eta} + f f_{,\eta\eta} = f_{,\tau\eta} + 2\tau(f_{,\tau} f_{,\eta\eta} - f_{,\eta} f_{\tau\eta}). \tag{24}$$

With the help of Maple [41], we can find an exact solution of Eq.24 as follows

$$f(\eta, \tau) = \frac{1}{5\tau^2} + \frac{\eta}{2\tau} + c_1\sqrt{\tau} + c_2 g_1 + c_3 h_1, \tag{25}$$

where c_1, c_2, c_3 are integral constants, the functions g_1 and h_1 are given in the Appendix 1.

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In the following, all flow velocity and stress field calculations are based on solution $f(\eta,\tau)$. Hence, we have the flow velocity components

$$u = U_0 f_{,\eta}, \tag{26}$$

$$v = (\frac{\nu U_0}{2x})^{1/2}(\eta f_{,\eta} - f + 2\tau f_{,\tau}). \tag{27}$$

To complete the calculations in the above relations, we need to have $f_{,\eta}$ and $f_{,\tau}$.

$$f_{,\eta} = \frac{1}{2\tau} + c_2 g_2 + c_3 h_2, \tag{28}$$

$$f_{,\tau} = -\frac{\eta}{2\tau^2} - \frac{2}{5\tau^3} + \frac{1}{2\sqrt{\tau}}c_1 + c_2 g_3 + c_3 h_3, \tag{29}$$

where the functions g_2, g_3 and h_2, h_3 are given in the Appendix 1.

Applying the boundary conditions to the solutions in u and v, the function $f(\eta,\tau)$ must satisfies the following conditions

$$\eta = \infty: \ u(\infty,\tau) = U_0 f_{,\eta}|_{\eta=\infty} = 0, \quad at \quad \tau = \infty, \tag{30}$$

which leads to

$$\eta = \infty: \ f_{,\eta} = 0, \quad at \quad \tau = \infty, \tag{31}$$

and

$$\eta = 0: \ u(0,\tau) = U_0 f_{,\eta}|_{\eta=0} = U_0, \quad at \quad \tau = \infty, \tag{32}$$

$$\eta = 0: \ v(0,\tau) = (\frac{\nu U_0}{2x})^{1/2}(\eta f_{,\eta} - f + 2\tau f_{,\tau}) = 0, \quad at \quad \tau = \infty, \tag{33}$$

which leads to following conditions

$$\eta = 0: \ f_{,\eta}(0,\tau) = 1, \quad at \quad \tau = \infty, \tag{34}$$

$$\eta = 0: \ -f(0,\tau) + 2\tau f_{,\tau}(0,\tau) = 0, \quad at \quad \tau = \infty. \tag{35}$$

Since $f_{,\eta}(\infty,\infty)$ undefined, to make $f_{,\eta}(\infty,\infty)$ be defined, we must have $c_2 = 0$. The condition in Eq.35 is satisfied due to the fact of $-f(0,\tau) + 2\tau f_{,\tau}(0,\tau) = \left(\eta[3\sqrt{\frac{3}{\pi}}\Gamma(\frac{2}{3})c_3 - c_2]\right)_{\eta=0} \equiv 0$. To find the constant c_1, we set condition $f(0,\infty) = 0$, leads to $5\Gamma(\frac{5}{6})c_1 + 4\sqrt{3\pi}c_3 = 0$.

For any value of η, when $\tau \to \infty$, we always have $f_{,\eta}(\eta,\infty) = -\left[3\sqrt{\frac{3}{\pi}}\Gamma(\frac{2}{3})c_3 - c_2\right]$. If applying the conditions in both Eq.31 and Eq.34 to $f_{,\eta}(\eta,\infty)$, We will obtain completely contradictory results. It means that these two conditions in Eq.31 and Eq.34 cannot be satisfied simultaneously.

