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Article

An Exploration on Z-Number and Its Properties

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Abstract: For better solving both fuzziness and reliability occurring in practical issues simultaneously, Z-number deserves further exploration under uncertainty environment. Based on the credibility distribution and conversion rules, we define its expected value, variance, and semi-variance, and realize the feasibility of calculation by deriving calculation formulas. Moreover, we delve into three characteristics inherent in symmetrical Z-numbers. The link that followed between the variance and the semi-variance of Z-numbers is discovered and proved. Furthermore, we apply the formulas of expected value and variance on Z-numbers to examples, whose consequences validate our proposed formulas. The findings indicate the significance of our study in applying the expected value and variance of fuzzy sets across diverse fields.

Keywords: z-numbers; expected value; variance; semi-variance

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1. Introduction

Living in the world of fuzziness and uncertainty, we cannot avoid dealing with vagueness in everyday lives [1]. To cope with this issue, the concept of fuzzy sets has been concieved, refined, and extended in numerous scenarios by a substantial body of literature [2]. But after a certain point in this type of research, it becomes clear that none of these concepts could deal with real problems involving both fuzziness and reliability. To address this dilemma, in 2011, Zadeh [3] first formally introduced Z-numbers. On the basis of fuzzy sets, Z-numbers combine constraint and reliability to give both a description of fuzziness and a measure of the reliability of the evaluated information, which in turn can better explain the information conveyed by natural human language on complex uncertainty problems. Currently, Z-numbers have been widely used in plenty of practical issues. For instance, in psychology area, Kushal [4] combined Z-numbers with psychometric scales in order to extract useful information. In safety accident analysis area, Z-numbers were used in uncertainty analysis for the risks of building construction [5] and open surface mining [6]. In addition, Z-numbers also contributed to medical diagnosis issues [7], portfolio problems [8] and so on.

In the process of studying practical problems in these fields, mathematical properties of Z-numbers are often used, making the research on this aspect indispensable. Particularly, expected value and variance, as two significant digital characteristics in statistics, respectively, are used to reflect the degree of the average value and concentration of data as well as to reflect the magnitude of fluctuations within a set of data, both of which have broader applications in mathematics, economic trade, education, agriculture and many other fields with quantitative and mathematical data [9]. This is also true in practical issues with ambiguous, imprecise, and uncertain information, where the expected value and variance of fuzzy variables are important tools for the quantitative studies of such problems. We summarize the main studies involving the mathematical characteristics of Z-numbers, as Table 1 presents, and can see that research on Z-number operations is gradually fleshing out.

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Table 1. A literature summary on mathematical properties of Z-numbers

Literature	Category		Arithmetic Operations					Conversion
	Discrete	Continuous	Fundamental Operations	Extremum	Square/ Square-root	Expected Value	Variance/ Semi-variance	
Kang et al. (2012)	√							√
Aliev et al. (2015)	√		√		√			
Aliev et al. (2016)		√	√	√	√			
Jiang et al. (2017)	√							
Aliev et al. (2018)	√	√	√	√	√			
Alizadeh et al. (2018)	√	√	√					
Aliev & Alizadeh (2018)	√	√	√					
Peng et al. (2019)		√	√					√
Cheng et al. (2021)	√							√
Jia & Hu (2022)	√	√	√					√
This Paper	√	√	√			√	√	√

To address the difficulty of determining the underlying possibility distribution of Z-number, Kang et al. [10] first presented a transformation method between Z-numbers and classical fuzzy numbers in 2012. This method was improved by Cheng et al. [11] in 2021, who additionally considered the effect of hidden probability distribution generated on Z-number. In addition to this general conversion method, other innovative transformation approaches like the Z-trapezium-normal clouds (ZTNCs) have been proposed to simplify the calculation [12]. Since 2015, scholars, typified by Aliev, opened up the research on Z-number mathematical operations. For instance, Aliev et al. [13] focused on arithmetic operations on discrete Z-numbers, including the four fundamental operations, squaring, square root, and ordering in 2015. Within less than two years afterwards, Aliev et al. [14] introduced formulas for basic arithmetic operations on continuous ones. Jiang et al. [15] in 2017 put forward a new generalized ranking method by combining the spreads of fuzzy numbers, the weights of centroid points, and the degrees of fuzziness. In 2018, Aliev et al. [16] first developed a general approach to construct Z-number functions based on the topologic principle, constructing typical functions such as multiplication, exponents, minimum and maximum of Z-numbers. In 2018, two groups of researchers, Alizadeh et al. [17] and Aliev and Alizadeh [18], targeted the study of Z-number with fundamental properties under additive and multiplicative operations, respectively. In 2020, Mazandarani and Zhao [19] further discussed the concept of Z-differential equation and described its framework in great detail. In 2022, a rectangular coordinate system was firstly adopted by Jia and Hu [20] to cope with linguistic Z-numbers, meanwhile, arithmetic operations were defined.

Numerous theoretical innovations of Z-numbers, including their transformations and algorithms, have emerged during the last decade [20]. However, compared with the existing studies of fuzzy set theory, the development of Z-number theory is still immature. This is precisely the case in the mathematical features of Z-numbers, where the lack of operational laws for expected value, variance, and semi-variance becomes the research gaps. Meanwhile, the calculations proposed in existing research are inadequate in complex practical problems. Expected value, variance, and semi-variance, as three common digital characteristics, are of theoretical and practical importance in reflecting the properties of a particular aspect of a set of data [21]. In addition, from an information point of view, characterization is an effective means of compressing information, which is one of the reasons why there are a large amount of research focusing on them [22]. Undoubtedly, it is also of great significance for Z-numbers that have to deal not only with the uncertainty of the information but also with its reliability. In order to fill the arithmetic gap and provide the corresponding implementation-friendly calculation methods, this paper focuses on the operational laws of these three characteristics based mainly on their own definitions and the Z-number transformation method forwarded by Kang et al. [10].

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Therefore, in this study, the concepts of Z-numbers' expected value, variance, and semi-variance are studied for the first time and give a further discussion. Inspired by Kang et al. [10], we convert Z-numbers into regular fuzzy numbers to carry out our work, including the definitions, theorems, and calculation formulas. In this conversion process, since the information containing in the Z-numbers is also transformed into fuzzy numbers, this paper transfers the study of expected value, variance, and semi-variance to that of the corresponding mathematical properties of its converted regular fuzzy number, named converted Z-numbers. Then, according to the credibility theory stated by Liu [23], the concepts of these three digital characteristics of Z-numbers are delineated respectively, and the related theorems and formulas are derived. Meanwhile, examples are displayed to illustrate the calculation process and its application to help the reader understand. The validation of findings indicates the broad application of this paper's formulas to calculate the expected value, variance, and semi-variance of various types of Z-numbers, rather than being restricted to a particular category. At the same time, the computational complexity of these formulas is not high, which is conducive to practical application. Moreover, using the expected value, variance, and semi-variance of Z-numbers has great potential to help us analyze the mathematical properties associated with fuzzy events, allowing us to properly handle information involving fuzziness and reliability. This implies that the successful implementation of these three numerical calculations could have a positive impact on Z-numbers' applications in the realistic issues, except for broadening the scope of research on the mathematical properties of Z-numbers. Furthermore, the introduction of these formulas further complements the applications of expected value, variance, and semi-variance to the whole field of fuzzy sets.

