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Article

Bounds on the Domination Numbers of δ -Complement Graphs

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Abstract

This study examines the δ -complements of graphs—a specific type of graph complement whose adjacency depends on the adjacency of the vertices with identical degrees in the original graph. More specifically, we study this type of complement regarding the domination number. We provide sharp Nordhaus-Gaddum-type bounds on the domination number of a graph and its δ -complement. We also provide sharp bounds on the domination numbers of the δ -complements of joined graphs and Cartesian product graphs.

Keywords: domination number; complement graph; Nordhaus-Gaddum type bound; join; Cartesian product

1. Introduction

A *dominating set* D of a simple undirected graph $G = (V, E)$ is a subset of V such that each vertex not in D is adjacent to a vertex in D . The smallest cardinality among the dominating sets of G is called the *domination number* of G , denoted by $\gamma(G)$. A dominating set with the smallest cardinality is called a *minimum dominating set*. Applications of dominating sets and domination numbers exist in various fields. For instance, in ad-hoc wireless networks, devices form a dominating set that ensures message delivery (Nguyen and Huynh (2006); Wu and Li (2001); Yu et al. (2013)); in social networks, we identify influential users based on their connections (Daliri Khomami et al. (2018); Wang et al. (2011)); and in facility location, we can select optimal locations to cover a region (Corcoran and Gagarin (2021); Sabarish et al. (2017)).

Many properties of the domination number have been investigated; for example, Nordhaus-Gaddum type bounds on the domination number in terms of the number of vertices were given individually by Borowiecki (Borowiecki (1976)) and Jaeger (Jaeger and Payan (1972)). Desormeaux et. al. (Desormeaux et al. (2018)) provided a Nordhaus-Gaddum type relation of the domination number in terms of minimum degrees of graphs. They also proved several results on the domination number in various parameters.

In 2022, Pai et. al. (Pai et al. (2022)) introduced a variant of graph complements in which the complement takes place only among the vertices of the same degree. They called it the δ -complement graph G_δ of G . Later in 2023, Vichitkunakorn et. al. (Vichitkunakorn et al. (2023)) gave a Nordhaus-Gaddum type relation on the chromatic number of a graph and its δ -complement. The Nordhaus-Gaddum type bounds are given in various parameters, including the number of vertices, degrees, and clique number.

In this work, we further study the δ -complement by investigating the domination numbers of the δ -complements of graphs. In particular, we provide Nordhaus-Gaddum type bounds, bounds on

the join of graphs, and bounds on Cartesian products of graphs. Examples showing that most of our bounds are sharp are also given.

Outline of the paper. Section 2 contains necessary notations, definitions, and some theorems about the domination number of a graph and its complement. Sections 3-5 contain our results. Specifically, a Nordhaus-Gaddum type bound on the domination number is provided in Section 3, some upper bounds and exact values on the domination number of the δ -complement of the join of two graphs are provided in Section 4, some upper bounds and exact values of the δ -complement of the Cartesian product of two graphs are provided in Section 5. We also provide examples relevant to each case.

2. Background

In this section, we provide necessary notations and definitions of a graph, its complement and δ -complement, a dominating set, and the domination number. We also present some results on bounds and Nordhaus-Gaddum type bounds on the domination number of the complement of a graph.

Let $G = (V(G), E(G))$ (or (V, E)) be a simple undirected graph where V is the vertex set and E is the edge set. The subgraph of G induced by a subset S of V is denoted by $G[S]$. For each vertex $u \in V$, the neighborhood of u in G is denoted by $N_G(u)$, and the degree of u in G is denoted by $\deg_G(u)$. Given two graphs G and H , the *join* of G and H , denoted by $G \vee H$, is a graph with $V(G \vee H) = V(G) \cup V(H)$ and $E(G \vee H) = E(G) \cup E(H) \cup \{(u, v) \in V(G) \times V(H)\}$. The *disjoint union* of G and H , denoted by $G + H$, is a graph with $V(G + H) = V(G) \cup V(H)$ and $E(G + H) = E(G) \cup E(H)$. The *Cartesian product* of G and H , denoted by $G \square H$, is a graph with $V(G \square H) = V(G) \times V(H)$ and $E(G \square H) = \{uv : u = (u_1, u_2) \in V(G \square H), v = (v_1, v_2) \in V(G \square H) \text{ where } u_1 = v_1 \text{ and } u_2 v_2 \in E(H), \text{ or } v_1 = v_2 \text{ and } u_1 u_2 \in E(G)\}$. In this paper, we denote a path and a cycle of order n by P_n and C_n , respectively. The complement of G is denoted by $\overline{G}$.

Many results on the bounds of the domination numbers of a graph and its complement have been discovered. The findings listed below serve as inspiration for our work.