The concept of τ approaching infinity in in both Eq.31 and Eq.34 only holds mathematical significance and cannot be practically calculated in real-life calculations, because it is already outside the boundary layer when $\eta = y/\delta > 1$. The conditions in Eq.34 and Eq.35 must be satisfied firstly since they are more important than the condition Eq.34, namely $3\sqrt{\frac{3}{\pi}}\Gamma(\frac{2}{3})c_3 - c_2 = 1, \quad c_2 = 0, \quad 5\Gamma(\frac{5}{6})c_1 + 4\sqrt{3\pi}c_3 = 0$. As a result, we obtain the following integral constants:

$$c_1 = \frac{4\pi}{15\Gamma(\frac{5}{6})\Gamma(\frac{2}{3})}, \quad c_2 = 0, \quad c_3 = -\frac{1}{3\Gamma(\frac{2}{3})}\sqrt{\frac{\pi}{3}}. \tag{36}$$

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With the above integral constants, we can prove that the condition Eq.31 is also satisfied due to the rapid decay ability of exp $\left(-\frac{(3\eta\tau+2)^2}{12\tau^3}\right)$. This means that the above integral constants in Eq.36 can make the solution satisfying all solution conditions.

The function $f(\eta,\tau)$, its derivatives $f_{,\eta}$, $f_{,\tau}$ and $f_{,\eta\eta}$ are depicted in Figure 2.

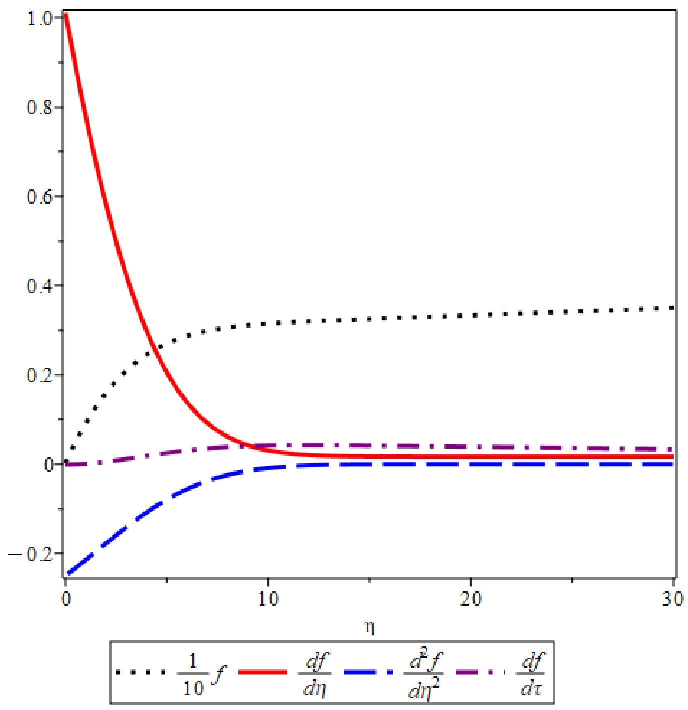

Figure 2. A semi-infinite flat plate impulsively started with velocity U_0: $f_{,\eta}$, $f_{,\eta\eta}$, f and $f_{,\tau}$.

The shear stress

$$\tau_{xy} = \mu\frac{\partial u}{\partial y} = \frac{\rho\nu}{\delta}\frac{\partial u}{\partial \eta} = \rho\left(\frac{\nu U_0}{2x}\right)^{1/2} f_{,\eta\eta}(\eta,\tau), \tag{37}$$

where

$$f_{,\eta\eta}(\eta,\tau) = -\frac{1}{\Gamma(\frac{2}{3})}\sqrt{\frac{\pi}{3}}\frac{3\eta\tau+2}{3\tau^2}\mathrm{KummerU}(\frac{5}{6},\frac{3}{2},\frac{(3\eta\tau+2)^2}{12\tau^3})\exp\left(-\frac{(3\eta\tau+2)^3}{12\tau^3}\right). \tag{38}$$

and wall shear stress $\tau_w = \tau_{xy}|_{\eta=0}$

$$\tau_w = \rho\left(\frac{\nu U_0}{2x}\right)^{1/2} f_{,\eta\eta}(0,\tau), \tag{39}$$

where

$$f_{,\eta\eta}(0,\tau) = -\frac{2}{3\Gamma(\frac{2}{3})}\sqrt{\frac{\pi}{3}}\frac{1}{\tau^2}\mathrm{KummerU}(\frac{5}{6},\frac{3}{2},\frac{1}{3\tau^3})\exp\left(-\frac{1}{3\tau^3}\right). \tag{40}$$

Clearly, the wall shear stress τ_w is a function of x, t, which is similar to the laminar boundary layer but totally different from 1D the planer turbulent flow, whose the wall shear stress τ_w is assumed a constant.

