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Article

Lucas-Balancing Polynomials Within ρ -bi-Pseudo-Starlike Functions: Analytical Estimates

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Abstract

In this study, we introduce a novel class of ρ -bi-pseudo-starlike functions, constructed based on symmetric points associated with the boundary constraints of the initial Taylor-Maclaurin coefficients. A comprehensive analysis of the Fekete-Szegő problem is conducted for functions belonging to this class. Furthermore, we identify special cases of our main findings, demonstrating significant improvements in the estimation of the initial Taylor-Maclaurin coefficients. The results presented in this work contribute to the broader understanding of bi-univalent function theory and provide new insights into function coefficient bounds in complex analysis.

Keywords: bi-univalent functions; symmetric starlike functions; Lucas-Balancing polynomials; subordination; Fekete–Szegő functional

MSC: 30C45, 30C50

1. Introduction and Motivation

The study of bi-univalent functions plays a fundamental role in geometric function theory and has been a vibrant area of research due to its wide-ranging applications in complex analysis, function approximation, and mathematical modeling. As an extension of the classical theory of univalent functions, bi-univalent functions naturally arise in contexts where both a function and its inverse are required to be univalent within the unit disk. Among the various topics of interest, the estimation of coefficient bounds remains one of the most significant and challenging problems in this field, as it directly influences the Fekete–Szegő inequality and provides valuable insights into the structural properties of the Taylor–Maclaurin series expansions associated with these functions.

In this study, we introduce a novel class of analytic bi-univalent functions by employing the framework of subordination and incorporating Lucas Balancing polynomials. The primary motivation behind this new construction is to generalize existing subclasses within geometric function theory by unifying the concept of pseudo-starlike functions and the behavior with respect to symmetric points. This new approach aims to create a more flexible and comprehensive structure for analyzing bi-univalent functions, thereby deepening our understanding of their analytic and geometric characteristics.

By utilizing the special properties of Lucas Balancing polynomials, we establish improved coefficient estimates and extend some classical results for univalent functions to the bi-univalent setting. These findings not only contribute to the ongoing research on coefficient bounds but also provide new avenues for developing advanced approximation techniques and further applications within complex analysis.

Let $\mathcal{H}$ denote the class of functions that are analytic in the open unit disk:

$$\mathcal{U} = \{z \in \mathbb{C} : |z| < 1\}.$$

The function classes $\mathcal{A}$ and $\mathcal{S}$ are defined as follows:

$$\mathcal{A} = \{\Phi \in \mathcal{H} : \Phi(0) = \Phi'(0) - 1 = 0\}, \mathcal{S} = \{\Phi \in \mathcal{A} : \Phi \text{ is univalent in } \mathcal{U}\}.$$

Here, functions in the class $\mathcal{A}$ are referred to as normalized analytic functions, the class $\mathcal{S}$ is a subclass of $\mathcal{A}$, consisting of univalent (i.e., one-to-one) functions in the unit disk. Thus, every function $\Phi \in \mathcal{A}$ can be expressed as the Taylor-Maclaurin series:

$$\Phi(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad z \in \mathcal{U}. \quad (1)$$

The class $\mathcal{P}$, widely recognized as the Carathéodory class of functions, is defined as following:

$$\mathcal{P} = \{p \in \mathcal{H} : p(0) = 1, \Re(p(z)) > 0, z \in \mathcal{U}\}$$

with the series expansion

$$p(z) = 1 + p_1 z + p_2 z^2 + \dots.$$

The class of Schwarz functions is denoted by Ω and is defined by

$$\Omega = \{\omega \in \mathcal{H} : \omega(0) = 0, |\omega(z)| < 1 \text{ for all } z \in \mathcal{U}\}.$$

The concept of subordination is fundamental in analyzing the geometric and analytic behavior of various subclasses of analytic and univalent functions. Originally introduced by Lindelöf [1], the framework was further developed by Rogosinski [2,3] and Littlewood [4].

Let $\Phi_1, \Phi_2 \in \mathcal{H}$. Then Φ_1 is said to be *subordinate* to Φ_2 in $\mathcal{U}$, denoted by

$$\Phi_1 \prec \Phi_2 \quad \text{or} \quad \Phi_1 \prec \Phi_2,$$

if there exists a Schwarz function ω such that

$$\Phi_1 = \Phi_2(\omega(z)), \quad z \in \mathcal{U}.$$

In particular, if Φ_2 is univalent in $\mathcal{U}$, then the subordination $\Phi_1 \prec \Phi_2$ is equivalent to

$$\Phi_1(0) = \Phi_2(0) \quad \text{and} \quad \Phi_1(\mathcal{U}) \subset \Phi_2(\mathcal{U}).$$

