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Application of local gauge theory to fluid mechanics

Thomas Merz[1]†

[1]no affiliation, 8055 Zürich, CH

PART ONE

Fluid mechanics as a local gauge theory and relation to Navier-Stokes equations

The problem of fluid mechanics can be greatly simplified if, for every point in space, the strain-rate tensor is diagonalized. This tensor is introduced into the Navier-Stokes equations via constitutive equation and divergence of the stress tensor. This article shows that local $SO(3) \times U(1)$ gauge fields can be used to locally diagonalize the diffusion components of the strain-rate tensor. The gauge fields resulting from the connection can be interpreted as convection components of the flow, they show properties of quasiparticles and can be understood as elementary vortices. Thus, the proposed approach not only offers new insights for the solution and situative simplification of the Navier-Stokes equations, it also uncovers hidden symmetries within the flow convection, allowing – depending on boundary conditions – further physical interpretation.

1. Introduction

1.1. *Idea and procedure*

The basic idea of the approach presented is to find the eigensystem of the strain-rate tensor in all space. The problem: In a turbulent flow, the eigensystem may be different at every position, as illustrated in Fig. 1.

Thus, to diagonalize the strain-rate tensor everywhere, a theory is needed that allows to perform an individual coordinate transformation for every point. Since coordinate transformations between Euclidean bases in $\mathbb{R}^3$ are represented by rotations, a theory which is able to replace a fixed rotation $\mathbf{R} \in SO(3)$ by a rotation field $\mathbf{R}(\mathbf{r})$ with location dependency should be suitable for the job.

With this idea in mind, the local gauge field theory of the rotation group $SO(3)$ is introduced: As a tool to perform locally adapting coordinate transformations,

† Email address for correspondence: t.merz@alumni.ethz.ch

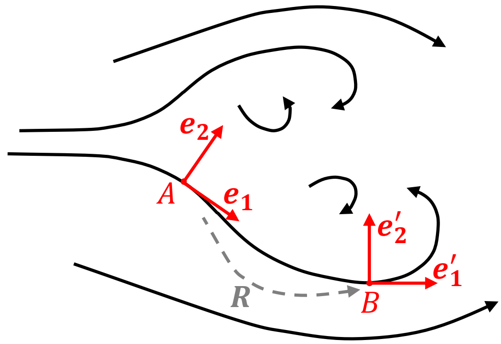

Figure 1: Illustrative look at a turbulent flow (black arrows): The strain-rate tensor is diagonalized respective to its eigensystem, which is different in points A and B. However, the transformation from eigensystem $\{\mathbf{e_i}\}$ to $\{\mathbf{e'_j}\}$ is a rotation $\mathbf{R}$ with $\mathbf{e'_j} = \sum_i \mathbf{e_i} R^i{}_j$ (gray dashed arrow). The local gauge field approach allows to stay in both eigensystems while separating the information on the connecting transformation into the gauge fields.

allowing to choose the eigensystem in every point, while separating the information on the connection to the surroundings into the gauge fields.

The procedure to formulate the theory consists of several steps which are described in Sec. 2: With a few introductory considerations (Sec. 2.1), the strain-rate tensor is put into a form suitable for gauge field theory. In particular a scalar expression is formed (Sec. 2.2) and the tensor is symmetrized by means of polar decomposition (Sec. 2.3). Based on this, the gauge fields for the strain-rate tensor are formulated in Sec. 2.4 and applied to the stress tensor in Sec. 2.5.

The gauge fields, abstract up to this point, are compared to the Navier-Stokes equations in Sec. 3, showing how they can be interpreted as flow components. Finally, the field equation is put into standard form for further use in Sec. 4. A short conclusion is found in Sec. 5.

1.2. *Notation and basics*

In the following, co- and contravariant vectors and second order tensors are written in bold, $\mathbf{I}$ and $\mathbf{0}$ denote the respective unit and zero matrix. Latin indexes run from 1 to 3, unless otherwise noted. Einstein summation convention is used in unambiguous cases. The separation of second order tensors $\mathbf{X} \in GL(3, \mathbb{R})$ or $\mathbf{X} \in GL(3, \mathbb{C})$ into isotropic, scalar parts $\mathbf{X}_{\parallel}$ and deviatory, trace-free parts $\mathbf{X}_{\perp}$ is carried out as follows:

$$\mathbf{X} = \mathbf{X}_{\parallel} + \mathbf{X}_{\perp} \qquad \mathbf{X}_{\parallel} = \frac{1}{3}\mathrm{tr}\left(\mathbf{X}\right)\mathbf{I} \qquad \mathbf{X}_{\perp} = \mathbf{X} - \mathbf{X}_{\parallel} \qquad \mathrm{tr}\left(\mathbf{X}_{\perp}\right) = 0. \quad (1.1)$$

All quantities used are considered to be dimensionless, quantities with dimensions are specifically marked with a superscript $()^{SI}$. The dimensionless quantities used are derived from the quantities with dimensions as follows:

$$\rho = \frac{\rho^{SI}}{\rho^{SI}_{\infty}} \qquad \mathbf{v} = \frac{\mathbf{v}^{SI}}{v^{SI}_{\infty}} \qquad \mathbf{r} = \frac{\mathbf{r}^{SI}}{l^{SI}_{c}} \qquad \boldsymbol{\nabla} = l^{SI}_{c}\boldsymbol{\nabla}^{SI} \qquad t = \frac{t^{SI}}{T^{SI}_{c}}, \quad (1.2)$$

using a characteristic density ρ^{SI}_{∞}, characteristic velocity v^{SI}_{∞}, characteristic length l^{SI}_{c} and characteristic time period T^{SI}_{c} (see e.g. Drazin & Riley 2006; Mei 2007).

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In addition, introducing the characteristic Reynolds number Re_C:

$$Re_C = \frac{\rho^{SI} v_\infty^{SI} l_c^{SI}}{\mu^{SI}}, \tag{1.3}$$

the dimensionless viscosities, i.e. the kinematic viscosity ν, the dynamic viscosity μ and the volume viscosity ζ, can be expressed as follows:

$$\nu = \frac{\nu^{SI}}{l_c^{SI} v_\infty^{SI}} = Re_C^{-1} \qquad \mu = Re_C^{-1}\rho \qquad \zeta = \frac{\zeta^{SI}}{\mu^{SI}} Re_C^{-1}\rho. \tag{1.4}$$

Throughout this manuscript, the characteristic Reynolds number Re_C and all viscosities are treated as constant within the region of interest.

The motion of viscous fluids is described by the Navier-Stokes equations. These are a special form of the Cauchy momentum equations, which are given in their dimensionless form by (e.g. Acheson 1990):

$$\rho\, Sr_C \frac{\partial \mathbf{v}}{\partial t} + \rho(\mathbf{v}\cdot\boldsymbol{\nabla})\mathbf{v} = \boldsymbol{\nabla}\cdot\boldsymbol{\sigma} - \boldsymbol{\nabla} p + \rho\mathbf{g}, \tag{1.5}$$

introducing the characteristic Strouhal number Sr_C which serves as a dimensionless frequency:

$$Sr_C = \frac{l_c^{SI}}{T_c^{SI} v_\infty^{SI}}. \tag{1.6}$$

The specific material properties are contained in the constitutive equation, which in the case of the Navier-Stokes equations is defined by the linear isotropic relationship between the strain-rate tensor $\dot{\boldsymbol{\varepsilon}}$ and the Cauchy stress tensor $\boldsymbol{\sigma}$:

$$\boldsymbol{\sigma} = 2\mu\, \dot{\boldsymbol{\varepsilon}}_\perp + 3\,\zeta\, \dot{\boldsymbol{\varepsilon}}_\parallel = Re_C^{-1}\rho\left(2\,\dot{\boldsymbol{\varepsilon}}_\perp + \frac{3\,\zeta^{SI}}{\mu^{SI}}\,\dot{\boldsymbol{\varepsilon}}_\parallel\right). \tag{1.7}$$

The linearized strain-rate tensor $\dot{\boldsymbol{\varepsilon}}$ is defined as the symmetrical component of the velocity gradient $\mathbf{F}$, given by the dyadic product $\mathbf{F} = (\boldsymbol{\nabla}\otimes\mathbf{v}) = \boldsymbol{\nabla}\mathbf{v}^{\mathsf{T}}$, whereby its components are usually written as:

$$\dot{\varepsilon}^i{}_j = \frac{1}{2}\left(\partial^i v_j + \partial^j v_i\right). \tag{1.8}$$

Inserting the constitutive equation Eq. 1.7 into the Cauchy momentum equations Eq. 1.5 using definition Eq. 1.8 yields the Navier-Stokes equations in their convective form:

$$\rho\, Sr_C \frac{\partial \mathbf{v}}{\partial t} + \rho(\mathbf{v}\cdot\boldsymbol{\nabla})\mathbf{v} = \mu(\boldsymbol{\nabla}\cdot\boldsymbol{\nabla})\mathbf{v} + \frac{1}{3}\mu\boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\mathbf{v}) - \boldsymbol{\nabla}\overline{p} + \rho\mathbf{g}. \tag{1.9}$$

Where the scalar components of the stress tensor are contained in the mechanical pressure:

$$\overline{p} = p - \zeta\,\boldsymbol{\nabla}\cdot\mathbf{v}. \tag{1.10}$$

In the following, only the stationary Navier-Stokes equations with $\frac{\partial \mathbf{v}}{\partial t} = \frac{\partial \rho}{\partial t} = 0$ will be considered. The system of equations also includes the continuity equation, which in the stationary case is reduced to:

$$\boldsymbol{\nabla}\cdot(\rho\mathbf{v}) = \boldsymbol{\nabla}\cdot\mathbf{j} = 0. \tag{1.11}$$

This defines the mass flow density $\mathbf{j} = \rho\mathbf{v}$.

