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Article

Wigner Functions of Time-Dependent Cat-Like Even/Odd Superpositions of Nonlinear Coherent States

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Abstract

We calculate and plot the Wigner functions of several families of highly excited even and odd superpositions of nonlinear coherent states, looking for conditions under which such superpositions can be interpreted as models of the “Schrödinger cat” states. It appears that the decisive factor is the form of the number distribution functions over the Fock basis: they must have well localized peaks. Otherwise, no “cat” structures are observed.

Keywords: coherent states; coherent phase states; nonlinear coherent states; Barut–Girardello coherent states; binomial states; negative binomial states

1. Introduction

The Wigner function [1] is a useful tool in many areas of quantum mechanics and quantum optics [2–5]. We use the definition (for one degree of freedom and $\hbar = 1$)

$$W(q, p) = \int_{-\infty}^{\infty} dz e^{ipz} \langle q - z/2 | \hat{\rho} | q + z/2 \rangle, \quad (1)$$

where $\hat{\rho}$ is a Hermitian positive-definite operator (the statistical operator) normalized as $\text{Tr}(\hat{\rho}) = 1$. In this case, Equation (1) defines a real function with the normalization

$$\int_{-\infty}^{\infty} dq \int_{-\infty}^{\infty} dp W(q, p) / (2\pi) = 1. \quad (2)$$

It is remarkable that the Wigner function is limited (in contrast to the wave function or the density matrix) [2,6–8]:

$$|W(q, p)| \leq 2. \quad (3)$$

The extremal values $W(0, 0) = \pm 2$ are achieved for pure quantum states with definite parity [2,9], whose wave functions obey the relations $\psi(x) = \pm \psi(-x)$. Moreover, the relation $W(q, p) = W(-q, -p)$ holds for all such states, as can be seen from Equation (1) after the replacement $z \rightarrow -z$, writing $\langle x | \hat{\rho} | y \rangle$ as $\langle x | \psi \rangle \langle \psi | y \rangle$.

Simple examples of states with definite parity are the Fock states $|n\rangle$ of the harmonic oscillator, i.e., eigenstates of the number operator $\hat{n} = \hat{a}^\dagger \hat{a}$, where $\hat{a}^\dagger$ and $\hat{a}$ are the bosonic creation and annihilation operators satisfying the commutation relation $[\hat{a}, \hat{a}^\dagger] = 1$. In this paper, we consider dimensionless canonical position and momentum operators, related to operators $\hat{a}^\dagger$ and $\hat{a}$ as follows,

$$\hat{q} = (\hat{a} + \hat{a}^\dagger) / \sqrt{2}, \quad \hat{p} = (\hat{a} - \hat{a}^\dagger) / (i\sqrt{2}). \quad (4)$$

The Wigner functions of the Fock states are well known [2–4,10–13]:

$$W_n(q, p) = 2(-1)^n \exp(-b^2) L_n(2b^2), \quad b^2 = q^2 + p^2, \quad (5)$$

where $L_n(z)$ is the Laguerre polynomial defined as [14,15]

$$L_n(z) = \frac{1}{n!} e^z \frac{d^n}{dz^n} (e^{-z} z^n) = \sum_{m=0}^n \frac{(-x)^m n!}{(m!)^2 (n-m)!}.$$

Functions (5) strongly oscillate when $n \gg 1$ (they have exactly n zeros). Consequently, they have approximately n circular regions of negativity in the phase plane (q, p) , in accordance with the property of the Fock states to be “the most nonclassical states”: see Figure 1.

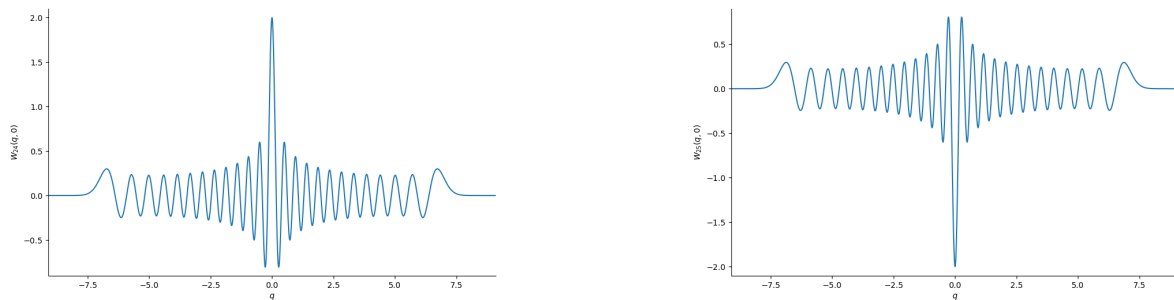


Figure 1. The sections $W(q, 0)$ of the Wigner functions of the Fock states with $n = 24$ (left) and $n = 25$ (right).

On the other hand, the infinite superpositions of the Fock states in the form

$$|\alpha\rangle = \exp(-|\alpha|^2/2) \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle, \quad \hat{a}|\alpha\rangle = \alpha|\alpha\rangle, \quad (6)$$

known as coherent states (CS) [16–18], possess non-oscillating (Gaussian) Wigner functions. The states with definite parity can be obtained as the following superpositions of coherent states:

$$|\Psi\rangle_{\pm;\alpha} = N_{\pm;\alpha} (|\alpha\rangle \pm |-\alpha\rangle), \quad N_{\pm;\alpha}^{-2} = 2 \left[1 \pm \exp(-2|\alpha|^2) \right]. \quad (7)$$

The states (7) (introduced in paper [19] under the name *even/odd coherent states*) turned out important in many applications to quantum mechanics, quantum optics and quantum information [20–25]. The Wigner functions of these states with $|\alpha| \gg 1$ have the form of two symmetric distant Gaussian peaks, *well separated* from the central *strongly oscillating* narrow peak (see Section 2). Due to this structure, the states (7) (as well as more general superpositions of *Gaussian* wave functions) are frequently considered as simple models of the famous “Schrödinger cat” states [26–36].

A lot of various generalizations of states (6) and their superpositions have been studied for the past decades [37]. In this connection, it seems natural to ask, whether all such superpositions can be considered as models of “cat” states or not? Searching for an answer to this question, we considered generalizations of states (7) of the form

$$|\Psi\rangle_{\pm;\varepsilon} = N_{\pm;\varepsilon} (|\varepsilon\rangle \pm |-\varepsilon\rangle), \quad N_{\pm;\varepsilon}^{-2} = 2(1 \pm \langle -\varepsilon | \varepsilon \rangle), \quad (8)$$

$$|\varepsilon\rangle = \mathcal{N}_f \sum_{n=0}^{\infty} \frac{\varepsilon^n}{f(n)} |n\rangle, \quad \mathcal{N}_f^{-2} = \sum_{n=0}^{\infty} \frac{|\varepsilon|^{2n}}{f^2(n)}, \quad (9)$$

where $f(n)$ is some positive function and $\varepsilon = |\varepsilon|e^{i\varphi}$ a complex number. The choice of the power dependence ε^n in the definition (9) is not accidental: due to the equidistant energy spectrum of the harmonic oscillator, the time evolution of the state (9) is reduced to the substitution $\varepsilon \rightarrow \varepsilon e^{-it}$ (in the dimensionless units with $\omega = 1$). Hence, the variations of phase φ are equivalent to the temporal

evolution of the quantum state (9). In view of this observation, we plot everywhere the sections $W(q, 0)$ for different values of the phase φ . Note that the choice $\varphi = \pi/2$ can be also interpreted as the transformation to the momentum representation: writing the Wigner function in the more detailed form as $W(q, p; \varphi)$ one can see that $W(q, 0; \pi/2) = W(0, p; 0)$.

