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Posted Date: 26 August 2025

doi: 10.20944/preprints202508.1874.v1

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Article

A Multivariate Copula Framework for Joint Modeling of Ranking Concordance, Dispersion, and Extremeness

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Abstract

Ranking analysis underpins decisions in many areas, from peer review to sports. Standard evaluation methods examine concordance, dispersion, and extremeness separately, often ignoring their interdependencies. This assumption risks biased inferences and a partial view of ranking patterns. The Concordance-Dispersion-Extremity Framework (CDEF) offers a new copula-based approach that jointly models these dimensions. Using Gumbel copulas to capture multivariate dependence, CDEF combines Kendall's W (concordance), concentration measures (hypergeometric or multinomial, selected via chi-square tests), and Gumbel extreme value distributions (extremeness). Monte Carlo estimation enables practical computation of the joint probabilities. Applying CDEF to NCAA football rankings showed striking differences from traditional univariate assessments. Standard methods indicated strong ranking properties—high concordance (Kendall's $W = 0.717$), significant concentration, and pronounced extremeness. However, the copula-based model revealed that these effects largely diminish when dependencies are accounted for ($P(\text{Concordance} | \text{Dispersion, Extremeness}) = 0.091$). Much of the perceived agreement arose from shared patterns, not genuine consensus. CDEF thus advances ranking analysis by revealing how apparent strength in one dimension depends on the others. This richer perspective helps detect bias, assess reliability, and support sound decisions in settings where rankings have serious consequences. Ignoring dependence can foster unwarranted confidence in ranking systems, highlighting the need for approaches that model these relationships explicitly.

Keywords: copula; ranking analysis; dependence modeling; extreme value theory; monte carlo methods; multivariate statistic

1. Introduction

1.1. The Problem of Ranking Analysis

Ranking analysis constitutes a fundamental component of decision-making processes across numerous domains [1,2]. In academic contexts, peer review systems rely on rankings to allocate research funding and evaluate scholarly contributions [3,4]. Sports competitions often depend on ranking methodologies to determine championships and seedings [5,6]. Medical diagnosis increasingly incorporates ranking approaches for risk stratification and treatment prioritization [7,8]. Business environments utilize rankings for performance evaluation, market analysis, and strategic planning [9,10].

Despite this ubiquity, the statistical evaluation of rankings remains methodologically fragmented [11,12]. Traditional approaches assess ranking characteristics—concordance among raters, dispersion of rank assignments, and extremeness of individual scores—as independent phenomena. This compartmentalized analysis fails to capture the complex interdependencies that characterize real-world ranking behaviors, potentially leading to misleading conclusions about ranking quality and structure.

1.2. Limitations of Current Approaches

While copula methods have appeared in various ranking contexts, to our knowledge, no framework has explicitly combined these three components within a unified copula-based model to quantify their joint dependence and reveal conditional relationships such as phantom concordance to our knowledge.

Existing ranking evaluation methods, such as Kendall's W for concordance [13] and Spearman's rank correlation [14], treat ranking properties in isolation. When multiple ranking characteristics are analyzed simultaneously, researchers typically apply separate statistical tests under independence assumptions. This approach suffers from several limitations.

Traditional methods have conceptual constraints in how they measure disagreement and distance between rankings [15], focusing on individual metrics without considering how one ranking property constrains others. For example, high concordance among raters naturally limits the dispersion of rank assignments and affects the likelihood of extreme scores.

Multiple testing issues also arise when components are analyzed separately without accounting for their joint distribution [16,17], potentially inflating Type I error rates and producing spurious significance findings. Additionally, independent analyses provide little insight into how ranking characteristics interact, reducing the interpretability of results for decision-makers.

1.3. The Copula Solution

Copula theory provides a mathematical framework for modeling complex dependence structures between random variables while preserving their individual distributional properties [18]. Originally formalized by Sklar [18], copulas enable the separation of marginal distributions from their dependence structure, offering flexibility in multivariate modeling [19,20]. To date, these capabilities have not been integrated into a unified ranking evaluation framework.

Recent advances in copula methodology have demonstrated their effectiveness across diverse applications, from financial risk modeling [21,22] to environmental science [23,24]. Copula-based approaches have shown particular promise for capturing asymmetric dependence and extreme co-movements [25,26], making them suitable for ranking analysis where tail behavior and dependency structures are central concerns.

1.4. Research Contribution and Novelty

This paper introduces the Concordance-Dispersion-Extremity Framework (CDEF), which represents a methodological advance in ranking analysis by providing insights not readily available through traditional methods. The novelty of CDEF lies not in merely combining existing statistical techniques, but in applying them in a unified tri-variate framework that uncovers previously unquantified dependencies among concordance, dispersion, and extremeness. This joint modeling enables diagnostic insights about phantom concordance that classical approaches do not provide in this integrated form.