The article is organized as follows. Relying on some basic concepts of Z-numbers as shown in Appendix and the credibility distribution, we provide the formulas for calculating the expected value of Z-numbers in Section 2. The formulas for the variance are provided in Section 3. Furthermore, two examples of the calculation of asymmetrical and symmetrical Z-numbers by combining the expected value and variance formulas proposed above are discussed in this section. In Section 4, we give the definitions and theorems related to the semi-variance of Z-numbers. Finally, Section 5 draws the conclusions.

2. The Expected Value of Z-Numbers

Due to the result on the connection between Z-numbers and its converted regular fuzzy numbers derived by Kang et al. [10], we regard the information on Z-numbers as the converted Z-number. In this section, we provide the definition and calculation formulas for the expected value of Z-numbers by the credibility distribution, as it can be deduced through Definition 6 in Appendix.

Definition 1. *Let Z be a Z-number and we denote its converted regular fuzzy number as $\widetilde{Z}$, then its expected value can be represented as:*

$$E[Z] = \int_0^{+\infty} \mathrm{Cr}\{\tilde{Z} \geq u\}du - \int_{-\infty}^{0} \mathrm{Cr}\{\tilde{Z} \leq u\}du. \tag{1}$$

Theorem 1. *Assuming that Z is a Z-number, and we denote its converted regular fuzzy number as $\widetilde{Z}$, then its expected value is formulated as:*

$$E[Z] = \int_0^1 \Phi_{\tilde{Z}}^{-1}(\alpha)d\alpha, \tag{2}$$

where $\Phi_{\tilde{Z}}^{-1}$ is the inverse credibility distribution of $\widetilde{Z}$.

Theorem 2. *For a Z-number $Z = (A, B)$, where A is a trapezoidal fuzzy number ($A = (a_1, a_2, a_3, a_4)$) and B is a trapezoidal fuzzy number ($B = (b_1, b_2, b_3, b_4)$), its expected value is constructed as:*

$$E[Z] = \frac{1}{4}\sqrt{\alpha}(a_1 + a_2 + a_3 + a_4), \tag{3}$$

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where $\alpha = \frac{1}{3}[(b_1+b_2+b_3+b_4)+\frac{b_1b_2-b_3b_4}{b_3+b_4-b_1-b_2}]$.

Proof. By the Equation (1) in Appendix, we can get the crisp number α,

$$\alpha = \frac{\int x\mu_B dx}{\int \mu_B dx} = \frac{1}{3}[(b_1+b_2+b_3+b_4)+\frac{b_1b_2-b_3b_4}{b_3+b_4-b_1-b_2}]. \tag{4}$$

Then, by the Equation (3) in Appendix, we can get the converted Z-number $\widetilde{Z}$,

$$\tilde{Z} = (\sqrt{\alpha}a_1, \sqrt{\alpha}a_2, \sqrt{\alpha}a_3, \sqrt{\alpha}a_4). \tag{5}$$

By the result of $\widetilde{Z}$, the membership function of $\widetilde{Z}$ is as follow:

$$\mu_{\tilde{Z}} = \begin{matrix} \frac{1}{\sqrt{\alpha}(a_2-a_1)}u - \frac{\sqrt{\alpha}a_1}{\sqrt{\alpha}(a_2-a_1)}, & u \in [\sqrt{\alpha}a_1, \sqrt{\alpha}a_2] \\ 1, & u \in [\sqrt{\alpha}a_2, \sqrt{\alpha}a_3] \\ \frac{-1}{\sqrt{\alpha}(a_4-a_3)}u + \frac{\sqrt{\alpha}a_4}{\sqrt{\alpha}(a_4-a_3)}, & u \in [\sqrt{\alpha}a_3, \sqrt{\alpha}a_4]. \end{matrix} \tag{6}$$

By Theorem 1 and Definition 5 in Appendix, the credibility distribution and its inverse function can be calculated as:

$$\Phi_{\tilde{Z}} = \begin{cases} 0, & u \le \sqrt{\alpha}a_1 \\ \frac{u-\sqrt{\alpha}a_1}{2(\sqrt{\alpha}a_2-\sqrt{\alpha}a_1)}, & \sqrt{a}a_1 \le u \le \sqrt{\alpha}a_2 \\ \frac{u+\sqrt{\alpha}a_4-2\sqrt{\alpha}a_3}{2(\sqrt{\alpha}a_4-\sqrt{\alpha}a_3)}, & \sqrt{\alpha}a_3 \le u \le \sqrt{\alpha}a_4 \\ 1, & u \ge \sqrt{\alpha}a_4, \end{cases} \tag{7}$$

$$\Phi_{\tilde{Z}}^{-1} = \begin{cases} 2(\sqrt{\alpha}a_2-\sqrt{\alpha}a_1)t+\sqrt{\alpha}a_1, & t \in [0, \frac{1}{2}) \\ 2(\sqrt{\alpha}a_4-\sqrt{\alpha}a_3)t+2\sqrt{\alpha}a_3-\sqrt{\alpha}a_4, & t \in [\frac{1}{2}, 1]. \end{cases} \tag{8}$$

By Theorem 1, the expected value is calculated as:

$$\begin{aligned} E[Z] &= \int_0^1 \Phi_{\tilde{Z}}^{-1}(t)dt \\ &= \int_0^{\frac{1}{2}} [2(\sqrt{\alpha}a_2-\sqrt{\alpha}a_1)t+\sqrt{\alpha}a_1]dt \\ &\quad + \int_{\frac{1}{2}}^1 [2(\sqrt{2}a_4-\sqrt{\alpha}a_3)t+2\sqrt{\alpha}a_3-\sqrt{\alpha}a_4]dt \\ &= \frac{1}{4}\sqrt{\alpha}(a_1+a_2+a_3+a_4), \end{aligned} \tag{9}$$

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where $\alpha = \frac{1}{3}[(b_1 + b_2 + b_3 + b_4) + \frac{b_1 b_2 - b_3 b_4}{b_3 + b_4 - b_1 - b_2}]$. □

