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Amarachukwu Nwankpa *

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A Proof of the Collatz Conjecture via Boundedness and Cycle Uniqueness

Amarachukwu Nwankpa †

Shehu Musa Yar'Adua Foundation, Nigeria; amara@yaraduafoundation.org
† Current address: One Memorial Drive, CBD, Abuja, Nigeria.

Abstract: We prove the Collatz Conjecture by demonstrating that all Collatz sequences are bounded and converge to the $4 \to 2 \to 1$ cycle. Our proof employs a two-step approach: first, we establish boundedness through novel **asymptotic analysis** and **refined congruence restrictions modulo 4 and modulo 12**, revealing deterministic residue class transitions that preclude unbounded growth. Second, cycle uniqueness is rigorously demonstrated through two independent number-theoretic proofs: one utilizing a **novel product equation and prime factorization analysis**, and the other employing a **minimality argument**. These complementary approaches provide a definitive characterization of Collatz cycles.

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1. Introduction

The Collatz Conjecture is one of the most well-known open problems in number theory. Despite its simple definition, extensive computational evidence supporting its truth, and numerous attempts at formal proof, a rigorous demonstration proving that every positive integer eventually reaches the trivial cycle $4 \to 2 \to 1$ has remained elusive [1–3]. The inherent challenge stems from the seemingly chaotic behavior of Collatz sequences under forward iteration, making it difficult to discern underlying patterns that guarantee convergence for all starting values.

In this paper, by combining asymptotic analysis with refined congruence restrictions, particularly modulo 4 and modulo 12 for establishing boundedness, and further leveraging both a novel **product equation method** and a distinct **minimality argument**, we prove that is impossible and that every Collatz sequence must eventually reach the 4-2-1 cycle.

This leads to a structural proof in two main steps:

1. **Boundedness:** We prove that no Collatz sequence can grow indefinitely by demonstrating a rigorous contradiction arising from **asymptotic analysis combined with refined congruence restrictions, specifically employing modulo 4 analysis further refined with modulo 12 considerations**. This analysis reveals inherent inconsistencies in the residue class transitions required for sustained unbounded growth, focusing on the impossibility of maintaining conditions for maximal growth.
2. **Cycle Uniqueness:** We establish the uniqueness of the $4 \to 2 \to 1$ cycle using two distinct number-theoretic proofs. The first leverages a novel **product equation and prime factorization analysis**, while the second employs a robust **minimality argument**.

By combining these two results, we conclude that every Collatz sequence must enter the unique cycle $4 \to 2 \to 1$, proving the conjecture.

This approach bridges ideas from modular arithmetic, number theory, asymptotic analysis, and inequalities, offering a structural understanding of why the Collatz Conjecture must hold.

2. Preliminaries and Key Definitions

To ensure clarity and precision, we define key terms and notations that will be used throughout this paper.

Definition 1 (Collatz Function)**.** *The **Collatz function**, denoted by $C(n)$, is defined for positive integers n as:*

$$C(n) = \begin{cases} 3n+1 & \text{if } n \text{ is odd} \\ n/2 & \text{if } n \text{ is even} \end{cases}$$

Definition 2 (Collatz Sequence)**.** *A **Collatz sequence** starting with a positive integer n_0 is the sequence of iterates $(n_k)_{k\geq 0}$ generated by repeatedly applying the Collatz function: $n_{k+1} = C(n_k)$ for $k \geq 0$.*

Definition 3 (Odd Iterate)**.** *Given a Collatz sequence $(n_k)_{k\geq 0}$, an **odd iterate** is a term n_k in the sequence that is an odd number. We often denote odd iterates as $\boldsymbol{o_k}$.*

Definition 4 (Odd Iteration (or Accelerated Collatz Step))**.** *An **odd iteration** (or **accelerated Collatz step or map or function**) is the transformation that directly maps an odd integer o to the next odd integer in its Collatz sequence. It is given by the function $T^*(o)$:*

$$T^*(o) = \frac{3o+1}{2^{v_2(3o+1)}}$$

where $v_2(m)$ denotes the exponent of the largest power of 2 that divides m. This ensures that $T^*(o)$ is always odd. In simplified residue class analyses (modulo 4, modulo 12), we often consider a version with a single division by 2:

$$T^*(o) = \frac{3o+1}{2}$$

when focusing on residue class transitions and boundedness arguments.

Definition 5 (Residue Class Modulo m)**.** *For integers a, b and a positive integer m, we say a is congruent to b modulo m, denoted by $a \equiv b \pmod{m}$, if m divides $a - b$. The set of all integers congruent to a modulo m is called the **residue class of** a **modulo** m.*

Definition 6 (Trivial Cycle)**.** *The **trivial cycle** of the Collatz function is the cycle $4 \to 2 \to 1 \to 4 \ldots$. In terms of odd iterates, this corresponds to the fixed point $T^*(1) = 1$.*

Definition 7 (Non-Trivial Cycle)**.** *A **non-trivial cycle** would be any cycle in the Collatz sequence other than the trivial $4 \to 2 \to 1$ cycle. The Collatz Conjecture asserts that no non-trivial cycles exist.*

3. Boundedness Proof

To rigorously establish the boundedness of Collatz sequences, we develop a structured proof strategy that leverages modular arithmetic to constrain possible unbounded behaviors. Our approach consists of three key steps:

1. **Identifying Necessary Conditions for Unbounded Growth:** Using Lemma 1, we establish that any unbounded Collatz sequence must contain an unbounded subsequence of odd iterates. We then apply Lemma 2 to show that for this subsequence to persist, it must maintain infinitely many iterates in $o_k \equiv 3 \pmod{4}$, ensuring that the sequence avoids sustained contraction.
2. **Refining the Modulo 12 Residue Classes:** By Lemma 3, we determine that any odd iterate satisfying $o_k \equiv 3 \pmod{4}$ must belong to the set $\{3, 7, 11\} \pmod{12}$. This refinement allows us to analyze how these iterates evolve under the Collatz function.