Theorem 1 (Desormeaux et al. (2018)). *If G is a graph with $\gamma(G) \geq 2$, then $\gamma(\overline{G}) \leq \left\lceil \frac{\delta(G)}{\gamma(G)-1} \right\rceil + 1$ where $\delta(G)$ is the minimum degree of G .*

Theorem 2 (Desormeaux et al. (2018)). *If G is a graph with $\gamma(G) = \gamma(\overline{G})$, then $\gamma(G) + \gamma(\overline{G}) < 4 + \sqrt{\delta(G)} + \sqrt{\delta(\overline{G})}$ where $\delta(G)$ is the minimum degree of G .*

Theorem 3 (Desormeaux et al. (2018)). *Let G be a graph. If $\gamma(G) \leq \gamma(\overline{G}) - 2$, then G contains $K_{r,r}$.*

Theorem 4 (Jaeger and Payan (1972)). *For a graph G with $n \geq 2$ vertices, we have*

$$3 \leq \gamma(G) + \gamma(\overline{G}) \leq n + 1$$

and

$$2 \leq \gamma(G) \cdot \gamma(\overline{G}) \leq n.$$

Theorem 5 (Borowiecki (1976)). *For a graph G with $n \geq 1$ vertices, we have*

$$2 \leq \gamma(G) + \gamma(\overline{G}) \leq n + 1$$

and

$$1 \leq \gamma(G) \cdot \gamma(\overline{G}) \leq n.$$

With a new variant of graph complements introduced by Pai et. al. [Pai et al. \(2022\)](#), we explore bounds on the domination numbers of this particular complement. A formal definition of the δ -complement of a graph is given below.

Definition 1 ([Pai et al. \(2022\)](#)). The δ -complement of a graph G , denoted by G_δ , is a graph obtained from G by using the same vertex set and the following edge conditions: $uv \in E(G_\delta)$ if

1. $\deg(u) = \deg(v)$ in G and $uv \notin E(G)$, or
2. $\deg(u) \neq \deg(v)$ in G and $uv \in E(G)$.

Some properties of the δ -complement of a graph were studied in [Pai et al. \(2022\)](#), [Tangjai et al. \(2024\)](#) and [Vichitkunakorn et al. \(2023\)](#). For instance, the following theorem will be used to prove one of our results in Section 5.

Theorem 6 ([Tangjai et al. \(2024\)](#)). For graphs G and H , we have $(G \square H)_\delta = (V, E)$ where $V = V(G \square H)$ and $E = E(G_\delta \square H_\delta) \cup S$ where $S = \{uv : u = (u_1, u_2) \in V(G \square H) \text{ and } v = (v_1, v_2) \in V(G \square H) \text{ where } u_1 \neq v_1, u_2 \neq v_2 \text{ and } \deg_{G \square H}(u) = \deg_{G \square H}(v)\}$.

3. Results on a Nordhaus-Gaddum Type Bound on the Domination Number

This section contains our results on a Nordhaus-Gaddum type bound on the domination number. Throughout the rest of the paper, we denote $n = |V(G)|$ the number of vertices in G and denote $m = m(G)$ the number of distinct degrees of vertices in G . We also denote the set of vertices of degree d in G by $V_d(G)$. Let $\mathcal{P}(G)$ be the partition of the vertex set of G as follows:

$$\mathcal{P}(G) = \{V_{d_1}(G), V_{d_2}(G), V_{d_3}(G), \dots, V_{d_m}(G)\}.$$

Let s_G be the number of singletons in $\mathcal{P}(G)$, i.e., $|V_{d_i}(G)| = 1$ for $i \leq s_G$ and $|V_{d_i}(G)| > 1$ for $s_G + 1 \leq i \leq m$. The ordering of the order of $V_{d_i}(G)$ and the number of singletons will be used later in Section 5.

We observe that $\gamma(G_\delta) = 1$ if and only if there exists a vertex v such that $N_G(v) = \{u \in V(G) : \deg(u) \neq \deg(v)\}$.

Lemma 1. *There are infinitely many graphs G that $\gamma(G) = \gamma(G_\delta) = 1$.*

Proof. For $n \geq 5$, let $G = u \vee C_{n-1}$. Since u is adjacent to all other vertices in G , we have $\gamma(G) = 1$. In addition, u is the only vertex of degree $n - 1$ in G , so u is also adjacent to all other vertices in G_δ . Hence $\gamma(G_\delta) = 1$. See Figure 1 for an example. $\square$

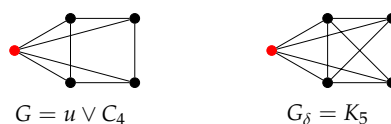


Figure 1. An example of $u \vee C_{n-1}$ and its δ -complement K_n , when $n = 5$, in the proof of Lemma 1. Their minimum dominating sets contain only one common vertex, which is indicated in red color.

Lemma 2. *There are infinitely many graphs G that $\gamma(G) = \gamma(G_\delta) = \frac{n+m}{2}$ where n is an order of G and m is the number of distinct degrees of G*

Proof. For an even number $n \geq 4$, let $G = K_r + \overline{K_r}$ where $r = \frac{n}{2}$. We have $G_\delta = \overline{K_r} + K_r \cong G$. Notice that G has r vertices of degree 0 and r vertices of degree $r - 1 > 0$. So $m = 2$. It is easy to see that $\gamma(G) = \gamma(G_\delta) = r + 1 = \frac{n+m}{2}$. $\square$

Theorem 7. Let G be a graph of order n having m distinct degrees. We have $2 \leq \gamma(G) + \gamma(G_\delta) \leq n + m$ and $1 \leq \gamma(G) \cdot \gamma(G_\delta) \leq \left(\frac{n+m}{2}\right)^2$. Furthermore, the bounds are sharp for infinitely many n .

