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Proof of Admissible Weights

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Abstract: Admissable weight is an important tool for studying spectral invariance in operator algebra. Common admissable weights include polynomial weights and sub exponential weights. This article mainly provides a proof that polynomial weights are permissible weights.

Keywords: weight function; polynomial weight; admissible weight

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1. Introduction

The Uniform Roe algebra and Roe algebra [1] originated from the index theory on non compact manifolds, reflecting the coarse structure of metric spaces These algebras play an important role in using the C^*algebra method to solve some geometric and topological problems (such as Novikov conjecture [2]) in differential topology Therefore, studying the spectral invariant subalgebras of consistent Roe algebras is of great significance There are many methods for studying spectral invariance. The traditional method is to start from the definition and verify spectral equality, but Hulanicki proposed a new method that no longer studies spectral equality but rather investigates whether spectral radii are equivalent. In 2007, Sun [3] provided a definition of admissable weights and briefly proved whether some common weight functions are admissable weights This article will provide a detailed proof in the following content that polynomial weights are admissable weights.

2. Weight Function

Definition 1 ([4])**.** *If function $w : G \longrightarrow [1, \infty)$ and*

$$w(xy) \le w(x)w(y), \quad \forall x, y \in G;$$

$$w\left(x^{-1}\right) = w(x), \quad \forall x \in G;$$

$$w(e) = 1.$$

Example 1. *Here are some examples of classic weight functions:*

(i) If $w_s, s \ge 0,$

$$w_s(x, y) = (1 + \rho_l(x, y))^s,$$

is a weight on G timesG, and this type of weight is called admissable weight

(ii) If $f_{\alpha,\beta}, \alpha \in (0, \infty)$ and $\beta \in (0, 1)$

$$f_{\alpha,\beta}(x, y) = \exp\left(\alpha\rho_l(x, y)^\beta\right),$$

is a weight on G timesG,this type of right is called a sub index right

Definition 2 ([5])**.** *Let G be a countable group and l be an appropriate length function on G For $\tau \in [1, \infty)$, let $|B(x, \tau)|$ represents the number of elements in the ball $B(x, \tau) = \{y \in G : \rho_l(x, y) < \tau\}$,We call it:*

The group G is polynomial growing. If C exists and d>0, then

$$|B(x, \tau)| \le C\tau^d \quad \text{for all } y \in G \text{ and } \tau \ge 1, \tag{1}$$

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3. Admissable Weight

Definition 3. *Let $1 \le p, r \le \infty$. We say that a weight ω is (p,r)-admissible if there exist another weight v and two positive constants $D \in (0, \infty)$ and $\theta \in (0,1)$ such that*

$$w(x,y) \le D(w(x,z)v(z,y) + v(x,z)w(z,y)) \quad \text{for all } x,y,z \in G, \tag{2}$$

$$\sup_{x\in G}\left\|(vw^{-1})(x,\cdot)\right\|_{p'} + \sup_{y\in G}\left\|(vw^{-1})(\cdot,y)\right\|_{p'} \le D, \tag{3}$$

and

$$\inf_{\tau>0} a_{r'}(\tau) + b_{p'}(\tau)t \le Dt^{\theta} \quad \text{for all } t \ge 1, \tag{4}$$

where $p' = p/(p-1)$, $r' = r/(r-1)$,

$$a_{r'}(\tau) = \sup_{x\in G}\left\|v(x,\cdot)\chi_{B(x,\tau)(\cdot)}\right\|_{r'} + \sup_{y\in G}\left\|v(\cdot,y)\chi_{B(y,\tau)(\cdot)}\right\|_{r'}, \tag{5}$$

$$b_{p'}(\tau) = \sup_{x\in G}\left\|(vw^{-1})(x,\cdot)\chi_{X\setminus B(x,\tau)(\cdot)}\right\|_{p'} + \sup_{y\in G}\left\|(vw^{-1})(\cdot,y)\chi_{X\setminus B(y,\tau)(\cdot)}\right\|_{p'}, \tag{6}$$

χ_E is the characteristic function on the set E, and $\|\cdot\|_p$ is the norm on ℓ^p, the space of all p-summable functions on G.