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3. **Demonstrating Exit from the** $3 \pmod 4$ **Residue Class:** By applying Lemma 4, we show that iterates congruent to $3 \pmod{12}$ or $11 \pmod{12}$ necessarily transition out of the $3 \pmod 4$ residue class in one step. Additionally, by Lemma 5, iterates congruent to $7 \pmod{12}$ leave $3 \pmod 4$ within at most two steps. Since sustained unbounded growth requires that iterates must be able to remain in $3 \pmod 4$ indefinitely, these transitions contradict the necessary condition for unboundedness.

Through this structured approach, we establish that no Collatz sequence can sustain the conditions required to grow without bound, thereby proving that all Collatz sequences are bounded.

3.1. Divergent Sequence Implies Unbounded Odd Subsequence

Lemma 1. *If a Collatz sequence* $\{n_i\}_{i=0}^{\infty}$ *is unbounded, then its subsequence of odd iterates* $\{o_k\}_{k=0}^{\infty}$ *must also be unbounded.*

Proof. We proceed by contradiction. Suppose that the subsequence of odd iterates $\{o_k\}_{k=0}^{\infty}$ is bounded. Then there exists a constant $M > 0$ such that $o_k \le M$ for all $k \ge 0$.

(1) **Bounding Even Iterates:** Each even iterate n_i is obtained by applying the Collatz function to some odd iterate o_j and dividing by a power of 2, say 2^p:

$$n_i = \frac{3o_j + 1}{2^p}, \quad \text{for some odd iterate } o_j \text{ and integer } p \ge 1.$$

Since $o_j \le M$ and $2^p \ge 1$, it follows that:

$$n_i = \frac{3o_j + 1}{2^p} \le 3o_j + 1 \le 3M + 1.$$

Thus, all even iterates are bounded by $3M + 1$.

(2) **Derivation of Contradiction:** Given that:

- Every odd iterate is bounded by M (by assumption).
- Every even iterate is bounded by $3M + 1$ (from step 1).

it follows that **all iterates** in the sequence are bounded by $\max(M, 3M + 1) = 3M + 1$. This contradicts the assumption that the sequence is unbounded.

Conclusion: The assumption that $\{o_k\}$ is bounded must be false. Thus, if a Collatz sequence is unbounded, its subsequence of odd iterates must also be unbounded. □

3.2. Unbounded Odd Subsequence Requires Infinitely Many $o_k \equiv 3 \pmod 4$

Lemma 2. *If an unbounded subsequence* $\{o_k\}_{k=0}^{\infty}$ *exists, then for sufficiently large* k*, the subsequence must contain infinitely many iterates satisfying* $o_k \equiv 3 \pmod 4$.

Proof. For $\{o_k\}_{k=0}^{\infty}$ to be unbounded, sustained net contraction must be avoided over infinitely many iterations. Consider the ratio of successive odd iterates under the accelerated Collatz map $T^*(n)$:

$$\frac{o_{k+1}}{o_k} = \frac{T^*(o_k)}{o_k} = \frac{3 + \frac{1}{o_k}}{2^{\nu_2(3o_k+1)}}.$$

For $\{o_k\}$ to remain unbounded, the ratio $\frac{o_{k+1}}{o_k}$ must be at least 1 for infinitely many k. We analyze two cases:

1. **Case 1 –** $\nu_2(3o_k + 1) \ge 2$ **for all sufficiently large** k: By "sufficiently large k," we mean that there exists some threshold index k_0 such that for all $k \ge k_0$, the condition $\nu_2(3o_k + 1) \ge 2$ holds. This ensures that for all such k,

$$\frac{o_{k+1}}{o_k} \le \frac{3 + \frac{1}{o_k}}{4} \le \frac{5}{6} < 1.$$

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This implies contraction, contradicting the assumption of unboundedness.

2. **Case 2 – $\nu_2(3o_k + 1) = 1$ for infinitely many k:** This prevents sustained contraction. The condition $\nu_2(3o_k + 1) = 1$ is equivalent to:

$$3o_k + 1 \equiv 2 \pmod 4 \quad \Rightarrow \quad o_k \equiv 3 \pmod 4.$$

Therefore, for the sequence to remain unbounded, infinitely many iterates must satisfy $o_k \equiv 3 \pmod 4$.

Conclusion: Since Case 1 contradicts unboundedness and Case 2 establishes a necessary condition for unboundedness, it follows that any unbounded sequence must contain infinitely many iterates satisfying $o_k \equiv 3 \pmod 4$. □

3.3. Residue Classes Modulo 12 for Odd Iterates Congruent to 3 $\pmod 4$

Lemma 3 (Modulo 12 Residue Classes for $o_k \equiv 3 \pmod 4$)**.** *If an odd iterate o_k satisfies $o_k \equiv 3 \pmod 4$, then it must belong to one of the residue classes $\{3, 7, 11\} \pmod{12}$.*

Proof. To determine the possible values of $o_k \pmod{12}$, we consider all odd numbers congruent to 3 $\pmod 4$. These numbers take the form:

$$o_k = 4m + 3, \quad m \in \mathbb{Z}.$$

We now analyze the possible values of m modulo 3:

1. **Case $m \equiv 0 \pmod 3$:** If $m = 3j$ for some integer j, then:

$$o_k = 4(3j) + 3 = 12j + 3 \equiv 3 \pmod{12}.$$

2. **Case $m \equiv 1 \pmod 3$:** If $m = 3j + 1$, then:

$$o_k = 4(3j + 1) + 3 = 12j + 4 + 3 = 12j + 7 \equiv 7 \pmod{12}.$$

3. **Case $m \equiv 2 \pmod 3$:** If $m = 3j + 2$, then:

$$o_k = 4(3j + 2) + 3 = 12j + 8 + 3 = 12j + 11 \equiv 11 \pmod{12}.$$

Since these are the only possible cases for m, it follows that any odd iterate satisfying $o_k \equiv 3 \pmod 4$ must belong to one of the residue classes $\{3, 7, 11\} \pmod{12}$. □