The states (9) are known under the name “nonlinear coherent states” [38–40] (although sometimes in slightly different forms), and their superpositions like (8) were considered, e.g., in papers [41–43]. It seems interesting to study how functions $f(n)$ different from $\sqrt{n!}$ influence the shape of the corresponding Wigner functions. Earlier, the Wigner functions of even/odd superpositions of different quantum states were plotted in papers [44–47], as well as in many papers cited above. However, the authors of those studies considered small values of parameters like $|\varepsilon|$ or $|\alpha|$, when the mean numbers of quanta in the superpositions are low (frequently, less than 1), due to technical difficulties in numeric calculations with big values of $|\varepsilon|$ or $|\alpha|$. On the other hand, the well known examples of the “usual” even/odd coherent states show that the most interesting behavior can be observed in superpositions with big mean numbers of quanta.

Using the program *Python*, we have succeeded to overcome the numeric challenges and plot the Wigner functions for highly excited superpositions of different families of states like (9). Numerous examples demonstrated that the “cat” structures are maintained for rapidly growing (faster than $\sqrt{n!}$) functions $f(n)$. On the other hand, choosing $f(n) \equiv 1$, we have discovered a total disappearance of the “cat” structure. Therefore, we investigated several families of well known nonlinear coherent states with different functions $f(n)$, trying to see, under which conditions the “cat” structures can be observed.

Our plan is as follows. In Section 2, we remind the well known properties of the usual even/odd coherent states. In Section 3, we bring general formulas for arbitrary functions $f(n)$. In Section 4, we show examples of superpositions with rapidly growing functions $f_g(n) = \sqrt{n!}\Gamma(n+g+1)$ (known under the name “Barut–Girardello coherent states”), which maintain the “cat” structure. Interesting special cases of the “binomial states” are demonstrated in Section 5. Superpositions of “coherent phase states” with $f(n) = 1$ are considered in Section 6. Plots of the Wigner functions for more general “negative binomial states” with $f_\nu(n) = [n!/\Gamma(n+\nu)]^{1/2}$ are shown in Section 7. Section 8 contains conclusions.

2. Wigner Functions of the Usual Even/Odd Coherent States

The wave function of the usual coherent state (6) in the coordinate representation has the well known Gaussian form

$$\langle x|\alpha\rangle = \pi^{-1/4} \exp\left(-\frac{1}{2}x^2 + \sqrt{2}x\alpha - \frac{1}{2}\alpha^2 - \frac{1}{2}|\alpha|^2\right). \quad (10)$$

In this case, the right-hand side of Equation (1) for the states (7) is reduced to the calculation of four simple Gaussian integrals according to the decomposition of the statistical operator

$$\hat{\rho} = N_{\pm;\alpha}^2(|\alpha\rangle\langle\alpha| + |-\alpha\rangle\langle-\alpha| \pm |\alpha\rangle\langle-\alpha| \pm |-\alpha\rangle\langle\alpha|).$$

The result can be expressed as follows,

$$W_{\pm;\alpha}(q, p) = 4N_{\pm;\alpha}^2 \left(\pm e^{-b^2} \cos[2\sqrt{2}b|\alpha| \sin \Phi] + e^{-2|\alpha|^2 \sin^2 \Phi} \left[e^{-(b-\sqrt{2}|\alpha| \cos \Phi)^2} + e^{-(b+\sqrt{2}|\alpha| \cos \Phi)^2} \right] \right), \quad (11)$$

where $\alpha = |\alpha|e^{i\varphi}$,

$$\Phi = \varphi - \chi, \quad q + ip = be^{i\chi}, \quad b = \sqrt{q^2 + p^2}. \quad (12)$$

If $|\alpha| \gg 1$ and $\Phi = 0$, one can see three Gaussian peaks of the same width: the central peak of the height (or depth) ± 2 and two symmetric distant positive peaks whose heights are twice lower than the height of the central one. The lateral peaks rapidly disappear with increase of the phase Φ (equivalent to the evolution in time), due to the amplitude factor $\exp(-2|\alpha|^2 \sin^2 \Phi)$. The evolving central peak acquires a rapidly oscillating “fine structure” with almost Gaussian envelope of the same width as in the case of $\Phi = 0$.

The right-hand side of Equation (11) depends on the difference of phases $\varphi - \chi$. Therefore, hereafter we fix the phase $\chi = 0$, studying the section $W(q, 0) = W(-q, 0)$ of the total Wigner function for different values of the phase φ of the complex parameter α (or parameter ε in the subsequent sections). Figure 2 shows the Wigner functions $W_{\pm;\alpha}(q, 0)$ for $|\alpha| = 5$ (when the mean number of quanta in each component equals $|\alpha|^2 = 25$) and two extreme phases: $\varphi = 0$ and $\varphi = \pi/2$.

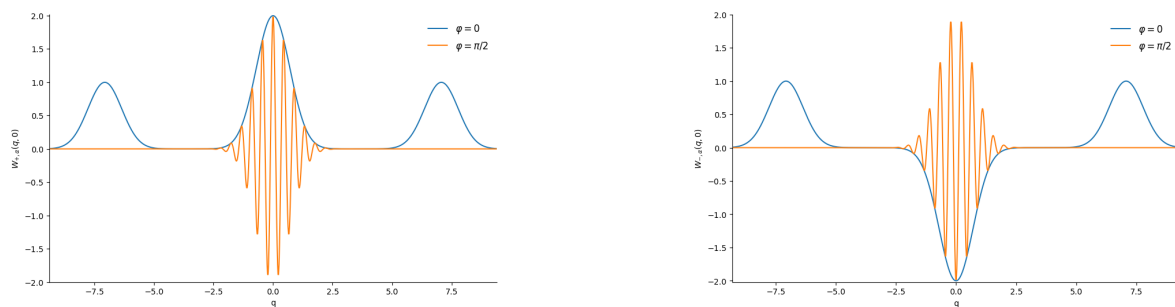


Figure 2. The sections $W(q, 0)$ of the Wigner functions of the even (left) and odd (right) coherent states with $|\alpha| = 5$ and two extreme phases: $\varphi = 0$ and $\varphi = \pi/2$.

The number distribution function $p_n = |\langle \psi | n \rangle|^2$ over the Fock states $|n\rangle$ has the following expressions for the even/odd coherent states:

$$p_{+;2k} = \frac{|\alpha|^{4k}}{\cosh(|\alpha|^2)(2k)!}, \quad p_{-;2k+1} = \frac{|\alpha|^{4k+2}}{\sinh(|\alpha|^2)(2k+1)!}. \quad (13)$$

Plots of these distributions as functions of $k = 0, 1, 2, \dots$ are shown in Figure 3 for $|\alpha| = 5$. Probably, it is worth remembering that even distributions are “super-Poissonian”, whereas odd distributions are “sub-Poissonian” for the distributions (13).

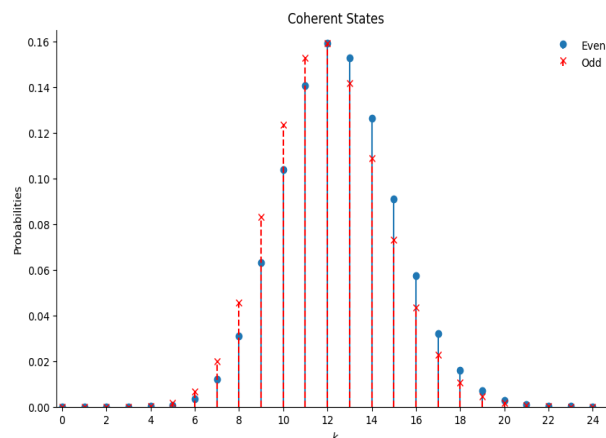


Figure 3. The number distribution functions of the even and odd coherent states with $|\alpha| = 5$.

Note that the maximal probabilities in the even and odd distributions practically coincide. Probably, this is a mere accidental coincidence. However, it is interesting that the same coincidences can be seen in almost all similar figures, except for the case of “mid-binomial” distributions in Figure 6.