The continuity equation allows the stationary, dimensionless Navier-Stokes equations to be written in their conservation form:

$$\boldsymbol{\nabla} \cdot (\rho\, \mathbf{v} \otimes \mathbf{v}) = \mu(\boldsymbol{\nabla} \cdot \boldsymbol{\nabla})\mathbf{v} + \frac{1}{3}\mu\boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \mathbf{v}) - \boldsymbol{\nabla}\overline{p} + \rho\mathbf{g}. \tag{1.12}$$

Previous attempts to use local gauge theory on viscous fluids can be found e.g. in Wyld Jr. (1961), Sulaiman (2005), Moulden (2012).

2. Formulation as a local gauge theory

2.1. *Diagonalization of the strain-rate tensor by local rotations*

The strain-rate tensor is symmetric by definition and can therefore, in situations where the tensor is uniform ($\dot{\boldsymbol{\varepsilon}} \neq \dot{\boldsymbol{\varepsilon}}(\mathbf{r})$), be diagonalized with $\dot{\boldsymbol{\varepsilon}}_{\mathbf{diag}} = \mathbf{R}^{\intercal}\dot{\boldsymbol{\varepsilon}}\,\mathbf{R}$, where $\mathbf{R} \in SO(3)$ is a rotation. One can therefore write:

$$\dot{\boldsymbol{\varepsilon}}_{\mathbf{diag}} = \begin{pmatrix} \partial^1 v_1 & 0 & 0 \\ 0 & \partial^2 v_2 & 0 \\ 0 & 0 & \partial^3 v_3 \end{pmatrix} \quad \text{and} \quad \mathrm{tr}\big[\dot{\boldsymbol{\varepsilon}}_{\mathbf{diag}}\big] = \boldsymbol{\nabla} \cdot \mathbf{v}. \tag{2.1}$$

However, if the strain-rate tensor depends on the location ($\dot{\boldsymbol{\varepsilon}} = \dot{\boldsymbol{\varepsilon}}(\mathbf{r})$), no general, global rotation can be found that diagonalizes the strain-rate tensor in all space. In order to be able to diagonalize the tensor in every point, the global rotation $\mathbf{R}$ is replaced by a local rotation $\mathbf{R}(\mathbf{r})$. This corresponds to the introduction of a local $SO(3)$ gauge field with $\dot{\boldsymbol{\varepsilon}}_{\mathbf{diag}}(\mathbf{r}) = \mathbf{R}^{\intercal}(\mathbf{r})\,\dot{\boldsymbol{\varepsilon}}(\mathbf{r})\,\mathbf{R}(\mathbf{r})$. The rotation field thus generated no longer commutes with the derivative in the strain-rate tensor, since now $\boldsymbol{\partial}(\mathbf{vR}) \neq (\boldsymbol{\partial}\mathbf{v})\mathbf{R}$.

The $SO(3)$ gauge field can be described as a Yang-Mills theory, making use of its extensively studied theoretical framework. However, the following constraints must be met first:

- The expression must be reduced to a scalar quantity, since Yang-Mills theories are applied to scalar quantities (usually Lagrange densities).
- By the symmetrization according to Eq. 1.8 two derivatives are formed in $\dot{\boldsymbol{\varepsilon}}$, each generating different, non-abelian and self-interacting gauge fields, whose interactions are unclear. The expression should be reduced to a single derivative.

These two points will be discussed in more detail in the next two paragraphs before formulating the theory.

2.2. *Scalar expression using the trace*

A simple way to obtain a scalar expression from the tensor $\dot{\boldsymbol{\varepsilon}}_{\mathbf{diag}}(\mathbf{r})$ is by forming its trace. The trace can be understood as a projection of the tensor onto a global orthonormal basis $\{\mathbf{e_i}\}$:

$$\mathrm{tr}\big[\dot{\boldsymbol{\varepsilon}}_{\mathbf{diag}}(\mathbf{r})\big] = \mathbf{e_i}\,\dot{\boldsymbol{\varepsilon}}_{\mathbf{diag}}(\mathbf{r})\,\mathbf{e^i} = \mathbf{e_i}\,\mathbf{R}^{\intercal}(\mathbf{r})\,\dot{\boldsymbol{\varepsilon}}(\mathbf{r})\,\mathbf{R}(\mathbf{r})\,\mathbf{e^i} = \mathbf{O}^{\intercal}_{\mathbf{i}}(\mathbf{r})\,\dot{\boldsymbol{\varepsilon}}(\mathbf{r})\,\mathbf{O^i}(\mathbf{r}). \tag{2.2}$$

In the last step the projection on the base components together with the rotation were combined to $\mathbf{O^i}(\mathbf{r}) = \mathbf{R}(\mathbf{r})\mathbf{e^i}$. The $\mathbf{O^i}$ thus transform like vectors $\mathbf{O'^i} = \mathbf{R'O^i}$ with respect to global changes of basis.

2.3. *Symmetrization using a phase*

To construct $\dot{\boldsymbol{\varepsilon}}$ in Eq. 2.2 from the velocity gradient $\mathbf{F} = (\boldsymbol{\nabla} \otimes \mathbf{v}) = \boldsymbol{\nabla}\mathbf{v}^{\intercal}$ and thus ultimately from a velocity field $\mathbf{v}(\mathbf{r})$, the velocity gradient $\mathbf{F}$ must be symmetrized in a way compatible with gauge fields.

To achieve this, at first the entries f_{jk} of the velocity gradient $\mathbf{F}$ are allowed to be complex with $f_{jk} \in \mathbb{C}$. As a condition, $\mathbf{F}$ should continue to be well-behaved, i.e. it is considered invertible and normal (unitary diagonalizable), thus $\mathbf{F} \in GL(3, \mathbb{C})$ and:

$$\mathbf{F}\mathbf{F}^{\dagger} = \mathbf{F}^{\dagger}\mathbf{F}. \tag{2.3}$$

With the extension to complex values, the velocity field can be constructed such that the real part of the velocity gradient contains the symmetric components s_{jk} and the imaginary part contains the skew-symmetric components a_{jk} of the initial velocity gradient, in summary:

$$f_{jk} = s_{jk} + a_{jk} \qquad \text{with} \quad f_{jk} \in \mathbb{C} \quad \text{and} \quad \begin{cases} s_{jk} = s_{kj}^{*}, & \mathrm{Im}(s_{jk}) = 0 \\ a_{jk} = -a_{kj}^{*}, & \mathrm{Re}(a_{jk}) = 0. \end{cases} \tag{2.4}$$

With this, $\mathbf{F} = \mathbf{HP}$ can be partitioned using polar decomposition, where $\mathbf{H}$ is in general a hermitian matrix ($h_{jk} = h_{kj}^{*}$) and $\mathbf{P}$ a unitary transformation. Since the construction in Eq. 2.4 ensures that only the real parts of $\mathbf{F}$ are Hermitian, $\mathbf{H}$ is not only Hermitian but symmetric $\mathbf{H} \to \mathbf{S}$ with real entries $s_{jk} \in \mathbb{R}$, and the polar decomposition reads:

$$\mathbf{F} = \mathbf{SP}. \tag{2.5}$$

Because $\mathbf{F}$ is invertible, $\mathbf{S}$ is positive definite and all its eigenvalues $\lambda_{S,j} > 0$. Moreover, since $\mathbf{F}$ is normal, $\mathbf{P}$ commutes with $\mathbf{S}^{\dagger}\mathbf{S}$ (inserted in Eq. 2.3):

$$\mathbf{P}^{\dagger}\left(\mathbf{S}^{\dagger}\mathbf{S}\right)\mathbf{P} = \mathbf{S}\left(\mathbf{P}\mathbf{P}^{\dagger}\right)\mathbf{S}^{\dagger} = \mathbf{S}\mathbf{S}^{\dagger} = \mathbf{S}^{\dagger}\mathbf{S} = \mathbf{P}^{\dagger}\mathbf{P}\left(\mathbf{S}^{\dagger}\mathbf{S}\right), \tag{2.6}$$

which means that $\mathbf{P}$ and $\mathbf{S}^{\dagger}\mathbf{S}$ are simultaneously diagonalizable. In addition, as a symmetric matrix, $\mathbf{S}$ can be diagonalized by applying a real rotational matrix $\mathbf{R} \in SO(3)$:

$$\mathbf{S}_{\mathbf{diag}} = \mathbf{R}^{\mathsf{T}}\mathbf{S}\mathbf{R}. \tag{2.7}$$

Any rotation $\mathbf{R}$ that diagonalizes $\mathbf{S}$ also diagonalizes $\mathbf{S}^{\dagger}\mathbf{S}$. Thus, $\mathbf{S}$ and $\mathbf{P}$ are simultaneously diagonalizable, and there exists a real rotation $\mathbf{R}$ which diagonalizes $\mathbf{S}$, $\mathbf{P}$ and $\mathbf{F}$ simultaneously.