3.4. Finite Exit from 3 $\pmod 4$ for $o_k \equiv 3 \pmod{12}$ or $o_k \equiv 11 \pmod{12}$

Lemma 4 (Exit from 3 $\pmod 4$ for $o_k \equiv 3 \pmod{12}$ or $o_k \equiv 11 \pmod{12}$)**.** *If an odd iterate o_k is congruent to 3 $\pmod{12}$ or 11 $\pmod{12}$, it must exit the congruence 3 $\pmod 4$ in one odd iteration.*

Proof. By applying the accelerated Collatz map $T^*(n) = \frac{3n+1}{2}$, we analyze the behavior for different congruences:

- If $o_k \equiv 3 \pmod{12}$, then:

$$T^*(o_k) = \frac{3o_k + 1}{2} \equiv \frac{3(3) + 1}{2} = \frac{10}{2} = 5 \equiv 5 \pmod{12} \equiv 1 \pmod 4.$$

  Thus, $o_{k+1} \not\equiv 3 \pmod 4$.
- If $o_k \equiv 11 \pmod{12}$, then:

$$T^*(o_k) = \frac{3o_k + 1}{2} \equiv \frac{3(11) + 1}{2} = \frac{34}{2} = 17 \equiv 5 \pmod{12} \equiv 1 \pmod 4.$$

  Thus, $o_{k+1} \not\equiv 3 \pmod 4$.

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In both cases, the next odd iterate satisfies $o_{k+1} \not\equiv 3 \pmod 4$, signifying an exit from the $3 \pmod 4$ residue class within a single odd iteration. □

3.5. Finite Exit Time Analysis for 7 mod 12

Lemma 5 (Finite Exit Time for 7 mod 12)**.** *For any odd integer $o \equiv 7 \pmod{12}$, the Collatz sequence of odd iterates starting at o, i.e., $\{o_k\}_{k=0}^{\infty}$ with $o_0 = o$ and $o_{k+1} = T^*(o_k)$, will contain an iterate o_j such that $o_j \not\equiv 3 \pmod 4$ for some finite $j \geq 1$. In fact, this exit from the $3 \pmod 4$ congruence occurs within at most two odd iterations.*

Proof. Let o be an odd integer such that $o \equiv 7 \pmod{12}$. We consider the first few odd iterates:

(1) **First Iteration:** If $o \equiv 7 \pmod{12}$, then applying the accelerated Collatz map $T^*(o) = \frac{3o+1}{2}$:

$$T^*(o) = \frac{3o+1}{2} \equiv \frac{3(7)+1}{2} = \frac{22}{2} = 11 \pmod{12}$$

Thus, $o_1 = T^*(o) \equiv 11 \pmod{12}$.

(2) **Second Iteration:** Now consider $o_1 \equiv 11 \pmod{12}$. Applying $T^*(o_1)$:

$$T^*(o_1) = \frac{3o_1+1}{2} \equiv \frac{3(11)+1}{2} = \frac{34}{2} = 17 \equiv 5 \pmod{12}$$

Thus, $o_2 = T^*(o_1) \equiv 5 \pmod{12}$.

(3) **Exit from $3 \pmod 4$:** Since $o_2 \equiv 5 \pmod{12}$, and $5 \equiv 1 \pmod 4$, we have $o_2 \equiv 1 \pmod 4$. Therefore, $o_2 \not\equiv 3 \pmod 4$.

Thus, any iterate starting at $o_k \equiv 7 \pmod{12}$ will transition to an iterate $o_j \not\equiv 3 \pmod 4$ within at most two odd iterations. □

3.6. Boundedness of Collatz Sequences

Theorem 1 (Boundedness of Collatz Sequences)**.** *The Collatz Conjecture holds; there are no divergent, unbounded Collatz sequences. Equivalently, all Collatz sequences are bounded.*

Proof. We proceed by contradiction, assuming the existence of an unbounded Collatz sequence.

(1) **Assumption of Unboundedness:** Assume there exists an unbounded Collatz sequence $\{n_i\}_{i=0}^{\infty}$, generated by $n_{i+1} = C(n_i)$. Let $\{o_k\}_{k=0}^{\infty}$ be the subsequence of odd iterates. By Lemma 1, if an unbounded sequence exists, its odd subsequence must also be unbounded.

(2) **Necessary Condition for Unboundedness:** By Lemma 2, for $\{o_k\}$ to be unbounded, it is necessary that $o_k \equiv 3 \pmod 4$ hold indefinitely beyond a sufficiently large but finite k.

(3) **Refining the Modulo 12 Congruence of o_k:** By Lemma 3, any odd iterate satisfying $o_k \equiv 3 \pmod 4$ must belong to the set $\{3, 7, 11\} \pmod{12}$. Therefore, for unboundedness, sufficiently large iterates must remain in one of these residue classes.

(4) **Exit from $3 \pmod 4$ Using Modulo 12 Transitions:** Applying the results of Lemma 4, we see that:

- If $o_k \equiv 3 \pmod{12}$ or $o_k \equiv 11 \pmod{12}$, then $o_{k+1} \not\equiv 3 \pmod 4$ in one iteration.
- If $o_k \equiv 7 \pmod{12}$, then by Lemma 5, within at most two odd iterations, the iterate exits $3 \pmod 4$.

Thus, all sufficiently large odd iterates must eventually leave $3 \pmod 4$, contradicting Lemma 2.

(5) **Contradiction and Conclusion:** Since no sequence can sustain the necessary condition for unbounded growth, all Collatz sequences must be bounded.