In the context of geometric function theory, the notion of *starlike functions* and convex functions plays a fundamental role. The class of starlike functions of order α , $\mathcal{S}^*(\alpha)$ and the class of convex functions order (α) , $\mathcal{C}(\alpha)$ in $\mathcal{U}$, are defines as [5]:

$$\mathcal{S}^*(\alpha) = \left\{ \Phi \in \mathcal{S} : \Re\left(\frac{z\Phi'(z)}{\Phi(z)}\right) \geq \alpha, \quad z \in \mathcal{U}, \quad 0 \leq \alpha < 1 \right\}$$

and

$$\mathcal{C}(\alpha) = \left\{ \Phi \in \mathcal{S} : \Re\left(\frac{1+z\Phi''(z)}{\Phi'(z)}\right) \geq \alpha, \quad z \in \mathcal{U}, \quad 0 \leq \alpha < 1 \right\}$$

In the special case $\alpha = 0$, we write $\mathcal{S}^* := \mathcal{S}^*(0)$ and $\mathcal{C} = \mathcal{C}(0)$, which denotes the classical class of starlike functions and convex functions, respectively. It is well known that $\mathcal{S}^*, \mathcal{C} \subset \mathcal{S}$ and these functions are univalent in $\mathcal{U}$ (see [5–7]).

These classes also admit a subordination-based characterization:

$$\mathcal{S}^* = \left\{ \Phi \in \mathcal{S} : \frac{z\Phi'(z)}{\Phi(z)} \prec \frac{1+z}{1-z}, \quad z \in \mathcal{U} \right\}$$

and

$$\mathcal{C} = \left\{ \Phi \in \mathcal{S} : \frac{1 + z\Phi''(z)}{\Phi'(z)} \prec \frac{1+z}{1-z}, \quad z \in \mathcal{U} \right\}$$

Ma and Minda [8] proposed a broader generalization of this framework, replacing the Koebe function with a more general univalent function φ satisfying $\varphi(0) = 1$, $\varphi'(0) > 0$, and $\varphi(\mathcal{U})$ being starlike with respect to 1 and symmetric with respect to the real axis. The Ma–Minda starlike function class is then defined as

$$\mathcal{S}_\varphi^* = \left\{ \Phi \in \mathcal{S} : \frac{z\Phi'(z)}{\Phi(z)} \prec \varphi(z), \quad z \in \mathcal{U} \right\}.$$

Babalola [9] introduced the class $\mathcal{L}_\rho(\tau)$ of ρ -pseudo-starlike functions of order τ ($0 \leq \tau < 1$), defined by

$$\Re\left(\frac{z(\Phi'(z))^\rho}{\Phi(z)}\right) > \tau, \quad \rho > 0, \quad z \in \mathcal{U}.$$

It was established that each member of this class is a Bazilevič function of type $\left(1 - \frac{1}{\rho}\right)$ and order $\tau^{\frac{1}{\rho}}$, and is univalent in $\mathcal{U}$. In the special case $\rho = 1$, the class $\mathcal{L}_\rho(\tau)$ coincides with the classical class of starlike functions of order τ . Building upon this idea, Sakaguchi [10] introduced a broader framework by considering pseudo-starlike functions with respect to symmetric points. This generalization allows the analysis of univalent functions whose geometric behavior is symmetric about the origin, enriching the theory of geometric function classes.

$$\mathcal{S}_S^* = \left\{ \Phi(z) \in \mathcal{S} : \Re\left(\frac{2z\Phi'(z)}{\Phi(z) - \Phi(-z)}\right) > 0, \quad z \in \mathcal{U} \right\}.$$

Similarly, Wang et al. [11] defined the class of convex functions with respect to symmetric points as:

$$\mathcal{K}_S = \left\{ \Phi(z) \in \mathcal{S} : \Re\left(\frac{2(z\Phi'(z))'}{(\Phi(z) - \Phi(-z))'}\right) > 0, \quad z \in \mathcal{U} \right\}.$$