Our literature review found that while copulas have found extensive application in finance [21,22], environmental science [23,24], and biostatistics [27,28], no existing work, to our knowledge, has applied a copula-based framework to jointly model concordance, dispersion, and extremeness for ranking evaluation. More critically, existing approaches to ranking evaluation are incapable of distinguishing between genuine ranking system properties and statistical artifacts created by unmodeled dependencies, a limitation that CDEF addresses.

CDEF demonstrates that apparent ranking system properties may be largely illusory when dependencies are properly modeled. For example, traditional methods analyzing the NCAA football rankings would conclude that the system exhibits strong concordance ($W = 0.717$), significant dispersion, and meaningful extreme behavior—suggesting a highly reliable ranking process. However, CDEF reveals that when these characteristics are modeled jointly, the conditional probability of concordance drops to just 0.091, indicating that much of the apparent agreement is explained by common biases rather than genuine consensus.

This capability to expose “phantom concordance”—apparent agreement that dissolves under joint analysis—offers a novel perspective. Classical approaches do not distinguish between genuine inter-rater agreement and agreement that merely reflects shared systematic patterns in dispersion and extremeness. While prior frameworks, including Generalizability Theory and rater bias modeling, address systematic sources of agreement, these approaches typically treat concordance separately from dispersion and extremeness. CDEF contributes a joint probabilistic framework that quantifies how dependence among ranking characteristics can inflate perceived agreement, providing a new diagnostic perspective with implications for decision-making.

2. Background and Related Work

2.1. Ranking Analysis Fundamentals

Ranking analysis encompasses the statistical evaluation of ordered preferences or performance assessments across multiple entities [29,30]. This field has evolved from simple pairwise comparisons to sophisticated multivariate frameworks capable of handling complex rating scenarios.

Three fundamental characteristics define the structure of any ranking system. Concordance measures the level of agreement among multiple raters when ranking the same set of entities. For m raters ranking n entities, concordance is quantified using Kendall’s W [13]:

$$W = \frac{12 \sum_{i=1}^n (R_i - \bar{R})^2}{m^2(n^3 - n)} \quad (1)$$

where R_i is the sum of ranks assigned to entity i across all raters, and $\bar{R} = \frac{m(n+1)}{2}$ is the expected rank sum under random assignment.

Dispersion (or concentration) evaluates the spread of rank assignments across the available ranking scale, independent of inter-rater agreement [31]. For a ranking system with k possible rank categories, dispersion is measured using the complement of the Herfindahl-Hirschman Index:

$$\text{Dispersion} = 1 - \sum_{j=1}^k p_j^2 \quad (2)$$

where p_j is the proportion of rankings assigned to category j . High dispersion (approaching 1) indicates rankings are widely distributed, while low dispersion (approaching 0) suggests concentration around specific rank values.

Extremeness captures the degree to which individual rankings deviate from expected distributions [32]. For an individual entity with standardized ranking score z , extremeness is defined as the upper tail probability under the fitted Gumbel distribution:

$$\text{Extremeness} = 1 - F_{\text{Gumbel}}(z) = 1 - \exp\left(-e^{-(z-\mu)/\beta}\right) \quad (3)$$

where μ and β are location and scale parameters estimated from the ranking data. Values approaching 1 indicate extreme deviations from expected patterns.

2.2. Traditional Ranking Evaluation Methods

2.2.1. Concordance Measures

Kendall’s W remains the standard measure for multi-rater concordance [13]. Alternative concordance measures include Fleiss’ Kappa [33] and Krippendorff’s Alpha [34], which assess agreement beyond chance levels. However, these measures suffer from non-normal asymptotic distributions and limited interpretability in ranking contexts [35,36].

2.2.2. Dispersion Analysis

Dispersion analysis in rankings typically employs discrete probability distributions [37]. The multinomial distribution models independent rank assignments:

$$P(X_1 = x_1, \dots, X_k = x_k) = \frac{N!}{x_1! \dots x_k!} p_1^{x_1} \dots p_k^{x_k} \quad (4)$$

where each x_i represents the count of rankings in category i , N is the total number of rankings, and p_i are the category probabilities.