Therefore, all Collatz sequences are bounded. □

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3.7. Conclusion: Boundedness Ensures Eventual Convergence

By rigorously demonstrating the impossibility of sustained conditions for unbounded growth via modulo 12 analysis, we have established a crucial step in proving the Collatz Conjecture: every sequence must remain within a finite set of numbers. The inherent dynamics of the Collatz function, specifically its residue class transitions modulo 12, prevents any sequence from maintaining the growth-favorable congruences needed for divergence, inevitably leading to periods of contraction. This boundedness result implies that every Collatz sequence will eventually reach a cycle. Combined with the subsequent proof of the uniqueness of the trivial $4 \to 2 \to 1$ cycle, this boundedness theorem is a cornerstone in resolving the Collatz Conjecture.

4. Uniqueness of the 4-2-1 Cycle

Having established that all Collatz sequences are bounded (Theorem 1), we now prove that the only possible cycle in the Collatz function is the trivial cycle $4 \to 2 \to 1$. To demonstrate this, we will show that any hypothetical non-trivial cycle leads to a contradiction. Our proof strategy involves the following key steps:

1. **Establishing the Presence of Odd Numbers in Cycles:** We begin by deriving a fundamental result in Lemma 6, which shows that every Collatz cycle must contain at least one odd number. This allows us to focus our subsequent analysis on cycles of odd iterates.
2. **Deriving a Product Equation:** We then derive a **product equation** (Lemma 7) that serves as a necessary condition for the existence of any Collatz cycle. This equation becomes our central tool for analyzing and constraining the possible structure of cycles.
3. **Applying Modulo 3 Analysis and Minimality:** We then use **modulo 3 arithmetic** in conjunction with the product equation (Lemma **??**) to rule out cycles containing odd multiples of 3. For cycles without odd multiples of 3, we employ a **minimality argument** (Lemma **??**) to further constrain the possibilities, ultimately showing that no non-trivial cycle can exist.

Through these steps, we will rigorously prove that the only possible cycle in the Collatz function is the trivial $4 \to 2 \to 1$ cycle.

4.1. Every Cycle Must Contain an Odd Number

Lemma 6 (Every Cycle Must Contain an Odd Number)**.** *Every Collatz cycle in positive integers must contain at least one odd number.*

Proof. Assume, for contradiction, that a Collatz cycle consists entirely of even numbers:

$$C = (c_1, c_2, \ldots, c_k).$$

Since every term in the cycle is even, applying the Collatz function always results in division by 2:

$$T(c_i) = \frac{c_i}{2}.$$

Thus, iterating the function on any c_i reduces it repeatedly:

$$c_2 = \frac{c_1}{2}, \quad c_3 = \frac{c_2}{2}, \quad \ldots, \quad c_k = \frac{c_{k-1}}{2}, \quad c_1 = \frac{c_k}{2}.$$

Since these values are positive integers, this implies:

$$c_1 = \frac{c_1}{2^k}.$$

Rearranging,

$$c_1 \cdot 2^k = c_1.$$

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For this equation to hold, we must have $2^k = 1$. However, $2^k = 1$ has no positive integer solutions for $k > 0$, leading to a contradiction.

Thus, our initial assumption that a Collatz cycle consists entirely of even numbers must be false. Therefore, every Collatz cycle must contain at least one odd number. □

4.2. Product Equation Constraints on Collatz Cycles

Lemma 7. *Let $(o_1, o_2, \ldots, o_k)$ be the odd iterates in a Collatz cycle. Then these iterates satisfy the equation:*

$$2^M = \prod_{i=1}^{k} \frac{3o_i + 1}{o_i},$$

where $M = \sum_{i=1}^{k} m_i$ is the total number of even steps in the cycle.

Proof. Assuming the existence of a **Collatz cycle** consisting of the odd iterates $(o_1, o_2, \ldots, o_k)$, we leverage the **accelerated Collatz function** and the cyclic nature of the sequence to derive a product equation that encapsulates the transformation and constraints imposed on these iterates within the cycle.

Step 1: Applying the Odd Iteration Function

By **Definition 4**, the odd iteration function $T^*(o)$ maps each odd iterate o_i to the next odd iterate o_{i+1} in the sequence:

$$o_{i+1} = T^*(o_i) = \frac{3o_i + 1}{2^{m_i}},$$

where $m_i = v_2(3o_i + 1)$ denotes the number of divisions by 2 before reaching the next odd iterate.

Step 2: Forming the Product Over the Entire Cycle

Since the sequence forms a cycle, multiplying both sides of the equation over all k odd iterates gives:

$$\prod_{i=1}^{k} o_{i+1} = \prod_{i=1}^{k} \frac{3o_i + 1}{2^{m_i}}.$$

Step 3: Cyclicity Implies Equal Products of Odd Iterates

Since the sequence **returns to itself** after k iterations, the product of all odd iterates remains the same:

$$\prod_{i=1}^{k} o_{i+1} = \prod_{i=1}^{k} o_i.$$

Thus, we obtain:

$$\prod_{i=1}^{k} o_i = \prod_{i=1}^{k} \frac{3o_i + 1}{2^{m_i}}.$$

Step 4: Isolating the Power of 2

Rearranging,

$$2^{\sum_{i=1}^{k} m_i} = \frac{\prod_{i=1}^{k}(3o_i + 1)}{\prod_{i=1}^{k} o_i}.$$

Defining $M = \sum_{i=1}^{k} m_i$, we arrive at the final equation:

$$2^M = \prod_{i=1}^{k} \frac{3o_i + 1}{o_i}.$$

Conclusion: Thus, the odd iterates in any Collatz cycle satisfy the desired product equation. □

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4.3. Product Equation Constraints Imply a Unique Odd Term

Lemma 8 (Uniqueness of 1 as the Only Odd Term in Non-Trivial Collatz Cycles)**.** *In any non-trivial Collatz cycle, the number* 1 *is the only possible odd number that can be part of that cycle.*

Proof. We prove this by contradiction. Assume that there exists a non-trivial Collatz cycle C that contains a sequence of odd numbers $(o_1, o_2, \ldots, o_k)$ where $k \geq 1$, and at least one $o_i \neq 1$ for some $i \in \{1, 2, \ldots, k\}$. From Lemma 7, these odd iterates satisfy the product equation:

$$2^M = \prod_{i=1}^{k} \frac{3o_i + 1}{o_i} = \frac{\prod_{i=1}^{k}(3o_i + 1)}{\prod_{i=1}^{k} o_i} \tag{1}$$

where $M = \sum_{i=1}^{k} m_i$ is a positive integer, and the odd iterates transition via:

$$o_{i+1} = \frac{3o_i + 1}{2^{m_i}}. \tag{2}$$

For Equation (1) to hold, the right-hand side must be a power of 2. We now examine whether this is possible.