Proof. Let $\gamma_i = \gamma(G[V_{d_i}(G)])$, $\bar{\gamma}_i = \gamma(\overline{G[V_{d_i}(G)]})$ and $n_i = |V_{d_i}(G)|$ for $i = 1, \dots, m$. Since $G_\delta[V_{d_i}(G)] = \overline{G[V_{d_i}(G)]}$, we have $2 \leq \gamma_i + \gamma(G_\delta[V_{d_i}(G)]) = \gamma_i + \bar{\gamma}_i \leq n_i + 1$ for $i = 1, \dots, m$, by Theorem 5. The union of a dominating set of $G[V_{d_i}(G)]$ gives a dominating set of G . Similarly, the union of a dominating set of $G_\delta[V_{d_i}(G)]$ gives a dominating set of G_δ . Thus $\gamma(G) + \gamma(G_\delta) \leq \sum_{i=1}^m (\gamma_i + \bar{\gamma}_i) \leq n + m$. Since $0 \leq (\gamma(G) - \gamma(G_\delta))^2$, it follows that $4\gamma(G) \cdot \gamma(G_\delta) \leq (\gamma(G) + \gamma(G_\delta))^2$. Hence, we have $\gamma(G) \cdot \gamma(G_\delta) \leq \left(\frac{n+m}{2}\right)^2$. By Lemma 1 and Lemma 2, we complete the proof. $\square$

Theorem 8. If G is not a complete graph and $\gamma(G) = 1$, then $\gamma(G_\delta) \leq 2$.

Proof. Consider $n \geq 2$. Since $\gamma(G) = 1$, there is a vertex v which is adjacent to all other vertices in G .

Case 1 v is the only vertex of degree $n - 1$ in G . Then v also has degree $n - 1$ in G_δ . Hence $\gamma(G_\delta) = 1$.

Case 2 v is not the only vertex of degree $n - 1$ in G . Pick $u \in V(G) \setminus V_{n-1}(G)$. In G_δ , we have v dominates $V(G) \setminus V_{n-1}(G)$ and u dominates $V_{n-1}(G)$. So $\{u, v\}$ dominates V in G_δ . Hence $\gamma(G_\delta) \leq 2$.

From these two cases, we have $\gamma(G_\delta) \leq 2$. $\square$

The converse of Theorem 8 is not necessarily true, as in the following example.

Example 1. If $G = K_3 + K_3$, then $G_\delta = K_{3,3}$. Hence, $\gamma(G) = 2$ and $\gamma(G_\delta) = 2$.

4. Join of Graphs

This section contains our results on upper bounds and exact values on the domination number of the δ -complement of the join of two graphs. Here is an overview of this section. Given two graph G and H , if they satisfy Condition (1) in Theorem 9, then $\gamma((G \vee H)_\delta) \leq 2$. If not, Theorem 10 provides the exact domination number of $(G \vee H)_\delta$ when both G and H are regular graphs. If one of the graphs is nonregular, a sharp upper bound is given in Theorem 11. Finally, Corollary 1 provides an upper bound when the two are nonregular. We summarize these results in Table 1 at the end of this section.

Theorem 9. Let G and H be graphs. If

$$\deg_G(u) + |V(H)| \neq \deg_H(v) + |V(G)| \text{ for all } u \in V(G) \text{ and } v \in V(H), \quad (1)$$

then $\gamma((G \vee H)_\delta) \leq 2$.

Proof. Since, in $(G \vee H)_\delta$, each vertex in $V(G)$ covers all vertices in $V(H)$ and vice-versa, it follows that $\gamma((G \vee H)_\delta) \leq 2$. $\square$

Example 2. Let $G = P_3 = u_1u_2u_3$ and $H = P_5 = v_1v_2v_3v_4v_5$. We see that

$$\deg_G(u_i) + |V(H)| = \begin{cases} 6 & \text{if } i = 1, 3, \\ 7 & \text{if } i = 2, \end{cases} \quad \text{and} \quad \deg_H(v_i) + |V(G)| = \begin{cases} 4 & \text{if } i = 1, 5, \\ 5 & \text{if } i = 2, 3, 4. \end{cases}$$

Graphs G and H satisfy the condition in Theorem 9. Hence, $\gamma((P_3 \vee P_5)_\delta) \leq 2$. In fact, $\gamma((P_3 \vee P_5)_\delta) = 1$ as vertex u_2 dominates $(P_3 \vee P_5)_\delta$.

When G and H are regular, $G \vee H$ is also regular, which implies that $(G \vee H)_\delta = \overline{G \vee H}$. Thus, $\gamma((G \vee H)_\delta) = \gamma(\overline{G \vee H})$. This yields the result in Theorem 10. Despite the simplicity of its proof, we have not found this result in any literature, to our knowledge. Hence, we present it here. (Same for the

case of the Cartesian product where $(G \square H)_\delta = \overline{G \square H}$ when G and H are regular and the results are present in Lemma 5, 6, 7 and Corollary 2.)