3. General Even/Odd Superpositions

If a pure state $|\psi\rangle$ is a superposition of the Fock states,

$$|\psi\rangle = \sum_{k=0}^{\infty} c_k |k\rangle, \quad (14)$$

the corresponding statistical operator $\hat{\rho}_\psi = |\psi\rangle\langle\psi|$ can be written as a double sum over the diadic operators $|l\rangle\langle r|$:

$$\hat{\rho}_\psi = \sum_{l,r=0}^{\infty} c_l c_r^* |l\rangle\langle r|. \quad (15)$$

The related Wigner function can be written as

$$W_\psi(q, p) = \sum_{l,r=0}^{\infty} c_l c_r^* W_{lr}(q, p), \quad (16)$$

where $W_{lr}(q, p)$ is the Weyl symbol [48] of the operator $|l\rangle\langle r|$. This symbol was calculated, e.g., in papers [10–13]:

$$W_{lr}(q, p) = 2(-1)^{\mu_{lr}} e^{i\chi(r-l)} \mathcal{L}(2b^2; \mu_{lr}, |l-r|), \quad \mu_{lr} = \min(l, r), \quad (17)$$

$$\mathcal{L}(z; n, k) = \sqrt{\frac{n!}{(n+k)!}} e^{-z/2} z^{k/2} L_n^{(k)}(z). \quad (18)$$

Here, $L_n^{(k)}(z)$ is the generalized Laguerre polynomial [14,15],

$$L_n^{(k)}(z) = \frac{1}{n!} e^z z^{-k} \frac{d^n}{dz^n} (e^{-z} z^{n+k}) = \sum_{m=0}^n \frac{(-x)^m (n+k)!}{m! (m+k)! (n-m)!}.$$

Note that the difference $l-r$ is an even number, both for the even and odd superpositions, whereas μ_{lr} is even number for even superpositions and odd number for odd superpositions. Expansions of superpositions (8) over the Fock basis have the form

$$|\Psi\rangle_{+;f} = \mathcal{N}_{+f} \sum_{n=0}^{\infty} \frac{\varepsilon^{2n}}{f(2n)} |2n\rangle, \quad (19)$$

$$|\Psi\rangle_{-;f} = \mathcal{N}_{-f} \sum_{n=0}^{\infty} \frac{\varepsilon^{2n+1}}{f(2n+1)} |2n+1\rangle, \quad (20)$$

$$\mathcal{N}_{+f}^{-2} = \sum_{n=0}^{\infty} \frac{|\varepsilon|^{4n}}{f^2(2n)}, \quad \mathcal{N}_{-f}^{-2} = \sum_{n=0}^{\infty} \frac{|\varepsilon|^{4n+2}}{f^2(2n+1)}. \quad (21)$$

The total Wigner function (16) of even (odd) states (19) or (20) can be written as $W_1 + W_2$, where the first series W_1 contains the terms with $m = n$ only (hence, it does not depend on phases φ and χ), whereas the summation in the double series W_2 over all numbers $l \neq r$ can be performed with respect to independent coefficients $\lambda = |n-m| = 1, 2, \dots$ and $\mu = 0, 1, 2, \dots$. The explicit expressions are as follows,

$$W_{+1;f}(q, 0) = 2\mathcal{N}_{+f}^2 e^{-q^2} \sum_{n=0}^{\infty} \frac{|\varepsilon|^{4n}}{[f(2n)]^2} L_{2n}(2q^2), \quad (22)$$

$$W_{-1;f}(q, 0) = -2\mathcal{N}_{-f}^2 e^{-q^2} \sum_{n=0}^{\infty} \frac{|\varepsilon|^{4n+2}}{[f(2n+1)]^2} L_{2n+1}(2q^2), \quad (23)$$

$$W_{+2;f}(q, 0) = 4\mathcal{N}_{+f}^2 \sum_{\mu=0}^{\infty} \sum_{\lambda=1}^{\infty} \frac{|\varepsilon|^{2(2\mu+\lambda)} \cos(2\lambda\varphi)}{f(2\mu)f(2\mu+2\lambda)} \mathcal{L}(2q^2; 2\mu, 2\lambda), \quad (24)$$

$$W_{-2;f}(q, 0) = -4\mathcal{N}_{-f}^2 \sum_{\mu=0}^{\infty} \sum_{\lambda=1}^{\infty} \frac{|\varepsilon|^{2(2\mu+\lambda+1)} \cos(2\lambda\varphi)}{f(2\mu+1)f(2\mu+2\lambda+1)} \mathcal{L}(2q^2; 2\mu+1, 2\lambda). \quad (25)$$

In the numeric calculations performed in the following sections, all series were calculated up to the maximal summation indexes $n_{max} = \mu_{max} = \lambda_{max} = 80$.

4. Even/Odd “Barut–Girardello” Coherent States

As an example of rapidly converging superpositions of the Fock states (faster than in the standard coherent states), we considered the so called “Barut–Girardello coherent states” [49] (studied also in [19]) with the nonlinearity function

$$f(n) = \sqrt{n! \Gamma(n+g+1)}, \quad (26)$$

where $\Gamma(z)$ is the Euler gamma-function and g arbitrary real parameter. The normalization constants (21) can be expressed in terms of the Bessel functions of the first and third kinds (in this section, we use letter γ instead of α and ε):

$$2\mathcal{N}_{\pm\gamma}^{-2} = |\gamma|^{-g} [I_g(2|\gamma|) \pm J_g(2|\gamma|)]. \quad (27)$$

Figure 4 shows the number distributions in the even/odd superpositions of the Barut–Girardello coherent states with $g = 0$ and $|\gamma| = 5$,

$$p_{+;2k} = \frac{2|\gamma|^{4k}}{[I_0(2|\gamma|) + J_0(2|\gamma|)](2k)!^2}, \quad p_{-;2k+1} = \frac{2|\gamma|^{4k+2}}{[I_0(2|\gamma|) - J_0(2|\gamma|)](2k+1)!^2}. \quad (28)$$

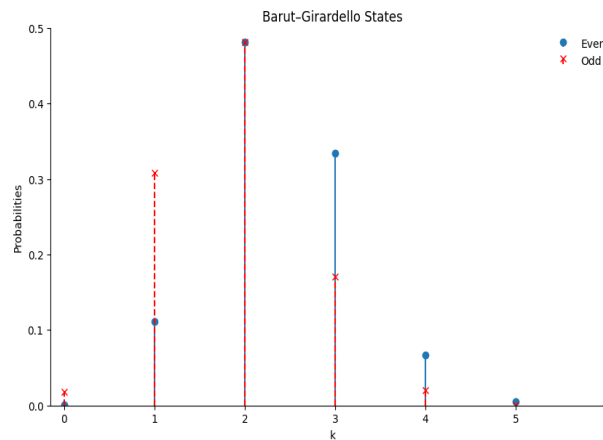


Figure 4. The number distribution functions of the even and odd superpositions of the Barut–Girardello coherent states with $|\gamma| = 5$.

Figure 5 shows practically the same “cat” structures of the Wigner functions as in Figure 2.

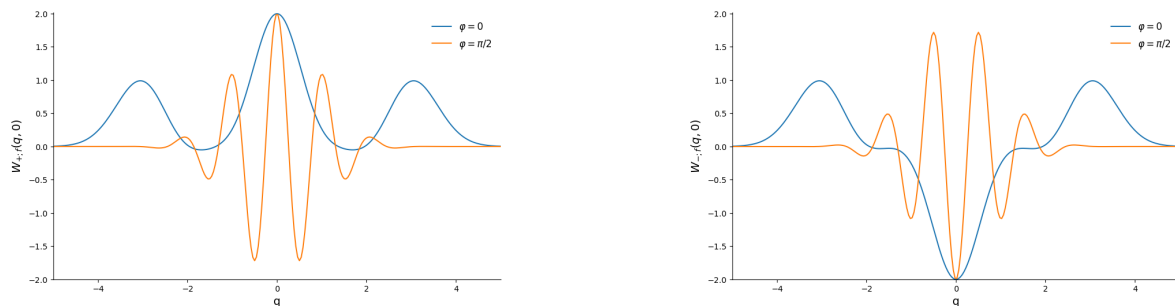


Figure 5. The sections $W(q, 0)$ of the Wigner functions of the even (left) and odd (right) Barut–Girardello coherent states with function $f(n)$ given by Equation (26); for $g = 0$, $|\gamma| = 5$ and two phases.