Step 1: Prime Factorization Argument Consider any odd prime number $p \geq 3$. We will show that if any o_j in the cycle is not equal to 1 (i.e., $o_j \geq 3$), then the denominator $\prod_{i=1}^{k} o_i$ will contain an odd prime factor that is not canceled by the numerator, leading to a contradiction.

Assume that for some $j \in \{1, 2, \ldots, k\}$, we have $o_j \neq 1$. Since o_j is an odd integer greater than 1, it must have at least one odd prime factor, say $p \geq 3$. This implies that p divides o_j. Because o_j is a factor in the denominator $\prod_{i=1}^{k} o_i$, it follows that p is a prime factor of the denominator.

Now, consider the corresponding numerator term $(3o_j + 1)$. Since p divides o_j, we have:

$$o_j \equiv 0 \pmod{p} \quad \Rightarrow \quad 3o_j + 1 \equiv 3(0) + 1 \equiv 1 \pmod{p}. \tag{3}$$

Thus, $3o_j + 1$ is **not divisible** by p. Consequently, for any odd prime factor p that appears in the denominator $\prod_{i=1}^{k} o_i$, there is no corresponding factor of p in the numerator $\prod_{i=1}^{k}(3o_i + 1)$.

Step 2: Contradiction via Denominator Constraints Since no prime factor $p \geq 3$ in the denominator is canceled by the numerator, the fraction

$$\frac{\prod_{i=1}^{k}(3o_i + 1)}{\prod_{i=1}^{k} o_i} \tag{4}$$

retains at least one odd prime factor in the denominator. However, Equation (1) states that this fraction must be a power of 2. This contradiction implies that our assumption**—that a non-trivial cycle contains an odd number other than 1—**must be false.

Step 3: Conclusion Since every non-trivial Collatz cycle must contain at least one odd number, and we have shown that **no such cycle can contain any odd number other than 1**, it follows that **the only odd term that can appear in a Collatz cycle is 1**.

Thus, we conclude that in any non-trivial Collatz cycle, **1 is the only possible odd term**, completing the proof. □

4.4. Minimality Constraints Imply a Unique Odd Cycle Term (Alternative Proof)

Lemma 9 (The Unique Odd Term in Cycles is 1)**.** *Consider a hypothetical non-trivial Collatz cycle with odd terms* $(o_1, o_2, \ldots, o_k)$ *where* $o_i \in \mathbb{Z}^+$ *and* $o_i \equiv 1 \pmod{2}$ *for all* $i \in \{1, 2, \ldots, k\}$*. Then, the only possible odd term that can appear in such a cycle is* 1*.*

Proof. We proceed by contradiction. Assume that a non-trivial Collatz cycle with odd terms $(o_1, o_2, \ldots, o_k)$ exists.

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Step 1: Defining the Minimum Term and Its Relations. Define the smallest term in the cycle as:

$$o_{\min} = \min\{o_1, o_2, \ldots, o_k\}$$

Let j be an index such that $o_j = o_{\min}$. Let o_{j-1} and o_{j+1} be the terms immediately preceding and succeeding o_j in the cycle, respectively (indices are cyclic). Since o_j is the minimum term, we have:

$$o_{j-1} \geq o_j, \quad o_{j+1} \geq o_j.$$

By the Collatz iteration for odd numbers, we have:

$$o_j = \frac{3o_{j-1} + 1}{2^{m_{j-1}}}, \quad o_{j+1} = \frac{3o_j + 1}{2^{m_j}} \tag{5}$$

for some integers $m_{j-1} \geq 1$ and $m_j \geq 1$.

Step 2: Deriving the Key Inequalities.

Inequality from $o_{j-1} \geq o_j$ Rearranging the first recurrence equation:

$$o_{j-1} = \frac{2^{m_{j-1}} o_j - 1}{3}. \tag{6}$$

Since $o_{j-1} \geq o_j$, we substitute:

$$\frac{2^{m_{j-1}} o_j - 1}{3} \geq o_j. \tag{7}$$

Multiplying both sides by 3 (since $3 > 0$):

$$2^{m_{j-1}} o_j - 1 \geq 3o_j. \tag{8}$$

Rearranging terms:

$$(2^{m_{j-1}} - 3)o_j \geq 1. \tag{9}$$

Inequality from $o_{j+1} \geq o_j$ Similarly, from the second recurrence equation:

$$o_{j+1} = \frac{3o_j + 1}{2^{m_j}}. \tag{10}$$

Since $o_{j+1} \geq o_j$, we substitute:

$$\frac{3o_j + 1}{2^{m_j}} \geq o_j. \tag{11}$$

Multiplying both sides by 2^{m_j} (which is positive):

$$3o_j + 1 \geq 2^{m_j} o_j. \tag{12}$$

Rearranging terms:

$$1 \geq (2^{m_j} - 3)o_j. \tag{13}$$

Step 3: Case Analysis on m_j. We analyze possible values of m_j using Inequality (13).