Theorem 10. Let G be an a -regular and H be a b -regular. If $a + |V(H)| = b + |V(G)|$, then $\gamma((G \vee H)_\delta) = \gamma(G_\delta) + \gamma(H_\delta)$.

Proof. Since all vertices in $G \vee H$ have the same degree, it follows that $(G \vee H)_\delta \cong G_\delta + H_\delta$. This completes the proof. $\square$

In the following example, we construct two graphs G and H that exhibit the sharp upper bound given in Theorem 11.

Example 3. For $n \geq 2$, let G be a graph such that $V(G) = U \cup W$ where $U = \{u_1, \dots, u_n\}$ and $W = \{w_1, \dots, w_{2n}\}$ in which U and W induce complete graphs K_n and K_{2n} , respectively, and u_i is adjacent to w_{2i-1} and w_{2i} for each $i = 1, \dots, n$. See Figure 2 for an example when $n = 2$. With this construction, we have $G_\delta = nP_3$, a disjoint union of n paths of order 3. Also $\gamma(G_\delta) = n$. Let H be an $(n+1)$ -regular graph with $3n$ vertices. We have $H_\delta = \overline{H}$.

In $G \vee H$, the degree of u_i is $4n+1$, the degree of w_i is $5n$, and the degree of $v \in V(H)$ is $4n+1$. Then

$$\begin{aligned} E((G \vee H)_\delta) &= E(G_\delta) \cup E(H_\delta) \cup \{wv : w \in W \text{ and } v \in V(H)\} \\ &= \{u_i w_{2i-1}, u_i w_{2i} : i = 1, \dots, n\} \cup E(\overline{H}) \cup \{wv : w \in W \text{ and } v \in V(H)\}. \end{aligned}$$

We see that $U \cup \{w_1\}$ is a dominating set of $G \vee H$ of size $n+1$.

Let S be a dominating set of $(G \vee H)_\delta$. We will show that $|S| \geq n+1$. Suppose that $|S| = n$. To cover u_i , at least one of u_i, w_{2i-1}, w_{2i} must be in S . Since $|S| = n$, exactly one of u_i, w_{2i-1}, w_{2i} is in S . If $S = U$, then S does not cover $V(H)$. If $S \neq U$, we assume that $u_1 \notin S$ and $w_1 \in S$. Then w_2 is not covered by S . Hence, S cannot be a dominating set, which is a contradiction.

Thus $\gamma((G \vee H)_\delta) = n+1 = \gamma(G_\delta) + 1$.

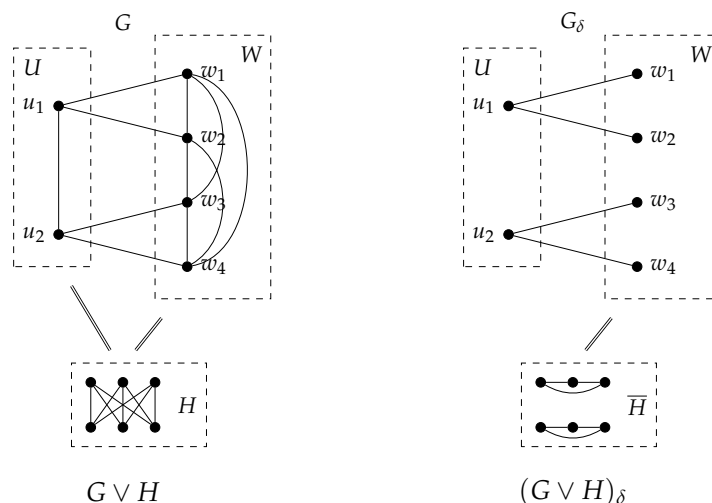


Figure 2. An example where $\gamma((G \vee H)_\delta) = \gamma(G_\delta) + 1$ as discussed in Example 3.

Theorem 11. For any graph G and H , if G is nonregular, then $\gamma((G \vee H)_\delta) \leq \gamma(G_\delta) + 1$. Furthermore, the bound is sharp for infinitely many graphs.

Proof. Let $m = m(G)$ and $l = m(H)$, and let $\mathcal{P}(G) = \{V_{d_1}(G), \dots, V_{d_m}(G)\}$ and $\mathcal{P}(H) = \{V_{d'_1}(H), \dots, V_{d'_l}(H)\}$ be the partitions of the vertex sets of G and H respectively. A vertex in $V_{d_i}(G)$

and a vertex in $V_{d'_j}(H)$ have the same degree in $G \vee H$ if and only if $d_i + |V(H)| = d'_j + |V(G)|$. Without loss of generality, we order d_i so that $d_i + |V(H)| = d'_i + |V(G)|$ for all $i = 1, \dots, k$ for some $k \leq \min\{m, \ell\}$.

We note that $(G \vee H)_\delta[V(G)] = G_\delta$ and $(G \vee H)_\delta[V(H)] = H_\delta$. In addition, in graph $(G \vee H)_\delta$, the vertices in $V_{d_i}(G)$ are adjacent to the vertices in $V_{d'_j}(H)$ for any $j \neq i$.