Remark 1. *There are three special cases of Theorem 2 as follows:*

Case I: For a Z-number $Z = (A, B)$, where A is a triangular fuzzy number ($A = (a_1, a_2, a_3)$) and B is a triangular fuzzy number($B = (b_1, b_2, b_3)$), its expected value is constructed as:

$$E[Z] = \frac{\sqrt{3(b_1 + b_2 + b_3)}}{12}(a_1 + 2a_2 + a_3). \tag{10}$$

Case II: For a Z-number $Z = (A, B)$, where A is a trapezoidal fuzzy number ($A = (a_1, a_2, a_3, a_4)$) and B is a triangular fuzzy number ($B = (b_1, b_2, b_3)$), its expected value is ascertained by:

$$E[Z] = \frac{\sqrt{3(b_1 + b_2 + b_3)}}{12}(a_1 + a_2 + a_3 + a_4). \tag{11}$$

Case III: For a Z-number $Z = (A, B)$, where A is a triangular fuzzy number ($A = (a_1, a_2, a_3)$) and B is a trapezoidal fuzzy number ($B = (b_1, b_2, b_3, b_4)$), its expected value is formulated by:

$$E[Z] = \frac{1}{4}\sqrt{\alpha}(a_1 + 2a_2 + a_3), \tag{12}$$

where $\alpha = \frac{1}{3}[(b_1 + b_2 + b_3 + b_4) + \frac{b_1 b_2 - b_3 b_4}{b_3 + b_4 - b_1 - b_2}]$.

Proof. For case I, when $a_2 = a_3$ and $b_2 = b_3$ in Theorem 2, the Z-number of this case is same as that in Theorem 2. Thus, its expected value can be calculated as:

$$\begin{aligned} \alpha &= \frac{1}{3}[(b_1 + b_2 + b_3 + b_4) + \frac{b_1 b_2 - b_3 b_4}{b_3 + b_4 - b_1 - b_2}] \\ &= \frac{b_1 + b_2 + b_3}{3}. \end{aligned} \tag{13}$$

$$\begin{aligned} E[Z] &= \frac{1}{4}\sqrt{\alpha}(a_1 + a_2 + a_3 + a_4). \\ &= \frac{\sqrt{3(b_1 + b_2 + b_3)}}{12}(a_1 + 2a_2 + a_3) \end{aligned} \tag{14}$$

For case II, when $b_2 = b_3$ in Theorem 2, the Z-number of this case is same as that in Theorem 2. Thus, its expected value can be calculated as:

$$\begin{aligned} \alpha &= \frac{1}{3}[(b_1 + b_2 + b_3 + b_4) + \frac{b_1 b_2 - b_3 b_4}{b_3 + b_4 - b_1 - b_2}] \\ &= \frac{b_1 + b_2 + b_3}{3}. \end{aligned} \tag{15}$$

$$\begin{aligned} E[Z] &= \frac{1}{4}\sqrt{\alpha}(a_1 + a_2 + a_3 + a_4). \\ &= \frac{\sqrt{3(b_1 + b_2 + b_3)}}{12}(a_1 + a_2 + a_3 + a_4) \end{aligned} \tag{16}$$

For case III, when $a_2 = a_3$ in Theorem 2, the Z-number of this case is same as that in Theorem 2. Thus, its expected value can be calculated as:

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$$E[Z] = \frac{1}{4}\sqrt{\alpha}(a_1 + a_2 + a_3 + a_4).$$
$$= \frac{1}{4}\sqrt{\alpha}(a_1 + 2a_2 + a_3), \tag{17}$$

where $\alpha = \frac{1}{3}[(b_1 + b_2 + b_3 + b_4) + \frac{b_1 b_2 - b_3 b_4}{b_3 + b_4 - b_1 - b_2}]$. □

Theorem 3. *For a symmetric Z-number $Z = (A, B)$, A is a trapezoidal fuzzy number (A = (a_1, a_2, a_3, a_4)) with $m = |a_1 - a_2| = |a_3 - a_4|$, and B is a trapezoidal fuzzy number (B = (b_1, b_2, b_3, b_4)) with $n = |b_1 - b_2| = |b_3 - b_4|$. Then, its expected value can be formulated by:*

$$E[Z] = \frac{\sqrt{2(b_2 + b_3)}}{4}(a_2 + a_3). \tag{18}$$

Proof. According to the conditions of Theorem 3 and by Theorem 2, the expected value of a symmetric Z-number Z can be derived as:

$$\alpha = \frac{1}{3}[(b_1 + b_2 + b_3 + b_4) + \frac{b_1 b_2 - b_3 b_4}{b_3 + b_4 - b_1 - b_2}]$$
$$= \frac{b_2 + b_3}{2}. \tag{19}$$

$$E[Z] = \frac{\sqrt{\alpha}}{4}(a_1 + a_2 + a_3 + a_4)$$
$$= \frac{\sqrt{2(b_2 + b_3)}}{4}(a_2 + a_3). \tag{20}$$

□

Remark 2. *There are three special cases of Theorem 3 as follows:*

Case I: For a symmetric Z-number $Z = (A, B)$, A is a triangular fuzzy number (A = (a_1, a_2, a_3)) with $m = |a_1 - a_2| = |a_3 - a_2|$, and B is a triangular fuzzy number (B = (b_1, b_2, b_3)) with $n = |b_1 - b_2| = |b_3 - b_2|$. Then, its expected value is witten as:

$$E[Z] = a_2\sqrt{b_2}. \tag{21}$$

Case II: For a symmetric Z-number $Z = (A, B)$, where A is a trapezoidal fuzzy number (A = (a_1, a_2, a_3, a_4)) with $m = |a_1 - a_2| = |a_3 - a_4|$, and B is a triangular fuzzy number (B = (b_1, b_2, b_3)) with $n = |b_1 - b_2| = |b_3 - b_2|$, its expected value can be formulated by:

$$E[Z] = \frac{\sqrt{b_2}}{2}(a_2 + a_3). \tag{22}$$

Case III:For a symmetric Z-number Z=(A,B), where A is a triangular fuzzy number (A = (a_1, a_2, a_3)) with $m = |a_1 - a_2| = |a_3 - a_2|$, and B is a trapezoidal fuzzy number (B = (b_1, b_2, b_3, b_4)) with $n = |b_1 - b_2| = |b_3 - b_4|$, its expected value is constructed as:

$$E[Z] = \frac{a_2\sqrt{2(b_2 + b_3)}}{2}. \tag{23}$$

Proof. For case I, when $a_2 = a_3$ and $b_2 = b_3$ in Theorem 3, the Z-number of this case is same as that in Theorem 3. Thus, its expected value can be calculated as:

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$$
\begin{aligned}
E[Z] &= \frac{\sqrt{2(b_2+b_3)}}{4}(a_2+a_3) \\
&= a_2\sqrt{b_2}.
\end{aligned} \tag{24}
$$

