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An Approximation to Riemann Hypothesis

Li An-Ping *

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An Approximation to Riemann Hypothesis

Li An-Ping

Independent Researcher, Beijing 100086, P.R.China; apli0001@sina.com

Abstract

There are added some matters for the estimation of $H(n,m)$ in the appendix.

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1. Introduction

Riemann zeta-function $\zeta(s)$ is originally defined as

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \qquad \text{for } \operatorname{Re} s > 1.$$

and it also can be expressed as the product form

$$\zeta(s) = \prod_p \frac{1}{1-1/p^s}, \qquad \text{for } \operatorname{Re} s > 1.$$

This formula is called Euler's product formula, which indicates the relation between $\zeta(s)$ and prime numbers. About $\zeta(s)$ there is a well-known Riemann hypothesis, states that all the non-trivial zeros of $\zeta(s)$ are on the critical line $\operatorname{Re} s = 1/2$. The researches on the conjecture are no doubt a most time-consuming one in mathematics, refer to see the survey paper [3].

The so-called trivial zeros of $\zeta(s)$ are $s = -2, -4, \cdots$, and nontrivial zeros of $\zeta(s)$ are known all in the critical strip $0 \le \operatorname{Re} s \le 1$.

Denote by $N(T)$ the number of zeros of $\zeta(\sigma + it)$ in the region $0 \le \sigma \le 1, 0 \le t \le T$, and by $N_0(T)$ the number of zeros on the critical line $\sigma = 1/2, 0 \le t \le T$. Riemann hypothesis is that

$$N_0(T) = N(T).$$

For $N(T)$, it is known that

$$N(T) = \frac{T}{2\pi}\log\frac{T}{2\pi} - \frac{T}{2\pi} + O(\log T). \tag{1.1}$$

And for $N_0(T)$, Hardy firstly shown that there are infinity many zeros on the critical line, and then he and Littlewood [5] and Selberg [8] proved that

$$\kappa = \frac{N_0(T)}{N(T)} > 0.$$

Levinson [6] proved

$$\kappa \ge \frac{1}{3}.$$

and then this result has been improved successively. Conrey [2], Feng [4] proved respectively

$$\kappa \ge 0.407, \quad \kappa \ge 0.412.$$

In this paper, we will prove that

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Theorem 1.1.
$$N(T) = N_0(T) + \mathscr{E}. \tag{1.2}$$
where $\mathscr{E} \ll T^{1/2}(\log T)^3$.

The main arguments in this paper are based the papers [6],[7] and [1], but instead of using Riemann-Siegel formula, it will be applied an auxiliary function $\omega(s, T_1, T_2)$ defined in Lemma 2.1, which will play a role of mollifier and ferry, it firstly used in [7] but here with a small modification.

2. Some Lemmas

In the following, it will be used an auxiliary function $\omega(s, T_1, T_2)$, which defined as following. Suppose that $T \le T_1 \le T_2 \le 2T$, $\lambda = T^{1-\epsilon}$, for any $\epsilon > 0$, $s_1 = \lambda + c + iu$,$(c \ge 0)$, $s = v + it$, define
$$g(w) = e^{\lambda}\Gamma(w)\lambda^{-w},$$
and
$$\omega(s, T_1, T_2) = \frac{1}{2\pi}\int_{T_1}^{T_2} g(s_1 - s)du. \tag{2.1}$$

Lemma 2.1. *Let* $\sigma = \lambda + c - v$, $\eta = \min\{|x - t|\,|x \in [T_1, T_2]\}$, *there is*
$$|\omega(s, T_1, T_2)| \ll \exp(((c - v)^2 - \eta^2)/2\sigma) \tag{2.2}$$
Suppose $\Delta \ge \Delta_0(= (2\beta\lambda\log T)^{1/2}, \beta > 0)$, *if* $t \in [T_1 + \Delta, T_2 - \Delta]$, *then*
$$|\omega(c + it, T_1, T_2) - 1| \ll T^{-\beta}. \tag{2.3}$$
And if $t \le T_1 - \Delta$, *or* $t \ge T_2 + \Delta$, *then*
$$|\omega(c + it, T_1, T_2)| \ll T^{-\beta}. \tag{2.4}$$
If $|t - u| = o(\sigma^{2/3})$, *then*
$$\arg g(s_1 - s) = \frac{t-u}{2\sigma} - \frac{(t-u)^3}{6\sigma^2} - \frac{(c-v)(t-u)}{\sigma} + O\left(\frac{1}{\sigma}\right). \tag{2.5}$$

Proof. By Stirling's formula, it has
$$\begin{aligned}\mathrm{Re}(\log\Gamma(\sigma + (u-t)i)) =& \left(\sigma - \frac{1}{2}\right)\frac{\log(\sigma^2 + (u-t)^2)}{2} - \sigma + \frac{1}{2}\log(2\pi) \\ &- (u-t)\arctan\left(\frac{u-t}{\sigma}\right) + O\left(\frac{1}{\sigma}\right)\end{aligned}$$
And
$$\begin{aligned}&\mathrm{Re}(\log(\Gamma(\sigma + (u-t)i))) + \mathrm{Re}(\log(\lambda^{-(\sigma+(u-t)i)})) + \lambda \\ &= \frac{(c-v)^2}{2\sigma} - \frac{(u-t)^2}{4\sigma^2} - \frac{(u-t)^2}{2\sigma} - \frac{1}{2}\log\sigma + \frac{1}{2}\log(2\pi) + O\left(\frac{1}{\sigma}\right)\end{aligned}$$
Hence,
$$|\omega(s, T_1, T_2)| \le e^{(c-v)^2/2\sigma}\frac{1}{\sqrt{2\pi\sigma}}\int_{T_1}^{T_2} e^{-(u-t)^2/2\sigma}du \ll e^{((c-v)^2-\eta^2)/2\sigma}.$$