PROOF 1. *In the case where $\mathbf{S}$ has three distinct eigenvalues $\lambda_{S,j}$, $\mathbf{S}^{\dagger}\mathbf{S}$ also has distinct eigenvalues. Thus $\mathbf{S}^{\dagger}\mathbf{S}$ and $\mathbf{P}$ have the same eigenvectors and a rotation $\mathbf{R}$ which diagonalizes $\mathbf{S}$ also diagonalizes $\mathbf{P}$.*

If $\mathbf{S}$ has two (or three) degenerate eigenvalues $\lambda_{S,j}$, one eigenvector of $\mathbf{S}^{\dagger}\mathbf{S}$ and $\mathbf{P}$ still coincides. The eigenvectors of $\mathbf{P}$ in the subspace with degenerate eigenvalues $\lambda_{S,j} = \lambda_{S,k}$ remain to be determined. However, in this subspace the symmetric part $\mathbf{S_{2x2}} = \lambda_{S,j}\mathbf{I_{2x2}}$ and its square behave as multiples of the unit matrix and commute with all similarity transformations. $\mathbf{F_{2x2}}$ is thus diagonalized when its skew-symmetric part a_{jk} is diagonalized. According to the construction Eq. 2.4, however, the skew-symmetric part is purely complex and therefore given by i times a symmetric matrix $\mathbf{S}'_{\mathbf{2x2}}$. This matrix can again be diagonalized by a rotation $\mathbf{R}'_{\mathbf{2x2}}$, giving $\mathbf{F}$ and $\mathbf{S}$ diagonal shapes. Then – due to Eq. 2.5 – $\mathbf{P}$ must be diagonal also, and the matrices $\mathbf{F}$, $\mathbf{S}$ and $\mathbf{P}$ are simultaneously diagonalized by a real rotation. □

The matrix $\mathbf{S}$ contains the requested symmetric components of $\mathbf{F}$ and corresponds to the strain-rate tensor $\mathbf{S} = \dot{\boldsymbol{\varepsilon}}$. In addition, there exists an expression for the velocity gradient $\mathbf{F} = \mathbf{SP}$, which gets diagonal shape simultaneously with the strain-rate tensor:

$$\mathbf{F_{diag}} = \mathbf{R}^{\mathsf{T}}\mathbf{F}\mathbf{R} = \dot{\boldsymbol{\varepsilon}}_{\mathbf{diag}}\mathbf{P_{diag}} = \begin{pmatrix} \lambda_{S,1}e^{-i\theta_1} & 0 & 0 \\ 0 & \lambda_{S,2}e^{-i\theta_2} & 0 \\ 0 & 0 & \lambda_{S,3}e^{-i\theta_3} \end{pmatrix}. \tag{2.8}$$

Where $\mathbf{F}$ is constructed according to Eq. 2.4, and with partial matrices:

$$\dot{\boldsymbol{\varepsilon}}_{\mathbf{diag}} = \mathbf{S_{diag}} = \begin{pmatrix} \lambda_{S,1} & 0 & 0 \\ 0 & \lambda_{S,2} & 0 \\ 0 & 0 & \lambda_{S,3} \end{pmatrix} \qquad \mathbf{P_{diag}} = \begin{pmatrix} e^{-i\theta_1} & 0 & 0 \\ 0 & e^{-i\theta_2} & 0 \\ 0 & 0 & e^{-i\theta_3} \end{pmatrix}. \tag{2.9}$$

Resolved to $\dot{\boldsymbol{\varepsilon}}_{\mathbf{diag}}$, expression Eq. 2.8 can be put into a form which can be inserted into Eq. 2.2:

$$\dot{\boldsymbol{\varepsilon}}_{\mathbf{diag}} = \mathbf{F_{diag}}\mathbf{P}^{\dagger}_{\mathbf{diag}} = \left(\mathbf{R}^{\mathsf{T}}\mathbf{F}\mathbf{R}\right)\mathbf{P}^{\dagger}_{\mathbf{diag}} = \left(\mathbf{R}^{\mathsf{T}}\left(\boldsymbol{\nabla}\mathbf{v}^{\mathsf{T}}\right)\mathbf{R}\right)\mathbf{P}^{\dagger}_{\mathbf{diag}}. \tag{2.10}$$

Remark 1. *The advantage of this approach becomes clear when formulating the interaction terms of the local gauge fields in the following section. When using the standard symmetrization approach with* $\dot{\varepsilon}^{j}_{\ k} = \frac{1}{2}\left(\partial^j v_k + \partial^k v_j\right)$, *the second summand leads to three additional non-abelian* $SO(3)$ *gauge fields with self-interaction, whereas the polar decomposition approach* $\mathbf{P_{diag}}(\mathbf{r}) = \mathrm{diag}\left(e^{-i\theta_1(\mathbf{r})}, e^{-i\theta_2(\mathbf{r})}, e^{-i\theta_3(\mathbf{r})}\right)$ *creates at most three abelian* $U(1)$ *gauge fields without self-interaction, which can be treated independently.*

Remark 2. *The construction of* $\mathbf{F}$ *according to Eq. 2.4 will be elaborated in more detail in the second part of this manuscript. A design is proposed which allows to directly enter the magnitude of shear and vorticity of the intended initial velocity field by making use of the exterior algebra.*

2.4. *Gauge invariant strain-rate tensor*

In summary, these considerations lead to the following expression for the trace of the strain-rate tensor (Eq. 2.10):

$$\begin{aligned} \mathrm{tr}\left[\dot{\boldsymbol{\varepsilon}}_{\mathbf{diag}}\right] &= \mathrm{tr}\left[\left(\mathbf{R}^{\mathsf{T}}\left(\boldsymbol{\nabla}\mathbf{v}^{\mathsf{T}}\right)\mathbf{R}\right)\mathbf{P}^{\dagger}_{\mathbf{diag}}\right] \\ &= \mathrm{tr}\left[\left(\mathbf{R}^{\mathsf{T}}\boldsymbol{\nabla}\right)\left(\mathbf{v}^{\mathsf{T}}\mathbf{R}\mathbf{P}^{\dagger}_{\mathbf{diag}}\right)\right]. \end{aligned} \tag{2.11}$$

As expected, this expression has a form on which Yang-Mills theory can be applied in order to diagonalize the strain-rate tensor $\dot{\boldsymbol{\varepsilon}}$ everywhere.

For this purpose, the rotation $\mathbf{R} = \mathbf{R}(\mathbf{r})$ and the phase $\mathbf{P}^{\dagger}_{\mathbf{diag}} = \mathbf{P}^{\dagger}_{\mathbf{diag}}(\mathbf{r})$ are promoted to position dependent, local rotation and phase fields. In doing so, it must be guaranteed that the expression obtained from local symmetrization and diagonalization describes the same physics as it did before. It must therefore remain invariant under the corresponding localized $SO(3)$ and $U(1)$ transformations. Their influence and interaction on the fields are considered in the following.

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2.4.1. *Gauge fields using local rotations*

At first, pure rotations $\mathbf{R}(\mathbf{r})$ with the phase angles set to zero ($\theta_i = 0$) are considered, such that $\mathbf{P}^\dagger_{\mathbf{diag}} = \mathbf{I}$:

$$\mathrm{tr}\left[\dot{\boldsymbol{\varepsilon}}_{\mathbf{diag}}(\mathbf{r})\right] = \mathrm{tr}\left[\mathbf{R}^{\mathsf{T}}(\mathbf{r})\,\boldsymbol{\nabla}\left(\mathbf{v}^{\mathsf{T}}(\mathbf{r})\,\mathbf{R}(\mathbf{r})\right)\right]. \tag{2.12}$$

As a prerequisite, it is noted that for global rotations, the trace is left invariant under similarity transformations $\mathrm{tr}\left[\mathbf{R}^{\mathsf{T}}\dot{\boldsymbol{\varepsilon}}\mathbf{R}\right] = \mathrm{tr}\left[\dot{\boldsymbol{\varepsilon}}\right]$, and the such applied rotations form a global symmetry of the expression. This global symmetry is promoted to a local symmetry. The rotations are represented as exponential functions:

$$\mathbf{R}(\mathbf{r}) = e^{i\phi_m(\mathbf{r})\mathbf{L}^{\mathbf{m}}} \tag{2.13}$$

where the generators are given by the Lie Algebra $so(3)$:

$$\mathbf{L}^{\mathbf{1}} = -\frac{i}{2}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \qquad \mathbf{L}^{\mathbf{2}} = -\frac{i}{2}\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} \qquad \mathbf{L}^{\mathbf{3}} = -\frac{i}{2}\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{2.14}$$

The generators are defined such that they fulfill the standard requirements regarding commutator and normalization for Yang Mills theories (e.g. Iliopoulos 2012):

$$\left[\mathbf{L}^{\mathbf{n}}\mathbf{L}^{\mathbf{l}}\right] = i f^{mnl}\mathbf{L}^{\mathbf{m}} \qquad f^{mnl} = -\frac{1}{2}\epsilon^{mnl} \qquad \mathrm{tr}\left(\mathbf{L}^{\mathbf{m}}\mathbf{L}^{\mathbf{n}}\right) = \frac{1}{2}\delta^{mn}, \tag{2.15}$$

where ϵ^{mnl} is the Levi-Civita symbol and f^{mnl} the structure constant defined from it.