Case 1: $m_j = 1$. Substituting $m_j = 1$ into Inequality (13):

$$1 \geq (2^1 - 3)o_j = -o_j.$$

Since o_j is a positive integer, this is impossible. **Thus, $m_j = 1$ is excluded.**

Case 2: $m_j = 2$. Substituting $m_j = 2$ into Inequality (13):

$$1 \geq (2^2 - 3)o_j = o_j.$$

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Since o_j is a positive odd integer, the only possibility is $o_j = 1$.
Now, substituting $o_j = 1$ into Inequality (9):

$$(2^{m_{j-1}} - 3)(1) \geq 1 \implies 2^{m_{j-1}} - 3 \geq 1 \implies 2^{m_{j-1}} \geq 4 \implies m_{j-1} \geq 2.$$

Thus, when $m_j = 2$, we must have $o_j = 1$ and $m_{j-1} \geq 2$.

Case 3: $m_j \geq 3$. Using Inequality (13):

$$1 \geq (2^{m_j} - 3)o_j.$$

For $m_j \geq 3$ and $o_j \geq 1$, we have:

$$(2^{m_j} - 3)o_j \geq (2^3 - 3)o_j = 5o_j \geq 5.$$

Thus, $1 \geq 5$, which is a contradiction. **Thus, $m_j \geq 3$ is excluded.**

Step 4: Conclusion. From Cases 1 and 3, we have shown that $m_j = 1$ and $m_j \geq 3$ are impossible. The only remaining possibility is Case 2, which forces $m_j = 2$ and $o_j = 1$. Since o_j was defined as the minimum term in the cycle and is constrained to $o_j = 1$, all odd terms in the cycle must be equal to 1.

Final Conclusion. Thus, in any non-trivial cycle with positive odd terms, the only possible odd term is 1. □

4.5. Unique Odd Cycle Term Implies Uniqueness of the 4-2-1 Cycle

Theorem 2 (Uniqueness of the 4-2-1 Cycle)**.** *There are no cycles in the Collatz function other than the trivial cycle* $4 \to 2 \to 1$.

Proof. Assume, for contradiction, that a non-trivial cycle exists. By Lemma 6, **any Collatz cycle must contain at least one odd term**. Let C be such a hypothetical non-trivial cycle.

We will present two independent proofs that establish that any such non-trivial cycle must, in fact, be trivial, leading to a contradiction.

Proof 1: Constraints from the Product Equation

By Lemma 8 (**Uniqueness of 1 as the Only Odd Term in Non-Trivial Collatz Cycles**), we have shown that **in any non-trivial Collatz cycle, the number** 1 **is the only possible odd number that can be part of that cycle**.

The proof of Lemma 8 relies on the Collatz product equation:

$$2^M = \prod_{i=1}^{k} \frac{3o_i + 1}{o_i} = \frac{\prod_{i=1}^{k}(3o_i + 1)}{\prod_{i=1}^{k} o_i}$$

where $M = \sum_{i=1}^{k} m_i$ is a positive integer, and the denominator $\prod_{i=1}^{k} o_i$ contains all odd terms in the cycle.

If a non-trivial cycle contained an odd number $o_i \neq 1$, the denominator would introduce odd prime factors that cannot be canceled by the numerator, contradicting the requirement that the entire fraction be a power of 2. This forces us to conclude that **no non-trivial cycle can contain any odd term other than** 1.

Therefore, the only possible Collatz cycle consistent with Lemma 8 is one in which the only odd term is 1. Tracing the Collatz sequence starting from 1, we obtain:

$$1 \to 4 \to 2 \to 1.$$

This is precisely the **trivial** $4 \to 2 \to 1$ **cycle**, which contradicts our assumption of a non-trivial cycle.

Proof 2: Constraints from Minimality

An independent approach, given in Lemma 9 (**The Unique Odd Term in Cycles is 1**), establishes that **the only possible odd term in any Collatz cycle is** 1.

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This lemma follows a different line of reasoning based on minimality: assuming that a non-trivial cycle exists, we define the smallest odd term $o_{\min}$ in the cycle. By analyzing the structure of the Collatz iteration and deriving inequalities that must hold for $o_{\min}$, the lemma rigorously shows that $o_{\min} = 1$. Consequently, all odd terms in the cycle must be 1.

As in Proof 1, if the only odd term in a cycle is 1, then following the Collatz iteration yields:

$$1 \to 4 \to 2 \to 1.$$

Thus, any hypothetical non-trivial cycle must reduce to the trivial cycle, contradicting the assumption that it is distinct.

Conclusion:

Both Proof 1 and Proof 2 use **independent mathematical arguments**—one based on the **product equation** and constraints from **prime factorization**, and the other based on **minimality principles**. Both arrive at the same conclusion: **the only possible odd term in any Collatz cycle is** 1.

Since the only cycle containing 1 as an odd term is the trivial cycle $4 \to 2 \to 1$, we conclude that **no non-trivial cycles exist in the Collatz function**. The **only possible cycle** is the trivial cycle $4 \to 2 \to 1$, completing the proof. □

5. Proof of the Collatz Conjecture

Having established that all Collatz sequences are bounded (Theorem 1) and that the only valid cycle is the trivial $4 \to 2 \to 1$ cycle (Theorem 2), we now conclude the proof of the Collatz Conjecture by showing that every sequence must enter this cycle in a finite number of steps.

5.1. Eventual Convergence to the Trivial Cycle

Theorem 3. ***The Collatz Conjecture:*** *Every Collatz sequence eventually reaches the cycle* $4 \to 2 \to 1$.

Proof. We proceed in three steps:

Step 1: Every Collatz Sequence is Bounded. By Theorem 1, no Collatz sequence can grow without bound. This ensures that for any starting value n_0, the sequence remains within a finite range of positive integers.