The technical assumption on the weight w, (p,r)-admissibility, plays very important role in our results,.

4. Proof of Admissable Weights for Polynomial Weights

Theorem 1. *Let G be a countable discrete group with a metric ρ_l and suppose G has polynomial growth. Define $w(x,y) = (1+\rho_l(x,y))^s$ as a left-invariant weight on $G \times G$. Then there exists another weight v such that (2)-(4), indicating that w is a (p,r)-admissible weight.*

Proof. We define the minimal rate of polynomial growth $d(G,\rho_l)$ as follows

$$d(G,\rho_l) = \inf\{d : (1) \text{ holds for some positive } C\}. \tag{7}$$

For $s \ge 0$, the weight w and v are defined as follows

$$w(x,y) = (1+\rho_l(x,y))^s \quad \text{for all } x,y \in G,$$

$$v(x,y) = (1+\rho_l(x,y))^0 = 1 \quad \text{for all } x,y \in G.$$

Then, we have

$$\begin{aligned} w(x,y) = (1+\rho_l(x,y))^s &\le (1+\rho_l(x,z)+\rho_l(z,y))^s \\ &\le (1+\rho_l(x,z))^s + (1+\rho_l(z,y))^s \\ &= w(x,z)\cdot 1 + w(z,y)\cdot 1 \\ &= w(x,z)v(z,y) + v(x,z)w(z,y), \end{aligned}$$

for all $x,y,z \in G$ with $D = 1$. Thus, we have the weights w and v satisfy (2).

Let $s > \frac{1}{p'}d(G,\rho_l)$. Using (1) and (7),

we obtain the following inequality for $\sum_{\rho_l(x,y)\ge\tau} w^{-p'}(x,y)$:

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$$
\begin{aligned}
\sum_{\rho_l(x,y)\geq\tau} w^{-p'}(x,y) &= \sum_{j=1}^{\infty} \sum_{2^{j-1}\tau\leq\rho_l(x,y)<2^j\tau} w^{-p'}(x,y) \\
&\leq \sum_{j=1}^{\infty}(1+2^{j-1}\tau)^{-p's}\left|B(x,2^j\tau)\right| \leq C\sum_{j=1}^{\infty}(1+2^{j-1}\tau)^{-p's}\cdot\left(2^j\tau\right)^{d(G,\rho_l)} \\
&\leq C2^{p's}\cdot\tau^{-p's+d(G,\rho_l)}\sum_{j=1}^{\infty}2^{j(d(G,\rho_l)-p's)} \\
&\leq C_\epsilon\tau^{-(p's-d(G,\rho_l))}<\infty \quad \text{for all } x\in G \text{ and } \tau\geq 1,
\end{aligned}
$$

where C is a positive constant independent of $x \in G$. Then, we get

$$
\begin{aligned}
\sup_{x\in G}\left\|vw^{-1}(x,\cdot)\right\|_{p'} &= \sup_{x\in G}\left(\sum_{y\in G}\left(w^{-p'}(x,y)\right)\right)^{\frac{1}{p'}} \\
&= \sup_{x\in G}\left(\sum_{0<\rho_l(x,y)<\tau} w^{-p'}(x,y)\right)^{\frac{1}{p'}} + \sup_{x\in G}\left(\sum_{\rho_l(x,y))\geq\tau} w^{-2}(x,y)\right)^{\frac{1}{p'}} \\
&\leq C+\sup_{x\in G}\left(\sum_{\rho_l(x,y))\geq\tau} w^{-p'}(x,y)\right)^{\frac{1}{p'}} < \infty.
\end{aligned}
$$

Similarly, we obtain

$$
\sup_{x\in G}\left\|vw^{-1}(\cdot,y)\right\|_{p'} < \infty.
$$

Therefore, we have

$$
\sup_{x\in G}\left\|vw^{-1}(x,\cdot)\right\|_{p'} + \sup_{y\in G}\left\|vw^{-1}(\cdot,y)\right\|_{p'} < \infty,
$$

which implies the weights w and v satisfy (3).

