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Sergey I. Kruglov *

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Exponential Scale Factor, $F(T)$ Teleparallel Gravity, Entropy of Apparent Horizon and Cosmology

Sergey I. Kruglov [1,2]

[1] Department of Physics, University of Toronto, 60 St. Georges St., Toronto, ON M5S 1A7, Canada; kruglov@rogers.com
[2] Canadian Quantum Research Center, 204-3002 32 Ave., Vernon, BC V1T 2L7, Canada

Abstract

The equation for the function $F(T)$ within the teleparallel gravity with torsion field T which provides the exponential scale factor is obtained and the function $F(T)$ was computed. It is shown that the deceleration parameter $q_0 \approx -0.535$ according to the Planck data at the current epoch, can not be realized for cosmology based on the exponential scale factor. Therefore, models of cosmology based on the exponential scale factor (studied in many papers) is ruled out. In the framework of entropic cosmology, the associated entropy was found.

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1. Introduction

The standard cosmological model (SCM) of Big Bang successfully describes the universe evolution from an early phase (inflation) to the present large-scale structure. The observational data of the cosmic microwave background (CMB) are conformed by SCM, and predictions of primordial nucleosynthesis corresponds to observed light element abundances. The large-scale distribution of galaxies also is conformed by SCM. Despite these successes of SCM there is a problem of the presence of an initial space-time singularity where physical quantities such as curvature and density of energy become infinity. Such singularity tells us not only a mathematical inconvenience but about the shortcoming of our physics understanding. Quantum gravitational effects become dominant at the Planck scale, but a complete theory of quantum gravity is not developed yet. To resolve the problem of the initial singularity, new alternative cosmological theories beyond the Einstein theory such as $F(R)$ gravity, teleparallel $F(T)$ gravity, teleparallel $F(Q)$ gravity based on non-metricity and others were appeared. One of the approaches is a cyclic cosmology where the universe undergoes a transition from a phase of contraction to an expanding phase leading to a universe that oscillates. In the bouncing cosmology the initial singular point is replaced by a nonsingular bounce, where the scale factor has a minimum value and then undergoes its evolution. Thus, the problem of the initial singularity within the SCM of the universe evolution can be solved in the bouncing or cyclic cosmology scenarios [1,2]. Various approaches of bouncing (cyclic) cosmology were investigated in [3–10]. The generalized scenarios of a periodic sequence of the universe contractions and expansions were considered in [11].

The $F(T)$ teleparallel gravity with the Weitzenböck connection [12] can explain the acceleration of the universe unlike the Einstein gravity. The $F(R)$ gravity leads to the fourth order equations but the field equations in the $F(T)$ gravity [14] are the second order that is an advantage. The space-time in the $F(T)$ gravity possesses the torsion T but not the curvature. In a bouncing scenario the Hubble parameter H is negative for the contracting phase before the bounce but in the expanding phase after the bounce the Hubble parameter H is positive. According to the continuity equation at the point of the bounce $H = 0$. In such scenario, throughout the transition from contracting to expanding we have $\dot{H} > 0$, but for the transition from expansion to contraction $\dot{H} < 0$. Here, we will consider the scale factor $a(t) = a_0 \exp(\beta t^2)$, derive the Hubble parameter $H(t)$ and compute the deceleration parameter

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q. Then the associated $F(T)$ gravity and entropic cosmology will be studied. We calculate the $F(T)$ function and entropy within boundary thermodynamics. We will use the units with $c = \hbar = 1$.

2. $F(T)$ Teleparallel Gravity

We consider here the Friedmann–Lemaître–Robertson–Walker (FLRW) spatially flat universe with the metric

$$ds^2 = -dt^2 + a(t)^2(dx^2 + dy^2 + dz^2). \tag{1}$$

The cosmological assumptions include the homogeneity and isotropy. We use the scale factor $a(t)$ in the exponential form

$$a(t) = a_0 \exp(\beta t^2). \tag{2}$$

For a bouncing cosmology a_0 is the scale factor at the bouncing point when $t = 0$, and β is a positive parameter. It will be shown that this approach possesses the analytic solutions. The cosmic time t belongs to the interval $(-\infty, +\infty)$. The Hubble parameter is given by $H = \dot{a}/a$, and we obtain $H(t) = 2\beta t$. The condition for a contraction of the universe is $\dot{H} < 0$ for $t < 0$ before the bounce, but it is not satisfied as $\dot{H} = 2\beta > 0$.