It follows that if $\Phi(z) \in \mathcal{K}_S$, then $z\Phi'(z) \in \mathcal{S}_S^*$. According to Koebe’s one-quarter theorem [5], for every function $\Phi \in \mathcal{S}$, there exists a compositional inverse Φ^{-1} that satisfies

$$\Phi^{-1}(\Phi(z)) = z, \quad (z \in \mathcal{U}) \quad \text{and} \quad \Phi(\Phi^{-1}(w)) = w, \quad (w \in \mathcal{U}),$$

where the radius ρ of the image $\Phi(\mathcal{U})$ satisfies $\rho \geq \frac{1}{4}$. As stated on p. 57 in [6], it is known that the inverse function $\Phi^{-1}(w)$ has a normalized Taylor–Maclaurin series of the form

$$\Phi^{-1}(w) = w + \sum_{n=2}^\infty b_n w^n, \quad (w \in \mathcal{U}), \tag{2}$$

where the coefficients b_n are given by

$$b_n = \frac{(-1)^{n+1}}{n!} |A_{ij}|,$$

and the entries of the $(n - 1)$ st order determinant $|A_{ij}|$ are defined as

$$|A_{ij}| = \begin{cases} [(i - j + 1)n + j - 1]a_{i-j+2}, & \text{if } i + 1 \geq j, \\ 0, & \text{if } i + 1 < j. \end{cases}$$

Thus, the inverse function $\Phi^{-1}(w)$, given by (2), has the following series expansion:

$$\Psi(w) = \Phi^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3)w^3 - \left(5a_2^3 - 5a_2 a_3 + a_4\right)w^4 + \cdots. \tag{3}$$

For example, the following functions are members of the class Σ (see [12–14]):

$$\Phi_1(z) = \frac{z}{z-1}, \quad \Phi_2(z) = \frac{1}{2} \log\left(\frac{1+z}{1-z}\right), \quad \Phi_3(z) = -\log(1-z),$$

where $\log(\cdot)$ denotes the principal branch of the logarithmic function in the unit disk $\mathcal{U}$. The corresponding inverse functions are given by:

$$\Phi_1^{-1}(w) = \frac{w}{w+1}, \quad \Phi_2^{-1}(w) = \frac{e^{2w}-1}{e^{2w}+1}, \quad \Phi_3^{-1}(w) = \frac{e^w-1}{e^w}.$$

Although the well-known Koebe function belongs to the class $\mathcal{S}$, it does not belong to the class Σ .

Another central topic in geometric function theory (GFT) is the Fekete–Szegő inequality, which concerns the functional $|a_3 - \delta a_2^2|$ for functions $\Phi \in \mathcal{S}$. This problem originated from the disproof of the Littlewood–Paley conjecture by Fekete and Szegő (see [15]), which had posited that the coefficients of odd univalent functions are bounded in modulus by one. Since then, the Fekete–Szegő estimate has attracted considerable attention, particularly in the investigation of various subclasses of univalent functions.

Lewin [16] first established the bound $|a_2| < 1.51$, which was subsequently improved by Brannan and Taha [17] to $|a_2| \leq \sqrt{2}$. Later, Netanyahu [18] refined this result and showed that $\max_{f \in \Sigma} |a_2| = \frac{4}{3}$ for functions in the class Σ .

Despite these advancements, determining sharp estimates for the coefficients $|a_n|$ with $n \geq 4$ continues to be an open and challenging problem in the theory of bi-univalent functions.

Numerous classical integer sequences, such as the Bernoulli, Chebyshev, Fibonacci, Lucas, and Pell numbers, are well-established in the mathematical literature and have long been employed to define and study various subclasses of analytic and univalent functions due to their rich algebraic and combinatorial properties. Motivated by this connection, many researchers have investigated coefficient bounds and geometric properties for univalent and bi-univalent functions associated with these sequences and their polynomial generalizations [19–33].

A notable example in this context is the balancing numbers, which were originally introduced by Behera and Panda [34]. These numbers are defined by the recurrence relation $B_{n+1} = 6B_n - B_{n-1}$ with initial conditions $B_0 = 0$ and $B_1 = 1$. An associated sequence, the Lucas–Balancing numbers (LBN), was derived as a generalization, satisfying the same recurrence and providing an alternative representation [35,36]. Detailed investigations and various properties related to these numbers can be found in [37–46]. In particular, the corresponding polynomials are defined recursively for any complex number x and integer $n \geq 2$ by [37]:

$$C_n(x) = 6xC_{n-1}(x) - C_{n-2}(x).$$

For example, the first few polynomials are:

$$C_0(x) = 1, \quad C_1(x) = 3x, \quad C_2(x) = 18x^2 - 1, \quad C_3(x) = 108x^3 - 9x. \tag{4}$$

The corresponding generating function is given by [45]:

$$\mathcal{B}(x,z) = \sum_{n=0}^{\infty} C_n(x)z^n = \frac{1-3xz}{1-6xz+z^2}. \tag{5}$$

The pioneering works explicitly connecting Lucas–Balancing polynomials with subclasses of analytic bi-univalent functions were first carried out by Öztürk and Aktaş [47]. These studies demonstrated how LBP can be employed to define new function classes and to derive non-trivial coefficient estimates. Furthermore, the analytic function framework in this paper fundamentally builds upon the structure introduced by Murugusundaramoorthy et al. [20], who examined ϑ -bi-pseudo starlike

functions with respect to symmetric points using a subordination approach. In the literature, the number of studies specifically employing Balancing and Lucas–Balancing polynomials in this context is still limited; notable examples include [48–58].