It is now necessary to determine the local dependence of the rotations. By using $(\mathbf{a}\otimes\mathbf{b}) = \mathbf{a}\mathbf{b}^{\mathsf{T}}$ and switching to component notation, the trace becomes (the position dependency is no longer written explicitly to allow for a compact notation):

$$\begin{aligned}
\left(\dot{\varepsilon}_{diag}\right)^j{}_j &= R^{Tj}{}_k\partial^k\left(v^T_l\; R^l{}_j\right) \\
&= R^{Tj}{}_k\partial^k\left(v^T_l\right)\; R^l{}_j + R^{Tj}{}_k v^T_l\;\partial^k\left(R^l{}_j\right) \\
&= R^{Tj}{}_k\partial^k\left(v^T_l\right)\; R^l{}_j + R^{Tj}{}_k v^T_l\;\left(i\partial^k\phi_m(\mathbf{r})\mathbf{L}^{\mathbf{m}}\right)^k R^l{}_j \\
&= R^{Tj}{}_k\partial^k\left(v^T_l\right)\; R^l{}_j + R^{Tj}{}_k\left(i\partial^k\phi_m(\mathbf{r})\mathbf{L}^{\mathbf{m}}\right)^k v^T_l\; R^l{}_j \\
&= R^{Tj}{}_k\partial^k\left(v^T_l\right)\; R^l{}_j + iR^{Tj}{}_k\,A^k_\perp\, v^T_l\; R^l{}_j \\
&= \left(\dot{\varepsilon}_{local}\right)^j{}_j + \left(\dot{\varepsilon}_{connect.\,SO(3)}\right)^j{}_j
\end{aligned} \tag{2.16}$$

where:

$$A^k_\perp = \left(A_{\perp m}\right)^k\mathbf{L}^{\mathbf{m}} \qquad \text{with} \qquad \left(A_{\perp m}\right)^k = \partial^k\phi_m(\mathbf{r}). \tag{2.17}$$

In vectorial notation this reads:

$$\begin{aligned}
\mathrm{tr}\left[\dot{\boldsymbol{\varepsilon}}_{\mathbf{diag}}(\mathbf{r})\right] &= \mathrm{tr}\left[\mathbf{R}^{\mathsf{T}}\boldsymbol{\nabla}\left(\mathbf{v}^{\mathsf{T}}\right)\mathbf{R}\right] + \mathrm{tr}\left[i\,\mathbf{R}^{\mathsf{T}}\,\mathbf{A}^{\mathbf{m}}_\perp\mathbf{v}^{\mathsf{T}}\,\mathbf{R}\right] \\
&= \mathrm{tr}\left[\boldsymbol{\nabla}\mathbf{v}^{\mathsf{T}}\right] + \mathrm{tr}\left[i\mathbf{v}\cdot\mathbf{A}^{\mathbf{m}}_\perp\right] \\
&= \mathrm{tr}\left[\dot{\boldsymbol{\varepsilon}}_{\mathbf{local}}(\mathbf{r})\right] + \mathrm{tr}\left[\dot{\boldsymbol{\varepsilon}}_{\mathbf{connect.\,SO(3)}}(\mathbf{r})\right].
\end{aligned} \tag{2.18}$$

Where $\mathbf{A}^{\mathbf{m}}_\perp$ denotes the vector with entries according to Eq. 2.17. It was used that the trace remains invariant under similarity transformations and that $\mathrm{tr}[\mathbf{a}\mathbf{b}^{\mathsf{T}}] =$

$\mathbf{a}\cdot\mathbf{b}$. By reordering the equation, one sees that the local component can be divided into a diagonal component $(\mathrm{tr}\left[\dot{\boldsymbol{\varepsilon}}_{\mathbf{diag}}(\mathbf{r})\right])$ and a Yang-Mills like component which is generated by the connection:

$$\mathrm{tr}\left[\dot{\boldsymbol{\varepsilon}}_{\mathbf{local}}(\mathbf{r})\right] = \mathrm{tr}\left[\dot{\boldsymbol{\varepsilon}}_{\mathbf{diag}}(\mathbf{r})\right] - \mathrm{tr}\left[\dot{\boldsymbol{\varepsilon}}_{\mathbf{connect.\,SO(3)}}(\mathbf{r})\right]. \tag{2.19}$$

Finally, the rotations commute with the homogeneous, scalar component $\mathbf{X}_{\|}$ of tensors in general and couple only to the deviatoric, trace-free parts $\mathbf{X}_{\perp}$:

$$\begin{aligned} \mathbf{X}_{\mathbf{diag}} &= \mathbf{R}^{\mathsf{T}}\left(\mathbf{X}_{\perp} + \mathbf{X}_{\|}\right)\mathbf{R} \\ &= \mathbf{R}^{\mathsf{T}}\,\mathbf{X}_{\perp}\,\mathbf{R} + \mathbf{X}_{\|}. \end{aligned} \tag{2.20}$$

And hence in the considered case, it holds:

$$\mathrm{tr}\left[\dot{\boldsymbol{\varepsilon}}_{\perp\mathbf{local}}(\mathbf{r})\right] = \mathrm{tr}\left[\dot{\boldsymbol{\varepsilon}}_{\perp\mathbf{diag}}(\mathbf{r})\right] - \mathrm{tr}\left[\dot{\boldsymbol{\varepsilon}}_{\perp\mathbf{connect.\,SO(3)}}(\mathbf{r})\right], \tag{2.21}$$

with the connecting strain-rate tensor $\dot{\boldsymbol{\varepsilon}}_{\perp\mathbf{connect.\,SO(3)}} = \dot{\boldsymbol{\varepsilon}}_{\mathbf{connect.\,SO(3)}}$ as in Eq. 2.18.

2.4.2. *Gauge fields using a local phase*

The procedure for the local phase field $\mathbf{P}^{\dagger}_{\mathbf{diag}}(\mathbf{r})$ is essentially the same. What needs to be examined additionally is the interaction between the real rotations $\mathbf{R}(\mathbf{r})$ and the phase field. Therefore, two cases are distinguished:

First, let all rotation angles be equal to zero ($\phi_m = 0$) and thus $\mathbf{R}(\mathbf{r}) = \mathbf{I}$. Then by definition the velocity gradient already is diagonalized $F^j{}_k = \delta^j_k \partial^j v^T_k$. The velocity components can be expressed as a magnitude and phase: $v^T_k = |v_k|\, e^{i\theta_k} = v_{\mathbb{R},k}\, e^{i\theta_k}$, and the components of the complex rotation can be reduced directly in the initial strain-rate tensor from Eq. 2.11:

$$\boldsymbol{\nabla}\left(\mathbf{v}^{\mathsf{T}}\mathbf{P}^{\dagger}_{\mathbf{diag}}\right) = \boldsymbol{\nabla}\left(\mathbf{v}_{\mathbb{R}}\mathbf{P}_{\mathbf{diag}}\mathbf{P}^{\dagger}_{\mathbf{diag}}\right) = \boldsymbol{\nabla}\mathbf{v}_{\mathbb{R}}. \tag{2.22}$$

This is also the expected result, since the eigenvalues of the strain-rate tensor are directly visible in its diagonal shape, and they must be real in a symmetrical tensor.

The problem has thus already been solved, without necessity to specifically consider the position dependency of the phase.

Second, as soon as real rotations $\mathbf{R}$ are introduced into the equation, the procedure above will no longer work. The complex rotations $\mathbf{P}^{\dagger}_{\mathbf{diag}}$ need to be taken into account as local gauge fields. They form a group P of diagonal unitary matrices:

$$\mathbf{P}^{\dagger}_{\mathbf{diag}} \in P \quad \text{with} \quad P = \{\mathbf{P} \in U(3) \mid P^j{}_k = \delta^j_k \lambda_j\}. \tag{2.23}$$

To furthermore get the most efficient approach, it must be ensured that only those complex rotations are applied which act orthogonally to the real rotations $\mathbf{R} \in SO(3)$.

To orthogonalize the two groups, we first remember that the combined operation of phase and rotation can be represented as (Eq. 2.10):

$$\mathbf{U} = \mathbf{R}\mathbf{P}^{\dagger}_{\mathbf{diag}}, \tag{2.24}$$

where the combined transformation $\mathbf{U} \in U(3)$ is still a unitary transformation, since both $SO(3)$ and P are subgroups of the unitary group $U(3)$. Considering

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that all these groups are Lie groups, Eq. 2.24 can be expressed as an exponential map of the corresponding Lie algebras:

$$e^{i(\varphi_k \boldsymbol{\lambda}^{\mathbf{k}} + \varphi_9 \mathbf{I})} = e^{i\phi_m \mathbf{L}^{\mathbf{m}}} e^{i\theta_n \mathbf{p}^{\mathbf{n}}}, \tag{2.25}$$

with the $\mathbf{L}^{\mathbf{m}}$ denoting the generators of the Lie algebra $so(3)$ as stated in Eq. 2.14, the Gell-Mann matrices $\boldsymbol{\lambda}^{\mathbf{k}}$, $k \in \{1, ..., 8\}$ as generators of the Lie algebra $su(3)$, and $\mathbf{p}^{\mathbf{n}}$ the generators of the Lie algebra to the group P according to Eq. 2.23, which are to be determined.

However, it is a property of the exponential map that group elements take the form $e^{\mathbf{X}}e^{\mathbf{Y}}$ (as on the right side of Eq. 2.25) only if the associated elements of the Lie algebra commute:

$$e^{\mathbf{X}+\mathbf{Y}} = e^{\mathbf{X}}e^{\mathbf{Y}} \quad \text{only if} \quad \left[\mathbf{X}, \mathbf{Y}\right] = 0. \tag{2.26}$$

Since the form $\mathbf{RP}^{\dagger}_{\mathbf{diag}}$ (Eq. 2.24) of the operators for diagonalization and symmetrization of $\mathbf{F}_{\mathbf{diag}}$ according to Eq. 2.8 suffices and a perturbation theory for rotations is to be developed, only operators commuting with all generators $\mathbf{L}^{\mathbf{m}}$ can be considered as the generators $\mathbf{p}^{\mathbf{n}}$. Otherwise the form $\mathbf{RP}^{\dagger}_{\mathbf{diag}}$ would be given up because of Eq. 2.26.