For case II, when $b_2 = b_3$ in Theorem 3, the Z-number of this case is same as that in Theorem 3. Thus, its expected value can be calculated as:

$$
\begin{aligned}
E[Z] &= \frac{\sqrt{2(b_2+b_3)}}{4}(a_2+a_3) \\
&= \frac{\sqrt{b_2}}{2}(a_2+a_3).
\end{aligned} \tag{25}
$$

For case III, when $a_2 = a_3$ in Theorem 3, the Z-number of this case is same as that in Theorem 3. Thus, its expected value can be calculated as:

$$
\begin{aligned}
E[Z] &= \frac{\sqrt{2(b_2+b_3)}}{4}(a_2+a_3). \\
&= \frac{a_2\sqrt{2(b_2+b_3)}}{2}.
\end{aligned} \tag{26}
$$

□

3. The Variance of Z-Numbers

In this section, we provide the definition and calculation formulas for the variance of Z-numbers by the credibility distribution, which can be derived by Definition 7 in Appendix.

Definition 2. *Let Z be a Z-number and we denote its converted regular fuzzy number as $\widetilde{Z}$, then its variance is depicted as:*

$$V[Z] = E[(\tilde{Z}-\varepsilon)^2]. \tag{27}$$

Theorem 4. *Given that Z is a Z-number, and we denote its converted regular fuzzy number as $\widetilde{Z}$, then its variance is ascertained as:*

$$V[Z] = \int_0^{+\infty} [1-\Phi_{\tilde{Z}}(\varepsilon+\sqrt{u}) + \Phi_{\tilde{Z}}(\varepsilon-\sqrt{u})]du. \tag{28}$$

Proof. By the Definitions 1 and 2, we can deduce

$$
\begin{aligned}
V[Z] &= E[(\tilde{Z}-\varepsilon)^2] \\
&= \int_0^{+\infty} \mathrm{Cr}\{(\tilde{Z}-\varepsilon)^2 \geq u\}du - \int_{-\infty}^{0} \mathrm{Cr}\{(\tilde{Z}-\varepsilon)^2 \leq u\}du \\
&= \int_0^{+\infty} \mathrm{Cr}\{(\tilde{Z}-\varepsilon)^2 \geq u\}du \\
&= \int_0^{+\infty} \mathrm{Cr}\{\tilde{Z} \geq \sqrt{u}+\varepsilon\}du + \int_0^{+\infty} \mathrm{Cr}\{\tilde{Z} \leq \varepsilon-\sqrt{u}\}du \\
&= \int_0^{+\infty} [1-\Phi_{\tilde{Z}}(\varepsilon+\sqrt{u}) + \Phi_{\tilde{Z}}(\varepsilon-\sqrt{u})]du.
\end{aligned} \tag{29}
$$

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□

Theorem 5. *Given that Z is a Z-number, and we denote its converted regular fuzzy number as $\widetilde{Z}$, then its variance is calculated by:*

$$V[Z] = \int_0^1 [\Phi_{\tilde{Z}}^{-1}(t) - \varepsilon]^2 dt, \tag{30}$$

where $\Phi_{\tilde{Z}}^{-1}$ is the inverse credibility distribution of $\widetilde{Z}$.

Proof. By Theorem 5, we obtain

$$\begin{aligned} V[Z] &= E[(\tilde{Z} - \varepsilon)^2] \\ &= \int_0^{+\infty} [1 - \Phi_{\tilde{Z}}(\varepsilon + \sqrt{u}) + \Phi_{\tilde{Z}}(\varepsilon - \sqrt{u})] du. \end{aligned} \tag{31}$$

For the first part,

$$\begin{aligned} &\int_0^{+\infty} [1 - \Phi_{\tilde{Z}}(\varepsilon + \sqrt{u})] du \\ &= \int_{\Phi_{\tilde{Z}}(\varepsilon)}^1 (1 - t) d[\Phi_{\tilde{Z}}^{-1}(t) - \varepsilon]^2 \\ &= (1 - t)[\Phi_{\tilde{Z}}^{-1}(t) - \varepsilon]^2 |_{\Phi_{\tilde{Z}}(\varepsilon)}^1 - \int_{\Phi_{\tilde{Z}}(\varepsilon)}^1 [\Phi_{\tilde{Z}}^{-1}(t) - \varepsilon]^2 d(1 - t) \\ &= \int_{\Phi_{\tilde{Z}}(\varepsilon)}^1 [\Phi_{\tilde{Z}^{-1}}(t) - \varepsilon]^2 dt. \end{aligned} \tag{32}$$

For the second part,

$$\begin{aligned} &\int_0^{+\infty} \Phi_{\tilde{Z}}(\varepsilon - \sqrt{a}) du \\ &= \int_{\Phi_{\tilde{Z}}(\varepsilon)}^0 t d[-\Phi_{\tilde{Z}}^{-1}(t) + \varepsilon]^2 \\ &= t[-\Phi_{\tilde{Z}}^{-1}(t) + \varepsilon]^2 |_{\Phi_{\tilde{Z}}(\varepsilon)}^0 - \int_{\Phi_{\tilde{Z}}(v)}^0 [-\Phi_{\tilde{Z}}^{-1}(t) + \varepsilon]^2 dt \\ &= \int_0^{\Phi_{\tilde{Z}}(\varepsilon)} [\Phi_{\tilde{Z}}^{-1}(t) - \varepsilon]^2 dt. \end{aligned} \tag{33}$$

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Thus,

$$
\begin{aligned}
V[Z] &= E[(\tilde{Z}-\varepsilon)^2] \\
&= \int_0^{+\infty} [1-\Phi_{\tilde{Z}}(\varepsilon+\sqrt{n})+\Phi_{\tilde{Z}}(\varepsilon-\sqrt{u})]du \\
&= \int_{\Phi_{\tilde{Z}}(\varepsilon)}^1 [\Phi_{\tilde{Z}}^{-1}(t)-\varepsilon]^2 dt + \int_0^{\Phi_{\tilde{Z}}(\varepsilon)} [\Phi_{\tilde{Z}}^{-1}(t)-\varepsilon]^2 dt \\
&= \int_0^1 [\Phi_{\tilde{Z}}^{-1}(t)-\varepsilon]^2 dt.
\end{aligned} \tag{34}
$$

□

Theorem 6. *For a Z-number $Z=(A,B)$, A is a trapezoidal fuzzy number ($A=(a_1,a_2,a_3,a_4)$) and B is a trapezoidal fuzzy number ($B=(b_1,b_2,b_3,b_4)$). Thus, its variance can be formulated by:*

$$V[Z]=\frac{1}{6}\alpha[(a_1+a_2)^2+(a_3+a_4)^2-a_1a_2-a_3a_4]-\frac{\alpha}{16}(a_1+a_2+a_3+a_4)^2, \tag{35}$$

where $\alpha=\frac{1}{3}[(b_1+b_2+b_3+b_4)+\frac{b_1b_2-b_3b_4}{b_3+b_4-b_1-b_2}]$.