For ranking systems where category counts are drawn without replacement from a finite pool (creating negative dependence), the multivariate hypergeometric distribution provides an appropriate model [38].

$$P(X_1 = x_1, \dots, X_k = x_k) = \frac{\prod_{i=1}^k \binom{N_i}{x_i}}{\binom{N}{n}} \quad (5)$$

This formulation expresses the probability of drawing counts x_1, \dots, x_k from categories of size N_1, \dots, N_k without replacement, where $N = \sum_{i=1}^k N_i$ is the total population size and $n = \sum_{i=1}^k x_i$ is the total number of draws.

The choice between these distributions significantly impacts dispersion assessment and subsequent statistical inference, yet traditional approaches often make this choice arbitrarily rather than through data-driven model selection.

Foundational work by Marden [29] has emphasized the critical role of selecting an appropriate statistical model for ranking dispersion analysis, noting that unsuitable model assumptions can distort measures of concentration. However, these studies have not examined how dispersion interacts with other ranking characteristics within joint dependency structures.

2.2.3. Extremeness Detection

Extreme value theory (EVT) offers the theoretical foundation for modeling ranking extremeness [39,40]. The Gumbel distribution, arising as the limiting distribution of maximum values under the Fisher-Tippett-Gnedenko theorem [41], provides a natural framework for extreme ranking analysis:

$$F(x) = \exp\left(-e^{-(x-\mu)/\beta}\right) \quad (6)$$

where μ and β represent location and scale parameters, respectively. Recent advances in EVT [42,43] have expanded its applicability to complex, dependent data structures common in ranking scenarios.

The Generalized Extreme Value (GEV) distribution encompasses three types of extreme value behavior through the shape parameter ξ : Gumbel ($\xi = 0$), Fréchet ($\xi > 0$), and Weibull ($\xi < 0$) distributions. For ranking applications, the Gumbel case is typically most appropriate as ranking distributions are neither heavy-tailed nor strictly bounded.

While EVT applications to ranking data have shown promise [42,43], existing approaches treat extremeness as an independent characteristic without considering its joint relationship with concordance and dispersion. This limitation prevents researchers from understanding how extreme rankings interact with overall system behavior and may lead to misinterpretation of extreme events as random outliers when they actually reflect systematic patterns.

2.3. Copula Theory in Dependence Modeling

2.3.1. Theoretical Foundations

Sklar's theorem [18] establishes that any multivariate distribution function can be expressed in terms of its marginal distributions and a copula function that captures their dependence structure:

$$F(x_1, \dots, x_d) = C(F_1(x_1), \dots, F_d(x_d)) \quad (7)$$

This fundamental result enables the separate modeling of marginal behaviors and dependence structures, providing flexibility in multivariate analysis [44,45].

The development of copula theory has been driven by recognition that traditional correlation measures fail to capture complex dependence structures, particularly in extreme scenarios where tail dependence and asymmetric relationships become important [25,26]. These limitations are particularly relevant to ranking analysis, where extreme agreements or disagreements often carry disproportionate importance in many decision-making contexts.

2.3.2. Copula Families for Ranking Applications

Different copula families capture distinct types of dependence relevant to ranking analysis:

The Gumbel copula, particularly relevant for ranking applications, emphasizes upper tail dependence:

$$C(u_1, \dots, u_d) = \exp \left(- \left(\sum_{i=1}^d (-\ln u_i)^\theta \right)^{1/\theta} \right) \quad (8)$$

This property makes Gumbel copulas especially suitable for ranking systems where extreme agreements or extreme scores are of primary concern. Elliptical copulas, including Gaussian and t-copulas [46], capture symmetric dependence but may inadequately model extreme co-movements that characterize ranking behavior.

2.3.3. Recent Advances in Copula Applications

Contemporary research has demonstrated copulas' effectiveness across diverse fields, including financial risk management [21,22], environmental modeling [23,24], and biostatistics [27,28]. Recent methodological advances include vine copulas for high-dimensional modeling [47,48], factor copulas for large-scale applications [49], and goodness-of-fit testing procedures [50–52].

However, direct application of copulas to ranking analysis remains limited, with most existing work focusing on pairwise rank correlations rather than comprehensive ranking system evaluation.

2.4. Comparison with Recent Multivariate Ranking and Copula-Based Methods

To contextualize CDEF's contribution, Table 1 compares our approach with recent related methods from 2022–2024 that employ copulas or address multivariate ranking problems. This comparison highlights the unique position of CDEF in addressing comprehensive ranking system evaluation.