Let D be a dominating set of G_δ . Since G is nonregular, $V(G) \setminus V_{d_i}(G)$ is nonempty for all $i = 1, \dots, k$. If $D \subseteq V_{d_i}(G)$ for some $i = 1, \dots, k$, then $D \cup \{v\}$ is a dominating set of $(G \vee H)_\delta$ where $v \in V(G) \setminus V_{d_i}(G)$. This is because D dominates $V(G \vee H) \setminus V_{d'_i}(H)$ and v dominates $V_{d'_i}(H)$. Otherwise, D is a dominating set of $(G \vee H)_\delta$.

The bound is sharp by Example 3. $\square$

Corollary 1. *Let G and H be nonregular graphs. Then*

$$\gamma((G \vee H)_\delta) \leq \min\{\gamma(G_\delta), \gamma(H_\delta)\} + 1.$$

Table 1. The summary of bounds on the join of two graphs. To use the table, first check whether G and H satisfy Condition (1), then proceed with the types of the two graphs.

Condition (1) in Theorem 9	G	H	Upper bound of $\gamma((G \vee H)_\delta)$	Reference
true	any graph	any graph	2	Theorem 9
false	regular	regular	$\gamma(G_\delta) + \gamma(H_\delta)$	Theorem 10
false	regular	nonregular	$\gamma(H_\delta) + 1$	Theorem 11
false	nonregular	regular	$\gamma(G_\delta) + 1$	Theorem 11
false	nonregular	nonregular	$\min\{\gamma(G_\delta), \gamma(H_\delta)\} + 1$	Corollary 1

5. Cartesian Product of Graphs

This section contains our results on upper bounds on the δ -complement of the Cartesian product of two graphs, in particular, when one of them is a regular graph or a cycle. Sharp bounds are obtained when both of the graphs are regular or one of them is a cycle. Recall that, for a graph G , the vertex partition of $V(G)$ by the degrees is $\mathcal{P}(G)$, and the number of singletons in $\mathcal{P}(G)$ is s_G .

There are a number of results of bounds on the domination number of the Cartesian product of graphs Gravier and Khelladi (1995); Hartnell (2004); Vizing (1963). We include one of the bounds below as Lemma 3, which, combining with Theorem 6, produces a rough bound on the δ -complement of the Cartesian product of any two graphs shown in Lemma 4.

Lemma 3 (Vizing (1963)). *Let G and H be graphs. Then*

$$\gamma(G \square H) \leq \min\{\gamma(G) \cdot |V(H)|, \gamma(H) \cdot |V(G)|\}.$$

Using Theorem 6, we get the following bound for $(G \square H)_\delta$.

Lemma 4. *Let G and H be graphs. Then,*

$$\gamma((G \square H)_\delta) \leq \min\{\gamma(G_\delta) \cdot |V(H)|, \gamma(H_\delta) \cdot |V(G)|\}.$$

Proof. From Theorem 6, we have $G_\delta \square H_\delta \subset (G \square H)_\delta$ and $V(G_\delta \square H_\delta) = V((G \square H)_\delta)$. We have $\gamma((G \square H)_\delta) \leq \gamma(G_\delta \square H_\delta) \leq \min\{\gamma(G_\delta) \cdot |V(H)|, \gamma(H_\delta) \cdot |V(G)|\}$. The last inequality follows from Lemma 3. $\square$

We note that the bound in Lemma 4 is sharp. Some examples include $G = P_3$ and $H = P_2$ or when one of the graphs is K_1 .

When one of the graphs in the product is regular, the bounds can be improved as shown in Theorem 12. We first give the following lemmas which will be used in Theorem 12.

Lemma 5. For $r \geq 3$ and $n \geq 3$, then $\gamma((K_r \square \overline{K_n})_\delta) = 2$.

Proof. Two vertices (u, v) and (u', v') are adjacent in $(K_r \square \overline{K_n})_\delta$ if and only if $v \neq v'$. Let $u \in V(K_r)$ and $v_1, v_2 \in V(K_n)$ where $v_1 \neq v_2$. Then (u, v_i) dominates $V(K_r) \times (V(K_n) \setminus \{v_i\})$. Hence, the set $\{(u, v_1), (u, v_2)\}$ is a dominating set. Since every vertex of $(K_r \square \overline{K_n})_\delta$ is of degree $r(n-1) < rn-1$, there is no dominating set of size 1. Hence, $\gamma((K_r \square \overline{K_n})_\delta) = 2$. $\square$

Lemma 6. For $r \geq 1$ and $n \geq 3$, we have

$$\gamma((K_r \square C_n)_\delta) = \begin{cases} 3 & \text{if } r \neq 2 \text{ and } n = 3 \\ 2 & \text{otherwise.} \end{cases}$$

Proof. Consider $r = 1$. Then, $K_r \square C_n \cong C_n$. Hence, $\gamma((K_r \square C_3)_\delta) = \gamma(\overline{C_3}) = 3$ and $\gamma((K_r \square C_n)_\delta) = \gamma(\overline{C_n}) = 2$ when $n \geq 4$.