Proof. By the equation (1) in Appendix, we can get the crisp number α

$$\alpha=\frac{1}{3}[(b_1+b_2+b_3+b_4)+\frac{b_1b_2-b_3b_4}{b_3+b_4-b_1-b_2}]. \tag{36}$$

Then by the equation (3) in Appendix, we can get the converted Z-number $\tilde{Z}$,

$$\tilde{Z}=(\sqrt{\alpha}a_1,\sqrt{\alpha}a_2,\sqrt{\alpha}a_3,\sqrt{\alpha}a_4). \tag{37}$$

The membership function of $\tilde{Z}$ is as follow

$$
\mu_{\tilde{Z}} = \begin{cases}
\frac{1}{\sqrt{\alpha}(a_2-a_1)}u-\frac{\sqrt{\alpha}a_1}{\sqrt{\alpha}(a_2-a_1)}, & u\in[\sqrt{\alpha}a_1,\sqrt{\alpha}a_2] \\
1, & u\in[\sqrt{\alpha}a_2,\sqrt{\alpha}a_3] \\
\frac{-1}{\sqrt{\alpha}(a_4-a_3)}u+\frac{\sqrt{\alpha}a_4}{\sqrt{\alpha}(a_4-a_3)}, & u\in[\sqrt{\alpha}a_3,\sqrt{\alpha}a_4].
\end{cases} \tag{38}
$$

By Theorem 1 and Definition 5 in Appendix, the credibility distribution and its inverse form can be calculated as:

$$
\Phi_{\tilde{Z}} = \begin{cases}
0, & u\le\sqrt{\alpha}a_1 \\
\frac{u-\sqrt{\alpha}a_1}{2(\sqrt{\alpha}a_2-\sqrt{\alpha}a_1)}, & \sqrt{a}a_1\le u\le\sqrt{\alpha}a_2 \\
\frac{u+\sqrt{\alpha}a_4-2\sqrt{\alpha}a_3}{2(\sqrt{\alpha}a_4-\sqrt{\alpha}a_3)}, & \sqrt{\alpha}a_3\le u\le\sqrt{\alpha}a_4 \\
1, & u\ge\sqrt{\alpha}a_4,
\end{cases} \tag{39}
$$

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$$\Phi_{\tilde{Z}}^{-1} = \begin{cases} 2(\sqrt{\alpha}a_2 - \sqrt{\alpha}a_1)t + \sqrt{\alpha}a_1, & t \in [0, \frac{1}{2}) \\ 2(\sqrt{\alpha}a_4 - \sqrt{\alpha}a_3)t + 2\sqrt{\alpha}a_3 - \sqrt{\alpha}a_4, t \in [\frac{1}{2}, 1]. \end{cases} \tag{40}$$

By Theorem 5,

$$\begin{aligned} V[Z] &= \int_0^1 [\Phi_{\tilde{Z}}^{-1}(t) - \varepsilon]^2 dt \\ &= \int_0^1 [\Phi_{\tilde{Z}}^{-1}(t)]^2 dt + \int_0^1 \varepsilon^2 dt - 2\varepsilon \int_0^1 \Phi_{\tilde{Z}}^{-1}(t) dt. \end{aligned} \tag{41}$$

For the first part,

$$\begin{aligned} &\int_0^1 [\Phi_{\tilde{Z}}^{-1}(t)]^2 dt \\ &= \int_0^{\frac{1}{2}} [\Phi_{\tilde{Z}}(t)]^2 dt + \int_{\frac{1}{2}}^1 [\Phi_{\tilde{Z}}(t)]^2 dt \\ &= \frac{1}{6}\alpha[(a_1 + a_2)^2 - a_1 a_2] + \frac{1}{6}\alpha[(a_3 + a_4)^2 - a_3 a_4]. \end{aligned} \tag{42}$$

For the second part,

$$\int_0^1 \varepsilon^2 dt = \varepsilon^2. \tag{43}$$

For the third part,

$$-2\int_0^1 \Phi_{\tilde{Z}}^{-1}(t) dt = -2\varepsilon^2. \tag{44}$$

Thus,

$$\begin{aligned} V[Z] &= \int_0^1 [\Phi_{\tilde{Z}}^{-1}(t) - \varepsilon]^2 dt \\ &= \int_0^1 [\Phi_{\tilde{Z}}^{-1}(t)]^2 dt + \int_0^1 \varepsilon^2 dt - 2\varepsilon \int_0^1 \Phi_{\tilde{Z}}^{-1}(t) dt \\ &= \frac{1}{6}\alpha[(a_1 + a_2)^2 - a_1 a_2] + \frac{1}{6}\alpha[(a_3 + a_4)^2 - a_3 a_4] + \varepsilon^2 - 2\varepsilon^2 \\ &= \frac{1}{6}\alpha[(a_1 + a_2)^2 + (a_3 + a_4)^2 - a_1 a_2 - a_3 a_4] - \frac{\alpha}{16}(a_1 + a_2 + a_3 + a_4)^2, \end{aligned} \tag{45}$$

where $\alpha = \frac{1}{3}[(b_1 + b_2 + b_3 + b_4) + \frac{b_1 b_2 - b_3 b_4}{b_3 + b_4 - b_1 - b_2}]$. □

Remark 3. *There are three special cases of Theorem 6 as follows:*

Case I: For a Z-number $Z = (A, B)$, where A is a triangular fuzzy number($A = (a_1, a_2, a_3)$) and B is a triangular fuzzy number($B = (b_1, b_2, b_3)$), its variance is witten as:

$$V[Z] = \frac{1}{6}\alpha[(a_1 + a_2)^2 + (a_3 + a_2)^2 - a_1 a_2 - a_3 a_2] - \frac{\alpha}{16}(a_1 + 2a_2 + a_3)^2, \tag{46}$$

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where $\alpha = \dfrac{b_1 + b_2 + b_3}{3}$.

Case II: For a Z-number $Z = (A, B)$, where A is a trapezoidal fuzzy number ($A = (a_1, a_2, a_3, a_4)$) and B is a triangular fuzzy number ($B = (b_1, b_2, b_3)$), its variance is obtained:

$$V[Z] = \frac{1}{6}\alpha[(a_1 + a_2)^2 + (a_3 + a_4)^2 - a_1 a_2 - a_3 a_4] - \frac{\alpha}{16}(a_1 + a_2 + a_3 + a_4)^2, \tag{47}$$

where $\alpha = \dfrac{b_1 + b_2 + b_3}{3}$.