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The skin-friction coefficient with the reference velocity U_0 is given by

$$c_f = -\frac{2}{3\Gamma(\frac{2}{3})}\sqrt{\frac{\pi}{3}}\frac{1}{U_0\sqrt{Re_x}}\frac{1}{\tau^2}\mathrm{KummerU}(\frac{5}{6},\frac{3}{2},\frac{1}{3\tau^3})\exp\left(-\frac{1}{3\tau^3}\right) \tag{41}$$

where the Reynolds number is defined as $Re_x = \frac{xU_0}{\nu}$. The dimensionless $K = U_0\sqrt{Re_x}c_f$ is depicted in Figure 3.

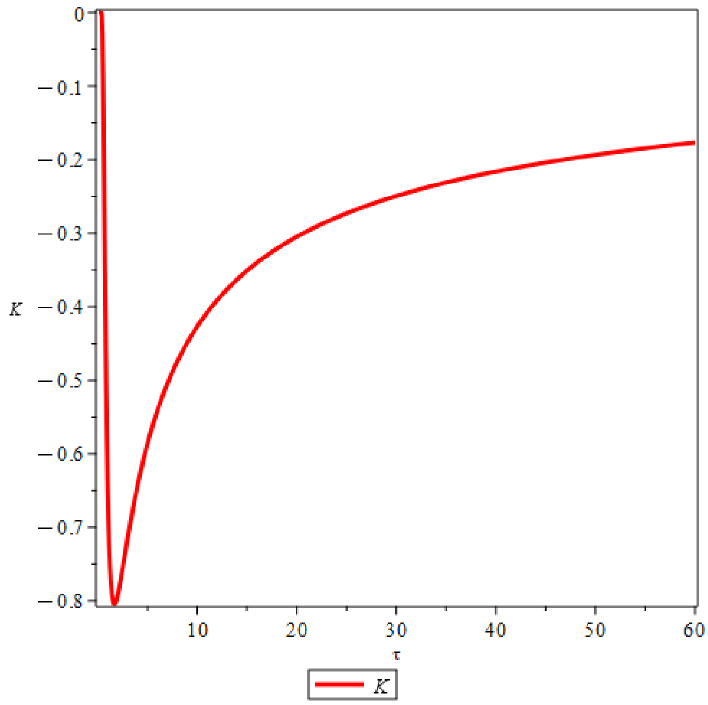

Figure 3. $K = U_0\sqrt{Re_x}c_f$. At $\tau \approx 0.8034$, the wall shear reaches its maximum value $K_{max} = U_0\sqrt{Re_x}c_f = 1.59545$, or $c_{f,max} \approx \frac{1.59545}{U_0\sqrt{Re_x}} = 1.59545U_0^{-3/2}\sqrt{\frac{\nu}{x}}$.

4. A semi-infinite flat plate in an uniform flow with velocity U_0

Exactly the opposite of the situation with the Stokes's first problem, it is not a flat plate moving but rather stationary, which refers to a stationary plate in a fluid with a velocity U_0. It is another version of the Stokes's first problem. In this case, the initial-boundary conditions become as follows:

$$\begin{aligned} &u = 0,\ v = 0: \quad \text{when } y = 0,\ x > 0,\ t > 0; \\ &u = 0: \quad \text{when } y > 0 \\ &\text{and either } t \geq 0,\ x = 0, \text{ or } x \geq 0,\ t = 0; \\ &u = U_0: \quad as\ y \to \infty,\ x \geq 0,\ t \geq 0. \end{aligned} \tag{42}$$

Obviously, the flow fields of these two problems are very similar, and the velocity components can be obtained as follows:

$$u = U_0[1 - f_{,\eta}], \tag{43}$$

$$v = (\frac{\nu U_0}{2x})^{1/2}(\eta f_{,\eta} - f + 2\tau f_{,\tau}). \tag{44}$$