Consider $r = 2$. We write $K_2 = u_1 u_2$. Let $v \in V(C_n)$. Then, $\{(u_1, v), (u_2, v)\}$ is a minimum dominating set of $(K_r \square C_n)_\delta$, see Figure 3 for an example.

Consider $r \geq 3$. Let $u \in V(K_r)$. For $n = 3$, we write $C_3 = v_1 v_2 v_3 v_1$. We have that $\{(u, v_1), (u, v_2), (u, v_3)\}$ is a minimum dominating set of $(K_r \square C_3)_\delta$, see Figure 3. For $n \geq 4$, we let $C_n = w_1 w_2 \dots w_n w_1$. Then $\{(u, w_1), (u, w_2)\}$ is a minimum dominating set of $(K_r \square C_n)_\delta$. $\square$

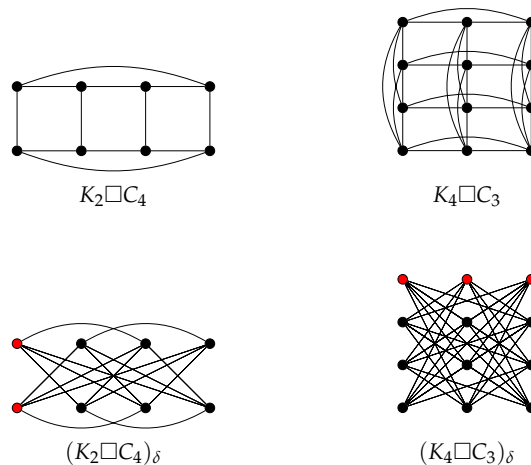


Figure 3. (Top figures) $K_2 \square C_4$ and $K_4 \square C_3$. (Bottom figures) The δ -complement of $K_2 \square C_4$ and $K_4 \square C_3$ with vertices of their minimum dominating sets indicated in red color.

Lemma 7. For $r \geq 3$ and $n \geq 3$, we have $\gamma((K_r \square K_n)_\delta) = 3$.

Proof. Two vertices (u, v) and (u', v') are adjacent in $(K_r \square K_n)_\delta$ if and only if $u \neq u'$ and $v \neq v'$. Let $u_1, u_2 \in V(K_r)$ and $v_1, v_2 \in V(K_n)$ be such that $u_1 \neq u_2$ and $v_1 \neq v_2$. Suppose $D = \{(u_1, v_1), (u_1, v_2), (u_2, v_1)\}$. Consider $(u, v) \notin D$. If $u = u_1$, then $v \neq v_1$. So, (u, v) is adjacent to (u_2, v_1) . If $u \neq u_1$, then (u, v) is adjacent to (u_1, v_1) or (u_1, v_2) . Hence, D is a dominating set.

Suppose there is a dominating set $S = \{(x, y), (x', y')\}$ of size 2. If $x = x'$, then (x, y'') where $y'' \notin \{y, y'\}$ is not adjacent to any vertices in S . If $y = y'$, then (x'', y) where $x'' \notin \{x, x'\}$ is not adjacent to any vertices in S . If $x \neq x'$ and $y \neq y'$, then (x, y') is not adjacent to any vertices in S . This is a contradiction. So, the domination number is 3. $\square$

Theorem 12. Let G and H be graphs, each with at least two vertices. If H is a regular graph, then

$$\gamma((G \square H)_\delta) \leq \begin{cases} 2m(G) - s_G & \text{if } \gamma(H_\delta) = 1, \\ 2m(G) & \text{if } \gamma(H_\delta) = 2, \\ 3m(G) + (\gamma(H_\delta) - 3)s_G & \text{otherwise.} \end{cases}$$

Furthermore, the bounds are sharp for infinitely many graphs.

Proof. Let $m = m(G)$. For a vertex partition $\mathcal{P}(G) = \{V_{d_1}(G), \dots, V_{d_m}(G)\}$, we order it so that $|V_{d_1}(G)| \leq |V_{d_2}(G)| \leq \dots \leq |V_{d_m}(G)|$. Since H is regular, we see that $\mathcal{P}(G \square H) = \{V_{d_1}(G) \times V(H), \dots, V_{d_m}(G) \times V(H)\}$ is a vertex partition of the vertices of $G \square H$ based on its degree.

Let $D \subseteq V(H)$ be a minimum dominating set of H_δ . For any $i \leq s_G$, we have that $V_{d_i}(G) \times D$ dominates $V_{d_i}(G) \times V(H)$ in $(G \square H)_\delta$. Thus, we need at most $\gamma(H_\delta) \cdot s_G$ vertices to cover $\bigcup_{i=1}^{s_G} V_{d_i}(G) \times V(H)$. Next, we consider each $i \geq s_G + 1$. Let $u_i \in V_{d_i}(G)$.

Case 1. $\gamma(H_\delta) = 1$. Let $v, w \in V(H)$ be such that v dominates H_δ and $w \neq v$. We have (u_i, v) dominates $\{u_i\} \times V(H)$ in $(G \square H)_\delta$. Let $(x, y) \in V_{d_i}(G) \times V(H)$ where $x \neq u_i$.