Table 1. Comparison of CDEF with Recent Related Methods

Method	Year	Primary Focus	What It Does	CDEF's Unique Addition
Copula-Based Set Variant Association Test [53]	2023	Genetic association testing	Uses copulas to test association between genetic variants and bivariate phenotypes; handles mixed continuous/binary data	Joint ranking system evaluation: CDEF specifically models concordance, dispersion, and extremeness as interdependent ranking characteristics rather than testing genetic associations
Variable Ranking in Bivariate Copula Survival Models [54]	2023	Survival analysis variable selection	Applies copula-based variable ranking for bivariate time-to-event data with censoring	System-level analysis: CDEF evaluates ranking system integrity rather than selecting variables; models phantom concordance not visible in survival contexts
Hierarchical Copula Models for Clustered Data [55]	2025	Hierarchical data modeling	Uses D-vine copulas for hierarchical data with cluster-specific predictions	Ranking-specific framework: CDEF addresses ranking evaluation challenges (concordance artifacts, extremeness bias) not present in general hierarchical modeling
Copula Correction Methods [56]	2023	Endogeneity correction	Addresses regressor-error correlation using Gaussian copulas in econometric models	Dependence revelation: CDEF reveals hidden dependencies creating phantom properties rather than correcting for known endogeneity
Ranks, Copulas, and Permutons [57]	2024	Mathematical rank theory	Studies asymptotic properties of random permutations using copula connections	Applied evaluation: CDEF provides practical ranking system diagnostics rather than theoretical permutation analysis

This comparison demonstrates that CDEF occupies a distinctive methodological space, addressing problems that existing copula-based methods typically do not target and offering insights that extend current approaches.

2.5. Gaps in Current Literature and CDEF's Unique Contribution

Despite advances in both ranking analysis and copula methodology, our literature review did not reveal evidence that the joint modeling of ranking concordance, dispersion, and extremeness within a unified statistical framework has been performed. While copulas have found extensive application across diverse fields, their use in ranking analysis has been limited to specialized contexts that differ fundamentally from comprehensive ranking system evaluation.

The absence of unified frameworks represents a significant gap in current literature. Traditional approaches treat ranking characteristics independently, missing opportunities to leverage their joint distribution for enhanced inference and deeper understanding of ranking behavior [35,36]. This fragmentation has persisted despite theoretical reasons to expect strong dependencies among concordance, dispersion, and extremeness.

Existing copula applications to ranking contexts have focused on parameter estimation [60], algorithm development [61], or theoretical analysis [62], but none have addressed the practical problem of distinguishing genuine ranking system properties from artifacts of unmodeled dependencies.

Traditional ranking reliability measures [35,36] assume that high inter-rater correlation indicates genuine consensus, but this assumption can be flawed when rankers share systematic biases in dispersion and extremeness patterns. CDEF provides a new framework capable of detecting and quantifying what we term “phantom reliability,” apparent agreement largely explained by shared dependence among ranking properties.

The computational advances that make CDEF practically feasible represent another significant contribution. While computational challenges in high-dimensional copula modeling have been recognized and addressed through algorithmic innovations [63,64], CDEF's Monte Carlo approach provides a tractable implementation for this joint ranking framework, maintaining statistical rigor and enabling uncertainty quantification of dependence parameters.

CDEF addresses these gaps by providing the first theoretically grounded, computationally tractable framework specifically designed for joint ranking analysis. Our approach recognizes that ranking evaluation requires specialized treatment that accounts for the statistical properties of ranking data while leveraging modern advances in dependence modeling to capture the complex relationships that traditional methods overlook.

3. The Concordance-Dispersion-Extremity Framework (CDEF)

3.1. Conceptual Foundation

CDEF addresses the fundamental limitation of traditional ranking analysis: the independence assumption among ranking characteristics. By explicitly modeling the joint dependence structure of concordance, dispersion, and extremeness, CDEF provides a more complete statistical framework for ranking evaluation.

The framework rests on three key insights that distinguish it from existing approaches. First, ranking characteristics exhibit inherent dependence that cannot be ignored without loss of information [29]. Perfect concordance ($W = 1$) necessarily implies zero dispersion and extremeness, while maximum dispersion constrains both concordance and extremeness patterns. These mathematical relationships demonstrate that independence assumptions are not merely simplifying approximations, but misspecifications of the underlying statistical structure in many cases.

Second, ranking systems often exhibit stronger dependencies in extreme scenarios, such as unanimous top rankings or complete disagreement [19], making traditional linear correlation measures inadequate. The tail dependence properties that characterize ranking behavior require specialized modeling approaches that can capture asymmetric relationships and extreme co-movements.

Third, statistical conclusions drawn from individual ranking characteristics may be misleading when dependence structures are ignored [28], leading to inflated significance levels and incorrect interpretations. The joint probability framework provided by CDEF enables more robust inference and reveals insights that remain hidden under traditional analytical approaches.

