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Article

First-Order Ordinary Differential Equations: Comprehensive Theory, Methods, and Modern Applications

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Abstract

This extensive review provides a thorough examination of first-order ordinary differential equations (ODEs), covering fundamental theoretical concepts, diverse analytical solution techniques, stability analysis methods, numerical approximation algorithms, and interdisciplinary applications. The paper systematically explores classical models including exponential growth, logistic dynamics, and cooling laws, while extending to advanced topics such as bifurcation analysis, stochastic extensions, and modern computational approaches. Special attention is given to the interplay between analytical and numerical methods, with practical examples drawn from ecology, physics, engineering, and biomedical sciences. The work serves as both an educational resource for students and a reference for researchers and practitioners working with dynamical systems across scientific domains.

Keywords: ordinary differential equations; dynamical systems; stability analysis; numerical methods; bifurcation theory; mathematical modeling; phase plane analysis; computational mathematics

1. Introduction and Historical Context

1.1. Historical Development

The study of differential equations originated in the 17th century with the work of Isaac Newton and Gottfried Leibniz, who developed calculus as a language for describing continuous change. Early applications focused on physical problems: planetary motion, optics, and mechanics. The 18th century saw systematic development by Euler, Bernoulli, and Lagrange, establishing many fundamental solution methods still in use today.

1.2. Mathematical Formulation

A first-order ordinary differential equation expresses a relationship between an unknown function $y(t)$, its derivative dy/dt , and the independent variable t :

$$F\left(t, y, \frac{dy}{dt}\right) = 0 \quad \text{or equivalently} \quad \frac{dy}{dt} = f(t, y) \tag{1.1}$$

The general solution typically contains one arbitrary constant, determined by an initial condition:

$$y(t_0) = y_0 \tag{1.2}$$

Theorem 1 (Picard–Lindelöf Existence and Uniqueness). *Consider the initial value problem (1.1)-(1.2). If $f(t, y)$ is continuous in a rectangle $R = \{(t, y) : |t - t_0| \leq a, |y - y_0| \leq b\}$ and satisfies a Lipschitz condition in y :*

$$|f(t, y_1) - f(t, y_2)| \leq L|y_1 - y_2| \quad \forall (t, y_1), (t, y_2) \in R$$

then there exists a unique solution $y(t)$ on some interval $|t - t_0| \leq h$, where $h = \min(a, b/M)$ with $M = \max_{(t,y) \in R} |f(t,y)|$.

1.3. Classification of First-Order ODEs

First-order ODEs can be categorized based on their structure:

Table 1. Classification of first-order ODEs.

Type	General Form	Characteristics
Separable	$\frac{dy}{dt} = g(t)h(y)$	Variables can be separated
Linear	$\frac{dy}{dt} + P(t)y = Q(t)$	Linear in y and dy/dt
Exact	$M(t,y)dt + N(t,y)dy = 0$	$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial t}$
Bernoulli	$\frac{dy}{dt} + P(t)y = Q(t)y^n$	Nonlinear but reducible to linear
Homogeneous	$\frac{dy}{dt} = f\left(\frac{y}{t}\right)$	Invariant under scaling

1.4. Modern Significance

In contemporary science and engineering, first-order ODEs model:

- Population dynamics in ecology
- Chemical kinetics and reaction rates
- Electrical circuits (RC, RL)
- Pharmacokinetics and drug metabolism
- Economic growth models
- Heat transfer and diffusion processes
- Machine learning (gradient descent dynamics)

2. Analytical Solution Methods: Comprehensive Treatment

2.1. Separation of Variables with Advanced Examples

The separation of variables technique applies when the function $f(t,y)$ factors as $g(t)h(y)$:

$$\frac{dy}{dt} = g(t)h(y) \Rightarrow \int \frac{dy}{h(y)} = \int g(t)dt + C$$

(2.1)

Example 2.1: Solve $\frac{dy}{dt} = y(1 - y^2)$ with $y(0) = 0.5$.