Let us consider the deceleration parameter with is defined by $q = -1 - \dot{H}/H^2$. At $q < 0$ the acceleration phase of the universe occurs and at $q > 0$ one has the deceleration phase. Making use of Eq. (2) we obtain $q = -1 - 1/(2\beta t^2 < 0$ for any parameters β and the cosmic time t. Thus, we have only the acceleration phase of the universe but for the bouncing cosmology we need also the deceleration phase of the universe. As a result, the scale factor (2) does not lead to a bouncing scenario of the universe. According to the Planck data [13] the value of the deceleration parameter at the current era is $q_0 = -0.535$. One easily obtains $-1 - 1/(2\beta t^2 < -1$, and therefore $q \neq -0.537$ for any parameter β and cosmic time t. As a result, the model of cosmology based on the scale factor (2) (considered in many papers) is not realistic. Nevertheless, we will study the question: what kind a gravity is realised for the scale factor (2)? Because the Einstein theory of general relativity does not lead to the acceleration of the universe, we will discuss the teleparallel theory of gravity.

Within Teleparallel Equivalent to General Relativity (TEGR), instead of the curvature the torsion is introduced. In such theory of gravity the dynamics is similar to the dynamics of the general relativity theory. Vierbein (tetrad) fields e^A_μ in TEGR describe the geometry of space-time and define a torsion tensor. Tetrad fields are the source of gravity and there is the antisymmetric contribution to the Christoffel connection. The torsion tensor defines the torsion scalar T which enters the gravitational action. In $F(T)$ gravity the Lagrangian density of TEGR is modified by using an arbitrary function of the torsion scalar. The space-time metric $g_{\mu\nu}$ is constructed from tetrads as $g_{\mu\nu} = \eta_{AB}e^A_{\ \mu}e^B_{\ \nu}$, where $\eta_{AB} = \mathrm{diag}(-1, +1, +1, +1)$ being the Minkowski metric in the local frame and indexes A and B label the orthonormal frame and μ, ν label space-time coordinates. In the teleparallel gravity the Weitzenböck connection is used, $\Gamma^\lambda_{\ \mu\nu} = e_A^{\ \lambda}\partial_\nu e^A_{\ \nu}$. In the Weitzenböck connection the Riemann curvature tensor vanishes ($R_{\alpha\beta\mu\nu} = 0$). The torsion tensor is defined as $T^\rho_{\ \mu\nu} = e_A^{\ \rho}\left(\partial_\mu e^A_{\ \nu} - \partial_\nu e^A_{\ \mu}\right)$, and the superpotential tensor is $S_\rho^{\ \mu\nu} = \frac{1}{2}\left(K^{\mu\nu}_{\ \ \rho} + \delta^\mu_\rho T^{\alpha\nu}_{\ \ \alpha} - \delta^\nu_\rho T^{\alpha\mu}_{\ \ \alpha}\right)$, where $K^{\mu\nu}_{\ \ \rho} = -\frac{1}{2}\left(T^{\mu\nu}_{\ \ \rho} - T^{\nu\mu}_{\ \ \rho} - T_\rho^{\ \mu\nu}\right)$ is the contorsion tensor. The torsion field T is given by $T = S_\rho^{\ \mu\nu}T^\rho_{\ \mu\nu}$. For FLRW metric (1) $e^A_\mu = \mathrm{diag}(1, a, a, a)$ and the torsion is $T = -6H^2$ [14]. From the Hubble parameter $H(t) = 2\beta t$ we obtain the torsion $T = -24\beta^2 t^2$ and the function $t(T) = \pm(1/(2\beta))\sqrt{-T/6}$. Within teleparallel gravity, one finds the Friedmann equation [14]