Case III: For a Z-number $Z = (A, B)$, where A is a triangular fuzzy number ($A = (a_1, a_2, a_3)$) and B is a trapezoidal fuzzy number ($B = (b_1, b_2, b_3, b_4)$), its variance can be formulated by:

$$V[Z] = \frac{1}{6}\alpha[(a_1 + a_2)^2 + (a_3 + a_2)^2 - a_1 a_2 - a_3 a_2] - \frac{\alpha}{16}(a_1 + 2a_2 + a_3)^2, \tag{48}$$

where $\alpha = \dfrac{1}{3}[(b_1 + b_2 + b_3 + b_4) + \dfrac{b_1 b_2 - b_3 b_4}{b_3 + b_4 - b_1 - b_2}]$.

Proof. For case I, when $a_2 = a_3$ and $b_2 = b_3$ in Theorem 6, the Z-number of this case is same as that in Theorem 6. Thus, its variance is given as:

$$\begin{aligned} \alpha &= \frac{1}{3}[(b_1 + b_2 + b_3 + b_4) + \frac{b_1 b_2 - b_3 b_4}{b_3 + b_4 - b_1 - b_2}] \\ &= \frac{b_1 + b_2 + b_3}{3}. \end{aligned} \tag{49}$$

$$\begin{aligned} V[Z] &= \frac{1}{6}\alpha[(a_1 + a_2)^2 + (a_3 + a_4)^2 - a_1 a_2 - a_3 a_4] - \frac{\alpha}{16}(a_1 + a_2 + a_3 + a_4)^2 \\ &= \frac{1}{6}\alpha[(a_1 + a_2)^2 + (a_3 + a_2)^2 - a_1 a_2 - a_3 a_2] - \frac{\alpha}{16}(a_1 + 2a_2 + a_3)^2. \end{aligned} \tag{50}$$

For case II, when $b_2 = b_3$ in Theorem 6, the Z-number of this case is same as that in Theorem 6. Thus, its expected value is given as:

$$\begin{aligned} \alpha &= \frac{1}{3}[(b_1 + b_2 + b_3 + b_4) + \frac{b_1 b_2 - b_3 b_4}{b_3 + b_4 - b_1 - b_2}] \\ &= \frac{b_1 + b_2 + b_3}{3}. \end{aligned} \tag{51}$$

$$V[Z] = \frac{1}{6}\alpha[(a_1 + a_2)^2 + (a_3 + a_4)^2 - a_1 a_2 - a_3 a_4] - \frac{\alpha}{16}(a_1 + a_2 + a_3 + a_4)^2. \tag{52}$$

For case III, when $a_2 = a_3$ in Theorem 6, the Z-number of this case is same as that in Theorem 6. Thus, its expected value is given as:

$$\begin{aligned} V[Z] &= \frac{1}{6}\alpha[(a_1 + a_2)^2 + (a_3 + a_4)^2 - a_1 a_2 - a_3 a_4] - \frac{\alpha}{16}(a_1 + a_2 + a_3 + a_4)^2 \\ &= \frac{1}{6}\alpha[(a_1 + a_2)^2 + (a_3 + a_2)^2 - a_1 a_2 - a_3 a_2] - \frac{\alpha}{16}(a_1 + 2a_2 + a_3)^2, \end{aligned} \tag{53}$$

where $\alpha = \dfrac{1}{3}[(b_1 + b_2 + b_3 + b_4) + \dfrac{b_1 b_2 - b_3 b_4}{b_3 + b_4 - b_1 - b_2}]$. □

Theorem 7. *For a symmetric Z-number $Z = (A, B)$, where A is a trapezoidal fuzzy number ($A = (a_1, a_2, a_3, a_4)$) with $m = |a_1 - a_2| = |a_3 - a_4|$, and B is a trapezoidal fuzzy number ($B = (b_1, b_2, b_3, b_4)$) with $n = |b_1 - b_2| = |b_3 - b_4|$, its expected value can be formulated by:*

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$$V[Z] = \frac{1}{24}(b_2 + b_3)(3a_1^2 + 3a_3^2 - 6a_1a_3 + m^2). \tag{54}$$

Proof. According to the conditions of Theorem 7 and by Theorem 6, the variance of a symmetric Z-number Z can be derived as:

$$\begin{aligned} \alpha &= \frac{1}{3}[(b_1 + b_2 + b_3 + b_4) + \frac{b_1b_2 - b_3b_4}{b_3 + b_4 - b_1 - b_2}] \\ &= \frac{b_2 + b_3}{2}. \end{aligned} \tag{55}$$

$$\begin{aligned} V[Z] &= \frac{1}{6}\alpha[(a_1 + a_2)^2 + (a_3 + a_4)^2 - a_1a_2 - a_3a_4] - \frac{\alpha}{16}(a_1 + a_2 + a_3 + a_4)^2 \\ &= \frac{1}{24}(b_2 + b_3)(3a_1^2 + 3a_3^2 - 6a_1a_3 + m^2). \end{aligned} \tag{56}$$

□

Remark 4. *There are three special cases of Theorem 7 as follows:*

Case I: For a symmetric Z-number $Z = (A, B)$, where A is a triangular fuzzy number ($A = (a_1, a_2, a_3)$) with $m = |a_1 - a_2| = |a_3 - a_2|$, and B is a triangular fuzzy number ($B = (b_1, b_2, b_3)$) with $n = |b_1 - b_2| = |b_3 - b_2|$, its variance is given by:

$$V[Z] = \frac{b_2}{12}(3a_1^2 + 3a_2^2 - 6a_1a_2 + m^2). \tag{57}$$

Case II: For a symmetric Z-number $Z = (A, B)$, where A is a trapezoidal fuzzy number ($A = (a_1, a_2, a_3, a_4)$) with $m = |a_1 - a_2| = |a_3 - a_4|$, and B is a triangular fuzzy number ($B = (b_1, b_2, b_3)$) with $n = |b_1 - b_2| = |b_3 - b_2|$, its variance is obtained as follows:

$$V[Z] = \frac{b_2}{12}(3a_1^2 + 3a_3^2 - 6a_1a_3 + m^2). \tag{58}$$

Case III: For a symmetric Z-number $Z = (A, B)$, where A is a triangular fuzzy number ($A = (a_1, a_2, a_3)$) with $m = |a_1 - a_2| = |a_3 - a_2|$, and B is a trapezoidal fuzzy number ($B = (b_1, b_2, b_3, b_4)$) with $n = |b_1 - b_2| = |b_3 - b_4|$, its variance is expressed as:

$$V[Z] = \frac{1}{24}(b_2 + b_3)(3a_1^2 + 3a_2^2 - 6a_1a_2 + m^2). \tag{59}$$

Proof. For case I, when $a_2 = a_3$ and $b_2 = b_3$ in Theorem 7, the Z-number of this case is same as that in Theorem 7. Thus, its variance is calculated as:

$$\begin{aligned} V[Z] &= \frac{1}{24}(b_2 + b_3)(3a_1^2 + 3a_3^2 - 6a_1a_3 + m^2) \\ &= \frac{b_2}{12}(3a_1^2 + 3a_2^2 - 6a_1a_2 + m^2). \end{aligned} \tag{60}$$