Solution: Separating variables:

$$\int \frac{dy}{y(1 - y^2)} = \int dt$$

Using partial fractions:

$$\frac{1}{y(1 - y^2)} = \frac{1}{y} + \frac{1}{2(1 - y)} - \frac{1}{2(1 + y)}$$

Integrating:

$$\ln |y| - \frac{1}{2} \ln |1 - y| - \frac{1}{2} \ln |1 + y| = t + C$$

Simplifying:

$$\ln \left| \frac{y}{\sqrt{1 - y^2}} \right| = t + C \Rightarrow \frac{y}{\sqrt{1 - y^2}} = Ae^t$$

With $y(0) = 0.5$, $A = \frac{0.5}{\sqrt{1 - 0.25}} = \frac{1}{\sqrt{3}}$. Final solution:

$$y(t) = \frac{e^t}{\sqrt{3 + e^{2t}}}$$

2.2. Linear Equations: Theory and Applications

The general linear first-order ODE has the form:

$$\frac{dy}{dt} + P(t)y = Q(t) \tag{2.2}$$

The integrating factor method provides a systematic solution:

$$\mu(t) = \exp\left(\int P(t)dt\right) \tag{2.3}$$

Multiplying (2.2) by $\mu(t)$ gives:

$$\frac{d}{dt}[\mu(t)y] = \mu(t)Q(t)$$

Integrating:

$$y(t) = \frac{1}{\mu(t)} \left[\int \mu(t)Q(t)dt + C \right] \tag{2.4}$$

Example 2.2: RL circuit equation: $L\frac{dI}{dt} + RI = V(t)$, where $V(t) = V_0 \sin(\omega t)$.

Solution: Rewrite as $\frac{dI}{dt} + \frac{R}{L}I = \frac{V_0}{L} \sin(\omega t)$. Integrating factor:

$$\mu(t) = \exp\left(\int \frac{R}{L}dt\right) = e^{Rt/L}$$

Solution:

$$I(t) = e^{-Rt/L} \left[\frac{V_0}{L} \int e^{Rt/L} \sin(\omega t)dt + C \right]$$

Evaluating the integral:

$$\int e^{Rt/L} \sin(\omega t)dt = \frac{e^{Rt/L} \left(\frac{R}{L} \sin(\omega t) - \omega \cos(\omega t) \right)}{\left(\frac{R}{L} \right)^2 + \omega^2}$$

Thus:

$$I(t) = \frac{V_0}{R^2 + \omega^2 L^2} (R \sin(\omega t) - \omega L \cos(\omega t)) + C e^{-Rt/L}$$

2.3. Exact Equations and Integrating Factors

The differential form $M(t,y)dt + N(t,y)dy = 0$ is exact if:

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial t} \tag{2.5}$$

If exact, there exists $\Psi(t,y)$ such that:

$$d\Psi = Mdt + Ndy = 0 \Rightarrow \Psi(t,y) = C$$

Example 2.3: Solve $(2ty + 3t^2)dt + (t^2 + 4y^3)dy = 0$.

Solution: Check exactness:

$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}(2ty + 3t^2) = 2t$$

$$\frac{\partial N}{\partial t} = \frac{\partial}{\partial t}(t^2 + 4y^3) = 2t$$

Equal, so exact. Find Ψ :

$$\frac{\partial \Psi}{\partial t} = 2ty + 3t^2 \Rightarrow \Psi = t^2y + t^3 + h(y)$$

$$\frac{\partial \Psi}{\partial y} = t^2 + h'(y) = t^2 + 4y^3 \Rightarrow h'(y) = 4y^3 \Rightarrow h(y) = y^4$$

Thus $\Psi(t, y) = t^2y + t^3 + y^4 = C$.

2.4. Bernoulli Equations and Transformations

The Bernoulli equation:

$$\frac{dy}{dt} + P(t)y = Q(t)y^n, \quad n \neq 0, 1 \quad (2.6)$$

Substituting $z = y^{1-n}$ gives a linear equation:

$$\frac{dz}{dt} + (1-n)P(t)z = (1-n)Q(t)$$

2.5. Riccati Equations

The Riccati equation:

$$\frac{dy}{dt} = a(t)y^2 + b(t)y + c(t) \quad (2.7)$$

If a particular solution $y_p(t)$ is known, the substitution $y = y_p + \frac{1}{z}$ transforms it to linear.