5. Superpositions of Binomial States

We consider normalized binomial states in the form

$$|\varepsilon; M\rangle = \left(1 + |\varepsilon|^2\right)^{-M/2} \sum_{n=0}^M \left[\frac{M!}{n!(M-n)!} \right]^{1/2} \varepsilon^n |n\rangle. \quad (29)$$

Such states (or their modifications and generalizations) were considered, e.g., in Refs. [50–52]. The mean number of quanta and its variance in state (29) are as follows,

$$\langle n \rangle_M = \frac{M|\varepsilon|^2}{1 + |\varepsilon|^2}, \quad \sigma_n^{(M)} = \frac{M|\varepsilon|^2}{(1 + |\varepsilon|^2)^2}. \quad (30)$$

Defining the nonlinearity function as

$$f(n) = \left[\frac{M!}{n!(M-n)!} \right]^{-1/2} \quad (31)$$

and using Newton's binomial expansion, one can find the normalization factors (21) of the even/odd superpositions:

$$\mathcal{N}_{\pm M}^{-2} = \frac{1}{2} \left[\left(1 + |\varepsilon|^2\right)^M \pm \left(1 - |\varepsilon|^2\right)^M \right]. \quad (32)$$

Phase-dependent parts of the Wigner functions are finite double sums:

$$W_{+2;M}(q, 0) = \sum_{\mu=0}^{\lfloor \frac{M}{2} \rfloor - 1} \sum_{\lambda=1}^{\lfloor \frac{M}{2} \rfloor - \mu} \frac{4\mathcal{N}_{+M}^2 e^{-q^2} M! |\varepsilon|^{2(2\mu+\lambda)} \cos(2\lambda\varphi) (2q^2)^\lambda L_{2\mu}^{2\lambda}(2q^2)}{(2\mu+2\lambda)! \sqrt{(M-2\mu)! (M-2\mu-2\lambda)!}},$$

$$W_{-2;M}(q, 0) = - \sum_{\mu=0}^{\lfloor \frac{M-3}{2} \rfloor} \sum_{\lambda=1}^{\lfloor \frac{M-1}{2} \rfloor - \mu} \frac{4\mathcal{N}_{-M}^2 e^{-q^2} M! |\varepsilon|^{2(2\mu+\lambda+1)} \cos(2\lambda\varphi) (2q^2)^\lambda L_{2\mu+1}^{2\lambda}(2q^2)}{(2\mu+2\lambda+1)! \sqrt{(M-2\mu-1)! (M-2\mu-2\lambda-1)!}}.$$

The phase-independent parts of the Wigner functions $W_{\pm 1;M}(q, 0)$ can be calculated analytically with the aid of formula 4.4.1.7 from Ref. [53]

$$\sum_{n=0}^M \frac{M!}{n!(M-n)!} z^n L_n(x) = (1+z)^M L_M\left(\frac{zx}{1+z}\right). \quad (33)$$

The results are as follows,

$$W_{\pm 1;M}(q, 0) = \mathcal{N}_{\pm M}^2 e^{-q^2} \left[\left(1 - |\varepsilon|^2\right)^M L_M\left(\frac{2q^2|\varepsilon|^2}{|\varepsilon|^2 - 1}\right) \pm \left(1 + |\varepsilon|^2\right)^M L_M\left(\frac{2q^2|\varepsilon|^2}{1 + |\varepsilon|^2}\right) \right].$$

These expressions can be simplified for the “mid-binomial” states with $|\varepsilon|^2 = 1$ (when $\langle n \rangle_M = M/2$, $\sigma_n^{(M)} = M/4$ and $\mathcal{N}_{+M}^2 = \mathcal{N}_{-M}^2 = 2^{1-M}$):

$$W_{\pm 1;M}^{mid}(q, 0) = \pm 2e^{-q^2} \left[L_M(q^2) \pm q^{2M}/M! \right]. \quad (34)$$

The probability distributions over the Fock states have the following forms for “mid-binomial” states with $|\varepsilon| = 1$:

$$p_{+;2k} = \frac{M!2^{1-M}}{(2k)!(M-2k)!}, \quad p_{-;2k+1} = \frac{M!2^{1-M}}{(2k+1)!(M-2k-1)!}. \quad (35)$$

These distributions are shown in Figure 6 for $M = 50$.

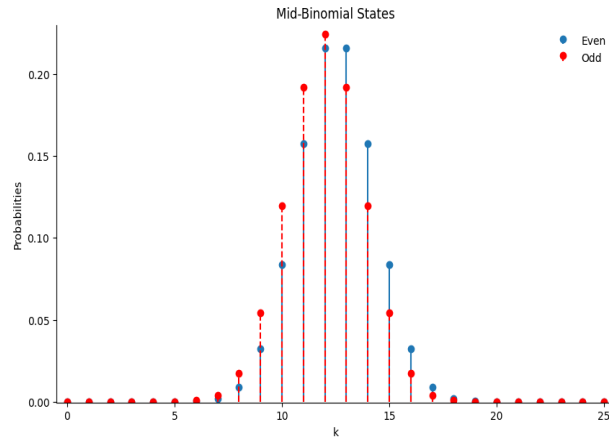


Figure 6. The number distribution functions of the even and odd superpositions of the “mid-binomial” states with $M = 50$.

Plots of the Wigner functions of the even/odd superpositions of the “mid-binomial” states with $M = 50$ are shown in Figure 7. They look very similar to the plots for usual even/odd coherent states in Figure 2.

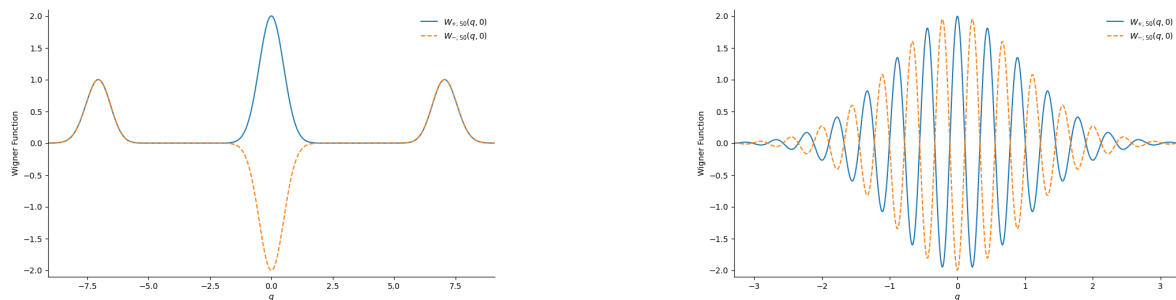


Figure 7. The sections $W(q, 0)$ of the Wigner functions of the even and odd superpositions of “mid-binomial” states with $M = 50$. (Left): for $\varphi = 0$. (Right): for $\varphi = \pi/2$.

6. Even/odd Coherent Phase States

Among an infinite number of possible functions $f(n)$ in Equation (9), the simplest choice seems to be $f(n) \equiv 1$. It gives the family of *coherent phase states* (CPS) [54–64] (called also as “harmonious” [65], “pseudothormal” [66] and “geometric” [67] states):

$$|\varepsilon\rangle = \sqrt{1 - |\varepsilon|^2} \sum_{n=0}^{\infty} \varepsilon^n |n\rangle, \quad \varepsilon = |\varepsilon|e^{i\varphi}, \quad |\varepsilon| < 1. \quad (36)$$

Even and odd superpositions of the CPS have the following normalized expansions in the Fock basis:

$$|\Psi\rangle_{+;\varepsilon} = \sqrt{1 - |\varepsilon|^4} \sum_{n=0}^{\infty} \varepsilon^{2n} |2n\rangle, \quad (37)$$

$$|\Psi\rangle_{-;\varepsilon} = |\varepsilon|^{-1} \sqrt{1 - |\varepsilon|^4} \sum_{n=0}^{\infty} \varepsilon^{2n+1} |2n+1\rangle. \quad (38)$$

In this case, we have thermal-like distributions (slowly and monotonously decaying),

$$p_{+,2k} = (1 - |\varepsilon|^4) |\varepsilon|^{4k} = p_{-,2k+1}, \quad (39)$$

which are shown in Figure 8 for $|\varepsilon|^2 = 25/26$.

