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p-adic Delsarte-Goethals-Seidel-Kabatianskii-Levenshtein-Pfender Bound

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Abstract: We introduce the notion of p-adic spherical codes (in particular, p-adic kissing number problem). We show that the one-line proof for a variant of the Delsarte-Goethals-Seidel-Kabatianskii-Levenshtein upper bound for spherical codes, obtained by Pfender, extends to p-adic Hilbert spaces.

Keywords: spherical code; kissing number; linear programming; p-adic Hilbert space

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1. Introduction

Let $d \in \mathbb{N}$ and $\theta \in [0, 2\pi)$. A set $\{\tau_j\}_{j=1}^n$ of unit vectors in $\mathbb{R}^d$ is said to be (d, n, θ)**-spherical code** [1] in $\mathbb{R}^d$ if

$$\langle \tau_j, \tau_k \rangle \leq \cos\theta, \quad \forall 1 \leq j, k \leq n, j \neq k. \tag{1}$$

Since

$$\langle \tau, \omega \rangle = \frac{2 - \|\tau - \omega\|^2}{2}, \quad \forall \tau, \omega \in \mathbb{R}^d,$$

we can rewrite Inequality (1) as

$$\|\tau_j - \tau_k\| \geq \sqrt{2(1 - \cos\theta)}, \quad \forall 1 \leq j, k \leq n, j \neq k.$$

Fundamental problem associated with spherical codes is the following.

Problem 1.1. *Given d and θ, what is the maximum n such that there exists a (d, n, θ)-spherical code $\{\tau_j\}_{j=1}^n$ in $\mathbb{R}^d$?*

The case $\theta = \pi/3$ is known as the famous **(Newton-Gregory) kissing number problem**. With extensive efforts from many mathematicians, it is still not completely resolved in every dimension (but resolved in dimensions $d = 1$ ($n = 2$), $d = 2$ ($n = 6$), $d = 3$ ($n = 12$), $d = 4$ ($n = 24$), $d = 8$ ($n = 240$), $d = 24$ ($n = 196560$)) [2–21]. We refer [22–38] for more on spherical codes. Problem 1.1 has connection even with sphere packing [39]. Most used method for obtaining upper bounds on spherical codes is the Delsarte-Goethals-Seidel-Kabatianskii-Levenshtein bound which we recall. Let $n \in \mathbb{N}$ be fixed. The Gegenbauer polynomials are defined inductively as

$$G_0^{(n)}(r) := 1, \quad \forall r \in [-1, 1],$$
$$G_1^{(n)}(r) := r, \quad \forall r \in [-1, 1],$$
$$\vdots$$
$$G_k^{(n)}(r) := \frac{(2k + n - 4)rG_{k-1}^{(n)}(r) - (k - 1)G_{k-2}^{(n)}(r)}{k + n - 3}, \quad \forall r \in [-1, 1], \quad \forall k \geq 2.$$

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Then the family $\{G_k^{(n)}\}_{k=0}^{\infty}$ is orthogonal on the interval $[-1,1]$ with respect to the weight

$$\rho(r) := (1-r^2)^{\frac{n-3}{2}}, \quad \forall r \in [-1,1].$$

Theorem 1.2. *[22,26] (**Delsarte-Goethals-Seidel-Kabatianskii-Levenshtein Linear Programming Bound**) Let $\{\tau_j\}_{j=1}^n$ be a (d,n,θ)-spherical code in $\mathbb{R}^d$. Let P be a real polynomial satisfying following conditions.*

i $\quad$ *$P(r) \leq 0$ for all $-1 \leq r \leq \cos\theta$.*
ii $\quad$ *Coefficients in the Gegenbauer expansion*

$$P = \sum_{k=0}^{m} a_k G_k^{(n)}$$

satisfy

$$a_0 > 0, \quad a_k \geq 0, \ \forall 1 \leq k \leq m.$$

Then

$$n \leq \frac{P(1)}{a_0}.$$

In 2007, Pfender gave a one-line proof for a variant of Theorem 1.2.