3. Qualitative Analysis and Stability Theory

3.1. Autonomous Equations and Phase Line Analysis

For autonomous equations $\frac{dy}{dt} = f(y)$, equilibrium points satisfy $f(y^*) = 0$.

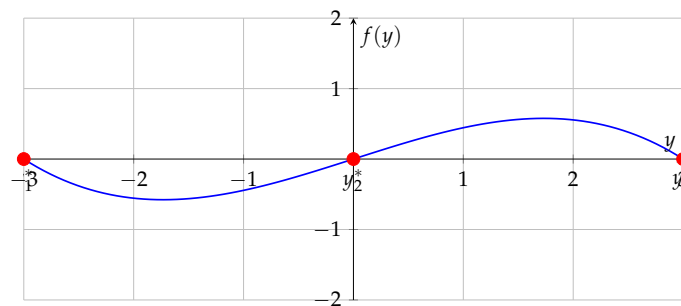


Figure 1. Graph of $f(y)$ showing equilibrium points

3.2. Linear Stability Analysis

For small perturbations $\eta = y - y^*$ around equilibrium:

$$\frac{d\eta}{dt} = f(y^* + \eta) \approx f'(y^*)\eta$$

Solution: $\eta(t) = \eta_0 e^{f'(y^*)t}$.

Stability classification:

- $f'(y^*) < 0$: asymptotically stable
- $f'(y^*) > 0$: unstable
- $f'(y^*) = 0$: linearization inconclusive

3.3. Lyapunov Functions for Nonlinear Stability

For systems where linear analysis fails, Lyapunov's direct method provides stability criteria.

Definition 1. A function $V(y)$ is a Lyapunov function for equilibrium y^* if:

1. $V(y^*) = 0$ and $V(y) > 0$ for $y \neq y^*$
2. $\frac{dV}{dt} = \nabla V \cdot f(y) \leq 0$ near y^*

Example 3.1: Analyze stability of $\frac{dy}{dt} = -y^3$.
Solution: Equilibrium: $y^* = 0$. Linearization: $f'(0) = 0$ (inconclusive). Choose $V(y) = \frac{1}{2}y^2$.

Then:

$$\frac{dV}{dt} = y \frac{dy}{dt} = y(-y^3) = -y^4 \leq 0$$

Thus $y^* = 0$ is asymptotically stable.

3.4. Bifurcation Analysis

Bifurcations occur when system behavior changes qualitatively with parameter variation.

Example 3.2: Transcritical bifurcation in $\frac{dy}{dt} = ry - y^2$.
Equilibria: $y^* = 0$ and $y^* = r$. Stability:

- For $r < 0$: $y^* = 0$ stable, $y^* = r$ unstable
- For $r > 0$: $y^* = 0$ unstable, $y^* = r$ stable
- At $r = 0$: bifurcation point

4. Canonical Models: Detailed Analysis

4.1. Exponential Growth and Decay

The fundamental model $\frac{dP}{dt} = kP$ has solution $P(t) = P_0e^{kt}$.
Applications:

- Population growth ($k > 0$)
- Radioactive decay ($k < 0$)
- Compound interest (k as interest rate)

Table 2. Half-life and doubling time calculations.

Process	Formula	Example
Half-life	$T_{1/2} = \frac{\ln 2}{ k }$	Carbon-14: $T_{1/2} = 5730$ years
Doubling time	$T_2 = \frac{\ln 2}{k}$	Bacteria: $T_2 = 20$ minutes

4.2. Logistic Growth: Complete Analysis

The logistic equation:

$$\frac{dP}{dt} = rP \left(1 - \frac{P}{K} \right) \tag{4.1}$$

Analytical solution via separation:

$$P(t) = \frac{K}{1 + \left(\frac{K - P_0}{P_0} \right) e^{-rt}} \tag{4.2}$$

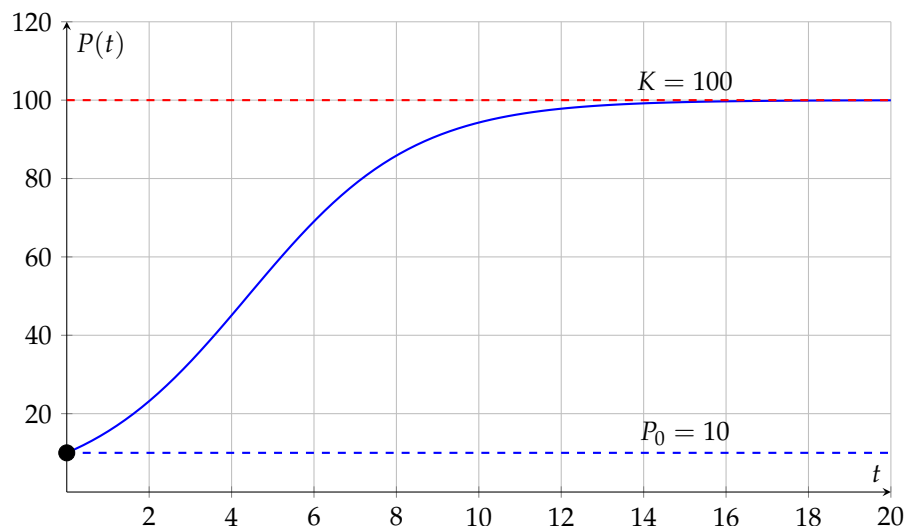


Figure 2. Logistic growth with $r = 0.5$, $K = 100$, $P_0 = 10$

4.3. Newton's Law of Cooling: Extended Models

General form: $\frac{dT}{dt} = -k(T - T_{\text{env}})$. Solution:

$$T(t) = T_{\text{env}} + (T_0 - T_{\text{env}})e^{-kt}$$

Example 4.1: Forensic application. A body found at 2 AM with temperature 27°C. Room temperature 21°C. At 3 AM, temperature is 26°C. Time of death?

Solution: Using two measurements:

$$27 = 21 + (T_0 - 21)e^{-k \cdot t_1}$$

$$26 = 21 + (T_0 - 21)e^{-k \cdot (t_1 + 1)}$$

Dividing: $\frac{27-21}{26-21} = e^k \Rightarrow k = \ln(6/5) \approx 0.1823$. Assuming body temperature at death = 37°C:

$$27 = 21 + (37 - 21)e^{-0.1823t_1} \Rightarrow t_1 \approx 5.3 \text{ hours}$$

Death occurred approximately 5.3 hours before 2 AM, around 8:42 PM.

4.4. Gompertz Growth Model

Used for tumor growth modeling:

$$\frac{dP}{dt} = rP \ln\left(\frac{K}{P}\right) \quad (4.3)$$

Solution: $P(t) = K \exp(\ln(P_0/K)e^{-rt})$.

5. Numerical Methods: Algorithms and Implementation

5.1. Euler's Method: Error Analysis

Forward Euler method:

$$y_{n+1} = y_n + hf(t_n, y_n) \quad (5.1)$$

Local truncation error: $LTE = \frac{h^2}{2} y''(\xi_n) = O(h^2)$. Global error: $|y(t_n) - y_n| \leq \frac{hM}{2L} (e^{L(t_n - t_0)} - 1)$, where $M = \max |y''(t)|$.

5.2. Runge-Kutta Methods Family

5.2.1. Second-Order Methods (RK2)

- Modified Euler: $y_{n+1} = y_n + hf(t_n + h/2, y_n + \frac{h}{2}f(t_n, y_n))$
- Heun's method: $y_{n+1} = y_n + \frac{h}{2}[f(t_n, y_n) + f(t_{n+1}, y_n + hf(t_n, y_n))]$

5.2.2. Classical Fourth-Order Runge-Kutta (RK4)

$$\begin{aligned} k_1 &= hf(t_n, y_n) \\ k_2 &= hf(t_n + h/2, y_n + k_1/2) \\ k_3 &= hf(t_n + h/2, y_n + k_2/2) \\ k_4 &= hf(t_n + h, y_n + k_3) \\ y_{n+1} &= y_n + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) \end{aligned} \quad (5.2)$$

Local error: $O(h^5)$, global error: $O(h^4)$.