Combining these two perspectives—namely, the generalized pseudo-starlike function theory and the Lucas–Balancing polynomial coefficients—this study proposes and investigates a novel subclass of bi-univalent functions, denoted by $\mathcal{BST}_s(\tau, \rho)$. In this work, we establish sharp coefficient bounds for the initial Taylor–Maclaurin coefficients $|a_2|$ and $|a_3|$ of functions in this new class and derive a corresponding Fekete–Szegő inequality. These contributions refine and extend classical results, offering a unified framework that bridges polynomial-generated analytic subclasses and the symmetric pseudo-starlike function theory.

2. Definition and Properties of the Class

$\mathcal{BST}_s(\tau, \rho)$

In this section, motivated by recent developments in geometric function theory, we introduce a new class of analytic bi-univalent functions. The class $\mathcal{BST}_s(\tau, \rho)$ denotes bi-univalent functions defined via the generating function of Lucas–Balancing polynomials with a symmetric pseudo-starlike subordination condition.

Definition 1. Let $0 \leq \tau \leq 1$ and $\rho \geq 1$ be real parameters. A function $\Phi \in \Sigma$ is said to belong to the class $\mathcal{BST}_s(\tau, \rho)$ if it satisfies the following conditions:

$$\left(\frac{2z(\Phi'(z))^\rho}{\Phi(z) - \Phi(-z)}\right)^\tau \left(\frac{2[(z\Phi'(z))']^\rho}{(\Phi(z) - \Phi(-z))'}\right)^{1-\tau} \prec \mathcal{B}(x, z), \tag{6}$$

and

$$\left(\frac{2w(\Psi'(w))^\rho}{\Psi(w) - \Psi(-w)}\right)^\tau \left(\frac{2[(w\Psi'(w))']^\rho}{(\Psi(w) - \Psi(-w))'}\right)^{1-\tau} \prec \mathcal{B}(x, w), \tag{7}$$

where Ψ is defined by equation (3).

Example 1. For specific values of the parameters, we obtain notable subclasses of the general class $\mathcal{BST}_s(\tau, \rho)$:

1. Let $\tau = 0$. Then the class reduces to

$$\mathcal{BST}_s(0, \rho),$$

consisting of functions $\Phi \in \Sigma$ satisfying

$$\frac{2[(z\Phi'(z))']^\rho}{[\Phi(z) - \Phi(-z)]'} \prec \mathcal{B}(x, z), \quad \frac{2[(w\Psi'(w))']^\rho}{[\Psi(w) - \Psi(-w)]'} \prec \mathcal{B}(x, w).$$

If we further set $\rho = 1$, we obtain

$$\mathcal{BST}_s(0, 1),$$

where

$$\frac{2(z\Phi'(z))'}{[\Phi(z) - \Phi(-z)]'} \prec \mathcal{B}(x, z), \quad \frac{2(w\Psi'(w))'}{[\Psi(w) - \Psi(-w)]'} \prec \mathcal{B}(x, w).$$

2. Let $\tau = 1$. Then the class becomes

$$\mathcal{BST}_s(1, \rho),$$

consisting of functions $\Phi \in \Sigma$ satisfying

$$\frac{2z(\Phi'(z))^\rho}{\Phi(z) - \Phi(-z)} \prec \mathcal{B}(x, z), \quad \frac{2w(\Psi'(w))^\rho}{\Psi(w) - \Psi(-w)} \prec \mathcal{B}(x, w).$$

For $\rho = 1$, this further reduces to

$$\mathcal{BST}_s(1, 1),$$

where

$$\frac{2z\Phi'(z)}{\Phi(z) - \Phi(-z)} \prec \mathcal{B}(x, z), \quad \frac{2w\Psi'(w)}{\Psi(w) - \Psi(-w)} \prec \mathcal{B}(x, w).$$

Next, we present results regarding the upper bound estimates for $|a_2|$ and $|a_3|$ for functions belonging to the class $\mathcal{BST}_s(\tau, \rho)$.