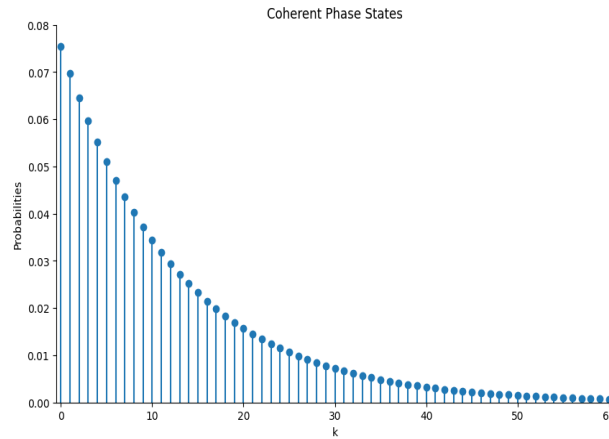


Figure 8. The number distribution functions of the even and odd superpositions of the coherent phase states with $|\varepsilon|^2 = 25/26$.

The superpositions (37) and (38) were considered in paper [68]. However, the corresponding Wigner functions were plotted there for small values of parameter $|\varepsilon|$ only, while the most interesting features can be observed in the limit $|\varepsilon| \rightarrow 1$, as we demonstrate below.

A detailed study of statistical properties of the even/odd coherent phase states (squeezing, the uncertainty products, Mandel's factor, etc.) was performed recently in paper [69]. Here, we show plots of the corresponding Wigner functions. Actually, their total difference from the case of usual even/odd coherent states induced writing the present paper.

The series (22) and (23) with $f(n) = 1$ can be calculated analytically as the even and odd parts $[S_x(z) \pm S_x(-z)]$ of the known generating function for the Laguerre polynomials,

$$S_x(z) = \sum_{n=0}^{\infty} L_n(x) z^n = (1-z)^{-1} \exp\left(\frac{xz}{z-1}\right). \quad (40)$$

The results are as follows,

$$W_{+1;\varepsilon}(q, 0) = \left(1 - |\varepsilon|^2\right) \exp\left(-q^2 \frac{1 - |\varepsilon|^2}{1 + |\varepsilon|^2}\right) + \left(1 + |\varepsilon|^2\right) \exp\left(-q^2 \frac{1 + |\varepsilon|^2}{1 - |\varepsilon|^2}\right), \quad (41)$$

$$W_{-1;\varepsilon}(q, 0) = |\varepsilon|^{-2} \left[\left(1 - |\varepsilon|^2\right) \exp\left(-q^2 \frac{1 - |\varepsilon|^2}{1 + |\varepsilon|^2}\right) - \left(1 + |\varepsilon|^2\right) \exp\left(-q^2 \frac{1 + |\varepsilon|^2}{1 - |\varepsilon|^2}\right) \right]. \quad (42)$$

One can see that $W_{\pm 1;\varepsilon}(0, 0) = \pm 2$, as it must be for even or odd states. The off-diagonal parts of the Wigner functions can be reduced to the following double series:

$$W_{+2;\varepsilon}(q, 0) = 4 \left(1 - |\varepsilon|^4\right) \sum_{\mu=0}^{\infty} \sum_{\lambda=1}^{\infty} |\varepsilon|^{2(2\mu+\lambda)} \cos(2\lambda\varphi) \mathcal{L}\left(2q^2; 2\mu, 2\lambda\right), \quad (43)$$

$$W_{-2;\varepsilon}(q, 0) = -4 \left(1 - |\varepsilon|^4\right) \sum_{\mu=0}^{\infty} \sum_{\lambda=1}^{\infty} |\varepsilon|^{2(2\mu+\lambda)} \cos(2\lambda\varphi) \mathcal{L}\left(2q^2; 2\mu + 1, 2\lambda\right). \quad (44)$$

Unfortunately, these series can be calculated only numerically.

Figure 9 shows the Wigner functions $W_{\pm 1;\varepsilon}(q, 0)$ for $|\varepsilon|^2 = 25/26$ (when the mean number of quanta in each component equals $\langle n \rangle_{CPS} = |\varepsilon|^2 / (1 - |\varepsilon|^2) = 25$) and different phases. We plot the figures for $q > 0$ in view of the relation $W(q, 0) = W(-q, 0)$. One can see a strong difference between Figs. 2 and 9. For example, there are no lateral peaks for $\varphi = 0$ in Figure 9. A probable explanation is the strong connection between the variance σ_x and the mean value $\langle x \rangle$ in the coherent phase states with $\varphi = 0$ [64]: $\sigma_x \approx 0.4 \langle n \rangle$ and $\langle x \rangle \approx 1.25 \sqrt{\langle n \rangle}$. Therefore, the peaks of two components of the even superposition of CPS merge into a single central peak, in contrast to the even superpositions of the

usual CS, whose peak widths do not depend on $\langle x \rangle$ and $\langle n \rangle$. Although something remotely similar to lateral peaks is visible for odd superpositions, these peaks are smooth continuations of the central peak, whereas lateral peaks in usual odd CS are well separated from the central peak. Another striking difference from the case of usual even/odd coherent states is the absence of any “fine structure” for the phase $\varphi = \pi/2$, as can be seen in Figure 10.

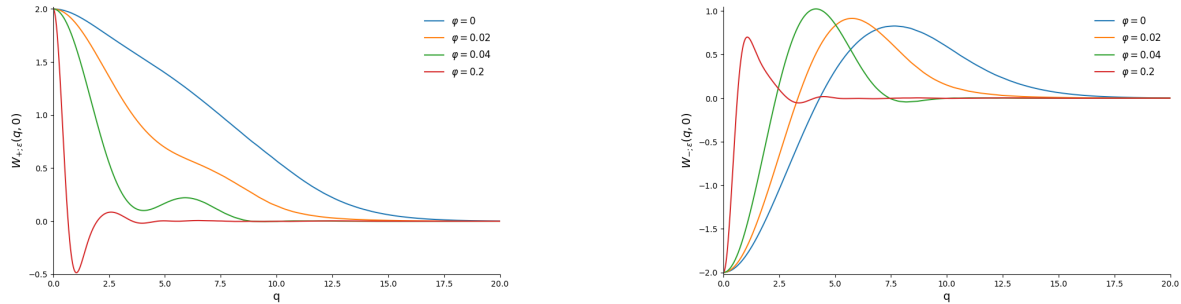


Figure 9. The sections $W(q, 0)$ of the Wigner functions of the even (left) and odd (right) coherent phase states with $|\varepsilon|^2 = 25/26$ and different phases.

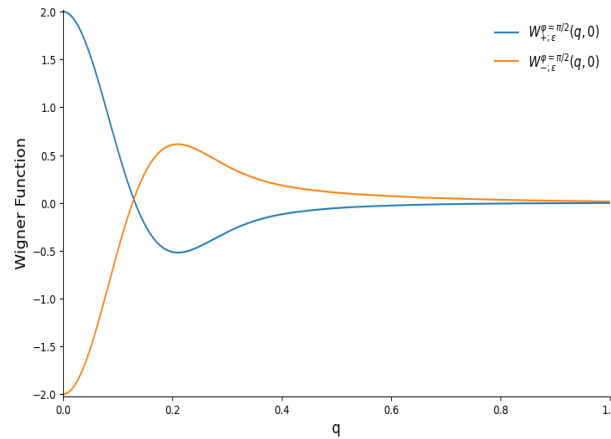


Figure 10. The section $W(q, 0)$ of the Wigner functions of the even and odd coherent phase states with $|\varepsilon|^2 = 25/26$ and $\varphi = \pi/2$.