5.3. Stiff Equations and Implicit Methods

For stiff equations like $\frac{dy}{dt} = -1000y + 3000 - 2000e^{-t}$, explicit methods require very small h .

5.3.1. Backward Euler Method

$$y_{n+1} = y_n + hf(t_{n+1}, y_{n+1}) \quad (5.3)$$

Unconditionally stable but requires solving nonlinear equation.

5.3.2. Trapezoidal Rule (Crank-Nicolson)

$$y_{n+1} = y_n + \frac{h}{2}[f(t_n, y_n) + f(t_{n+1}, y_{n+1})] \quad (5.4)$$

Second-order accurate, A-stable.

5.4. Adaptive Step Size Control

Step size adjustment based on error estimation:

Listing 1: Adaptive RK45 in pseudocode

```
function adaptive_rk45(t0, y0, tf, tol)
    t = t0; y = y0; h = initial_h;
    while t < tf
        y1, error = rk45_step(t, y, h);
        if error < tol
            t = t + h; y = y1; % accept step
            save_result(t, y);
        end
        h = 0.9 * h * (tol/error)^(1/5); % adjust h
    end
end
```


6. Advanced Topics and Extensions

6.1. Allee Effect Models

Strong Allee effect:

$$\frac{dP}{dt} = rP\left(1 - \frac{P}{K}\right)\left(\frac{P}{A} - 1\right), \quad 0 < A < K \tag{6.1}$$

Three equilibria: $P^* = 0$ (stable), $P^* = A$ (unstable threshold), $P^* = K$ (stable).

6.2. Delay Differential Equations (DDEs)

Include historical dependence: $\frac{dy}{dt} = f(t, y(t), y(t - \tau))$.

Example: Hutchinson’s delayed logistic equation:

$$\frac{dP}{dt} = rP(t)\left(1 - \frac{P(t - \tau)}{K}\right) \tag{6.2}$$

Can exhibit oscillatory behavior and stability switches.

6.3. Stochastic Differential Equations (SDEs)

Add random fluctuations: $dY_t = f(t, Y_t)dt + g(t, Y_t)dW_t$.

Example: Stochastic logistic growth:

$$dP = rP\left(1 - \frac{P}{K}\right)dt + \sigma PdW_t \tag{6.3}$$

6.4. Fractional Differential Equations

Generalize to fractional order: $D^\alpha y = f(t, y), 0 < \alpha \leq 1$.

7. Applications Across Disciplines

7.1. Ecology and Population Dynamics

- Lotka-Volterra predator-prey models
- Metapopulation dynamics
- Species invasion models
- Harvesting and management strategies

7.2. Biomedical Sciences

7.2.1. Pharmacokinetics: One-Compartment Model

$$\frac{dC}{dt} = -\frac{CL}{V}C + \frac{R(t)}{V}, \quad C(0) = C_0 \tag{7.1}$$

where C = concentration, CL = clearance, V = volume, $R(t)$ = infusion rate.

7.2.2. Epidemiology: SI Model

$$\frac{dS}{dt} = -\beta SI, \quad \frac{dI}{dt} = \beta SI - \gamma I \tag{7.2}$$

Basic reproduction number: $R_0 = \frac{\beta S_0}{\gamma}$.

7.3. Physics and Engineering

7.3.1. Electrical Circuits

RC circuit: $\frac{dV}{dt} = -\frac{1}{RC}V + \frac{1}{C}I(t)$. RL circuit: $\frac{dI}{dt} = -\frac{R}{L}I + \frac{1}{L}V(t)$.

7.3.2. Mechanics with Drag

$$m \frac{dv}{dt} = mg - kv - cv^2 \quad (7.3)$$

Linear drag ($cv^2 = 0$): $v(t) = \frac{mg}{k}(1 - e^{-kt/m})$. Quadratic drag: analytical solution involves hyperbolic functions.

7.4. Economics and Finance

7.4.1. Solow Growth Model

$$\frac{dk}{dt} = sf(k) - (n + \delta)k \quad (7.4)$$

where k = capital per worker, s = savings rate, n = population growth, δ = depreciation.