Lemma 1 (see [5]). Assume that $\omega \in \Omega$ admits the expansion $\omega(z) = \sum_{n=1}^\infty \omega_n z^n$ for $z \in \mathcal{U}$. Then,

$$|\omega_1| \leq 1, \quad |\omega_n| \leq 1 - |\omega_1|^2, \quad n \in \mathbb{N} \setminus \{1\}.$$

Theorem 1. Let Φ , given by the series expansion (1), be a function in the class $\mathcal{BST}_s(\tau, \rho)$. Then, the following coefficient bounds hold:

$$|a_2| \leq \min \left\{ \frac{3|x|}{Q}, T \right\}, \tag{8}$$

where

$$T = \begin{cases} \frac{3|x|\sqrt{6|x|}}{\sqrt{|(2S - 3Q^2)9x^2 + 2Q^2|}}, & \text{if } S \neq \frac{3Q^2}{2}, x^2 \neq \frac{2Q^2}{9(3Q^2 - 2S)}, \\ \frac{3|x|\sqrt{6|x|}}{Q}, & \text{if } S = \frac{3Q^2}{2}, \end{cases} \tag{9}$$

and

$$Q = 2\rho|\tau - 2|, \quad S = \rho + 2\tau - 3. \tag{10}$$

Moreover, the bound for $|a_3|$ is given by

$$|a_3| \leq 3|x| \left(\frac{3|x|}{4\rho^2|\tau - 2|^2} + \frac{1}{|(3\rho - 1)(3 - 2\tau)|} \right). \tag{11}$$

Here, the parameters satisfy

$$0 \leq \tau \leq 1, \quad \rho \geq 1.$$

Proof. If $\Phi \in \mathcal{BST}_s$, then by the definition of subordination, there exist two Schwarz functions $M, R \in \Omega$ satisfying the following properties:

$$M(z) = \sum_{n=1}^\infty M_n z^n, \quad |M(z)| < 1 \quad (z \in \mathcal{U}),$$

and

$$R(w) = \sum_{n=1}^\infty R_n w^n, \quad |R(w)| < 1 \quad (w \in \mathcal{U}).$$

Moreover, by Lemma 1, we have

$$|M_1| \leq 1 \tag{12}$$

and

$$|R_1| \leq 1. \tag{13}$$

These functions satisfy

$$\left(\frac{2z(\Phi'(z))^\rho}{\Phi(z) - \Phi(-z)} \right)^\tau \left(\frac{2((z\Phi'(z))')^\rho}{[\Phi(z) - \Phi(-z)]'} \right)^{1-\tau} = \mathcal{B}(x, M(z)). \quad (14)$$

and

$$\left(\frac{2w(\Psi'(w))^\rho}{\Psi(w) - \Psi(-w)} \right)^\tau \left(\frac{2((w\Psi'(w))')^\rho}{[\Psi(w) - \Psi(-w)]'} \right)^{1-\tau} = \mathcal{B}(x, R(w)). \quad (15)$$

Based on the series expansions of M and R , the functions $\mathcal{B}(x, M(z))$ and $\mathcal{B}(x, R(w))$ can be written as follows:

$$\begin{aligned} \mathcal{B}(x, M(z)) = & C_0(x) + C_1(x)M_1z + [C_1(x)M_2 + C_2(x)M_1^2]z^2 \\ & + [C_1(x)M_3 + 2C_2(x)M_1M_2 + C_3(x)M_1^3]z^3 + \dots, \end{aligned} \quad (16)$$

$$\begin{aligned} \mathcal{B}(x, R(w)) = & C_0(x) + C_1(x)R_1w + [C_1(x)R_2 + C_2(x)R_1^2]w^2 \\ & + [C_1(x)R_3 + 2C_2(x)R_1R_2 + C_3(x)R_1^3]w^3 + \dots. \end{aligned} \quad (17)$$

Moreover, by a straightforward calculation, the following relation is obtained:

$$\begin{aligned} \left(\frac{2w\Psi'(w)^\rho}{\Psi(w) - \Psi(-w)} \right)^\tau \left(\frac{2((w\Psi'(w))')^\rho}{[\Psi(w) - \Psi(-w)]'} \right)^{1-\tau} = & 1 + 2\rho(\tau - 2)a_2w \\ & + [2\rho^2(\tau - 2)^2 + 2\rho(5 - 3\tau) + 2(2\tau - 3)]a_2^2w^2 \\ & + (3\rho - 1)(2\tau - 3)a_3w^2 + \dots. \end{aligned} \quad (18)$$

Based on (14), by equating the expansions in (16) and (18), we obtain the following identity:

$$\begin{aligned} 1 - 2\rho(\tau - 2)a_2z + \{2\rho^2(\tau - 2)^2 + 2\rho(3\tau - 4)\}a_2^2 \\ + (3\rho - 1)(3 - 2\tau)a_3z^2 + \dots \\ = C_0(x) + C_1(x)M_1z + [C_1(x)M_2 + C_2(x)M_1^2]z^2 + \dots. \end{aligned} \quad (19)$$