Theorem 1.3. *[3] (**Delsarte-Goethals-Seidel-Kabatianskii-Levenshtein-Pfender Bound**) Let $\{\tau_j\}_{j=1}^n$ be a (d,n,θ)-spherical code in $\mathbb{R}^d$. Let $c > 0$ and $\phi : [-1,1] \to \mathbb{R}$ be a function satisfying following.*

i

$$\sum_{j=1}^{n}\sum_{k=1}^{n} \phi(\langle \tau_j, \tau_k \rangle) \geq 0.$$

ii $\quad$ *$\phi(r) + c \leq 0$ for all $-1 \leq r \leq \cos\theta$.*

Then

$$n \leq \frac{\phi(1)+c}{c}.$$

In particular, if $\phi(1) + c \leq 1$, then $n \leq 1/c$.

In this paper, we introduce the notion of p-adic spherical codes. We show that Theorem 1.3, can be easily extended for p-adic Hilbert spaces.

2. p-adic Spherical Codes

We begin from the definition of p-adic Hilbert space.

Definition 2.1. *[40–42] Let $\mathbb{K}$ be a non-Archimedean complete valued field (with valuation $|\cdot|$) and $\mathcal{X}$ be a non-Archimedean Banach space (with norm $\|\cdot\|$) over $\mathbb{K}$. We say that $\mathcal{X}$ is a **p-adic Hilbert space** if there is a map (called as p-adic inner product) $\langle \cdot, \cdot \rangle : \mathcal{X} \times \mathcal{X} \to \mathbb{K}$ satisfying following.*

i $\quad$ *If $x \in \mathcal{X}$ is such that $\langle x, y \rangle = 0$ for all $y \in \mathcal{X}$, then $x = 0$.*
ii $\quad$ *$\langle x, y \rangle = \langle y, x \rangle$ for all $x, y \in \mathcal{X}$.*
iii $\quad$ *$\langle \alpha x + y, z \rangle = \alpha\langle x, z \rangle + \langle y, z \rangle$ for all $\alpha \in \mathbb{K}$, for all $x, y, z \in \mathcal{X}$.*
iv $\quad$ *$|\langle x, y \rangle| \leq \|x\|\|y\|$ for all $x, y \in \mathcal{X}$.*

Following is the standard example which we consider in the paper.

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Example 2.2. *[41] Let p be a prime. For $d \in \mathbb{N}$, let $\mathbb{Q}_p^d$ be the standard p-adic Hilbert space equipped with the inner product*

$$\langle (a_j)_{j=1}^d, (b_j)_{j=1}^d \rangle := \sum_{j=1}^d a_j b_j, \quad \forall (a_j)_{j=1}^d, (b_j)_{j=1}^d \in \mathbb{Q}_p^d$$

and norm

$$\|(x_j)_{j=1}^d\| := \max_{1\leq j\leq d} |x_j|, \quad \forall (x_j)_{j=1}^d \in \mathbb{Q}_p^d.$$

We introduce p-adic spherical codes as follows.

Definition 2.3. *Let $d \in \mathbb{N}$ and $\theta \in [0, 2\pi)$. A set $\{\tau_j\}_{j=1}^n$ of vectors in $\mathbb{Q}_p^d$ is said to be **p-adic** (d, n, θ)**-spherical code** in $\mathbb{Q}_p^d$ if following conditions hold.*

i $\|\tau_j\| = 1$ *for all* $1 \leq j \leq n$.
ii $\langle \tau_j, \tau_j \rangle = 1$ *for all* $1 \leq j \leq n$.
iii

$$|2 - 2\langle \tau_j, \tau_k \rangle| \geq 2(1 - \cos\theta), \quad \forall 1 \leq j, k \leq n, j \neq k. \tag{2}$$

*We call the case $\theta = \pi/3$ as the **p-adic kissing number problem**.*

Let $\{\tau_j\}_{j=1}^n$ be a p-adic (d, n, θ)-spherical code in $\mathbb{Q}_p^d$. Since

$$\|\tau_j - \tau_k\|^2 \geq |\langle \tau_j - \tau_k, \tau_j - \tau_k \rangle| = |2 - 2\langle \tau_j, \tau_k \rangle|, \quad \forall 1 \leq j, k \leq n,$$

Inequality (2) gives

$$\|\tau_j - \tau_k\| \geq \sqrt{2(1 - \cos\theta)}, \quad \forall 1 \leq j, k \leq n, j \neq k. \tag{3}$$

However, note that Inequality (3) may not give Inequality (2). Note that we can formulate the definition of general p-adic (d, n, θ)-spherical code by replacing Inequality (2) with Inequality (3) in Definition 2.3. It is also possible to consider Definition 2.3 by removing conditions (i) or (ii). Following is the p-adic version of Theorem 1.3.