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The shear stress

$$\tau_{xy} = -\rho \left(\frac{\nu U_0}{2x} \right)^{1/2} f_{,\eta\eta}(\eta, \tau), \tag{45}$$

and wall shear stress $\tau_w = \tau_{xy}|_{\eta=0}$

$$\tau_w = -\rho \left(\frac{\nu U_0}{2x} \right)^{1/2} f_{,\eta\eta}(0, \tau), \tag{46}$$

The skin-friction coefficient with the reference velocity U_0 is given by

$$c_f = \frac{2}{3\Gamma(\frac{2}{3})} \sqrt{\frac{\pi}{3}} \frac{1}{U_0\sqrt{Re_x}} \frac{1}{\tau^2} \text{KummerU}(\frac{5}{6}, \frac{3}{2}, \frac{1}{3\tau^3}) \exp\left(-\frac{1}{3\tau^3} \right) \tag{47}$$

The function $f(\eta, \tau)$, its derivatives $f_{,\eta}$, $f_{,\tau}$ and $f_{,\eta\eta}$ are depicted in Figure 4.

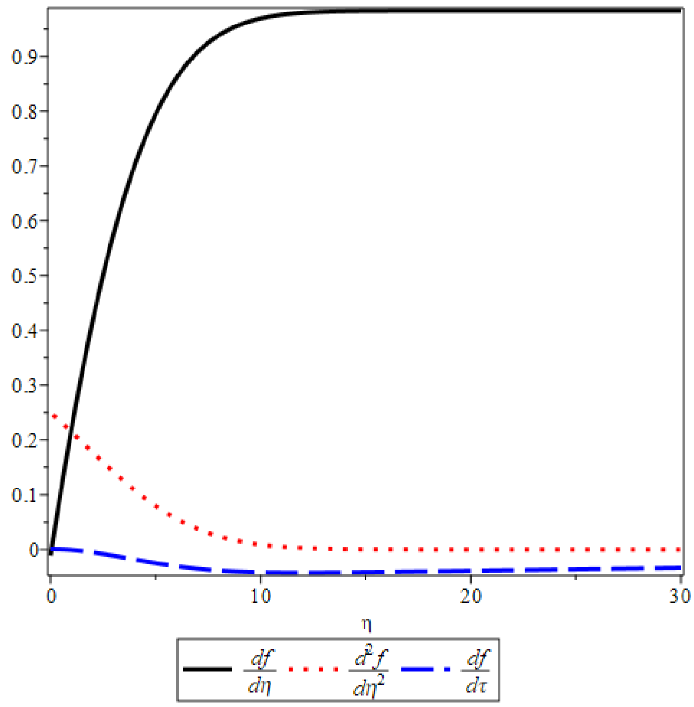

Figure 4. A semi-infinite flat plate in an uniform flow with velocity U_0: $f_{,\eta}$, $f_{,\eta\eta}$, f and $f_{,\tau}$.

The solutions in this section can be viewed as Blasius type solution, which are obtained for the first time for unsteady laminar boundary layer.

5. From Rayleigh solution to Blasius solution

After solving Eq.24, let's take a look at the transitional problem from Rayleigh solution [2] to Blasius solution [18] proposed by Stewartson [4].

Introducing a new variable $\sigma = 1/\tau$, then $f_{,\tau} = -\sigma^2 f_{,\sigma}$, the Eq.24 can be converted into

$$f_{,\eta\eta\eta} + f f_{,\eta\eta} = -\sigma^2 f_{,\sigma\eta} + 2\sigma(f_{,\eta} f_{,\sigma\eta} - f_{,\sigma} f_{,\eta\eta}). \tag{48}$$

It is not difficult to see that when $\tau \to \infty$, namely $\sigma \to 0$, the Eq.48 and/or Eq.24 becomes $f_{,\eta\eta\eta} + f f_{,\eta\eta} = 0$, which is the Blasius equation [18]. This means that the Blasius solution is contained in Eq.24.