8. Computational Implementation Examples

8.1. Python Implementation of RK4

Listing 2: RK4 implementation in Python

```
import numpy as np

def rk4(f, t0, y0, tf, n):
    """
    RK4 method for solving  $dy/dt = f(t, y)$ 

    Parameters:
    f: function  $f(t, y)$ 
    t0: initial time
    y0: initial value
    tf: final time
    n: number of steps

    Returns:
    t: array of time points
    y: array of solution values
    """
    h = (tf - t0) / n
    t = np.linspace(t0, tf, n+1)
    y = np.zeros(n+1)
    y[0] = y0

    for i in range(n):
        k1 = h * f(t[i], y[i])
        k2 = h * f(t[i] + h/2, y[i] + k1/2)
        k3 = h * f(t[i] + h/2, y[i] + k2/2)
        k4 = h * f(t[i] + h, y[i] + k3)
        y[i+1] = y[i] + (k1 + 2*k2 + 2*k3 + k4) / 6

    return t, y

# Example: Logistic equation
def logistic(t, P, r=0.5, K=100):
```

```

    return r * P * (1 - P/K)

# Solve and plot
t, P = rk4(lambda t, y: logistic(t, y), 0, 10, 20, 100)

```

8.2. MATLAB Implementation for Stiff Problems

Listing 3: MATLAB code for stiff ODE

```

function stiff_example
% Solve stiff ODE: dy/dt = -1000y + 3000 - 2000*exp(-t)
tspan = [0 2.5];
y0 = 0;

% Using ode15s for stiff problems
options = odeset('RelTol',1e-6,'AbsTol',1e-9);
[t, y] = ode15s(@stiff_ode, tspan, y0, options);

plot(t, y, 'b-', 'LineWidth', 2);
xlabel('Time'); ylabel('y(t)');
title('Solution of Stiff ODE');

function dydt = stiff_ode(t, y)
    dydt = -1000*y + 3000 - 2000*exp(-t);
end
end

```

9. Conclusions and Future Directions

First-order ODEs continue to be indispensable in scientific modeling. Future directions include:

- **Machine Learning Integration:** Neural ODEs that parameterize $f(t, y)$ with neural networks
- **High-Dimensional Systems:** Applications in data science and network dynamics
- **Stochastic Methods:** Improved numerical methods for SDEs
- **Hybrid Systems:** Combining continuous ODEs with discrete events
- **Uncertainty Quantification:** Propagating parameter uncertainties through ODE models

The interplay between analytical insight and computational power continues to drive innovation in both theoretical understanding and practical applications of first-order differential equations.

Appendix A. Useful Integrals and Transformations

Appendix A.1. Common Integrals

$$\begin{aligned}
 \int \frac{dx}{x} &= \ln|x| + C \\
 \int \frac{dx}{a^2 + x^2} &= \frac{1}{a} \arctan\left(\frac{x}{a}\right) + C \\
 \int \frac{dx}{a^2 - x^2} &= \frac{1}{2a} \ln\left|\frac{a+x}{a-x}\right| + C \quad (|x| \neq a) \\
 \int \frac{dx}{\sqrt{a^2 - x^2}} &= \arcsin\left(\frac{x}{a}\right) + C \quad (|x| < a)
 \end{aligned}$$

Appendix A.2. Integration by Parts Formula

$$\int u \, dv = uv - \int v \, du$$

Appendix B. Stability Criteria Summary

Table A1. Stability conditions for common equilibrium types.

Equilibrium Type	Condition	Stability
Node	$f'(y^*) < 0$, real	Asymptotically stable
Source	$f'(y^*) > 0$, real	Unstable
Saddle	$f'(y^*) = 0, f''(y^*) \neq 0$	Semistable
Degenerate	$f'(y^*) = f''(y^*) = 0$	Higher-order analysis

Appendix C. Appendix C: Numerical Methods Error Comparison

Table A2. Comparison of numerical methods for ODEs.

Method	Order	Stability	Computational Cost
Forward Euler	1	Conditional	Low
Backward Euler	1	Unconditional	Medium (implicit)
Trapezoidal Rule	2	Unconditional	Medium (implicit)
Heun’s Method	2	Conditional	Medium
Classical RK4	4	Conditional	High
Adams-Bashforth	Variable	Conditional	Medium

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