- If $y \neq v$, then (x, y) is not adjacent to (u_i, v) in $G \square H$.
- If $y = v$, then (x, y) is not adjacent to (u_i, w) in $G \square H$.

We have $\{(u_i, v), (u_i, w)\}$ dominates $V_{d_i}(G) \times V(H)$ in $(G \square H)_\delta$ for $s_G + 1 \leq i \leq m$. Thus, we need at most $2(m - s_G)$ vertices to cover $\bigcup_{i=s_G+1}^m V_{d_i}(G) \times V(H)$. Hence, in this case, $\gamma((G \square H)_\delta) \leq s_G + 2(m - s_G) = 2m - s_G$.

To show that this bound is sharp, for $m \geq 2$ and $n \geq 3$, we let $G = K_1 + K_3 + K_4 + \dots + K_{m+1}$ and H be a null graph $\overline{K_n}$. We see that $s_G = 1$ and $\gamma(H_\delta) = 1$. Note that

$$G \square H = (K_1 \square \overline{K_n}) + (K_3 \square \overline{K_n}) + \dots + (K_{m+1} \square \overline{K_n}).$$

Since $G \square H$ contains no edge between two vertices of different degrees, it follows that $(G \square H)_\delta = (K_1 \square \overline{K_n})_\delta + (K_3 \square \overline{K_n})_\delta + \dots + (K_{m+1} \square \overline{K_n})_\delta$. By Lemma 5, we have $\gamma((K_r \square \overline{K_n})_\delta) = 2$ for $r \geq 3$. We note $\gamma((K_1 \square \overline{K_n})_\delta) = \gamma(K_n) = 1$. Thus, $\gamma((G \square H)_\delta) = 1 + 2(m - 1) = 2m - 1 = 2m - s_G$.

Case 2. $\gamma(H_\delta) = 2$. Let $v, w \in V(H)$ be such that $v \neq w$ and $\{v, w\}$ dominates H_δ . We have $\{(u_i, v), (u_i, w)\}$ dominates $\{u_i\} \times V(H)$ in $(G \square H)_\delta$. Consider $(x, y) \in V_{d_i}(G) \times V(H)$ where $x \neq u_i$.

- If $y \neq v$, then (x, y) is not adjacent to (u_i, v) in $G \square H$.
- If $y \neq w$, then (x, y) is not adjacent to (u_i, w) in $G \square H$.

It follows that $\{(u_i, v), (u_i, w)\}$ dominates $V_{d_i}(G) \times V(H)$ in $(G \square H)_\delta$ for $s_G + 1 \leq i \leq m$. Thus, we need at most $2(m - s_G)$ vertices to cover $\bigcup_{i=s_G+1}^m V_{d_i}(G) \times V(H)$. Hence, in this case, $\gamma((G \square H)_\delta) \leq 2s_G + 2(m - s_G) = 2m$.

To show that this bound is sharp, for $m \geq 1$ and $n \geq 4$, we let $G = K_1 + \dots + K_m$ and $H = C_n$. We see that $s_G = 1$ and $\gamma(H_\delta) = 2$. Note that

$$G \square H = (K_1 \square C_n) + \dots + (K_m \square C_n).$$

Since $G \square H$ contains no edge between two vertices of different degrees, it follows that $(G \square H)_\delta = (K_1 \square C_n)_\delta + \dots + (K_m \square C_n)_\delta$. By Lemma 6, we have $\gamma((K_r \square C_n)_\delta) = 2$ for $r \geq 1$. Thus, $\gamma((G \square H)_\delta) = 2m$.

Case 3. $\gamma(H_\delta) \geq 3$. For each $i \geq s_G + 1$, pick any two vertices $a_i, b_i \in V_{d_i}(G)$ and any two vertices $v_1, v_2 \in V(H)$. Let $(x, y) \in V_{d_i}(G) \times V(H)$.

- If $x \neq a_i$ and $y \neq v_1$, then (x, y) is not adjacent to (a_i, v_1) in $G \square H$.
- If $x = a_i$ and $y \neq v_1$, then (x, y) is not adjacent to (b_i, v_1) in $G \square H$.
- If $x \neq a_i$ and $y = v_1$, then (x, y) is not adjacent to (a_i, v_2) in $G \square H$.

Hence, $\{(a_i, v_1), (b_i, v_1), (a_i, v_2)\}$ dominates $V_{d_i}(G) \times V(H)$ in $(G \square H)_\delta$ for $s_G + 1 \leq i \leq m$. Thus, we need at most $3(m - s_G)$ vertices to cover $\bigcup_{i=s_G+1}^m V_{d_i}(G) \times V(H)$. Hence, in this case, $\gamma((G \square H)_\delta) \leq \gamma(H_\delta) \cdot s_G + 3(m - s_G) = 3m + (\gamma(H_\delta) - 3)s_G$.