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Another question is whether Eq.24 also contains the Rayleigh solution [2]. If the Rayleigh solution is not a solution of Eq.24, then the transitional problem from Rayleigh solution to Blasius solution proposed by Stewartson [4] would be meaningless. If the Rayleigh solution [2] is a solution of Eq.24, then it answers the challenging question raised by Stewartson [4].

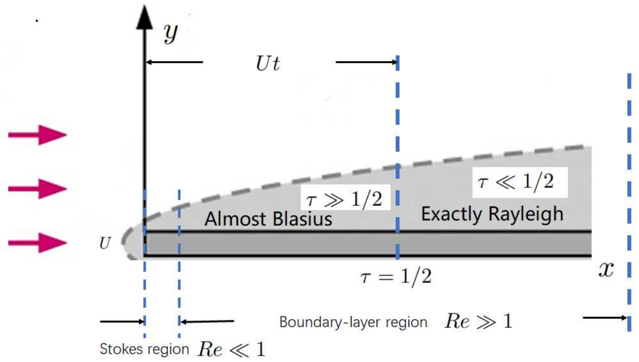

Figure 5. Region of unsteady laminar boundary layers past a flat plate: Reylaigh state [2] and Blasius state [18].

From Figure 5, for Rayleigh state, since $\tau \ll \frac{1}{2}$, Eq.24 can be reduced to $f_{,\eta\eta\eta} + f f_{,\eta\eta} = f_{,\tau\eta}$, and can be further approximated as $f_{,\eta\eta\eta} \approx f_{,\tau\eta}$, whose solution can be easily obtained as follows

$$f_{Rayleigh} = \eta - \eta \mathrm{erf}(\frac{\eta}{2\sqrt{\tau}}) - 2\sqrt{\frac{\tau}{\pi}}\left[\exp(-\frac{\eta^2}{4\tau}) - 1\right], \tag{49}$$

together with relations $f_{,\eta} = 1 - \mathrm{erf}(\frac{\eta}{2\sqrt{\tau}})$ and $f_{,\tau} = -\frac{1}{\sqrt{\pi\tau}}[\exp(-\frac{\eta^2}{4\tau}) - 1]$.

The solution in Eq.49 satisfies Eq.24 and all corresponding boundary conditions: $f(0,\tau) = 0$, $f_{,\eta}(0,\tau) = 1$ and $f_{,\eta}(\infty,\tau) = 1$, therefore the solution in Eq.49 is also an exact solution of Eq.24.

The flow velocity components are given by

$$u = U_0 f_{,\eta} = U_0[1 - \mathrm{erf}(\frac{\eta}{2\sqrt{\tau}})], \tag{50}$$

$$v = (\frac{\nu U_0}{2x})^{1/2}(\eta f_{,\eta} - f + 2\tau f_{,\tau}) \equiv 0. \tag{51}$$

From the velocity components, we can identify the solution in Eq.**??** must be the Rayleigh solution of flat plate due to impulsive motion along the direction of its plane.

Based on the discussions here, the equation Eq.24 we derived indeed includes both Rayleigh and Blasius solutions, and their transformation is natural, without the need for the singularity assistance mentioned by Stew. In particular, in the general case, we obtain two exact solutions, which are continuous with respect to spatial coordinates and time, without any singularities. This resolves the mathematical problem proposed by Stew in 1951, which has remained unsolved for over 70 years.

Comparing the solutions in this article with Rayleigh's solution as shown in Figure 6, it can be observed that the decay rate of the solution in this article is much faster as η increases. The reason behind this is that the decay rate of $\exp\left(-\frac{(3\eta\tau+2)^3}{12\tau^3}\right)$ is much faster than $[1 - \mathrm{erf}(\frac{\eta}{2\sqrt{\tau}})]$.