It is easily found that none of the diagonal Gell-Mann matrices commutes with all the rotational generators $\mathbf{L}^{\mathbf{m}}$. Thus, the only generator of the Lie algebra of P acting orthogonally to $SO(3)$ is the unit matrix, and the following form for $\mathbf{P}^{\dagger}_{\mathbf{diag}}$ is obtained:

$$\mathbf{P}^{\dagger}_{\mathbf{diag}} = e^{i\theta \mathbf{I}} = \mathbf{I}e^{i\theta}. \tag{2.27}$$

This result fits seamlessly into the previous findings, since as stated in Eq. 2.20, real rotations leave scalars invariant and therefore also the scalar component of the tensors considered. Furthermore, the scalar component forms an invariant one-dimensional subspace, onto which the group $\mathbf{P}_{\mathbf{diag}}$ of complex rotations orthogonal to the real rotations must act as a phase. Hence, the separated equation is:

$$\begin{aligned} \mathrm{tr}\left[\dot{\boldsymbol{\varepsilon}}_{\mathbf{diag}}\right] &= \mathrm{tr}\left[\mathbf{R}^{\mathsf{T}}\,\dot{\boldsymbol{\varepsilon}}_{\perp}\,\mathbf{R} + \dot{\boldsymbol{\varepsilon}}_{\|}e^{i\theta}\right] \\ &= \mathrm{tr}\left[\mathbf{R}^{\mathsf{T}}\,\dot{\boldsymbol{\varepsilon}}_{\perp}\,\mathbf{R}\right] + \mathrm{tr}\left[\dot{\boldsymbol{\varepsilon}}_{\|}e^{i\theta}\right] \\ &= \mathrm{tr}\left[\dot{\boldsymbol{\varepsilon}}_{\perp\mathbf{diag}}\right] + \mathrm{tr}\left[\dot{\boldsymbol{\varepsilon}}_{\|\mathbf{diag}}\right]. \end{aligned} \tag{2.28}$$

In summary: If real rotations $\mathbf{R}(\mathbf{r})$ are introduced, the actions of $\mathbf{R}(\mathbf{r})$ and $\mathbf{P}(\mathbf{r})$ must be orthogonalized to avoid the introduction of redundant symmetry operations. The group of rotations $\mathbf{P}(\mathbf{r})$ orthogonal to $\mathbf{R}(\mathbf{r})$ then acts only as a phase on the one-dimensional subspace of scalar components of the strain-rate tensor.

Remark 3 (Group structure). *$U(1)$ is the normal subgroup of $U(3)$, defined as the subgroup that commutes with each element of the group. Therefore, $U(1) \rtimes SO(3)$ forms a semidirect product, the two subgroups have a trivial intersection $U(1) \cap SO(3) = \mathbf{I}$, and $U(1)$ is again the normal subgroup of this composite group.*

With the orthogonalization, the scalar component can be treated separately. To introduce the local dependency of the phase, one needs an expression that remains invariant under global phase transformations. This can be found by squaring the scalar part. One can then write for local phases $\theta(\mathbf{r})$:

$$\begin{aligned}
\dot{\boldsymbol{\varepsilon}}^*_{\|\mathbf{diag}} \cdot \dot{\boldsymbol{\varepsilon}}_{\|\mathbf{diag}} &= \frac{1}{9}\mathbf{I}\left(\boldsymbol{\nabla}\cdot\left(\mathbf{v}^* e^{-i\theta}\right)\right)\left(\boldsymbol{\nabla}\cdot\left(\mathbf{v}e^{i\theta}\right)\right) \\
&= \frac{1}{9}\mathbf{I}\left((\boldsymbol{\nabla}\cdot\mathbf{v}^*)e^{-i\theta} - i\mathbf{v}^*\cdot(\boldsymbol{\nabla}\theta)e^{-i\theta}\right)\left((\boldsymbol{\nabla}\cdot\mathbf{v})e^{i\theta} + i\mathbf{v}\cdot(\boldsymbol{\nabla}\theta)e^{i\theta}\right) \\
&= \frac{1}{9}\mathbf{I}\left(\boldsymbol{\nabla}\cdot\mathbf{v}^* - i\mathbf{v}^*\cdot(\boldsymbol{\nabla}\theta)\right)\left(\boldsymbol{\nabla}\cdot\mathbf{v} + i\mathbf{v}\cdot(\boldsymbol{\nabla}\theta)\right) \\
&= \frac{1}{9}\mathbf{I}\left(\boldsymbol{\nabla}\cdot\mathbf{v} + i\mathbf{v}\cdot(\boldsymbol{\nabla}\theta)\right)^2 \\
&= \frac{1}{9}\mathbf{I}\left(\boldsymbol{\nabla}\cdot\mathbf{v} + i\mathbf{v}\cdot\mathbf{A}_{\|}\right)^2,
\end{aligned} \tag{2.29}$$

for which the definition $\mathbf{A}_{\|} = \boldsymbol{\nabla}\theta(\mathbf{r})$ was used in the last step.

This results in two complex conjugated roots for the scalar part of the strain-rate tensor, whereby in the following only one root is evaluated first (the second root and implications caused by squaring are dealt with in Part 2 of this manuscript). In continuation:

$$\begin{aligned}
\mathrm{tr}\left[\dot{\boldsymbol{\varepsilon}}_{\|\mathbf{diag}}\right] &= \mathrm{tr}\left[\boldsymbol{\nabla}\cdot\mathbf{v} + i\mathbf{v}\cdot\mathbf{A}_{\|}\right] \\
&= \mathrm{tr}\left[\boldsymbol{\nabla}\mathbf{v}^{\mathbf{T}}\right] + \mathrm{tr}\left[i\mathbf{v}\cdot\mathbf{A}_{\|}\right] \\
&= \mathrm{tr}\left[\dot{\boldsymbol{\varepsilon}}_{\|\mathbf{local}}(\mathbf{r})\right] + \mathrm{tr}\left[\dot{\boldsymbol{\varepsilon}}_{\|\mathbf{connect.\,U(1)}}(\mathbf{r})\right],
\end{aligned} \tag{2.30}$$

which again can be reordered to:

$$\mathrm{tr}\left[\dot{\boldsymbol{\varepsilon}}_{\|\mathbf{local}}(\mathbf{r})\right] = \mathrm{tr}\left[\dot{\boldsymbol{\varepsilon}}_{\|\mathbf{diag}}(\mathbf{r})\right] - \mathrm{tr}\left[\dot{\boldsymbol{\varepsilon}}_{\|\mathbf{connect.\,U(1)}}(\mathbf{r})\right]. \tag{2.31}$$

The full local strain-rate tensor can be composed of the scalar and deviatoric components:

$$\mathrm{tr}\left[\dot{\boldsymbol{\varepsilon}}_{\mathbf{local}}\right] = \mathrm{tr}\left[\dot{\boldsymbol{\varepsilon}}_{\perp\mathbf{local}}\right] + \mathrm{tr}\left[\dot{\boldsymbol{\varepsilon}}_{\|\mathbf{local}}\right], \tag{2.32}$$

and inserted from Eqs. 2.21 and 2.31, as well as using Eq. 2.28, this finally results in the trace of the strain-rate tensor being:

$$\begin{aligned}
\mathrm{tr}\left[\dot{\boldsymbol{\varepsilon}}_{\mathbf{local}}(\mathbf{r})\right] &= \mathrm{tr}\left[\dot{\boldsymbol{\varepsilon}}_{\perp\mathbf{diag}}(\mathbf{r})\right] + \mathrm{tr}\left[\dot{\boldsymbol{\varepsilon}}_{\|\mathbf{diag}}(\mathbf{r})\right] - \mathrm{tr}\left[\dot{\boldsymbol{\varepsilon}}_{\perp\mathbf{connect.\,SO(3)}}(\mathbf{r})\right] - \mathrm{tr}\left[\dot{\boldsymbol{\varepsilon}}_{\|\mathbf{connect.\,U(1)}}(\mathbf{r})\right] \\
&= \mathrm{tr}\left[\dot{\boldsymbol{\varepsilon}}_{\mathbf{diag}}(\mathbf{r})\right] - \mathrm{tr}\left[i\mathbf{v}\cdot\mathbf{A}^{\mathbf{m}}_{\perp}\right] - \mathrm{tr}\left[i\mathbf{v}\cdot\mathbf{A}_{\|}\right].
\end{aligned} \tag{2.33}$$

Where $\dot{\boldsymbol{\varepsilon}}_{\mathbf{diag}}$ is as in Eq. 2.1.

2.4.3. *Full strain-rate tensor*

In order to obtain the full dynamics of the gauge fields, their kinetic terms are introduced. For this purpose the gauge fields $\mathbf{A}_{\|}$ and $\mathbf{A}^{\mathbf{m}}_{\perp}$ are written as vector potentials (Sanchez-Monroy & Quimbay 2010):

$$\begin{aligned}
\mathbf{B}_{\|}(\mathbf{r}) &= \boldsymbol{\nabla}\times\mathbf{A}_{\|} \\
\mathbf{B}^{\mathbf{m}}_{\perp}(\mathbf{r}) &= \boldsymbol{\nabla}\times\mathbf{A}^{\mathbf{m}}_{\perp} - \frac{1}{2}\mathbf{f}_{\mathbf{mnl}}\left(\mathbf{A}^{\mathbf{n}}_{\perp}\times\mathbf{A}^{\mathbf{l}}_{\perp}\right).
\end{aligned} \tag{2.34}$$

With the help of these fields, respective field strength tensors are defined:

$$\begin{aligned} F_{jk} &= -\epsilon_{ijk}(B_{\parallel})^{i} \\ G^{m}_{jk} &= -\epsilon_{ijk}(B^{m}_{\perp})^{i}. \end{aligned} \tag{2.35}$$

$\mathbf{B}_{\parallel}(\mathbf{r})$ and $\mathbf{B}^{\mathbf{m}}_{\perp}(\mathbf{r})$ are axial vector fields and $\mathbf{B}^{\mathbf{m}}_{\perp}(\mathbf{r})$ has bivectors as components. $\mathbf{F_{jk}}(\mathbf{r})$ and $\mathbf{G}^{\mathbf{m}}_{\mathbf{jk}}(\mathbf{r})$ are the respective bivector fields associated with the axial vector fields.