To show that this bound is sharp, for $m \geq 2$ and $n \geq 3$, we let $G = K_1 + K_3 + K_4 + \cdots + K_{m+1}$ and $H = K_n$. We see that $s_G = 1$ and $\gamma(H_\delta) = n$. Note that

$$G \square H = (K_1 \square K_n) + (K_3 \square K_n) + \cdots + (K_{m+1} \square K_n).$$

Since $G \square H$ contains no edge between two vertices of different degrees, it follows that $(G \square H)_\delta = (K_1 \square K_n)_\delta + (K_3 \square K_n)_\delta + \cdots + (K_{m+1} \square K_n)_\delta$. By Lemma 7, we have $\gamma((K_r \square K_n)_\delta) = 3$ for $r \geq 3$. We note $\gamma((K_1 \square K_n)_\delta) = \gamma(\overline{K_n}) = n$. Thus, $(G \square H)_\delta = n + 3(m - 1) = 3m + (\gamma(H_\delta) - 3)s_G$. $\square$

Consequently, we obtain sharp upper bounds when both graphs in the product are regular.

Corollary 2. *Let G and H be regular graphs, each with at least two vertices. Then*

$$\gamma((G \square H)_\delta) \leq \begin{cases} 2 & \text{if } \min\{\gamma(G_\delta), \gamma(H_\delta)\} \leq 2, \\ 3 & \text{otherwise.} \end{cases}$$

Furthermore, the bounds are sharp for infinitely many graphs.

Proof. Note that $m(G) = m(H) = 1$ and $s_G = s_H = 0$. The result follows directly from Theorem 12.

Next, we show that the bounds are sharp. For $n \geq 3$, $\min\{\gamma((K_2)_\delta), \gamma((C_n)_\delta)\} \leq 2$. Lemma 6 implies that $\gamma((K_2 \square C_n)_\delta) = 2$. In addition, $\min\{\gamma((K_n)_\delta), \gamma((C_3)_\delta)\} = 3$. Lemma 6 implies that $\gamma((K_n \square C_3)_\delta) = 3$. Hence, both bounds are sharp. $\square$

Finally, the following theorem is another special case of Theorem 12 where one of the graphs in the product is a cycle. In this case, sharp bounds can also be obtained.

Theorem 13. *Let G be a graph. Then*

$$\gamma((G \square C_n)_\delta) \leq \begin{cases} 3m(G) & \text{if } n = 3, \\ 2m(G) & \text{if } n \geq 4. \end{cases}$$

Furthermore, the bounds are sharp for infinitely many graphs.

Proof. Let $m = m(G)$. If $|V(G)| = 1$, then $G = K_1$ and $m = 1$. From Lemma 6, we get $\gamma((G \square C_3)_\delta) = 3$ and $\gamma((G \square C_n)_\delta) = 2$ for $n \geq 4$.

For $|V(G)| \geq 2$, Theorem 12 implies the result. To show that the bounds are sharp, we let $G = K_3 + K_4 + \cdots + K_{m+2}$. Note that

$$G \square C_n = (K_3 \square C_n) + \cdots + (K_{m+2} \square C_n).$$

So, $G \square C_n$ contains no edge between two vertices of different degrees, hence

$$(G \square C_n)_\delta = (K_3 \square C_n)_\delta + \cdots + (K_{m+2} \square C_n)_\delta.$$

In the case that $n = 3$, by Lemma 6, $\gamma((K_r \square C_3)_\delta) = 3$ for $3 \leq r \leq m + 2$. Hence, $\gamma((G \square C_3)_\delta) = 3m$. In the case that $n \geq 4$, by Lemma 6, $\gamma((K_r \square C_n)_\delta) = 2$ for $3 \leq r \leq m + 2$. Hence, $\gamma((G \square C_n)_\delta) = 2m$. $\square$

Table 2. The summary of bounds on the Cartesian product of two graphs. Note that, other than the first case, G and H are required to contain at least two vertices.

Condition	G	H	Upper bound of $\gamma((G \square H)_\delta)$	Reference
-	any graph	any graph	$\min\{\gamma(G_\delta) \cdot V(H) , \gamma(H_\delta) \cdot V(G) \}$	Lemma 4
$\min\{\gamma(G_\delta), \gamma(H_\delta)\} \leq 2$	regular	regular	2	Corollary 2
$\min\{\gamma(G_\delta), \gamma(H_\delta)\} > 2$	regular	regular	3	Corollary 2
$\gamma(H_\delta) = 1$	any graph	regular	$2m(G) - s_G$	Theorem 12
$\gamma(H_\delta) = 2$	any graph	regular	$2m(G)$	Theorem 12
$\gamma(H_\delta) > 2$	any graph	regular	$3m(G) + (\gamma(H_\delta) - 3)s_G$	Theorem 12

6. Discussion and Conclusion

In this research, we investigate the domination numbers of the δ -complements of graphs. We provide Nordhaus-Gaddum type bounds on the domination number between a graph and its δ -complement, bounds on the domination number of the δ -complement of joined graphs and Cartesian products of graphs. The bounds are in terms of the number of distinct degrees, the domination number of the δ -complement of each graph in the product, or the number of degrees with one vertex, and the given bounds are sharp.

Many interesting aspects of the domination numbers of the δ -complements of graphs are yet to be explored, for example, bounds when both graphs in the product are non-regular, bounds on other types of graph products, and bounds on other types of domination numbers of the δ -complement graphs.

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