$$\frac{1}{6}\left[F(T) - 2TF'(T)\right] = \left(\frac{8\pi G}{3}\right)\rho, \tag{3}$$

where ρ is the matter density. By virtue of the continuity equation

$$\dot{\rho} = -3H(\rho + p), \tag{4}$$

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and utilizing Eq. (2) at $a_0 = 1$, we obtain the density of the matter

$$\rho = \rho_0 \exp(-3(1+w)\beta t^2), \tag{5}$$

where ρ_0 is the matter density at the time $t = 0$, and $w = p/\rho$ being the equation of state (EoS) for the matter and p is the matter pressure. For the matter in the form of a dust $w = 0$ and for a radiation field $w = 1/3$. It is convenient also to introduce the redshift $z = 1/a(t) - 1$. Then we obtain the density energy as $\rho = \rho_0(1+z)^{3(1+w)}$. In the presence of the matter as a dust ($w = 0$) and a radiation field ($w = 1/3$) we have $\rho = \rho_m + \rho_r = \rho_{m0}(1+z)^3 + \rho_{r0}(1+z)^4$. The ρ_{m0} corresponds to the energy density of the matter at the current era, $z = 0$, and ρ_{r0} denotes the energy density of radiation at the current era.

By virtue of Eqs. (4) and (5), equation (3) becomes

$$F'(T) - \frac{F}{2T} = -\frac{\rho_0}{TM_{Pl}^2} \exp\left(\frac{(1+w)T}{8\beta}\right), \tag{6}$$

where $M_{Pl} = 1/\sqrt{8\pi G}$ is the reduced Planck mass. The solution to Eq. (6) is

$$F(T) = \frac{2\rho_0}{M_{Pl}^2}\left[\exp\left(\frac{(1+w)T}{8\beta}\right) - \sqrt{\frac{-(1+w)T}{8\beta}}\,\Gamma\left(\frac{1}{2}, -\frac{(1+w)T}{8\beta}\right)\right], \tag{7}$$

where $\Gamma(a, x)$ is the incomplete gamma function.

3. Dark Energy

Models of dark energy have to be in agreement with the Hubble horizon for the flat FLRW background. Therefore, we consider the generalized Friedmann equation which is given by

$$H^2 = \frac{8\pi G}{3}(\rho + \rho_D), \tag{8}$$

where ρ_D is the dark energy density. Making use of Eqs. (5), (8) and Hubble parameter $H = 3\beta t$, we obtain the dark energy density

$$\rho_D = \frac{3}{8\pi G}H^2 - \rho = \frac{3}{2\pi G}\beta^2 t^2 - \rho_0 \exp(-3(1+w)\beta t^2). \tag{9}$$

By virtue of the continuity equation (ordinary conservation law) for dark energy $\dot{\rho_D} = -3H(\rho_D + p_D)$ (assuming that there are not mutual interactions between the components of the cosmos) and Eq. (8), we obtain the second generalized Friedmann equation as follows:

$$H^2 + \frac{2}{3}\dot{H} = -\frac{8\pi G}{3}(p + p_D). \tag{10}$$

From Eq. (10) we find the pressure p_D corresponding to dark energy

$$p_D = -\frac{3\beta}{2\pi G}\left(\beta t^2 + \frac{1}{3}\right) - w\rho_0 \exp(-3(1+w)\beta t^2). \tag{11}$$

Equations (9) and (11) allow us to obtain EoS of dark energy $w_D = p_D/\rho_D$ and the sound velocity $v_s^2 = dp_D/d\rho_D$. When the requirement $0 < v_s^2 < 1$ is satisfied the model of dark energy is stable.

One can introduce the normalized density parameter of the matter $\Omega_m = \rho/(3M_P^2H^2)$ and the normalized density parameter of dark energy $\Omega_D = \rho_D/(3M_P^2H^2)$, with $M_P = 1/\sqrt{8\pi G}$ being the reduced Planck mass. Then Eq. (8) becomes $\Omega_m + \Omega_D = 1$. According to the Planck data for the current era $\Omega_{m0} \approx 0.315$ [13].