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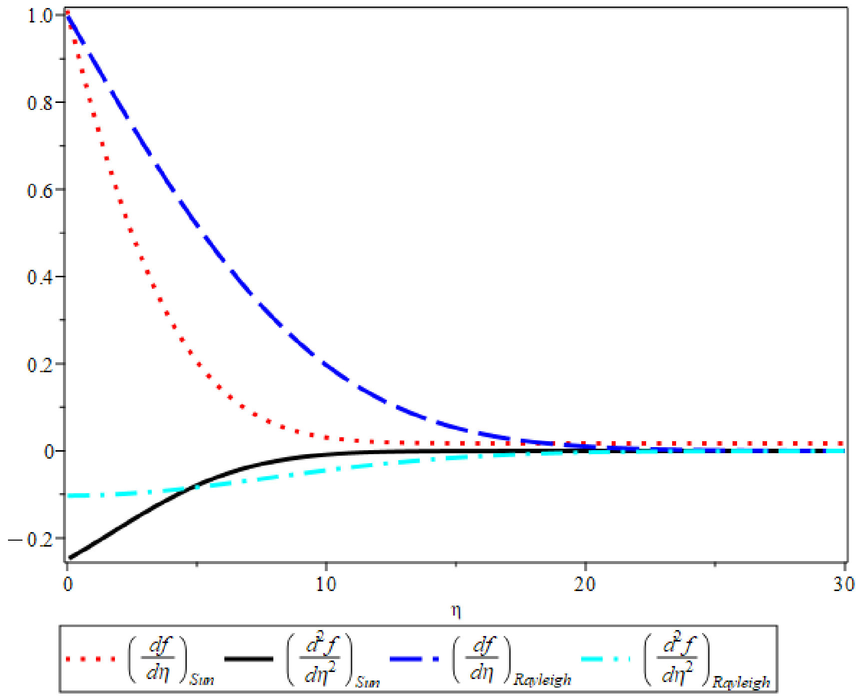

Figure 6. Comparisons between the Rayleigh solution and the solution of this paper.

6. Conclusions

The transformation of the 2D unsteady laminar boundary layer equations into a single partial differential equation is a clever approach that allows for the exact solution. This single partial differential equation is easy to be numerically integrated, especially for the semi-infinite plate impulsively moving with a velocity, where we have obtained an exact solution. This is the first exact solution for an unsteady laminar boundary layer. Using our equation and the exact solution, we have provided a clear explanation for Stewartson's problem of how the Rayleigh solution can be transformed into the Blasius solution, which was proposed 70 years ago by Stewartson [4]. Our interpretation for Stewartson's problem is that the transformation from the Rayleigh solution to the Blasius solution does not require passing through a singularity. It has been proven that $\tau = 1/2$ is not a singularity point. The similar transformations and the single equation obtained here can be used to solve other problems, such as thermal boundary layers, convergent and wedge boundary layers. In summary, the results of this article represent a breakthrough in the study of unsteady boundary layers, and they will also contribute to understanding the transition from laminar to turbulence, and even the origin of turbulence.

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Availability of data: The data supporting the findings of this study are available from the corresponding author upon reasonable request.

Declaration of Competing Interest: The author declares that he has no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Appendix 1 The functions g_1, g_2, g_3 and h_1, h_2, h_3

$$\begin{aligned} g_1 = & \frac{9(\eta\tau+2/3)}{10\tau^4}(\eta^2\tau^2+\frac{2}{9}\tau^3+\frac{4}{9}\eta\tau+\frac{4}{9})\exp\left(-\frac{(3\eta\tau+2)^2}{12\tau^3}\right)\mathrm{KummerM}(\frac{5}{6},\frac{3}{2},\frac{(3\eta\tau+2)^2}{12\tau^3}) \\ & +\frac{4}{5\tau}(\eta\tau+\frac{2}{3})\exp\left(-\frac{(3\eta\tau+2)^2}{12\tau^3}\right)\mathrm{KummerM}(-\frac{1}{6},\frac{3}{2},\frac{(3\eta\tau+2)^2}{12\tau^3}), \end{aligned} \tag{1}$$

$$\begin{aligned} h_1 = & \frac{27(\eta\tau+2/3)}{10\tau^4}(\eta^2\tau^2+\frac{2}{9}\tau^3+\frac{4}{9}\eta\tau+\frac{4}{9})\exp\left(-\frac{(3\eta\tau+2)^2}{12\tau^3}\right)\mathrm{KummerU}(\frac{5}{6},\frac{3}{2},\frac{(3\eta\tau+2)^2}{12\tau^3}) \\ & -\frac{18}{5\tau}(\eta\tau+\frac{2}{3})\exp\left(-\frac{(3\eta\tau+2)^2}{12\tau^3}\right)\mathrm{KummerU}(-\frac{1}{6},\frac{3}{2},\frac{(3\eta\tau+2)^2}{12\tau^3}), \end{aligned} \tag{2}$$

in which, KummerM and KummerU are the Kummer functions [40] and their calculations can be done by symbolic code Maple [41].