7. Negative Binomial States

Coherent phase states can be considered as the special case ($\nu = 1$) of *negative binomial states* (NBS), whose normalized expansion over the Fock states has the form

$$|\varepsilon, \nu\rangle = \left(1 - |\varepsilon|^2\right)^{\nu/2} \sum_{n=0}^{\infty} \left[\frac{\Gamma(n + \nu)}{\Gamma(\nu)n!} \right]^{1/2} \varepsilon^n |n\rangle, \quad (45)$$

with the following simple expressions for the mean number of quanta and its variance [67]:

$$\langle n \rangle_{\nu} = \frac{\nu |\varepsilon|^2}{1 - |\varepsilon|^2}, \quad \sigma_n^{(\nu)} = \frac{\nu |\varepsilon|^2}{(1 - |\varepsilon|^2)^2}. \quad (46)$$

The early history of studies of such states goes to papers [19,55,67,70–74]. Even and odd NBS were introduced in paper [75]. The normalization factor in Equation (45) follows from the formula

$$\sum_{n=0}^{\infty} \frac{\Gamma(n + \nu)}{\Gamma(\nu)n!} z^n = (1 - z)^{-\nu}. \quad (47)$$

Consequently, using the function

$$f(n) = \left[\frac{n!}{\Gamma(n+\nu)} \right]^{1/2}, \quad (48)$$

we obtain the following normalization coefficients in Equation (21):

$$\mathcal{N}_{\pm,\nu}^2 = \frac{2}{\Gamma(\nu)} \left[\left(1 - |\varepsilon|^2\right)^{-\nu} \pm \left(1 + |\varepsilon|^2\right)^{-\nu} \right]^{-1}. \quad (49)$$

The number distributions are as follows,

$$p_{+,2k} = \frac{2|\varepsilon|^{4k}\Gamma(2k+\nu)}{\Gamma(\nu) \left[(1 - |\varepsilon|^2)^{-\nu} + (1 + |\varepsilon|^2)^{-\nu} \right] (2k)!},$$

$$p_{-,2k+1} = \frac{2|\varepsilon|^{4k+2}\Gamma(2k+1+\nu)}{\Gamma(\nu) \left[(1 - |\varepsilon|^2)^{-\nu} - (1 + |\varepsilon|^2)^{-\nu} \right] (2k+1)!}.$$

To plot the Wigner functions, we calculated numerically the following series:

$$W_{+, \nu}(q, 0) = 2\mathcal{N}_{+, \nu}^2 e^{-q^2} \sum_{n=0}^{\infty} \frac{\Gamma(2n+\nu)}{(2n)!} |\varepsilon|^{4n} L_{2n}(2q^2),$$

$$W_{-, \nu}(q, 0) = -2\mathcal{N}_{-, \nu}^2 e^{-q^2} \sum_{n=0}^{\infty} \frac{\Gamma(2n+1+\nu)}{(2n+1)!} |\varepsilon|^{4n+2} L_{2n+1}(2q^2),$$

$$W_{+, \nu}(q, 0) = 4\mathcal{N}_{+, \nu}^2 e^{-q^2} \sum_{\mu=0}^{\infty} \sum_{\lambda=1}^{\infty} |\varepsilon|^{2(2\mu+\lambda)} (2q^2)^{\lambda}$$

$$\times \frac{\sqrt{\Gamma(2\mu+\nu)\Gamma(2\mu+2\lambda+\nu)}}{(2\mu+2\lambda)!} \cos(2\lambda\varphi) L_{2\mu}^{(2\lambda)}(2q^2),$$

$$W_{-, \nu}(q, 0) = -4\mathcal{N}_{-, \nu}^2 e^{-q^2} \sum_{\mu=0}^{\infty} \sum_{\lambda=1}^{\infty} |\varepsilon|^{2(2\mu+\lambda+1)} (2q^2)^{\lambda}$$

$$\times \frac{\sqrt{\Gamma(2\mu+1+\nu)\Gamma(2\mu+2\lambda+1+\nu)}}{(2\mu+2\lambda+1)!} \cos(2\lambda\varphi) L_{2\mu+1}^{(2\lambda)}(2q^2).$$

Numeric calculations for the values $\nu \neq 1$ resulted in figures looking qualitatively like Figures 9 and 10, provided the difference $|\nu - 1|$ is not very big. Figure 11 illustrates the case of $\nu = 2$.

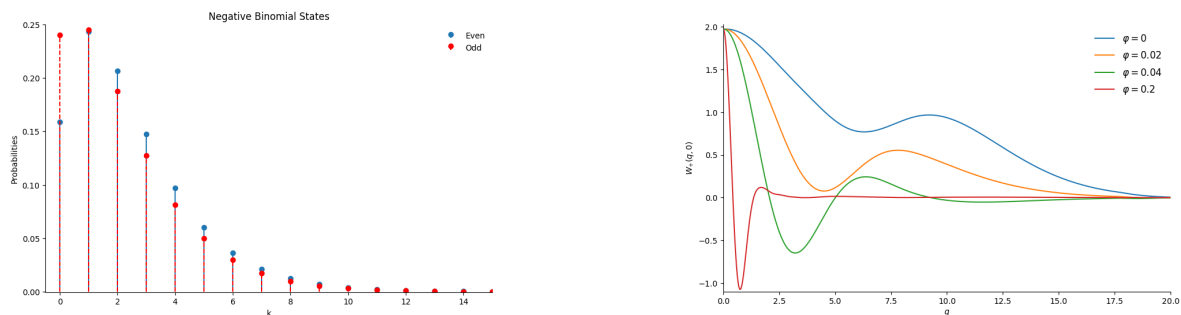


Figure 11. Illustrations for the even and odd superpositions of the negative binomial states with $|\varepsilon|^2 = 5/7$ and $\nu = 2$. (Left): the number distribution functions. (Right): Function $W(q, 0)$ for the even superpositions with different values of phase φ .

Figures for different values of parameter ν have similar shapes for odd superpositions, both for $\nu > 1$ and $\nu < 1$. A small difference between the cases of $\nu > 1$ and $\nu < 1$ was observed for even superpositions: their Wigner functions show practically no negative regions for $\nu < 1$, especially for $\nu \ll 1$. Such a behavior can be understood, if one takes into account that even superpositions go to the vacuum state $|0\rangle$ in the limit $\nu \rightarrow 0$ (while odd superpositions continue to have an infinite number of terms).

If parameter ν increases, the number distribution function acquires more and more pronounced peaks: see Figure 12. The related plots for two different values of parameter ν are shown in Figure 12.

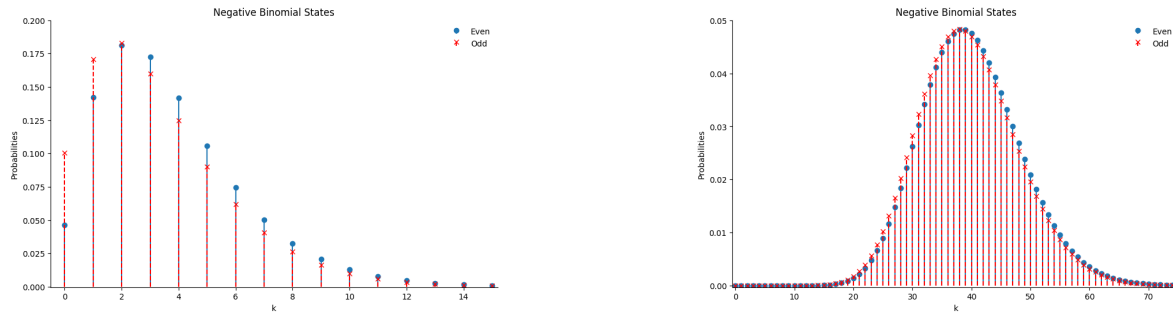


Figure 12. The number distribution functions of the even and odd superpositions of the negative binomial states with $|\varepsilon|^2 = 5/7$ for $\nu = 3$ (left) and $\nu = 32$ (right).

Note that Equation (46) yields the mean number of quanta in each component of the superposition $\langle n \rangle = 80$ for $\nu = 32$ and $|\varepsilon|^2 = 5/7$.

Finally, the “cat” structure totally revives for $\nu \gg 1$, as can be seen in Figure 13.