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4. Entropic Cosmology

The modification of the standard approach is the holographic principle. According to holographic principle the degrees of freedom of a spatial region are in the boundary but not in the bulk. Therefore, in a certain sense the world is two-dimensional but not three dimensional as supposed [15]. The holography based on the apparent horizon obeys the first law of thermodynamics. Let us consider the apparent horizon thermodynamics with the exponential scale factor (2). The unified first law of thermodynamics can be used to the apparent horizon of the FLRW universe which corresponds to a system of non-equilibrium thermodynamics. Here, the fundamental Clausius relation $\delta Q = TdS$ in the equilibrium thermodynamics, where T is interpreted as the Unruh temperature seen by an accelerated observer just inside the horizon [16], is generalized. The radius of the apparent horizon is defined by $R_h = 1/H = 1/(2\beta t$. For FLRW spatially flat universe the apparent horizon radius is equal to the Hubble horizon radius. The apparent horizon in FLRW universe represents a proper causal boundary [17] and the laws of thermodynamics are satisfied on this boundary [18]. The Hubble horizon can be considered as the IR cutoff [19]. There is the mutual relation between the UV cutoff and the entropy of the system [20]. The first law of apparent horizon thermodynamics reads

$$dE = -T_h dS_h + WdV_h, \tag{12}$$

where S_h represents the apparent horizon entropy. The work density is given by $W = (\rho - p)/2$ [17,21] and $E = (4\pi/3)\rho R_h^3$ defines the total energy inside the space. The work density at the apparent horizon corresponds to the work done by a change of the apparent horizon. The energy flux passes through the apparent horizon ($-dE$ denotes the amount of energy crossing the apparent horizon during an infinitesimal time interval), $dE = -dQ$, and the heat flow dQ is related to the change of energy inside the apparent horizon E. The apparent horizon temperature is given by [22]

$$T_h = \frac{H}{2\pi}\left|1 + \frac{\dot{H}}{2H^2}\right|. \tag{13}$$

By virtue of first law of apparent horizon thermodynamics (12), and making use of Eqs. (13) and (4) we find [23,24]

$$\dot{S}_h = -\frac{8\pi^2\dot{\rho}}{3H^4}, \tag{14}$$

where the dot over the variable denotes the derivative with respect of the time. From Eq. (5) by integrating Eq. (14) we obtain the entropy

$$S_h = \frac{\pi^2(1+w)\rho_0}{\beta^3}\int \frac{\exp(-3(1+w)\beta t^2)}{t^3}dt$$
$$= -\frac{\pi^2(1+w)\rho_0}{2\beta^3}\left[3(1+w)\beta Ei(-3(1+w)\beta t^2) + \frac{\exp(-3(1+w)\beta t^2)}{t^2}\right], \tag{15}$$

where $Ei(x)$ is the exponential integral. Note that $Ei(-x) = -\Gamma(0,x)$ for $x > 0$. Equation (15) defines the apparent horizon entropy for arbitrary parameters β and w. The entropy (15) of our model of dark energy with the exponential scale factor, represents non-extensive entropy.

5. Summary

In summery, we analytically have been studied the evolution of universe in a model with the exponential scale factor leading to dark energy which does not have any mutual interactions. The equation for the function $F(T)$ within the teleparallel gravity with torsion field T corresponding to the exponential scale factor was obtained. The entropic cosmology with entropy (15) corresponds to the teleparallel gravity with the Lagrangian function F(T). Thus, the bouncing cosmology as well as standard Big Bang cosmology is ill defined in the scenario with the exponential scale factor. It was shown that the deceleration parameter according to the Planck data $q_0 \approx -0.535$ at the current epoch is not reproduced for any parameter β within exponential scale factor. Therefore, models of cosmology (studied in many papers) based on the exponential scale factor are ruled out. The realistic bouncing model of cosmology was studied in [25].

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