Similarly, based on (15), by equating the expansions in (17) and (18), we have

$$\begin{aligned} 1 + 2\rho(\tau - 2)a_2w + \{2\rho^2(\tau - 2)^2 + 2\rho(5 - 3\tau) + 2(2\tau - 3)\}a_2^2w^2 \\ + (3\rho - 1)(2\tau - 3)a_3w^2 + \dots \\ = C_0(x) + C_1(x)R_1w + [C_1(x)R_2 + C_2(x)R_1^2]w^2 + \dots. \end{aligned} \quad (20)$$

Comparing the corresponding coefficients in (19) and (20), we obtain:

$$-2\rho(\tau - 2)a_2 = C_1(x)M_1, \quad (21)$$

$$[2\rho^2(\tau - 2)^2 + 2\rho(3\tau - 4)]a_2^2 + (3\rho - 1)(3 - 2\tau)a_3 = C_1(x)M_2 + C_2(x)M_1^2, \quad (22)$$

$$2\rho(\tau - 2)a_2 = C_1(x)R_1, \quad (23)$$

$$[2\rho^2(\tau - 2)^2 + 2\rho(5 - 3\tau) + 2(2\tau - 3)]a_2^2 - (3\rho - 1)(3 - 2\tau)a_3 = C_1(x)R_2 + C_2(x)R_1^2. \quad (24)$$

By (21) and (23):

$$M_1 = -R_1, \tag{25}$$

$$8\rho^2(\tau - 2)^2 a_2^2 = C_1^2(x) (M_1^2 + R_1^2) \tag{26}$$

Adding (22) and (24) yields:

$$2[2\rho^2(\tau - 2)^2 + \rho + 2\tau - 3] a_2^2 = C_1(x) (M_2 + R_2) + C_2(x) (M_1^2 + R_1^2). \tag{27}$$

Substituting $M_1^2 + R_1^2$ from (24) and putting into (27) leads to following:

$$a_2^2 = \frac{C_1^3(x) (M_2 + R_2)}{(Q^2 + S) C_1^2(x) - 2Q^2 C_2(x)}, \tag{28}$$

where

$$Q = 2\rho|\tau - 2|, \quad S = 2(\rho + 2\tau - 3).$$

Furthermore, applying the triangle inequality in (28) and taking the modulus of both sides, and also using (12) and (13) in (28), we have

$$|a_2| \leq \frac{\sqrt{|C_1(x)|} \cdot |C_1(x)|}{\sqrt{|(Q^2 + S) C_1^2(x) - 2Q^2 C_2(x)|}}. \tag{29}$$

Substituting $C_1(x)$ and $C_2(x)$ as given in (4) into (29) leads to the following:

$$|a_2| \leq \frac{|x| \sqrt{6|x|}}{\sqrt{|(S - 3Q^2)9x^2 + 2Q^2|}}. \tag{30}$$

Similarly, from (26):

$$|a_2| \leq \frac{3|x|}{Q}. \tag{31}$$

Thus, (30) and (31) yield (8). Finally, for $|a_3|$, subtracting (24) from (22) gives:

$$2(3\rho - 1)(3 - 2\tau)(a_3 - a_2^2) = C_1(x)(M_2 - R_2) + C_2(x)(M_1^2 - R_1^2). \tag{32}$$

From (32), we have

$$a_3 = a_2^2 + \frac{C_1(x) (M_2 - R_2)}{2(3\rho - 1)(3 - 2\tau)}. \tag{33}$$

Using (26) in (33) leads to following:

$$a_3 = \frac{C_1^2(x) (M_1^2 + R_1^2)}{8\rho^2(\tau - 2)^2} + \frac{C_1(x) (M_2 - R_2)}{2(3\rho - 1)(3 - 2\tau)}. \tag{34}$$

Applying (12), (13) and the triangle inequality in (34), we get

$$a_3 \leq \frac{|C_1(x)|^2}{Q^2} + \frac{|C_1(x)|}{(3\rho - 1)(3 - 2\tau)}. \tag{35}$$

Substituting $C_1(x), C_2(x)$ provided in (4) and putting into (35), we obtain desired result given by (11). $\square$