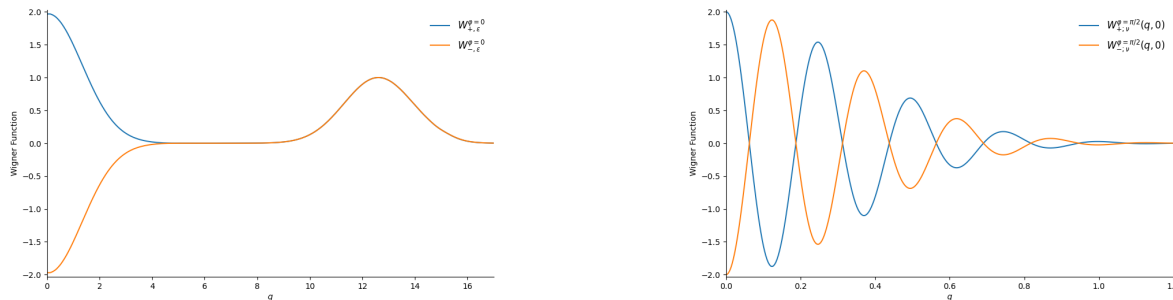


Figure 13. The section $W(q, 0)$ of the Wigner functions of the even and odd negative binomial states with $\nu = 32$ and $|\varepsilon|^2 = 5/7$, for $\varphi = 0$ (left) and $\varphi = \pi/2$ (right).

8. Conclusions

The plots demonstrated in this paper show that the family of possible “cat-like” quantum states is significantly bigger than it was thought (probably) earlier. It includes, in particular, even and odd highly excited superpositions of several types of nonlinear coherent states. The decisive factor is the form of the number distribution functions over the Fock basis: they must have relatively narrow peaks. Otherwise, no “cat” structures can be observed. In all examples, the heights of lateral peaks of the cat-like Wigner functions are exactly twice smaller than the height of the central interference peak. Since all these lateral peaks have certainly non-Gaussian shapes, the discoveries of this paper open the new direction for the future studies: to try to understand how the shapes of new cat-like states can influence the decoherence, “classicalization” and thermalization times in the presence of different reservoirs (comparing, e.g., with the results of studies [21,46,76–78]).

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References

1. Wigner, E. On the quantum corrections for thermodynamic equilibrium. *Phys. Rev.* **1932**, *40*, 749–759.
2. Tatarskii, V.I. The Wigner representation of quantum mechanics. *Sov. Phys. – Uspekhi* **1983**, *26*, 311–327.
3. Hillery, M.; O’Connell, R.F.; Scully, M.O.; Wigner, E.P. Distribution functions in physics: fundamentals. *Phys. Rep.* **1984**, *106*, 121–167.
4. Schleich, W.P. *Quantum Optics in Phase Space*; Wiley: New York, USA, 2001.
5. Vourdas, A. Analytic representations in quantum mechanics. *J. Phys. A: Math. Gen.* **2006**, *39*, R65–R141.
6. Takabayasi, T. The formulation of quantum mechanics in terms of ensemble in phase space. *Prog. Theor. Phys.* **1954**, *11*, 341–373.
7. Cahill, K.E.; Glauber, R.J. Density operators and quasiprobability distributions. *Phys. Rev.* **1969**, *177*, 1882–1902.
8. Royer, A. Wigner function as the expectation value of a parity operator. *Phys. Rev. A* **1977**, *15*, 449–450.
9. Dodonov, V.V. Wigner functions and statistical moments of quantum states with definite parity. *Phys. Lett. A* **2007**, *364*, 368–371.
10. Groenewold, H.J. On the principles of elementary quantum mechanics. *Physica* **1946**, *12*, 405–460.
11. Bartlett, M.S.; Moyal, J.E. The exact transition probabilities of quantum-mechanical oscillator calculated by the phase-space method. *Proc. Camb. Phil. Soc.* **1949**, *45*, 545–553.
12. Cahill, K.E.; Glauber, R.J. Ordered expansions in boson amplitude operators. *Phys. Rev.* **1969**, *177*, 1857–1881.
13. Dodonov, V.V.; Man’ko, V.I. Phase space eigenfunctions of multidimensional quadratic Hamiltonians. *Physica A* **1986**, *137*, 306–316.
14. Erdélyi, A. (Ed.) *Bateman Manuscript Project: Higher Transcendental Functions*; McGraw-Hill: New York, NY, USA, 1953; vol. 2.
15. Gradshteyn, I.S.; Ryzhik, I.M. *Table of Integrals, Series, and Products*, 7th edition; Academic: Amsterdam, The Netherlands, 2007.
16. Klauder, J.R. The action option and a Feynman quantization of spinor fields in terms of ordinary c -numbers. *Ann. Phys. (NY)* **1960**, *11*, 123–168.
17. Glauber, R.J. Photon correlations. *Phys. Rev. Lett.* **1963**, *10*, 84–86.
18. Sudarshan, E.C.G. Equivalence of semiclassical and quantum mechanical descriptions of statistical light beams. *Phys. Rev. Lett.* **1963**, *10*, 277–279.
19. Dodonov, V.V.; Malkin, I.A.; Man’ko, V.I. Even and odd coherent states and excitations of a singular oscillator. *Physica* **1974**, *72*, 597–615.
20. Hillery, M. Amplitude-squared squeezing of the electromagnetic field. *Phys. Rev. A* **1987**, *36*, 3796–3802.
21. Bužek, V.; Vidiella-Barranco, A.; Knight, P.L. Superpositions of coherent states: Squeezing and dissipation. *Phys. Rev. A* **1992**, *45*, 6570–6585.
22. Gerry, C.C. Nonclassical properties of even and odd coherent states. *J. Mod. Opt.* **1993**, *40*, 1053–1071.
23. Ansari, N.A.; Di Fiore, L.; Manko, M.A.; Manko, V.I.; Solimeno, S.; Zaccaria, F. Quantum limits in interferometric gravitational-wave antennas in the presence of even and odd coherent states. *Phys. Rev. A* **1994**, *49*, 2151–2156.
24. Bužek, V.; Knight, P.L. Quantum interference, superposition states of light, and nonclassical effects. In *Progress in Optics*, vol. XXXIV; Wolf, E., Ed.; North-Holland: Amsterdam, The Netherlands, 1995; pp. 1–158.
25. de Matos Filho, R.L.; Vogel, W. Even and odd coherent states of the motion of a trapped ion. *Phys. Rev. Lett.* **1996**, *76*, 608–611.
26. Yurke, B.; Stoler, D. Generating quantum mechanical superpositions of macroscopically distinguishable states via amplitude dispersion. *Phys. Rev. Lett.* **1986**, *57*, 13–16.
27. Schleich, W.; Pernigo, M.; Kien, F.L. Nonclassical state from two pseudoclassical states. *Phys. Rev. A* **1991**, *44*, 2172–2187.