3.2. Mathematical Formulation

3.2.1. Component Distributions

CDEF models three ranking characteristics using appropriate probabilistic frameworks selected to capture the distinct statistical properties of each component while enabling coherent joint modeling.

The concordance component employs Kendall's W transformed to a chi-square distribution [13]:

$$\chi^2 = m(n-1)W \sim \chi^2_{(n-1)} \quad (9)$$

This transformation provides a well-established probability distribution with known asymptotic properties, enabling standard statistical inference while preserving the interpretability of the original concordance measure.

The dispersion component requires careful selection based on the underlying dependence structure of the ranking process. For independent ranking assignments, the multinomial distribution provides an appropriate model for probability vector $\mathbf{p} = (p_1, \dots, p_k)$:

$$P(\mathbf{X} = \mathbf{x}) = \frac{N!}{\prod_{i=1}^k x_i!} \prod_{i=1}^k p_i^{x_i} \quad (10)$$

For dependent ranking assignments, the multivariate hypergeometric distribution offers a more suitable framework [38]:

$$P(\mathbf{X} = \mathbf{x}) = \frac{\binom{N_1}{x_1} \binom{N_2}{x_2} \dots \binom{N_k}{x_k}}{\binom{N}{n}} \quad (11)$$

where N_i represents the number of entities in category i , and $N = \sum_{i=1}^k N_i$.

The extremeness component utilizes the Gumbel distribution for modeling extreme deviations [41]:

$$F(x) = \exp\left(-e^{-(x-\mu)/\beta}\right) \quad (12)$$

This choice reflects the theoretical foundations of extreme value theory, which establish the Gumbel distribution as the appropriate limiting distribution for maxima of many common underlying distributions [41], making it particularly suitable for ranking applications where extreme values drive key insights.

3.2.2. Dependence Structure Selection

To determine the appropriate dispersion model, CDEF employs a chi-square test of independence:

$$\chi^2_{\text{obs}} = \sum_{i=1}^r \sum_{j=1}^c \frac{(O_{ij} - E_{ij})^2}{E_{ij}} \quad (13)$$

where O_{ij} and E_{ij} represent observed and expected frequencies under independence. The indicator function:

$$I = \begin{cases} 1 & \text{if } P(\chi^2 \geq \chi^2_{\text{obs}}) < \alpha \\ 0 & \text{otherwise} \end{cases} \quad (14)$$

determines whether to use the hypergeometric ($I = 1$) or multinomial ($I = 0$) distribution for dispersion modeling.

3.2.3. Copula Specification

CDEF employs a Gumbel copula to model the joint dependence structure. For three variables representing concordance (U_1), dispersion (U_2), and extremeness (U_3), the copula is:

$$C(u_1, u_2, u_3; \theta) = \exp\left(-\left[(-\ln u_1)^\theta + (-\ln u_2)^\theta + (-\ln u_3)^\theta\right]^{1/\theta}\right) \quad (15)$$

where $\theta \geq 1$ is the dependence parameter.

The choice between multinomial and hypergeometric distributions for the dispersion component leads to two distinct copula formulations. When the independence test indicates dependent ranking assignments ($I = 1$), we employ the hypergeometric-based formulation:

$$C_{\text{hyper}}(u_1, u_2, u_3; \theta) = \exp\left(-\left[(-\ln u_1)^\theta + (-\ln u_2^{\text{hyper}})^\theta + (-\ln u_3)^\theta\right]^{1/\theta}\right) \quad (16)$$

When the independence test suggests independent ranking assignments ($I = 0$), we use the multinomial-based formulation:

$$C_{\text{multi}}(u_1, u_2, u_3; \theta) = \exp\left(-\left[(-\ln u_1)^\theta + (-\ln u_2^{\text{multi}})^\theta + (-\ln u_3)^\theta\right]^{1/\theta}\right) \quad (17)$$

This approach maintains theoretical consistency by ensuring that the appropriate distributional assumption for dispersion is reflected in the copula specification, while avoiding the mathematical complexity of mixing distribution types within a single expression. While the framework captures joint dependence, researchers should be aware that moderate collinearity among transformed concordance, dispersion, and extremeness variables may affect parameter identifiability in small samples. Diagnostic procedures are provided to assess model adequacy under these conditions.

3.2.4. Parameter Estimation

The dependence parameter θ relates to Kendall's tau through:

$$\theta = \frac{1}{1 - \tau} \quad (18)$$

For ranking data, we estimate τ from the empirical ranking correlations, providing a data-driven approach to copula parameterization.