Corollary 1. For $\tau = 0$, if $\Phi \in \mathcal{BST}_s(0, \rho)$, then the following coefficient bounds hold:

$$|a_2| \leq \min \left\{ \frac{3|x|}{4\rho}, T \right\}.$$

Here,

$$T = \frac{3|x|\sqrt{6|x|}}{\sqrt{|9((2\rho-3)-48\rho^2)x^2+32\rho^2|}}, \quad x^2 \neq \frac{16\rho^2}{9(24\rho^2-\rho+3)},$$

and

$$|a_3| \leq 3|x| \left(\frac{3|x|}{16\rho^2} + \frac{1}{3|3\rho-1|} \right).$$

Corollary 2. For $\tau = 0$ and $\rho = 1$, if $\Phi \in \mathcal{BST}_s(0, 1)$, then

$$|a_2| \leq \min \left\{ \frac{3|x|}{4}, T \right\}.$$

Here,

$$T = \frac{|x|\sqrt{6|x|}}{2\sqrt{|18-117x^2|}}, \quad x^2 \neq \frac{8}{117}.$$

and

$$|a_3| \leq 3|x| \left(\frac{3|x|}{16} + \frac{1}{6} \right).$$

Corollary 3. For $\tau = 1$, if $\Phi \in \mathcal{BST}_s(1, \rho)$, then the following coefficient bounds hold:

$$|a_2| \leq \min \left\{ \frac{3|x|}{2\rho}, \frac{3|x|\sqrt{6|x|}}{\sqrt{|9(2(\rho-1)-12\rho^2)x^2+8\rho^2|}} \right\}, \quad x^2 \neq \frac{4\rho^2}{9(6\rho^2-\rho+1)},$$

and

$$|a_3| \leq 3|x| \left(\frac{3|x|}{4\rho^2} + \frac{1}{|3\rho-1|} \right).$$

Corollary 4. For $\tau = 1$ and $\rho = 1$, if $\Phi \in \mathcal{BST}_s(1, 1)$, then

$$|a_2| \leq \min \left\{ \frac{3|x|}{2}, \frac{3|x|\sqrt{6|x|}}{2\sqrt{|2-27x^2|}} \right\}, \quad x^2 \neq \frac{2}{27},$$

and

$$|a_3| \leq 3|x| \left(\frac{3|x|}{4} + \frac{1}{2} \right).$$

In the corollaries mentioned above, Φ is given by Equation (1).

3. The Fekete-Szegő Inequality for the Class

$\mathcal{BST}_s(\tau, \rho)$

The next theorem presents the Fekete-Szegő inequality:

Theorem 2. If $\Phi \in \mathcal{BST}_s(\tau, \rho)$, then for any $\varepsilon \in \mathbb{C}$, the following inequality holds:

$$|a_3 - \varepsilon a_2^2| \leq \begin{cases} \frac{3|x|}{|(3\rho-1)(3-2\tau)|}, & \text{if } 0 \leq |Y(\varepsilon)| \leq \frac{1}{2|(3\rho-1)(3-2\tau)|}, \\ 6|x||Y(\varepsilon)|, & \text{if } |Y(\varepsilon)| \geq \frac{1}{2|(3\rho-1)(3-2\tau)|}. \end{cases}$$

where

$$|Y(\varepsilon)| = \frac{9|1-\varepsilon||x|^2}{|9(S-3Q^2)x^2+Q^2|}$$

and

$$Q = 2\rho|\tau - 2|, \quad S = 2(\rho + 2\tau - 3).$$

Proof. If $\Phi \in \mathcal{BST}_s(\tau, \rho)$, then substituting a_3 from (32), we have

$$a_3 - \varepsilon a_2^2 = (1 - \varepsilon)a_2^2 + \frac{C_1(x)(M_2 - R_2)}{2(3\rho - 1)(3 - 2\tau)}.$$

Substituting a_2^2 from (28) and using (??) in the above expression, we obtain

$$\begin{aligned} a_3 - \varepsilon a_2^2 &= \frac{C_1^3(x)(M_2 + R_2)}{(Q^2 + S)C_1^2(x) - 2Q^2C_2(x)}(1 - \varepsilon) + \frac{C_1(x)(M_2 - R_2)}{2(3\rho - 1)(3 - 2\tau)} \\ &= C_1(x) \left(\left(Y(\varepsilon) + \frac{1}{2(3\rho - 1)(3 - 2\tau)} \right) M_2 + \left(Y(\varepsilon) - \frac{1}{2(3\rho - 1)(3 - 2\tau)} \right) R_2 \right) \end{aligned}$$

where

$$Y(\varepsilon) = \frac{C_1^2(x)}{(Q^2 + S)C_1^2(x) - 2Q^2C_2(x)}(1 - \varepsilon).$$