28. Brune, M.; Haroche, S.; Raimond, J.M.; Davidovich, L.; Zagury, N. Manipulation of photons in a cavity by dispersive atom-field coupling: quantum-nondemolition measurements and generation of "Schrödinger cat" states. *Phys. Rev. A* **1992**, *45*, 5193–5214.
29. Tara, K.; Agarwal, G.S.; Chaturvedi, S. Production of Schrödinger macroscopic quantum-superposition states in a Kerr medium. *Phys. Rev. A* **1993**, *47*, 5024–5029.
30. Janszky, J.; Domokos, P.; Adam, P. Coherent states on a circle and quantum interference. *Phys. Rev. A* **1993**, *48*, 2213–2219.
31. Janszky, J.; Vinogradov, A.V.; Kobayashi, T.; Kis, Z. Vibrational Schrödinger-cat states. *Phys. Rev. A* **1994**, *50*, 1777–1284.
32. Dodonov, V.V.; Kalmykov, S.Y.; Man'ko, V.I. Statistical properties of Schrödinger real and imaginary cat states. *Phys. Lett. A* **1995**, *199*, 123–130.
33. Castaños, O.; López-Peña, R.; Man'ko, V.I. Crystallized Schrödinger cat states. *J. Russ. Laser Res.* **1995**, *16*, 477–525.
34. Monroe, C.; Meekhof, D.M.; King, B.E.; Wineland, D.J. A "Schrödinger cat" superposition state of an atom. *Science* **1996**, *272*, 1131–1136.
35. Delgado, A.; Klimov, A.B.; Retamal, J.C.; Saavedra, C. Macroscopic field superpositions from collective interactions. *Phys. Rev. A* **1998**, *58*, 655–662.
36. Malbouisson, J.M.C.; Baseia, B. Higher-generation Schrödinger cat states in cavity QED. *J. Mod. Opt.* **1999**, *46*, 2015–2041.
37. Dodonov, V.V. 'Nonclassical' states in quantum optics: a 'squeezed' review of first 75 years. *J. Opt. B: Quantum Semiclass. Opt.* **2002**, *4*, R1–R33.
38. de Matos Filho, R.L.; Vogel, W. Nonlinear coherent states. *Phys. Rev. A* **1996**, *54*, 4560–4563.
39. Man'ko, V.I.; Marmo, G.; Sudarshan, E.C.G.; Zaccaria, F. *f*-oscillators and nonlinear coherent states. *Phys. Scripta* **1997**, *55*, 528–541.
40. Sivakumar, S. Studies on nonlinear coherent states. *J. Opt. B: Quantum Semiclass. Opt.* **2000**, *2*, R61–R75.
41. Mancini, S. Even and odd nonlinear coherent states. *Phys. Lett. A* **1997**, *233*, 291–296.
42. Sivakumar, S. Even and odd nonlinear coherent states. *Phys. Lett. A* **1998**, *250*, 257–262.
43. Roy, B.; Roy, P. Time dependent nonclassical properties of even and odd nonlinear coherent states. *Phys. Lett. A* **1999**, *263*, 48–52.
44. Zhang, J.; Wu, W.; Meng, S.-Y. Quantum properties of two-mode squeezed even and odd coherent states. *Commun. Theor. Phys.* **2007**, *47*, 317–324.
45. Darwish, M. Even and odd nonlinear negative binomial states. *Int. J. Theor. Phys.* **2008**, *47*, 3035–3056.
46. Mohammadbeigi, A.; Tavassoly, M.K. The decoherence of the quantum excitation of even/odd coherent states in a photon-loss channel. *J. Phys. B: At. Mol. Opt. Phys.* **2014**, *47*, 165502.
47. Ren, G.; Du, J.-m.; Zhang, W.-h.; Yu, H.-j. Nonclassicality generated by applying Hermite-polynomials photon-added operator on the even/odd coherent states. *Int. J. Theor. Phys.* **2017**, *56*, 1537–1549.
48. Weyl, H. Quantenmechanik und Gruppentheorie. *Z. Phys.* **1928**, *46*, 1–46.
49. Barut, A.O.; Girardello, L. New "coherent" states associated with non-compact groups. *Commun. Math. Phys.* **1971**, *21*, 41–55.
50. Stoler, D.; Saleh, B.E.A.; Teich, M.C. Binomial states of the quantized radiation field. *Optica Acta* **1985**, *32*, 345–355.
51. Fu, H.C.; Sasaki, R. Generalized binomial states: ladder operator approach. *J. Math. Phys.* **1996**, *29*, 5637–5644.
52. Vidiella Barranco, A. The binomial states of light. In *Theory of Non-classical States of Light*; Dodonov, V.V., Man'ko, V.I., Eds.; Taylor & Francis: London, England, 2003; pp. 241–276.
53. Prudnikov, A.P.; Brychkov, Y.A.; Marichev, O.I. *Integrals and Series*, vol. II; Gordon and Breach: London, England, 1986.
54. Lerner, E.C.; Huang, H.W.; Walters, G.E. Some mathematical properties of oscillator phase operator. *J. Math. Phys.* **1970**, *11*, 1679–1684.
55. Aharonov, Y.; Lerner, E.C.; Huang, H.W.; Knight, J.M. Oscillator phase states, thermal equilibrium and group representations. *J. Math. Phys.* **1973**, *14*, 746–756.
56. Lévy-Leblond, J.-M. Who is afraid of nonhermitian operators? A quantum description of angle and phase. *Ann. Phys. (NY)* **1976**, *101*, 319–341.
57. Vourdas, A. SU(2) and SU(1;1) phase states. *Phys. Rev. A* **1990**, *41*, 1653–1661.

58. Shapiro, J.H.; Shepard, S.R. Quantum phase measurement: a system-theory perspective. *Phys. Rev. A* **1991**, *43*, 3795–3818.
59. Hall, M.J.W. Phase resolution and coherent phase states. *J. Mod. Opt.* **1993**, *40*, 809–824.
60. Vourdas, A.; Brif, C.; Mann, A. Factorization of analytic representations in the unit disc and number-phase statistics of a quantum harmonic oscillator. *J. Phys. A: Math. Gen.* **1996**, *29*, 5887–5898.
61. Wünsche, A. A class of phase-like states. *J. Opt. B: Quantum Semiclass. Opt.* **2001**, *3*, 206–218.
62. Dodonov, V.V.; Mizrahi, S.S. Creating quanta with an ‘annihilation’ operator. *J. Phys. A: Math. Gen.* **2002**, *35*, 8847–8857.
63. Gazeau, J.-P.; Hussin, V.; Moran, J.; Zelaya, K. Generalized Susskind–Glogower coherent states. *J. Math. Phys.* **2021**, *62*, 072104.
64. de Freitas, M.C.; Dodonov, V.V. Coherent phase states in the coordinate and Wigner representations. *Quantum Rep.* **2022**, *4*, 509–522.
65. Sudarshan, E.C.G. Diagonal harmonious state representation. *Int. J. Theor. Phys.* **1993**, *32*, 1069–1076.
66. Dodonov, V.V.; Mizrahi, S.S. Uniform nonlinear evolution equations for pure and mixed quantum states. *Ann. Phys. (NY)* **1995**, *237*, 226–268.
67. Fu, H.C.; Sasaki, R. Negative binomial states of quantized radiation fields. *J. Phys. Soc. Jpn.* **1997**, *66*, 1989–1994.
68. Obada, A.-S.F.; Darwish, M. Partial phase state as a nonlinear coherent state and some of its properties. *J. Mod. Opt.* **2004**, *51*, 209–222.
69. de Freitas, M.C.; Dodonov, V.V. Statistical properties of superpositions of coherent phase states with opposite arguments. *Entropy* **2024**, *26*, 977.
70. Aharonov, Y.; Huang, H.W.; Knight, J.M.; Lerner, E.C. Generalized destruction operators and a new method in group theory. *Lett. Nuovo Cim.* **1971**, *2*, 1317–1320.
71. Onofri, E.; Pauri, M. Analyticity and quantization. *Lett. Nuovo Cim.* **1972**, *3*, 35–42.
72. Joshi, A.; Lawande, S.V. The effects of negative binomial field distribution on Rabi oscillations. *Opt. Commun.* **1989**, *70*, 21–24.
73. Matsuo, K. Glauber–Sudarshan P-representation of negative binomial states. *Phys. Rev. A* **1990**, *41*, 519–522.
74. Stephen, S.M. Negative binomial states of the quantized radiation field. *J. Mod. Opt.* **1998**, *45*, 2201–2205.
75. Joshi, A.; Obada, A.-S.F. Some statistical properties of the even and the odd negative binomial states. *J. Phys. A: Math. Gen.* **1997**, *30*, 81–97.
76. Dodonov, V.V.; Mizrahi, S.S.; de Souza Silva, A.L. Decoherence and thermalization dynamics of a quantum oscillator. *J. Opt. B: Quantum Semiclass. Opt.* **2000**, *2*, 271–281.
77. Dodonov, V.V.; de Souza, L.A. Decoherence of superpositions of displaced number states. *J. Opt. B: Quantum Semiclass. Opt.* **2005**, *7*, S490–S499.
78. Dodonov, V.V.; Valverde, C.; Souza, L.S.; Baseia, B. Classicalization times of parametrically amplified “Schrödinger cat” states coupled to phase-sensitive reservoirs. *Phys. Lett. A* **2011**, *375*, 3668–3676.

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