$$\begin{aligned} g_2 = & \frac{3}{2\tau^3}(\eta^2\tau^2-\frac{2}{9}\tau^3+\frac{4}{9}\eta\tau+\frac{4}{9})\exp\left(-\frac{(3\eta\tau+2)^2}{12\tau^2}\right)\mathrm{KummerM}(\frac{5}{6},\frac{3}{2},\frac{(3\eta\tau+2)^2}{12\tau^3}) \\ & +\frac{4}{3}\exp\left(-\frac{(3\eta\tau+2)^2}{12\tau^3}\right)\mathrm{KummerM}(-\frac{1}{6},\frac{3}{2},\frac{(3\eta\tau+2)^2}{12\tau^3}), \end{aligned} \tag{3}$$

$$\begin{aligned} h_2 = & \frac{9}{2\tau^3}(\eta^2\tau^2-\frac{2}{9}\tau^3+\frac{4}{9}\eta\tau+\frac{4}{9})\exp\left(-\frac{(3\eta\tau+2)^2}{12\tau^2}\right)\mathrm{KummerU}(\frac{5}{6},\frac{3}{2},\frac{(3\eta\tau+2)^2}{12\tau^3}) \\ & -6\exp\left(-\frac{(3\eta\tau+2)^2}{12\tau^3}\right)\mathrm{KummerU}(-\frac{1}{6},\frac{3}{2},\frac{(3\eta\tau+2)^2}{12\tau^3}), \end{aligned} \tag{4}$$

$$\begin{aligned} g_3 = & -\frac{3}{10\tau^5}(\frac{16}{9}-\frac{8}{9}\eta\tau^4+(\eta^3-\frac{4}{3})\tau^3+\frac{16}{3}\eta^2\tau^2+\frac{52}{9}\eta\tau)\exp\left(-\frac{(3\eta\tau+2)^2}{12\tau^3}\right)\mathrm{KummerM}(\frac{5}{6},\frac{3}{2},\frac{(3\eta\tau+2)^2}{12\tau^3}) \\ & -\frac{4}{15\tau^2}(\eta\tau+4)\exp\left(-\frac{(3\eta\tau+2)^2}{12\tau^3}\right)\mathrm{KummerM}(-\frac{1}{6},\frac{3}{2},\frac{(3\eta\tau+2)^2}{12\tau^3}), \end{aligned} \tag{5}$$

$$\begin{aligned} h_3 = & -\frac{9}{10\tau^3}(\frac{16}{9}-\frac{8}{9}\eta\tau^4+(\eta^3-\frac{4}{3})\tau^3+\frac{16}{3}\eta^2\tau^2+\frac{52}{9}\eta\tau)\exp\left(-\frac{(3\eta\tau+2)^2}{12\tau^3}\right)\mathrm{KummerU}(\frac{5}{6},\frac{3}{2},\frac{(3\eta\tau+2)^2}{12\tau^3}) \\ & +\frac{6}{5\tau^2}(\eta\tau+4)\exp\left(-\frac{(3\eta\tau+2)^2}{12\tau^3}\right)\mathrm{KummerU}(-\frac{1}{6},\frac{3}{2},\frac{(3\eta\tau+2)^2}{12\tau^3}), \end{aligned} \tag{6}$$

Appendix 2 Kummer functions

The Kummer functions KummerM(μ,ν,z) and KummerU(μ,ν,z) are the solution of following differential equation:

$$zy''+(\nu-z)y'+\mu y=0. \tag{7}$$

Detail please referred to Abramowitz, M., and Stegun, I., eds. Handbook of Mathematical Functions. New York: Dover, 1972.

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