The inherent dynamics of the gauge fields can be expressed in their covariant form as a function of the field strength tensors and is incorporated into the trace of the strain-rate tensor as follows:

$$\begin{aligned} \mathrm{tr}\left[\dot{\boldsymbol{\varepsilon}}_{\mathbf{local}}\right] &= \mathrm{tr}\left[\dot{\boldsymbol{\varepsilon}}_{\mathbf{diag}} - \dot{\boldsymbol{\varepsilon}}_{\mathbf{connect.\,U(1)}} - \dot{\boldsymbol{\varepsilon}}_{\mathbf{connect.\,SO(3)}} + \dot{\boldsymbol{\varepsilon}}_{\mathbf{U(1)}} + \dot{\boldsymbol{\varepsilon}}_{\mathbf{SO(3)}}\right] \\ &= \mathrm{tr}\left[\dot{\boldsymbol{\varepsilon}}_{\mathbf{diag}} - i\mathbf{v}\cdot\mathbf{A}_{\parallel} - i\mathbf{v}\cdot\mathbf{A}^{\mathbf{m}}_{\perp}\right] + \frac{1}{4}\mathbf{F_{jk}}\mathbf{F^{jk}} + \frac{1}{4}\mathbf{G^{m}}_{\mathbf{jk}}\mathbf{G^{m\,jk}}. \end{aligned} \tag{2.36}$$

The kinetic terms of the gauge fields can be rewritten such that they would be included in the trace (using the relation of the Levi-Civita symbol to the Kronecker-Delta $\epsilon_{ijk}\epsilon^{njk} = 2\delta^{n}_{i}$):

$$\frac{1}{4}F_{jk}F^{jk} = \frac{1}{4}\epsilon_{ijk}\epsilon^{njk}B^{i}_{\parallel}\,B_{\parallel n} = \frac{1}{4}\,2\delta^{n}_{i}B^{i}_{\parallel}\,B_{\parallel n} = \frac{1}{2}\mathrm{tr}\left[B^{i}_{\parallel}\,B_{\parallel n}\right], \tag{2.37}$$

as well as

$$\frac{1}{4}G^{m}{}_{jk}G^{mjk} = \frac{1}{4}\epsilon_{ijk}\epsilon^{njk}B^{m\,i}_{\perp}\,B^{m}_{\perp n} = \frac{1}{4}\,2\delta^{n}_{i}B^{m\,i}_{\perp}\,B^{m}_{\perp n} = \frac{1}{2}\mathrm{tr}\left[B^{m\,i}_{\perp}\,B^{m}_{\perp n}\right]. \tag{2.38}$$

In vectorial notation this becomes:

$$\mathrm{tr}\left[\dot{\boldsymbol{\varepsilon}}_{\mathbf{U(1)}}\right] = \frac{1}{4}\mathbf{F_{jk}}\mathbf{F^{jk}} = \frac{1}{2}\mathrm{tr}\left[\mathbf{B}_{\parallel}\otimes\mathbf{B}_{\parallel}\right], \tag{2.39}$$

and

$$\mathrm{tr}\left[\dot{\boldsymbol{\varepsilon}}_{\mathbf{SO(3)}}\right] = \frac{1}{4}\mathbf{G^{m}}_{\mathbf{jk}}\mathbf{G^{m\,jk}} = \frac{1}{2}\mathrm{tr}\left[\mathbf{B}^{\mathbf{m}}_{\perp}\otimes\mathbf{B}^{\mathbf{m}}_{\perp}\right]. \tag{2.40}$$

In this form, the trace can be removed on both sides of Eq. 2.36, to obtain the strain-rate tensor (determined up to a similarity transformation):

$$\dot{\boldsymbol{\varepsilon}}_{\mathbf{local}} = \dot{\boldsymbol{\varepsilon}}_{\mathbf{diag}} - i\mathbf{v}\mathbf{A}^{\mathsf{T}}_{\parallel} - i\mathbf{v}(\mathbf{A}^{\mathbf{m}}_{\perp})^{\mathsf{T}} + \frac{1}{2}\left(\mathbf{B}_{\parallel}\otimes\mathbf{B}_{\parallel}\right) + \frac{1}{2}\left(\mathbf{B}^{\mathbf{m}}_{\perp}\otimes\mathbf{B}^{\mathbf{m}}_{\perp}\right). \tag{2.41}$$

Where $\dot{\boldsymbol{\varepsilon}}_{\mathbf{diag}}$ is as in Eq. 2.1, and it was used that the scalar product can be expressed as the trace of a dyadic product $\mathbf{a}\cdot\mathbf{b} = \mathrm{tr}[\mathbf{a}\mathbf{b}^{\mathsf{T}}]$.

2.5. *Gauge invariant stress-tensor*

The relation between stress-tensor and strain-rate tensor is expressed by the constitutive equation Eq. 1.7. Thus, the local stress tensor can be obtained by

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inserting the local strain-rate tensor Eq. 2.41:

$$\begin{aligned}\boldsymbol{\sigma}_{\mathbf{local}} =& 2\mu\,\dot{\boldsymbol{\varepsilon}}_{\perp\mathbf{local}} + 3\zeta\,\dot{\boldsymbol{\varepsilon}}_{\|\mathbf{local}} \\ =& \boldsymbol{\sigma}_{\perp\mathbf{diag}} + \boldsymbol{\sigma}_{\|\mathbf{diag}} - \boldsymbol{\sigma}_{\mathbf{connect.\,SO(3)}} - \boldsymbol{\sigma}_{\mathbf{connect.\,U(1)}} + \boldsymbol{\sigma}_{\mathbf{SO(3)}} + \boldsymbol{\sigma}_{\mathbf{U(1)}} \\ =& 2\mu\,\dot{\boldsymbol{\varepsilon}}_{\mathbf{diag}} + \left(\zeta - \frac{2\mu}{3}\right)(\boldsymbol{\nabla}\cdot\mathbf{v})\mathbf{I} \\ & - 2\mu\, i\mathbf{v}(\mathbf{A}_{\perp}^{\mathbf{m}})^{\mathbf{T}} - 3\zeta\, i\mathbf{v}\mathbf{A}_{\|}^{\mathbf{T}} \\ & + \mu\left(\mathbf{B}_{\perp}^{\mathbf{m}} \otimes \mathbf{B}_{\perp}^{\mathbf{m}}\right) + \frac{3\zeta}{2}\left(\mathbf{B}_{\|} \otimes \mathbf{B}_{\|}\right),\end{aligned} \tag{2.42}$$

with $\dot{\boldsymbol{\varepsilon}}_{\mathbf{diag}}$ as in Eq. 2.1.

3. Identification of gauge fields and relation to the Navier-Stokes equations

The location-dependent diagonalization and symmetrization of the local velocity gradient give rise to the gauge fields $\mathbf{B}_{\perp}^{\mathbf{m}}$ and $\mathbf{B}_{\|}$ on the connection between different points. These fields are abstract until now. The goal of this section is to show that these gauge fields can be identified with components of the velocity field of the Navier-Stokes equations, and thus do not add complexity, but rather involve a rearrangement of the existing fields.

For the comparison, the derived tensor 2.42 is brought into the same form as in the Navier-Stokes equations by by calculating its divergence. The divergence is first determined for the individual components, then combined and compared with the Navier-Stokes equations.

3.1. *Divergence of the diffusion component*

The diffusion component $\boldsymbol{\sigma}_{\mathbf{diag}}$ is already diagonalized, such that the divergence can be expressed as:

$$\boldsymbol{\nabla}\cdot\boldsymbol{\sigma}_{\mathbf{diag}} = 2\mu\,\boldsymbol{\nabla}\cdot\dot{\boldsymbol{\varepsilon}}_{\mathbf{diag}} + \left(\zeta - \frac{2\mu}{3}\right)\boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\mathbf{v}). \tag{3.1}$$

To simplify this expression somewhat further, it can be used that $\dot{\boldsymbol{\varepsilon}}_{\mathbf{diag}}$ is rotation-free ($\boldsymbol{\nabla}\times\mathbf{v} = 0$), since its off-diagonal components are zero everywhere $\dot{\varepsilon}_{diag}{}^{j}{}_{k} = \partial^j v_k^T = 0$ for $j \neq k$, and its eigenvalues are real $\dot{\varepsilon}_{diag}{}^{j}{}_{k} = \delta_k^j \partial^j v_k^T \in \mathbb{R}$:

$$\begin{aligned}\boldsymbol{\nabla}\cdot\boldsymbol{\sigma}_{\mathbf{diag}} &= 2\mu\,\boldsymbol{\nabla}\cdot\dot{\boldsymbol{\varepsilon}}_{\mathbf{diag}} + \left(\zeta - \frac{2\mu}{3}\right)\boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\mathbf{v}) \\ &= 2\mu\,\boldsymbol{\nabla}^2\mathbf{v} + \left(\zeta - \frac{2\mu}{3}\right)\left(\boldsymbol{\nabla}^2\mathbf{v} + \boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\mathbf{v})\right) \\ &= \left(\zeta + \frac{4\mu}{3}\right)\boldsymbol{\nabla}^2\mathbf{v}.\end{aligned} \tag{3.2}$$