We claim that

$$
\inf_{\tau\geq 1}\left(a_{r'}(\tau)+b_{p'}(\tau)\cdot t\right) \leq Ct^\theta,
$$

for all $t \geq 1$ and $\theta = \frac{d(G,\rho_l)}{d(G,\rho_l)+(s-\frac{d(G,\rho_l)}{2})(\frac{r}{r-1})} < 1$.

Firstly, we have

$$
\begin{aligned}
\sup_{x\in G}\left\|v(x,\cdot)\chi_{B(x,\tau)(\cdot)}\right\|_{r'} &= \sup_{x\in G}\left[\sum_{y\in G}(\chi_{B(x,\tau)(y)})^{r'}\right]^{\frac{1}{r'}} \\
&\leq \sup_{x\in G}(|B(x,\tau)|)^{\frac{1}{r'}} \\
&\leq C_1\tau^{\frac{d(G,\rho_l)}{r'}}.
\end{aligned}
$$

Similarly, we have

$$
\begin{aligned}
\sup_{x\in G}\left\|(vw^{-1})(x,\cdot)\chi_{X\setminus B(x,\tau)(\cdot)}\right\|_{p'} &= \sup_{x\in G}\Big(\sum_{y\in G,\rho_l(x,y)\geq\tau}(1+\rho_l(x,y))^{-sp'}\Big)^{\frac{1}{p'}} \\
&\leq \sup_{x\in G}\Big(\sum_{y\in G,\rho_l(x,y)\geq\tau} w^{-p'}(x,y)\Big)^{\frac{1}{p'}} \\
&\leq C_1\tau^{-(s-\frac{d(G,\rho_l)}{p'})}.
\end{aligned}
$$

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Then, we get

$$\inf_{\tau\geq 1}\left(a_{r'}(\tau)+b_{p'}(\tau)\cdot t\right)\leq 2\inf_{\tau\geq 1}\left(C_1\tau^{\frac{d(G,\rho_I)}{r'}}+C_1\tau^{-(s-\frac{d(G,\rho_I)}{p'})}\cdot t\right)$$
$$=\inf_{\tau\geq 1}\left(C\tau^{\frac{d(G,\rho_I)}{r'}}+C\tau^{-(s-\frac{d(G,\rho_I)}{2})}\cdot t\right)$$

where $p'=p/(p-1), r'=r/(r-1), 1\leq r\leq\infty$..

Indeed, let $\tau=t^{\alpha}, \alpha\geq 0$, it is enough to prove

$$t^{\frac{\alpha d(G,\rho_I)}{r'}}\leq Ct^{\frac{d(G,\rho_I)}{d(G,\rho_I)+(s-\frac{d(G,\rho_I)}{p'})(\frac{r}{r-1})}} \tag{8}$$

$$t^{-\alpha(s-\frac{d(G,\rho_I)}{p'})}\cdot t\leq Ct^{\frac{d(G,\rho_I)}{d(G,\rho_I)+(s-\frac{d(G,\rho_I)}{p'})(\frac{r}{r-1})}}. \tag{9}$$

By setting$\alpha=\frac{1}{(1-\frac{1}{r})d(G,\rho_I)+(s-\frac{d(G,\rho_I)}{p'})}$, we obtain inequalities (8) and (9).

Hence, we have

$$\inf_{\tau\geq 1}\left(a_{r'}(\tau)+b_{p'}(\tau)\cdot t\right)\leq\inf_{\tau\geq 1}\left(C\tau^{\frac{d(G,\rho_I)}{r'}}+C\tau^{-(s-\frac{d(G,\rho_I)}{p'})}\cdot t\right)$$
$$\leq Ct^{\frac{d(G,\rho_I)}{d(G,\rho_I)+(s-\frac{d(G,\rho_I)}{p'})(\frac{r}{r-1})}}=Ct^{\theta},$$

which means the weights w and v satisfy (4). □

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