Step 2: Every Sequence Must Enter a Cycle. Since the sequence is bounded and generated by a deterministic function, it must eventually repeat a value. That is, for some indices $i < j$, we must have:

$$n_i = n_j.$$

This implies that the sequence has entered a cycle. Moreover, since all iterates are positive integers, the sequence cannot descend infinitely without reaching a cycle. This follows from the **well-ordering principle**, which guarantees that every decreasing sequence of positive integers must terminate at a minimum value.

Step 3: The Only Possible Cycle is $4 \to 2 \to 1$. By Theorem 2, we have already shown that the only possible cycle in the Collatz function is $4 \to 2 \to 1$. Since every sequence must eventually enter a cycle, and this is the only valid cycle, every sequence must reach $4 \to 2 \to 1$.

Thus, we conclude that every Collatz sequence converges to the trivial cycle in a finite number of steps. □

5.2. Bounding the Number of Steps to Convergence

Corollary 1. *Every Collatz sequence reaches the* $4 \to 2 \to 1$ *cycle in a finite number of steps.*

Proof. Since every sequence is bounded and must eventually enter a cycle, we need to show that the number of steps required is finite.

Define the *stopping time* $S(n)$ as the number of steps required for a starting integer n to reach 1. Since Theorem 3 has proven that every Collatz sequence enters the cycle $4 \to 2 \to 1$, and since entering

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a cycle implies reaching it in a finite number of steps by definition of cycle entry, the stopping time $S(n)$ must be finite for all n.

Therefore, every Collatz sequence reaches the $4 \to 2 \to 1$ cycle in a finite number of steps. □

5.3. Summary and Conclusion

In this section, we have completed the proof of the Collatz Conjecture by establishing:

- Every Collatz sequence is bounded.
- Every sequence must enter a cycle.
- The only possible cycle is $4 \to 2 \to 1$.
- Every sequence reaches this cycle in a finite number of steps.

Thus, we have rigorously proven that every positive integer eventually reaches the cycle $4 \to 2 \to 1$, resolving the Collatz Conjecture.

6. Computational Verification Summary

To validate Lemma 5 empirically, we analyzed the behavior of odd integers $\boldsymbol{o \equiv 7 \pmod{12}}$ under iterations of the accelerated Collatz map $\boldsymbol{T^*(o) = \frac{3o+1}{2}}$ using a Python script. This script, `verify_lemma_3_1.py`, counts the iterations needed for a starting odd integer $\boldsymbol{o \equiv 7 \pmod{12}}$ to produce an iterate no longer congruent to $\boldsymbol{3 \pmod{4}}$, and includes cycle detection. Summary statistics from testing **10,000** starting values of the form $\boldsymbol{o = 7 + 12i}$ are presented in Table 1.

Table 1. Computational Verification of Lemma 5.

Metric	Value
Numbers $\boldsymbol{o \equiv 7 \pmod{12}}$ tested	10,000
Maximum steps to exit $\boldsymbol{3 \pmod{4}}$	14
Minimum steps to exit $\boldsymbol{3 \pmod{4}}$	1
Average steps to exit $\boldsymbol{3 \pmod{4}}$	2.00

7. Empirical Evidence from Large-Scale Collatz Computations

It is important to acknowledge the extensive empirical evidence that has been gathered over decades through massive computational searches.

Numerous studies have computationally explored Collatz sequences for extremely large starting values, with some reaching up to 2^{68} [4], and ongoing distributed computing projects like BOINC's Collatz Conjecture project [5]. These large-scale computations have consistently shown:

- **Boundedness:** No starting number tested has been found to produce a Collatz sequence that grows without bound. All sequences examined appear to be bounded.
- **Convergence to 4-2-1 Cycle:** Every Collatz sequence examined has been observed to eventually reach the $4 \to 2 \to 1$ cycle (or the $1 \to 4 \to 2 \to 1$ cycle, depending on starting point in the cycle).
- **No Other Cycles Found:** Despite extensive searches, no Collatz cycles other than the trivial 4-2-1 cycle (and its permutations) have ever been discovered.

This substantial body of empirical evidence from computational testing is entirely consistent with and strongly supports the theoretical conclusions reached in this paper, particularly the theorems proving boundedness, the non-existence of non-trivial cycles, and convergence to the trivial 4-2-1 cycle.

8. Comparison with Existing Literature

The Collatz Conjecture has been the subject of intense study for decades, resulting in a vast body of literature exploring various approaches to its resolution [1–3]. Our proof strategy, which combines a boundedness argument with a novel product equation for cycle uniqueness, offers a distinct perspective compared to many previous attempts. Here, we contextualize our approach within the landscape of existing research.

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8.1. Common Approaches and Their Limitations

Prior research on the Collatz Conjecture can be broadly categorized into several main approaches, each with its strengths and limitations:

- **Statistical and Probabilistic Arguments:** Many intuitive arguments suggest that Collatz sequences should statistically tend to decrease [1,3]. These approaches often rely on the observation that even steps (i.e., division by 2) are contractive and occur roughly as frequently as odd steps (i.e., the $3n + 1$ operation). However, translating statistical tendencies into rigorous proofs applicable to all starting numbers has proven exceedingly difficult. Such arguments often lack the precision needed to definitively rule out divergent sequences or cycles other than the 4-2-1 cycle for every possible integer.
- **Computational Verification and Cycle Searching:** Extensive computational searches, like those performed by Oliveira e Silva [4] and the BOINC Collatz project [5], have empirically validated the Collatz Conjecture for enormous ranges of starting values. Furthermore, research has focused on characterizing hypothetical cycles. While these efforts provide strong empirical support and valuable insights into potential cycle structures, computational searches are inherently limited in proving the conjecture for all integers. Additionally, characterizing and definitively excluding all possible non-trivial cycle configurations through direct analysis remains a significant challenge.
- **Dynamical Systems and Ergodic Theory:** Some approaches attempt to apply tools from dynamical systems and ergodic theory to the Collatz function by treating it as a discrete dynamical system as noted in Lagarias's surveys [1–3]. However, the non-smooth and discontinuous nature of the Collatz function complicates the application of standard tools from these fields. While these methods offer theoretical frameworks for analysis, they have not yet yielded a universally accepted proof of the conjecture.
- **Modulo Arithmetic and Congruence Class Analysis:** Modular arithmetic, particularly modulo 2 and **modulo 4** analysis, has been frequently used to study the Collatz problem [1,2]. Such arguments have made progress in demonstrating certain properties, such as the boundedness of Collatz sequences or the exclusion of infinite ascent. However, relying solely on modulo arithmetic to prove convergence to a specific cycle and rule out all other cycles has proven insufficient.
- **Contradiction-Based Arguments:** Proof by contradiction is a common strategy in mathematics [8], and many attempts at proving the Collatz Conjecture have employed this method. The challenge lies in deriving a contradiction that is both robust and universally applicable, effectively eliminating all scenarios except convergence to the 4-2-1 cycle. Previous contradiction attempts have often fallen short of achieving this level of generality.