3.2. *Divergence of the interaction terms*

The divergence of the interaction terms disappears if the directional derivatives of the gauge fields along the unperturbed velocity vector $(\mathbf{v}\cdot\boldsymbol{\nabla})\,\mathbf{A}_{\perp}^{\mathbf{m}} \approx 0$ and $(\mathbf{v}\cdot\boldsymbol{\nabla})\,\mathbf{A}_{\|} \approx 0$ vanish. These become negligible if the perturbation along the axis of flow changes slowly, i.e. $\phi_m(\mathbf{r}) \approx \phi_m(\mathbf{r}+\mathbf{v}dt)$ and $\theta(\mathbf{r}) \approx \theta(\mathbf{r}+\mathbf{v}dt)$. This is the case for a sufficiently large unperturbed flow $|\mathbf{v}| = \frac{dr}{dt} \gg 1$ (and therefore $dr \gg dt$) with location-independent shear or vorticity, to which small location-dependent perturbations are applied. By additionally using the continuity equation Eq. 1.11, one has:

$$\begin{aligned}
\boldsymbol{\nabla}\cdot\boldsymbol{\sigma}_{\mathbf{connection}} &= \boldsymbol{\nabla}\cdot\left(2\mu\, i\mathbf{v}(\mathbf{A}^{\mathbf{m}}_{\perp})^{\mathsf{T}}\right) + \boldsymbol{\nabla}\cdot\left(3\zeta\, i\mathbf{A}_{\parallel}\mathbf{v}^{\mathsf{T}}\right)\\
&= \boldsymbol{\nabla}\cdot\left(2\, Re_{_C}^{-1}\rho\, i\mathbf{v}(\mathbf{A}^{\mathbf{m}}_{\perp})^{\mathsf{T}}\right) + \boldsymbol{\nabla}\cdot\left(3\frac{\zeta^{SI}}{\mu^{SI}}Re_{_C}^{-1}\rho\, i\mathbf{A}_{\parallel}\mathbf{v}^{\mathsf{T}}\right)\\
&= 2i\, Re_{_C}^{-1}\Big(\left(\boldsymbol{\nabla}\cdot\rho\mathbf{v}\right)\mathbf{A}^{\mathbf{m}}_{\perp} + \rho\left(\mathbf{v}\cdot\boldsymbol{\nabla}\right)\mathbf{A}^{\mathbf{m}}_{\perp}\Big) \qquad (3.3)\\
&\quad + 3i\frac{\zeta^{SI}}{\mu^{SI}}Re_{_C}^{-1}\Big(\left(\boldsymbol{\nabla}\cdot\rho\mathbf{v}\right)\mathbf{A}_{\parallel} + \rho\left(\mathbf{v}\cdot\boldsymbol{\nabla}\right)\mathbf{A}_{\parallel}\Big)\\
&= 0.
\end{aligned}$$

3.3. *Divergence of the kinetic terms of the gauge fields*

The divergence of the kinetic terms of the gauge fields is remaining, which is considered in the following. The components are rescaled:

$$\boldsymbol{\sigma}_{\mathbf{SO(3)}} = \mu\,\mathbf{B}^{\mathbf{m}}_{\perp}\otimes\mathbf{B}^{\mathbf{m}}_{\perp} = \rho\left(Re_{_C}^{-1}\,\mathbf{B}^{\mathbf{m}}_{\perp}\otimes\mathbf{B}^{\mathbf{m}}_{\perp}\right) =: \rho\left(\mathbf{v}^{\mathbf{m}}_{\perp}\otimes\mathbf{v}^{\mathbf{m}}_{\perp}\right) \qquad (3.4)$$

and

$$\boldsymbol{\sigma}_{\mathbf{U(1)}} = \frac{3\zeta}{2}\mathbf{B}_{\parallel}\otimes\mathbf{B}_{\parallel} = \rho\left(\frac{3\zeta^{SI}}{2\mu^{SI}}Re_{_C}^{-1}\,\mathbf{B}_{\parallel}\otimes\mathbf{B}_{\parallel}\right) =: \rho\left(\mathbf{v}_{\parallel}\otimes\mathbf{v}_{\parallel}\right). \qquad (3.5)$$

In the respective last steps, the following scaled gauge fields were defined:

$$\mathbf{v}^{\mathbf{m}}_{\perp} = \sqrt{Re_{_C}^{-1}}\;\mathbf{B}^{\mathbf{m}}_{\perp} \qquad \text{and} \qquad \mathbf{v}_{\parallel} = \sqrt{\frac{3\zeta^{SI}}{2\mu^{SI}}Re_{_C}^{-1}}\;\mathbf{B}_{\parallel}. \qquad (3.6)$$

Next, the divergence is calculated, taking advantage of the fact that by construction the $\mathbf{v}^{\mathbf{m}}_{\perp}$ are orthogonal to $\mathbf{v}_{\parallel}$ (e.g. Huybrechts 2004):

$$\begin{aligned}
\boldsymbol{\nabla}\cdot\left(\boldsymbol{\sigma}_{\mathbf{SO(3)}} + \boldsymbol{\sigma}_{\mathbf{U(1)}}\right) &= \boldsymbol{\nabla}\cdot\left[\rho\left(\mathbf{v}^{\mathbf{m}}_{\perp}\otimes\mathbf{v}^{\mathbf{m}}_{\perp} + \mathbf{v}_{\parallel}\otimes\mathbf{v}_{\parallel}\right)\right]\\
&= \boldsymbol{\nabla}\cdot\left[\rho\left(\mathbf{v}^{\mathbf{m}}_{\perp} + \mathbf{v}_{\parallel}\right)\otimes\left(\mathbf{v}^{\mathbf{m}}_{\perp} + \mathbf{v}_{\parallel}\right)\right].
\end{aligned} \qquad (3.7)$$

This expression has the same form as the convection term of the Navier-Stokes equations in their conservation form, Eq. 1.12 :

$$\boldsymbol{\nabla}\cdot\left(\rho\,\mathbf{v}\otimes\mathbf{v}\right). \qquad (3.8)$$

Furthermore, the gauge field components $\mathbf{v}^{\mathbf{m}}_{\perp}$ and $\mathbf{v}_{\parallel}$ from Eq. 3.6 transform like axial vectors, and thus, up to space inversion, the same as the velocity field $\mathbf{v}$ in Eq. 3.8.

The issue of space inversion is addressed in the next paragraph, and solved by changing the sign of the coupling constant when inverting space.

As a result, the scaled gauge fields $\mathbf{v}^{\mathbf{m}}_{\perp}$ and $\mathbf{v}_{\parallel}$ coincide both in form and transformation properties with the convection term of the Navier-Stokes equations and can therefore be identified with each other. According to Eq. 3.7 and Eq. 3.8 the identification reads:

$$\mathbf{v} = \mathbf{v}_{\perp}^{\mathbf{m}} + \mathbf{v}_{\parallel}, \tag{3.9}$$

with the gauge fields $\mathbf{v}_{\perp}^{\mathbf{m}}$ and $\mathbf{v}_{\parallel}$ forming the convection velocity field.

3.4. *Divergence of the overall stress tensor*

The divergence of the stress tensor becomes in summary (inserted from Eqs. 3.2, 3.3 and 3.7):

$$\begin{aligned} \boldsymbol{\nabla}\cdot\boldsymbol{\sigma}_{\mathbf{local}} &= \boldsymbol{\nabla}\cdot\boldsymbol{\sigma}_{\mathbf{diag}} - \boldsymbol{\nabla}\cdot\boldsymbol{\sigma}_{\mathbf{connection}} + \boldsymbol{\nabla}\cdot\left(\boldsymbol{\sigma}_{\mathbf{SO(3)}} + \boldsymbol{\sigma}_{\mathbf{U(1)}}\right) \\ &= 2\mu\,\boldsymbol{\nabla}\cdot\dot{\boldsymbol{\varepsilon}}_{\mathbf{diag}} + \left(\zeta - \frac{2\mu}{3}\right)\boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\mathbf{v}) + \boldsymbol{\nabla}\cdot(\rho\,\mathbf{v}\otimes\mathbf{v}) \\ &= \left(\zeta + \frac{4\mu}{3}\right)\boldsymbol{\nabla}^2\mathbf{v} + \boldsymbol{\nabla}\cdot(\rho\,\mathbf{v}\otimes\mathbf{v})\,. \end{aligned} \tag{3.10}$$

3.5. *Insertion into the Cauchy momentum equations and comparison with Navier-Stokes equations*

The divergence Eq. 3.10 is inserted into the stationary Cauchy momentum equations (from Eq. 1.5):

$$\rho(\mathbf{v}\cdot\boldsymbol{\nabla})\mathbf{v} = \boldsymbol{\nabla}\cdot\boldsymbol{\sigma}_{\mathbf{local}} - \boldsymbol{\nabla}p + \rho\mathbf{g}. \tag{3.11}$$

Bringing the material derivative on the left side to its convective form yields:

$$\begin{aligned} \boldsymbol{\nabla}\cdot(\rho\,\mathbf{v}\otimes\mathbf{v}) &= \left(\zeta + \frac{4\mu}{3}\right)\boldsymbol{\nabla}^2\mathbf{v} + \boldsymbol{\nabla}\cdot(\rho\,\mathbf{v}\otimes\mathbf{v}) - \boldsymbol{\nabla}p + \rho\mathbf{g} \\ 0 &= \left(\zeta + \frac{4\mu}{3}\right)\boldsymbol{\nabla}^2\mathbf{v} - \boldsymbol{\nabla}p + \rho\mathbf{g}. \end{aligned} \tag{3.12}$$

As required at the start, with Eq. 3.12 the Navier-Stokes equations are obtained in the Eigensystem of the stress tensor at each point. These equations based on the diagonalized stress tensor contain only diffusive flow components, while the nonlinear, convective information from the material derivative is separated and absorbed by the divergence of the gauge fields.

At the same time, with the transfer of the convective flow components into the gauge fields, these components are suitably structured to apply perturbation theory.

The stress tensor $\boldsymbol{\sigma}_{\mathbf{local}}$ from Eq. 2.42 thus contains – up to a global similarity transformation – the information on all inner forces of the Navier-Stokes equations including the convective parts in the case of a base flow with location-dependent perturbations according to Par. 3.2.