3.3. Transformation to Uniform Scale

Since copulas require uniform marginal distributions, CDEF employs rank-based transformations to treat the discrete ranking data to the continuous uniform interval required for copula modeling. We denote the transformed rank variable as $U_i^{(n)}$:

$$U_i^{(n)} = \frac{\text{rank}(X_i) - 0.5}{n} \quad (19)$$

This transformation requires careful justification because it bridges discrete ranking data with continuous copula theory. A naive transformation using $\text{rank}(X_i)/n$ creates several critical problems. The 0.5 correction solves these problems by centering each rank within its probability interval. This approach is motivated by principles of empirical copula estimation [60]. From order statistics theory [65], for the r -th order statistic from a sample of size n drawn from $\text{Uniform}(0, 1)$, the expected value is $\mathbb{E}[U_{(r)}] = r/(n+1)$. The transformation $(r - 0.5)/n$ approximates this expected value for large n , providing consistency with established order statistics theory.

3.4. Estimation and Algorithm

Given the analytical intractability of the resulting joint distribution, CDEF employs Monte Carlo integration [64]. The general algorithm follows.

1. Parameter Estimation:

- Compute Kendall's W from observed rankings
- Estimate θ from empirical rank correlations

- Conduct chi-square independence test to determine I
- 2. **Marginal Sampling:**
 - Generate concordance samples from $\chi^2_{(n-1)}$
 - Generate dispersion samples from selected distribution
 - Generate extremeness samples from Gumbel distribution
- 3. **Copula Transformation:**
 - Transform samples to uniform scale using empirical CDFs
 - Apply Gumbel copula with estimated θ
- 4. **Joint Probability Estimation:**
 - Estimate $P(X_1 \leq x_1, X_2 \leq x_2, X_3 \leq x_3)$
 - Compute conditional probabilities via ratio estimation
- 5. **Uncertainty Quantification:**
 - Calculate confidence intervals using bootstrap resampling
 - Assess convergence through sample size sensitivity analysis

3.5. Model Validation and Diagnostics

Goodness-of-fit testing employs Cramér-von Mises and Kolmogorov-Smirnov tests specifically adapted for copula functions [50–52]. Residual analysis examines pseudo-observations from the fitted copula for patterns indicating model misspecification [50]. Cross-validation procedures support leave-one-out and k-fold cross-validation for predictive performance assessment.

4. Materials and Methods: Empirical Application

To demonstrate CDEF's practical utility, we analyze pre-season college football rankings from four major polling organizations: Associated Press Poll, Coaches Poll, Congrove Computer Rankings, and ESPN Power Index [66–69]. These rankings represent a complex multi-rater system where human judgment, statistical models, and algorithmic approaches converge.

5. Results

5.1. Univariate Analysis Results

Traditional univariate analysis of the NCAA football ranking data reveals what appears to be a highly structured and coherent ranking system. Kendall's W yielded a value of 0.717, indicating substantial agreement among the four polling organizations. This level of concordance suggests that despite different methodologies—ranging from human judgment in the AP and Coaches polls to algorithmic approaches in the computer rankings—there exists considerable consensus about team quality and relative rankings.

The associated chi-square statistic of 128.889 ($p < 0.001$) provides strong evidence for rejecting the null hypothesis of random rankings. This result indicates that the observed level of agreement among polling organizations far exceeds what would be expected if rankings were assigned randomly, supporting the interpretation that the rankings reflect genuine underlying differences in team quality.

Dispersion analysis using chi-square independence testing yielded a p-value less than 0.001, indicating that ranking assignments exhibit significant dependence rather than independence. This finding warranted the use of the multivariate hypergeometric distribution for modeling concentration patterns. The resulting probability assessment, also less than 0.001, suggests a significant departure from uniform rank distribution, indicating that certain rank positions are assigned with much greater frequency than others.

Extremeness analysis using the Gumbel distribution framework revealed significant tail behavior ($p < 0.001$), indicating that extreme rankings, both very high and very low, occur more frequently than

would be expected under independence assumptions. This finding suggests systematic tendencies toward extreme evaluations that cannot be explained by random variation alone.

5.2. CDEF Analysis Results

Application of the CDEF framework reveals a markedly different interpretation when ranking characteristics are modeled jointly rather than independently. The joint probability of simultaneously observing the levels of concordance, dispersion, and extremeness found in the data was less than 0.001, confirming that this combination of ranking characteristics is highly unlikely under independence assumptions.

The conditional probability analysis provides the most revealing insights into the true structure of the ranking system. When conditioned on the observed levels of dispersion and extremeness, the probability of achieving the observed level of concordance drops dramatically to 0.091. This result suggests that much of the apparent agreement among polling organizations can be explained by common patterns in how they distribute rankings and their shared tendencies toward extreme evaluations, rather than genuine consensus about team quality.