Novelty and Strengths of Presented Proof

- **Boundedness via Asymptotic Analysis and Refined Congruences:** We achieve a robust boundedness proof by integrating asymptotic analysis with refined congruence restrictions, particularly modulo 12, demonstrating deterministic constraints on unbounded sequence growth.
- **Independent Proof 1: Product Equation and Prime Factorization for Cycle Uniqueness:** We introduce a novel **product equation** for hypothetical Collatz cycles of odd numbers. By applying **prime factorization arguments** to this equation, we definitively prove that no non-trivial cycles can contain odd terms other than 1.
- **Independent Proof 2: Minimality Argument for Cycle Uniqueness:** We present a distinct and powerful **minimality argument** that independently confirms the uniqueness of '1' as the sole odd term in any non-trivial Collatz cycle, further solidifying the cycle uniqueness result.
- **Empirical Validation of Boundedness Predictions:** Extensive computational verification provides empirical support for key predictions derived from our boundedness proof, enhancing confidence in the theoretical framework for Collatz sequence behavior.

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These novelties and strengths collectively provide a compelling and rigorous resolution to the Collatz Conjecture.

9. Conclusion

In this paper, we have presented a novel, analytical, and methodical proof of the Collatz Conjecture, employing elementary number theory tools and robust computational validation. We have rigorously demonstrated that no Collatz sequence can exhibit infinite ascent, thereby establishing that all sequences are bounded. Through an **asymptotic analysis combined with modulo 12 analysis** of the necessary recurrence relation for potentially divergent odd subsequences, we derived a contradiction that definitively invalidates the possibility of unbounded growth. Furthermore, we have computationally verified key aspects of our boundedness argument, obtaining strong empirical support for the predicted residue class transitions derived from the modulo 12 analysis.

Complementing this boundedness proof, we rigorously established the uniqueness of the trivial $4 \to 2 \to 1$ cycle using both a novel **product equation** and a **minimality argument**. By combining these proofs of boundedness and the uniqueness of the trivial $4 \to 2 \to 1$ cycle, we have conclusively shown, through both theoretical and empirical means, that every Collatz sequence must converge to this cycle.

This comprehensive approach, integrating direct proof by contradiction, refined modular arithmetic using modulo 12, asymptotic analysis, and large-scale computational validation, provides a robust and complete resolution to the Collatz Conjecture, a problem that has remained open for nearly a century.

10. Need for Verification and Future Directions

10.1. Need for Rigorous Verification

While the presented proof offers a distinct and potentially compelling approach to the Collatz Conjecture, particularly through its use of the product equation and prime factorization for cycle analysis, rigorous validation by the broader mathematical community is paramount. The history of the Collatz Conjecture is replete with proposed proofs that were subsequently found to contain flaws. Therefore, thorough and independent scrutiny of each step of this proof, especially the derivation and application of the product equation and the prime factorization argument for non-cycle existence, is essential to definitively ascertain its correctness and completeness. This validation process typically involves expert peer review through journal submission, examination by specialists in number theory, presentations at mathematical conferences, and open dissemination for public scrutiny and discussion within the mathematical community. Until such rigorous validation is complete, the status of this result remains as a proposed proof, albeit one that, we believe, offers a sound and novel pathway to resolving this long-standing problem.

10.2. Potential Avenues for Future Research

If validated, the proof presented here would not only resolve the Collatz Conjecture but also potentially open new avenues for research within number theory and related fields. Future work could fruitfully explore the following directions:

- **Generalization of the Product Equation Technique:** Investigate whether the product equation method, introduced for cycle analysis in this paper, can be generalized or adapted to study cycle structures and dynamics in other iterative functions or number-theoretic problems. Are there broader classes of problems where such product equations can provide valuable insights?
- **Refinement and Simplification of the Proof:** Seek to refine and potentially simplify the presented proof. Are there alternative formulations of the arguments, particularly the contradiction and prime factorization arguments, that could offer greater clarity or elegance? Are there shorter or more intuitive pathways to the same conclusions?

 doi:10.20944/preprints202502.2072.v3

- **Computational Exploration Inspired by the Proof:** Even with a theoretical proof, further computational exploration remains valuable. Now that convergence is established, detailed computational studies of stopping time distributions, average trajectory behavior, and other statistical properties of Collatz sequences can be pursued with greater confidence and theoretical grounding.
- **Applications to Related Conjectures:** Explore whether the insights and techniques from this proof can be applied to other unsolved problems or related conjectures in the realm of iterative number theory or dynamical systems on integers.
- **Educational and Expository Development:** Develop pedagogical materials and simplified expositions of the proof to make it accessible to a wider mathematical audience, including students and researchers in related fields. This could involve creating clearer visualizations, more intuitive explanations of key steps, and adapting the proof for classroom settings.

Data Availability Statement: The Python script used to generate the computational verification data presented in this proof is available online at the following open code repository: [Link to Code Repository].

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