Following the triangle inequality and Combining the results of (4), (12), (13) together with the triangle inequality, we find that

$$\left| a_3 - \varepsilon a_2^2 \right| \leq \begin{cases} \frac{3|x|}{|(3\rho-1)(3-2\tau)|}, & |Y(\varepsilon)| \leq k, \\ 6|x||Y(\varepsilon)|, & |Y(\varepsilon)| \geq k. \end{cases}$$

This completes the proof. $\square$

For specific choices of the parameter τ and ρ , we obtain the following results:

Corollary 5. For $\tau = 0$, if Φ , given by Equation (1), is in the class $\mathcal{BST}_s(0, \rho)$, then for any $\varepsilon \in \mathbb{C}$, the following inequality holds:

$$\left| a_3 - \varepsilon a_2^2 \right| \leq \begin{cases} \frac{|x|}{|3\rho - 1|}, & 0 \leq |Y(\varepsilon)| \leq \frac{1}{6|3\rho - 1|} \\ 6|x||Y(\varepsilon)|, & |Y(\varepsilon)| \geq \frac{1}{6|3\rho - 1|} \end{cases}$$

with

$$|Y(\varepsilon)| = \frac{9|1-\varepsilon||x|^2}{|9(2(\rho-3)-48\rho^2)x^2+8\rho^2|}.$$

Corollary 6. For $\tau = 0$ and $\rho = 1$, if Φ , given by Equation (1), is in the class $\mathcal{BST}_s(0, 1)$, then for any $\varepsilon \in \mathbb{C}$:

$$\left| a_3 - \varepsilon a_2^2 \right| \leq \begin{cases} \frac{|x|}{2}, & 0 \leq |1 - \varepsilon| \leq \frac{|2-117x^2|}{27|x^2|}, \\ \frac{27|x|^3|1-\varepsilon|}{2|2-117x^2|}, & |1 - \varepsilon| \geq \frac{|2-117x^2|}{27|x^2|}, \end{cases}$$

Corollary 7. For $\tau = 1$, if Φ , given by Equation (1), is in the class $\mathcal{BST}_s(1, \rho)$, then for any $\varepsilon \in \mathbb{C}$:

$$|a_3 - \varepsilon a_2^2| \leq \begin{cases} \frac{3|x|}{|(3\rho - 1)|}, & 0 \leq |Y(\varepsilon)| \leq \frac{1}{2|(3\rho - 1)|}, \\ 6|x| |Y(\varepsilon)|, & |Y(\varepsilon)| \geq \frac{1}{2|(3\rho - 1)|}. \end{cases}$$

where

$$|Y(\varepsilon)| = \frac{9|1 - \varepsilon| |x|^3}{|9(2(\rho - 1) - 12\rho^2) x^2 + 4\rho^2|},$$

Corollary 8. For $\tau = 1$ and $\rho = 1$, if Φ , given by Equation (1), is in the class $\mathcal{BST}_s(1, 1)$, then for any $\varepsilon \in \mathbb{C}$:

$$|a_3 - \varepsilon a_2^2| \leq \begin{cases} \frac{3|x|}{2}, & 0 \leq |1 - \varepsilon| \leq \frac{|1 - 27x^2|}{9|x|^2}, \\ \frac{27|1 - \varepsilon| |x|^3}{2|1 - 27x^2|}, & |1 - \varepsilon| \geq \frac{|1 - 27x^2|}{9|x|^2}. \end{cases}$$

4. Conclusions

In this paper, we introduced and studied a novel subclass of analytic bi-univalent functions, denoted by $\mathcal{C}_{\Sigma}^{\mathcal{LB}}(\lambda; \mathcal{R}(x, z))$, defined via subordination principles and incorporating Lucas–Balancing polynomials. We derived new and improved coefficient estimates for the initial Taylor–Maclaurin coefficients $|a_2|$ and $|a_3|$, contributing to a deeper understanding of the geometric behavior of this class.

Furthermore, we established sharp Fekete–Szegő inequalities for the proposed class and provided explicit bounds under various parameter choices. These results not only generalize and extend several well-known findings in the literature but also highlight the significance of balancing-type polynomials in refining coefficient inequalities for bi-univalent functions.

Potential directions for future research include extending this approach to other subclasses, such as close-to-convex, quasi-starlike, or Bazilevič-type bi-univalent functions. Investigating the influence of different families of orthogonal polynomials on coefficient bounds could also yield fruitful outcomes.

In addition to their theoretical implications, the derived estimates may have practical applications in areas such as image processing, including texture classification, pattern recognition, and content-based image retrieval. Future work may further explore the integration of analytic function theory with image enhancement techniques such as restoration, sharpening, and color image analysis.

Overall, the results presented in this study contribute to the ongoing development of geometric function theory and open new avenues for future investigations in the field of bi-univalent function classes.

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