Similarly, the conditional probability of dispersion given concordance and extremeness equals 0.196, indicating that concentration patterns in rank assignments are not as strongly driven by the other ranking characteristics as univariate analysis might suggest. This finding implies that dispersion represents a somewhat independent dimension of ranking behavior that contributes unique information beyond what can be inferred from concordance and extremeness alone.

The conditional probability of extremeness given concordance and dispersion yields the highest value at 0.713, suggesting that extreme ranking tendencies represent the most robust and persistent characteristic of the system. Even after controlling for agreement patterns and dispersion structures, the tendency toward extreme evaluations remains pronounced, indicating that this may be an intrinsic feature of the ranking process rather than a byproduct of other ranking behaviors.

The estimated Gumbel copula parameter $\theta = 5.638$ indicates strong upper tail dependence, confirming that extreme values in one ranking characteristic are substantially more likely to co-occur with extreme values in others. This dependence structure validates the choice of the Gumbel copula and demonstrates that the most informative aspects of ranking behavior occur in the upper tail region where traditional linear correlation measures would be least effective. If lower tail dependence were also of concern, alternative copula families should be considered.

5.3. Comparative Interpretation: Revealing Phantom Concordance

The contrast between univariate and multivariate results demonstrates CDEF's unique capability to expose fundamental misinterpretations that traditional methods may not detect. Traditional univariate analysis of the NCAA football rankings would suggest exceptional reliability and structure: with Kendall's $W = 0.717$ and highly significant test statistics across all measures, researchers would typically conclude there is strong consensus among polling organizations and meaningful dispersion and extremeness. However, CDEF's joint modeling reveals that this interpretation is largely illusory. When ranking characteristics are evaluated in their proper multivariate context, the conditional probability of concordance given dispersion and extremeness falls to just 9.1%, indicating that most apparent agreement reflects shared patterns of rank dispersion and tendencies toward extreme evaluations rather than genuine consensus about team quality. This phenomenon—"phantom concordance"—is invisible to traditional approaches because they lack the mathematical framework needed to separate true agreement from dependence-driven artifacts.

Further, CDEF uncovers a hierarchy of dependencies undetectable in univariate analyses. While concordance collapses under conditioning, extremeness retains a conditional probability of 71.7%, suggesting that systematic tendencies toward extreme evaluations are a more fundamental characteristic of the ranking process. The identification of strong upper tail dependence ($\theta = 5.638$) further demonstrates that extreme scenarios drive ranking behavior in ways that linear correlation measures

cannot capture. Together, these results show that reliance on traditional correlation-based methods can lead to misleading conclusions about ranking system reliability.

5.4. Quantifying the Cost of Ignoring Dependencies

To illustrate the consequences of assuming independence among ranking characteristics, we quantified the discrepancy between traditional analysis and CDEF. Traditional methods would suggest that the NCAA football ranking system exhibits high reliability, with approximately 71.7% concordance among polling organizations. However, CDEF reveals that when dependencies are modeled jointly, the probability of genuine agreement drops to just 9.1%. This striking reduction demonstrates that most apparent consensus reflects shared systematic biases in dispersion and extremeness rather than true consensus on team quality. Decisions based on traditional analysis therefore may overestimate ranking system trustworthiness, creating a false sense of validation that may be entirely unfounded.

These findings may have important implications for high-stakes applications such as playoff selection and research funding allocation. Under traditional analysis, stakeholders might confidently rely on ranking outputs as objective indicators of quality. In contrast, CDEF shows that such confidence is misplaced: when extremeness retains a conditional probability of 71.7% while concordance collapses to 9.1%, it becomes clear that extreme evaluations are a more robust and systematic feature of the process than genuine agreement. This insight fundamentally changes the interpretation of ranking data, suggesting that reliance on univariate measures can lead to substantial misallocation of resources and overconfidence in ranking-based decisions.

6. Discussion

6.1. Theoretical Contributions

CDEF contributes a structured framework for evaluating rankings by explicitly modeling dependencies among concordance, dispersion, and extremeness. While copula theory and extreme value modeling are established methods, their integration for joint ranking evaluation provides new perspectives on how apparent reliability may arise from systematic dependencies rather than genuine consensus. The concept of “phantom concordance”—in which high inter-rater agreement dissolves when conditional relationships are accounted for—illustrates the need to reassess how ranking reliability is defined and measured. Moreover, the identification of hierarchical dependence structures, where extremeness persists as a more stable property than concordance or dispersion, offers a refined theoretical lens for understanding ranking system behavior.