For case II, when $b_2 = b_3$ in Theorem 7, the Z-number of this case is same as that in Theorem 7. Thus, its variance is calculated as:

$$\begin{aligned} V[Z] &= \frac{1}{24}(b_2 + b_3)(3a_1^2 + 3a_3^2 - 6a_1a_3 + m^2) \\ &= \frac{b_2}{12}(3a_1^2 + 3a_3^2 - 6a_1a_3 + m^2). \end{aligned} \tag{61}$$

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For case III, when $a_2 = a_3$ in Theorem 7, the Z-number of this case is same as that in Theorem 7. Thus, its variance is calculated as:

$$\begin{aligned} V[Z] &= \frac{1}{24}(b_2+b_3)(3a_1^2+3a_3^2-6a_1a_3+m^2) \\ &= \frac{1}{24}(b_2+b_3)(3a_1^2+3a_2^2-6a_1a_2+m^2). \end{aligned} \tag{62}$$

□

Example 1. *Given that a case that Robert's math has low probability to be very poor. Calculate its expected value and variance.*

Solution. *Firstly, convert the natural language to Z-valuation Z = (degree of Robert's math, very poor, low).*

Secondly, convert it to Z-number by Table S1 and S2 as shown in Appendix, which is Z = [(0, 0, 0.1, 0.1), (0.05, 0.2 0.35)].

Thirdly, By Equations (11) and (47), we derive the expected value and variance of it,

$$\begin{aligned} E[Z] &= \frac{\sqrt{3(b_1+b_2+b_3)}}{12}(a_1+a_2+a_3+a_4). \\ &= \frac{\sqrt{3(0.05+0.2+0.35)}}{12}(0+0+0.1+0.2) \\ &= 0.03354. \end{aligned} \tag{63}$$

$$\alpha = \frac{b_1+b_2+b_3}{3} = 0.2. \tag{64}$$

$$\begin{aligned} V[Z] &= \frac{1}{6}\alpha[(a_1+a_2)^2+(a_3+a_4)^2-a_1a_2-a_3a_4] - \frac{\alpha}{16}(a_1+a_2+a_3+a_4)^2 \\ &= \frac{0.2}{6}[(0+0)^2+(0.1+0.2)^2-0-0.02] - \frac{0.2}{16}(0+0+0.1+0.2)^2 \\ &= 0.00121. \end{aligned} \tag{65}$$

Example 2. *Given that a case that the source of desert has high probability to be a little bit poor. Calculate its expected value and variance.*

Solution. *Firstly, convert the natural language to Z-valuation Z=(the resource of forest, a little bit poor, high).*

Secondly, convert it to Z-number by Table S1 and S2, which is Z=[(0.2, 0.3, 0.4, 0.5), (0.65, 0.8 0.95)].

Thirdly, By Equations (22) and (58), we derive the expected value and variance of it,

$$\begin{aligned} E[Z] &= \frac{\sqrt{b_2}}{2}(a_2+a_3) \\ &= \frac{\sqrt{0.2}}{2}(0.3+0.4) \\ &= 0.1565. \end{aligned} \tag{66}$$

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$$
\begin{aligned}
V[Z] &= \frac{b_2}{12}(3a_1^2 + 3a_3^2 - 6a_1a_3 + m^2) \\
&= \frac{0.8}{12}(0.12 + 0.48 - 0.48 + 0.01) \\
&= 0.00867.
\end{aligned} \tag{67}
$$

4. The Semi-Variance of Z-Numbers

In this section, we provide the definitions and theorems for the semi-variance of Z-numbers by the credibility distribution, relying on Definitions 8 and 9 in Appendix.

Definition 3. *Supposing that Z is a Z-number, we denote its converted regular fuzzy number as $\tilde{Z}$. If its expected value is ε, thus its upside semi-variance is derived by:*

$$S_{v^+}[Z] = E[(\tilde{Z} - \varepsilon)^+)^2], \tag{68}$$

where

$$(\tilde{Z} - \varepsilon))^+ = \begin{matrix} \tilde{Z} - \varepsilon, & \text{if } \tilde{Z} > \varepsilon \\ 0, & \text{if } \tilde{Z} \leq \varepsilon \end{matrix} \tag{69}$$

Definition 4. *Supposing that Z is a Z-number, we denote its converted regular fuzzy number as $\tilde{Z}$. If its expected value is ε, thus its downside semi-variance is derived by:*

$$S_{v^-}[Z] = E[((\tilde{Z} - \varepsilon)^-)^2], \tag{70}$$

where

$$(\tilde{Z} - \varepsilon)^- = \begin{cases} 0, & \text{if } \tilde{Z} > \varepsilon \\ \tilde{Z} - \varepsilon, & \text{if } \tilde{Z} \leq \varepsilon. \end{cases} \tag{71}$$

Theorem 8. *Supposing that Z is a Z-number, we denote its converted regular fuzzy number as $\tilde{Z}$. Then the credibility distribution is $\Phi_{\tilde{Z}}$ and the expected value is ε. Its upside semi-variance is derived by:*

$$S_{v^+}[Z] = \int_0^{+\infty} [1 - \Phi_{\tilde{Z}}(\varepsilon + \sqrt{u})]du. \tag{72}$$

Proof. By Definition 3,

$$
\begin{aligned}
S_{v^+}[Z] &= E[((\tilde{Z} - \varepsilon)^+)^2] \\
&= \int_0^{+\infty} \mathrm{Cr}\{(\tilde{Z} - \varepsilon)^2 \geq u, \tilde{Z} > \varepsilon\}du \\
&= \int_0^{+\infty} \mathrm{Cr}\{\tilde{Z} \geq \varepsilon + \sqrt{u}\}du \\
&= \int_0^{+\infty} [1 - \Phi_{\tilde{Z}}(\varepsilon + \sqrt{u})]du
\end{aligned} \tag{73}
$$

□

Theorem 9. *Supposing that Z is a Z-number, we denote its converted regular fuzzy number as $\tilde{Z}$. Then its credibility distribution is $\Phi_{\tilde{Z}}$ and the expected value is ε. Its downside semi-variance is derived by:*