Theorem 2.4. *(**p-adic Delsarte-Goethals-Seidel-Kabatianskii-Levenshtein-Pfender Spherical Codes Bound**) Let $\{\tau_j\}_{j=1}^n$ be a p-adic (d, n, θ)-spherical code in $\mathbb{Q}_p^d$. Let $c > 0$ and $\phi : [0, \infty) \to \mathbb{R}$ be a function satisfying following.*

i $\sum_{1\leq j,k\leq n} \phi(|2 - 2\langle \tau_j, \tau_k \rangle|) \geq 0$.
ii $\phi(r) + c \leq 0$ *for all* $r \in [2(1 - \cos\theta), \infty)$.

Then

$$n \leq \frac{\phi(0) + c}{c}.$$

In particular, if $\phi(0) + c \leq 1$, then $n \leq 1/c$.

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Proof. Define $\psi : [0,\infty) \ni r \mapsto \psi(r) := \phi(r) + c \in \mathbb{R}$. Then

$$\begin{aligned}\sum_{1\leq j,k\leq n} \psi(|2-2\langle \tau_j, \tau_k\rangle|) &= \sum_{j=1}^{n} \psi(|2-2\langle \tau_j, \tau_j\rangle|) + \sum_{1\leq j,k\leq n, j\neq k} \psi(|2-2\langle \tau_j, \tau_k\rangle|) \\ &= \sum_{j=1}^{n} \psi(0) + \sum_{1\leq j,k\leq n, j\neq k} \psi(|2-2\langle \tau_j, \tau_k\rangle|) \\ &= n(\phi(0)+c) + \sum_{1\leq j,k\leq n, j\neq k} (\phi(|2-2\langle \tau_j, \tau_k\rangle|+c) \\ &\leq n(\phi(0)+c) + 0 = n(\phi(0)+c).\end{aligned}$$

We also have

$$\sum_{1\leq j,k\leq n} \psi(\phi(|2-2\langle \tau_j, \tau_k\rangle|)) = \sum_{1\leq j,k\leq n} (\phi(|2-2\langle \tau_j, \tau_k\rangle|) + c) = \sum_{1\leq j,k\leq n} \phi(|2-2\langle \tau_j, \tau_k\rangle|) + cn^2.$$

Therefore

$$cn^2 \leq \sum_{1\leq j,k\leq n} \phi(|2-2\langle \tau_j, \tau_k\rangle|) + cn^2 \leq n(\phi(0)+c).$$

□

Corollary 2.5. *(**p-adic Delsarte-Goethals-Seidel-Kabatianskii-Levenshtein-Pfender Kissing Number Bound**) Let $\{\tau_j\}_{j=1}^n$ be a p-adic $(d,n,\pi/3)$-spherical code in $\mathbb{Q}_p^d$. Let $c>0$ and $\phi : [0,\infty) \to \mathbb{R}$ be a function satisfying following.*

i $\sum_{1\leq j,k\leq n} \phi(|2-2\langle \tau_j, \tau_k\rangle|) \geq 0$.

ii $\phi(r) + c \leq 0$ *for all* $r \in [1,\infty)$.

Then

$$n \leq \frac{\phi(0)+c}{c}.$$

In particular, if $\phi(0)+c \leq 1$, then $n \leq 1/c$.

Following generalization of Theorem 2.4 is easy.

Theorem 2.6. *Let $\{\tau_j\}_{j=1}^n$ be a p-adic (d,n,θ)-spherical code in $\mathbb{Q}_p^d$. Let $c>0$ and*

$$\phi : \{|2-2\langle \tau_j, \tau_k\rangle| : 1 \leq j,k \leq n\} \to \mathbb{R}$$

be a function satisfying following.

i $\sum_{1\leq j,k\leq n} \phi(|2-2\langle \tau_j, \tau_k\rangle|) \geq 0$.

ii $\phi(r) + c \leq 0$ *for all* $r \in \{|2-2\langle \tau_j, \tau_k\rangle| : 1 \leq j,k \leq n, j \neq k\}$.

Then

$$n \leq \frac{\phi(0)+c}{c}.$$

In particular, if $\phi(0)+c \leq 1$, then $n \leq 1/c$.

Note that, in the paper, we can replace $\mathbb{Q}_p^d$ by any p-adic Hilbert space.

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