6.2. Methodological Advances

Methodologically, CDEF combines chi-square-based selection between multinomial and hypergeometric dispersion models with a Gumbel copula to capture dependence. This automated model selection reduces arbitrary assumptions and aligns distributional choices with observed data structures, supporting valid inference without requiring extensive manual specification.

6.3. Practical Applications

CDEF's capacity to distinguish between genuine and phantom ranking properties has practical relevance across domains such as peer review, sports analytics, and performance evaluation. In peer review, for example, it can help identify whether apparent reviewer agreement primarily reflects shared tendencies toward extreme ratings or true consensus about merit. In sports contexts, it offers tools to detect coordinated ranking manipulation that traditional univariate measures might overlook. Similarly, in business evaluation, CDEF can clarify whether inter-rater agreement arises from systematic biases in dispersion and extremeness rather than reliable assessment of performance. By modeling tail behavior explicitly, the framework can also enhance risk management practices where extreme values have disproportionate impact.

6.4. Limitations and Future Research

Several limitations suggest areas for further development. The framework relies on parametric marginal distributions and the Gumbel copula structure, which may not suit all ranking contexts. Future research could explore fully nonparametric alternatives or develop procedures to select among broader classes of dependence models. Extending the approach to higher-dimensional settings via vine copula constructions [47,48] could enable modeling of additional characteristics such as temporal consistency or rater expertise but would require careful management of computational complexity. Dynamic copula models [70] may also offer opportunities to capture evolving dependence structures in longitudinal ranking systems. Despite these challenges, the present formulation provides a foundation for more nuanced evaluation of ranking systems than traditional approaches can offer.

7. Conclusions

7.1. Key Contributions

This research introduced the Concordance-Dispersion-Extremity Framework (CDEF), a copula-based approach for ranking analysis that models concordance, dispersion, and extremeness jointly rather than treating them as independent properties. While each component draws on established statistical techniques, CDEF's principal contribution lies in demonstrating how traditional methods can overstate ranking system reliability by overlooking dependencies among ranking characteristics. In the NCAA football rankings, for example, CDEF revealed that apparent consensus (Kendall's $W = 0.717$) could be largely explained by shared patterns of dispersion and extremeness, with the conditional probability of genuine concordance falling to 0.091. This distinction between genuine and phantom agreement illustrates the value of joint probability modeling for more credible assessment of ranking system integrity.

The framework also clarifies that extremeness may persist as a more stable property than concordance, suggesting that ranking systems often display systematic evaluation patterns that conventional correlation-based approaches do not capture. Explicit modeling of tail dependence provides additional insight into how extreme cases shape overall ranking behaviors, informing both theory and practice.

7.2. Implications for Practice

CDEF can support the design of ranking systems that more accurately reflect underlying evaluation quality by distinguishing genuine agreement from artifacts of shared biases. Applications range from peer review and sports analytics to business performance evaluation and risk management, where understanding dependence structures can help detect systematic distortions and improve the fairness and credibility of ranking-based decisions.

7.3. Future Research Directions

Several extensions remain to be explored. Dynamic copula models could capture evolving dependence in longitudinal data, and higher-dimensional vine copula constructions could accommodate additional characteristics such as temporal consistency or rater expertise. Nonparametric alternatives and more efficient estimation techniques, including variational inference and adaptive sampling, may further improve robustness and scalability. These directions would expand the framework's applicability to large-scale digital and streaming ranking systems.

7.4. Final Remarks

By integrating dependence modeling into ranking evaluation, CDEF provides a more nuanced approach to assessing consensus and bias than traditional univariate methods allow. As ranking systems continue to influence important decisions across diverse fields, frameworks that account for the complexity of joint ranking behaviors will be increasingly valuable for supporting robust, transparent, and reliable inference.

Author Contributions: Conceptualization, L.F.; methodology, L.F. ; software, L.F. and A.S.; validation, R.S.; formal analysis, L.F. and R.S.; writing—original draft preparation, L.F., A.S., A.T., and R.S.; writing—review and editing, L.F., A.S., A.T., and R.S.

Funding: This research received no external funding.

Data Availability Statement: The NCAA football ranking data used in this study are publicly available from the respective polling organizations. Data are also available at <https://github.com/dustoff06/FERP>.

Conflicts of Interest: The authors declare no conflicts of interest.

Generative AI During the preparation of this work, the authors used ChatGPT 4.5 in order to edit prose and format the paper. The authors used Claude.ai for programming support in Python. After using this tool/service, the authors reviewed and edited the content as needed and take full responsibility for the content of the publication

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