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$$S_{v^-}[Z] = \int_0^{+\infty} \Phi_{\tilde{Z}}(\varepsilon - \sqrt{u})du. \tag{74}$$

Proof. By the Definition 4,

$$\begin{aligned} S_{v^-}[\tilde{Z}] &= E[((\tilde{Z} - \varepsilon)^-)^2] \\ &= \int_0^{+\infty} \text{Cr}\{(\tilde{Z} - \varepsilon)^2 \geq u, \tilde{Z} > \varepsilon\}du \\ &= \int_0^{+\infty} \text{Cr}\{\tilde{Z} \leq \varepsilon - \sqrt{u}\}du \\ &= \int_0^{+\infty} \Phi_{\tilde{Z}}(\varepsilon - \sqrt{u})du \end{aligned} \tag{75}$$

□

Theorem 10. *Given that $\widetilde{Z}$ is a converted Z-number. Then its credibility distribution is $\Phi_{\widetilde{Z}}$ and the expected value is ε. Its upside semi-variance is derived by:*

$$s_{v^+}[Z] = \int_{\Phi_{\tilde{Z}}(\varepsilon)}^{1} [\Phi_{\tilde{Z}}^{-1}(t) - \varepsilon]^2 dt, \tag{76}$$

where $\Phi_{\widetilde{Z}}^{-1}$ is the inverse credibility distribution of $\widetilde{Z}$.

Proof. By the Definition 3 and Theorem 7,

$$\begin{aligned} S_{v^+}[\tilde{Z}] &= E[((\tilde{Z} - \varepsilon)^+)^2] \\ &= \int_0^{+\infty} [1 - \Phi_{\tilde{Z}}(\varepsilon + \sqrt{u})]du \\ &= \int_{\Phi_{\tilde{Z}}(\varepsilon)}^{1} (1-t)d[\Phi_{\tilde{Z}}^{-1}(t) - \varepsilon]^2 \\ &= (1-t)[\Phi_{\tilde{Z}}^{-1}(t) - \varepsilon]^2|_{\Phi_{\tilde{Z}}(\varepsilon)}^{1} - \int_{\Phi_{\tilde{Z}}(\varepsilon)}^{1} [\Phi_{\tilde{Z}}^{-1}(t) - \varepsilon]^2 d(1-t) \\ &= \int_{\Phi_{\tilde{Z}}(\varepsilon)}^{1} [\Phi_{\tilde{Z}}^{-1}(t) - \varepsilon]^2 dt. \end{aligned} \tag{77}$$

□

Theorem 11. *Given that $\widetilde{Z}$ is a converted Z-number. Then its credibility distribution is $\Phi_{\widetilde{Z}}$ and the expected value is ε. Thus, its downside semi-variance is derived by:*

$$S_{v^-}[\tilde{Z}] = \int_0^{\Phi_{\tilde{Z}}(\varepsilon)} [\Phi_{\tilde{Z}}^{-1}(t) - \varepsilon]^2 dt, \tag{78}$$

where $\Phi_{\widetilde{Z}}^{-1}$ is the inverse credibility distribution of $\widetilde{Z}$.

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Proof. By the Definition 4 and Theorem 10, we derive

$$\begin{aligned} S_{v^-}[\tilde{Z}] &= E[((\tilde{Z}-\varepsilon)^-)^2] \\ &= \int_0^{+\infty} \Phi_{\tilde{Z}}(\varepsilon - \sqrt{u})du \\ &= \int_{\Phi_{\tilde{Z}}(\varepsilon)}^0 td[-\Phi_{\tilde{Z}}^{-1}(t)+\varepsilon]^2 \\ &= t[-\Phi_{\tilde{Z}}(t)+\varepsilon]^2|_{\Phi_{\tilde{Z}}(\varepsilon)}^0 - \int_{\Phi_{\tilde{Z}(\varepsilon)}}^0 [-\Phi_{\tilde{Z}}^{-1}(t)+\varepsilon]^2 dt \\ &= \int_0^{\Phi_{\tilde{Z}}(\varepsilon)} [\Phi_{\tilde{Z}}^{-1}(t)-\varepsilon]^2 dt \end{aligned} \tag{79}$$

□

Theorem 12. *Given that $\tilde{Z}$ is a converted Z-number. Its relationship between semi-variance and variance is as follow:*

$$S_{v^+}[\tilde{Z}] + S_{v^-}[\tilde{Z}] = V[\tilde{Z}]. \tag{80}$$

Proof. By Theorem 6, Theorem 10, and Theorem 11, we derive

$$\begin{aligned} V[\tilde{Z}] &= \int_0^1 [\Phi_{\tilde{Z}}^{-1}(t)-\varepsilon]^2 dt \\ &= \int_0^{\Phi_{\tilde{Z}}(\varepsilon)} [\Phi_{\tilde{Z}}^{-1}(t)-\varepsilon]^2 dt + \int_{\Phi_{\tilde{Z}}(\varepsilon)}^1 [\Phi_{\tilde{Z}}^{-1}(t)-\varepsilon]^2 dt \\ &= S_{v^-}[\tilde{Z}] + S_{v^+}[\tilde{Z}]. \end{aligned} \tag{81}$$

□

5. Conclusions

Z-number as a ubiquitous pair of fuzzy numbers ignites a large number of academics' enthusiasm, which can not only depict the fuzziness and reliability simutaneously, but also better to explain human judgments in decision making. In the present article, three properties of Z-numbers are explored, such as the expected value, variance, and semi-variance. Initially, the information on Z-numbers is transformed to its regular fuzzy numbers by the particular process named converting Z-numbers. By the following, based on the related theories, we defined the three properties of Z-numbers, and derived the formulas of general and symmetrical ones' expected value and variance. Subsequently, relationships between the variance of Z-numbers and the semi-variance were obtained. Thereafter, two concrete examples were shown to illustrate the steps of calculating the general and symmetrical Z-numbers' expected value and variance, which confirmed the validity of the formulas inferred in this paper. Obviously, our research is valuable for the in-depth study of Z-number theory, especially for solving realistic problems such as accident risk analysis in the field of building construction and portfolio selection evaluation in the financial issues.

Some limitations are also exsiting. Firstly, the type of Z-numbers in this paper is restricted to a regular fuzzy number. Given this, our findings can be further studied and extended to other types of Z-numbers. Secondly, this paper was to convert the information of a Z-number into a fuzzy number and then derived its expectation and variance. During this transformation process, it indeed produced

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certain information loss to some extent. Hence, in the future, we need to further study how to reduce and avoid information loss to get more accurate results. Finally, there is still a gap in this paper and existing studies on the practical applications of Z-numbers' these three properties, which calls for more thorough research in future work.

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