4. Normalized scalar form

In this section, the field equation defined in Sec. 3 is expressed in a standardized form suitable for further treatment as a field theory. From the scaled vector fields

Eq. 3.6, scaled field tensors and vector potentials are defined:

$$F'_{jk} = -\epsilon_{ijk}(v_{\|})^i = \sqrt{\frac{3\zeta^{SI}}{\mu^{SI}\, Re_C}}\, F_{jk}$$
$$G'^m_{jk} = -\epsilon_{ijk}(v^m_{\perp})^i = \sqrt{\frac{2}{Re_C}}\, G^m_{jk} \tag{4.1}$$

$$\mathbf{A}'_{\|} = \sqrt{\frac{3\zeta^{SI}}{\mu^{SI}\, Re_C}}\, \mathbf{A}_{\|} = \sqrt{3\zeta\rho^{-1}}\mathbf{A}_{\|}$$
$$\mathbf{A}'^{\mathbf{m}}_{\perp} = \sqrt{\frac{2}{Re_C}}\, \mathbf{A}^{\mathbf{m}}_{\perp} = \sqrt{2\mu\rho^{-1}}\mathbf{A}^{\mathbf{m}}_{\perp}. \tag{4.2}$$

Whereas for the convective velocity fields Eq. 3.6 it holds that:

$$\mathbf{v}_{\|} = \frac{1}{\sqrt{2}}\boldsymbol{\nabla} \times \mathbf{A}'_{\|}$$
$$\mathbf{v}^{\mathbf{m}}_{\perp} = \frac{1}{\sqrt{2}}\left(\boldsymbol{\nabla} \times \mathbf{A}'^{\mathbf{m}}_{\perp} - \frac{1}{2}\mathbf{f}_{\mathbf{mnl}}\mathbf{A}'^{\mathbf{n}}_{\perp} \times \mathbf{A}'^{\mathbf{l}}_{\perp}\right). \tag{4.3}$$

Going back to the stress tensor Eq. 2.42 and looking at its trace, this reads in terms of the scaled fields, using that $\dot{\varepsilon}_{diag}{}^j{}_k = \delta^j_k \partial^j v^T_k \in \mathbb{R}$ (as in Eq. 3.2) and applying Eq. 1.4 for the constants:

$$\begin{aligned} \mathrm{tr}\big[\boldsymbol{\sigma}_{\mathbf{local}}\big] =& \left(\zeta + \frac{4\mu}{3}\right) \mathrm{tr}\big[\boldsymbol{\nabla}\mathbf{v}^{\mathrm{T}}\big] - 2\mu\, i\, \mathrm{tr}\big[\mathbf{v}\cdot\mathbf{A}^{\mathbf{m}}_{\perp}\big] - 3\zeta\, i\, \mathrm{tr}\big[\mathbf{v}\cdot\mathbf{A}_{\|}\big] \\ &+ \frac{\mu}{2}\mathbf{G^m}_{\mathbf{jk}}\mathbf{G}^{\mathbf{m}\,\mathbf{jk}} + \frac{3\zeta}{4}\mathbf{F}_{\mathbf{jk}}\mathbf{F}^{\mathbf{jk}} \\ =& \left(\zeta + \frac{4\mu}{3}\right) \boldsymbol{\nabla}\cdot\mathbf{v} - \sqrt{\frac{2}{Re_C}}\,\rho\, i\, \mathbf{A}'^{\mathbf{m}}_{\perp}\cdot\mathbf{v} - \sqrt{\frac{3\zeta^{SI}}{\mu^{SI}\, Re_C}}\,\rho\, i\, \mathbf{A}'_{\|}\cdot\mathbf{v} \\ &+ \frac{1}{4}\rho\, \mathbf{G'^m}_{\mathbf{jk}}\mathbf{G}'^{\mathbf{m}\,\mathbf{jk}} + \frac{1}{4}\rho\, \mathbf{F}'_{\mathbf{jk}}\mathbf{F}'^{\mathbf{jk}}. \end{aligned} \tag{4.4}$$

This expression can be brought into the standard form of Yang-Mills theories with normalized kinetic terms according to $\frac{1}{4}\mathbf{G'^m}_{\mathbf{jk}}\mathbf{G}'^{\mathbf{m}\,\mathbf{jk}}$ and $\frac{1}{4}\mathbf{F}'_{\mathbf{jk}}\mathbf{F}'^{\mathbf{jk}}$ by dividing through the density:

$$\rho^{-1}\,\mathrm{tr}\big[\boldsymbol{\sigma}_{\mathbf{local}}\big] = \left(\frac{\zeta^{SI}}{\mu^{SI}} + \frac{4}{3}\right) Re_C^{-1}\,\mathbf{D}\cdot\mathbf{v} + \frac{1}{4}\mathbf{G'^m}_{\mathbf{jk}}\mathbf{G}'^{\mathbf{m}\,\mathbf{jk}} + \frac{1}{4}\mathbf{F}'_{\mathbf{jk}}\mathbf{F}'^{\mathbf{jk}}. \tag{4.5}$$

Finally defining the gauge covariant derivative $\mathbf{D}$ and the coupling constants g and g' between the locally diagonalized diffusion term and the convection fields:

$$\mathbf{D} = \boldsymbol{\nabla} - ig\mathbf{A}'^{\mathbf{m}}_{\perp} - ig'\mathbf{A}'_{\|}, \tag{4.6}$$

and therefore (using that $\boldsymbol{\nabla}\cdot\mathbf{v} = \mathrm{tr}\big[\dot{\boldsymbol{\varepsilon}}_{\mathbf{diag}}\big]$ from Eq. 2.1):

$$\mathbf{D}\cdot\mathbf{v} = \mathrm{tr}\big[\dot{\boldsymbol{\varepsilon}}_{\mathbf{diag}}\big] - ig\mathbf{A}'^{\mathbf{m}}_{\perp}\cdot\mathbf{v} - ig'\mathbf{A}'_{\|}\cdot\mathbf{v}, \tag{4.7}$$

with

$$g = \frac{\sqrt{2}}{\left(\frac{\zeta^{SI}}{\mu^{SI}} + \frac{4}{3}\right)} \sqrt{Re_C} \qquad \text{and} \qquad g' = \frac{\sqrt{\frac{3\zeta^{SI}}{\mu^{SI}}}}{\left(\frac{\zeta^{SI}}{\mu^{SI}} + \frac{4}{3}\right)} \sqrt{Re_C}. \tag{4.8}$$

Please note that as stated in Section 3, the sign of the coupling constants is inverted upon space inversion in order to achieve the correct transformation properties of the underlying gauge fields: $\breve{g} = -g$ and $\breve{g}' = -g'$.

5. Conclusion

A method is presented to locally diagonalize the strain-rate tensor found in fluid mechanics in the entire space, using $SO(3) \times U(1)$ gauge fields. From this, a locally diagonal stress tensor is determined which is coupled through these gauge fields to the surrounding stresses. It is derived that the diagonal component contains the diffusion terms of the flow, whereas the gauge fields contain the convective parts.

In addition, the stress tensor includes interaction terms which describe the interaction between diffusion and convection terms. Taken the material-related pre-factors in the gauge coupling constants as a given, the interaction terms couple proportionally to the square root of the Reynolds number $\sqrt{Re_C}$ to the diffusion terms (Eq. 4.8), thus exhibiting the expected property that flows behave according to the Reynolds similarity law.

The introduced gauge fields describe the symmetry properties of convective flows to a previously unknown degree: The convection is composed of four skew-symmetric flow fields, of which three are volume preserving and one is not. The volume preserving fields are non-abelian and show self-interactions of higher order.

It is shown that the stress tensor constructed according to Eq. 4.5 contains all information of the Navier-Stokes equations up to a global similarity transformation in the situation of a stationary flow with location-independent shear or vorticity to which location-dependent perturbations are applied. This equation can thus be used as an alternative basic equation for many stationary problems in fluid mechanics, and offers advantages over the Navier-Stokes equations in terms of interpretation and/or solution approaches, depending on the application.

Since the structure of Eq. 4.5 is very similar to minimal coupling field theories in quantum mechanics, it can be expected, that – with favorable boundary conditions – the gauge fields can be quantized using the second quantization formalism.

This makes it possible to interpret the gauge fields as quasi-particles with properties depending on the Reynolds number. Due to the properties of the underlying symmetry groups, the quasi-particles have bosonic spin 1 character, and thus contain an inherent angular momentum. They can therefore be interpreted as elementary vortices. The $SO(3)$ fields can be understood as elementary vortices with orientation in the three Cartesian spatial directions, which interact with each other. It is interesting to note that in two-dimensional flows the self-interaction disappears, because in this case the system of equations is reduced to a $SO(2) \times U(1)$ theory with the abelian rotating group $SO(2)$.

The $U(1)$ convection field interacts due to volume viscosity and disappears in non-compressible fluids.

Since the gauge fields in field equation Eq. 4.5 leave the trace $\mathrm{tr}\,[\dot{\boldsymbol{\varepsilon}}]$ of the strain-rate tensor (or the trace $\mathrm{tr}\,[\dot{\boldsymbol{\varepsilon}}^{*}\dot{\boldsymbol{\varepsilon}}]$ in the compressible case) invariant, the developed equation is particularly suitable to investigate flows with globally constant trace – including the case of incompressible flows with constant trace $\mathrm{tr}\,[\dot{\boldsymbol{\varepsilon}}] = 0$. The gauge fields then provide the information about possible perturbations and degrees of freedom within the framework set by the trace.

However, more detailed examinations on these topics must be left for further work.

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