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Besides, it is familiar that

$$e^{-\lambda} = \frac{1}{2\pi i} \int_{(c)} \Gamma(s_1 - s)\lambda^{s-s_1} ds.$$

Hence,

$$1 - \omega(c + it, T_1, T_2) = R_1 + R_2,$$

where

$$R_1 = \frac{e^{\lambda}}{2\pi} \int_{T_2}^{\infty} \Gamma(\lambda + (u-t)i)\lambda^{-(\lambda+(u-t)i)} du,$$

$$R_2 = \frac{e^{\lambda}}{2\pi} \int_{-\infty}^{T_1} \Gamma(\lambda + (u-t)i)\lambda^{-(\lambda+(u-t)i)} du,$$

Hence, if $t \in [T_1 + \Delta, T_2 - \Delta]$, then

$$\begin{aligned} |R_1| &\ll \int_{T_2}^{\infty} \left| e^{\lambda}\Gamma(\lambda + (u-t)i)\lambda^{-(\lambda+(u-t)i)} \right| du \\ &\ll \lambda^{-1/2} \int_{T_2}^{\infty} \exp(-(u-t)^2/2\lambda) du \\ &\ll T^{-\beta}. \end{aligned}$$

and similarly,

$$|R_2| \ll \lambda^{-1/2} \int_{-\infty}^{T_1} \exp(-(u-t)^2/2\lambda) du \ll T^{-\beta}.$$

if $t \le T_1 - \Delta$, or $t \ge T_2 + \Delta$, then

$$|\omega(c + it, T_1, T_2)| \ll \lambda^{-1/2} \int_{T_1}^{T_2} \exp(-(u-t)^2/2\lambda) du \ll T^{-\beta}.$$

If $|t - u| = o(\sigma^{2/3})$, then

$$\begin{aligned} \mathrm{Im}(\log(\blacksquare(s_1 - s)^{s-s_1})) =& \frac{t-u}{2\sigma} + \frac{(t-u)^3}{3\sigma^2} - \frac{(t-u)^3}{2\sigma^2} \\ &- \frac{(c-v)(t-u)}{\sigma} + O\left(\frac{1}{\sigma}\right) \\ =& \frac{t-u}{2\sigma} - \frac{(t-u)^3}{6\sigma^2} - \frac{(c-v)(t-u)}{\sigma} + O\left(\frac{1}{\sigma}\right). \end{aligned}$$

□

Lemma 2.2. *Let $L = \log(T/2\pi)$, $G_0(s) = \zeta(s) + \zeta'(s)/L$, and $0 < a \le 1/2$, $1 < b < \beta - 1$, $u \in [T_1, T_2]$, $s_1 = \lambda + c + iu$, then*

$$\begin{aligned} &\frac{1}{2\pi i} \int_{a+(u-\Delta)i}^{a+(u+\Delta)i} g(s_1 - s)G_0(s)G_0(2a - s) ds \\ &= \frac{1}{2\pi i} \int_{b+(u-\Delta)i}^{b+(u+\Delta)i} g(s_1 - s)G_0(s)G_0(2a - s) ds + O(b/T) \end{aligned} \tag{2.6}$$

Proof. Let $\mathscr{B}$ be the rectangle with vertices $a+(u-\Delta)i$, $a+(u+\Delta)i$, $b+(u-\Delta)i$ and $b+(u+\Delta)i$. Take the integral $\int_{\mathscr{B}} g(s_1-s)G(s)G(2a-s)ds$, with the residue theorem, it has

$$\begin{aligned}&\frac{1}{2\pi i}\int_{b+(u-\Delta)i}^{b+(u+\Delta)i} g(s_1-s)G_0(s)G_0(2a-s)ds\\ =&\frac{1}{2\pi i}\int_{a+(u-\Delta)i}^{a+(u+\Delta)i} g(s_1-s)G_0(s)G_0(2a-s)ds\\ +&\frac{1}{2\pi i}\int_{a+(u+\Delta)i}^{b+(u+\Delta)i} g(s_1-s)G_0(s)G_0(2a-s)ds\\ +&\frac{1}{2\pi i}\int_{b+(u-\Delta)i}^{a+(u-\Delta)i} g(s_1-s)G_0(s)G_0(2a-s)ds.\end{aligned}$$

On the upper side and the lower side, as $|\mathrm{Im}(s_1-s)|=\Delta$, by Lemma 2.1, it has

$$|g(s_1-s)| \ll 1/T^{\beta}.$$

Besides, by the functional equation, it is easy to know

$$|G_0(v+(u\pm\Delta)i)G_0(2a-(v+(u\pm\Delta)i))| \ll T^b, \quad (a\le v\le b).$$

Hence, the two integrals on the upper side and the lower side of $\mathscr{B}$

$$\frac{1}{2\pi i}\int_{a+(u\pm\Delta)i}^{b+(u\pm\Delta)i} g(s_1-s)G_0(s)G_0(2a-s)ds \ll b/T,$$

and the Lemma is followed.

□

Lemma 2.3. *Suppose that* $0<\alpha<\beta-2$. *For* $r>0$, *let*

$$J(r)=\frac{1}{2\pi}\int_{T_1}^{T_2}\int_{u-\Delta}^{u+\Delta}\left(\frac{t}{2\pi}\right)^{\alpha} e^{-(t-u)^2/2\sigma}\exp\left(it\log\left(\frac{t}{re}\right)\right)dtdu$$

Then for $T_1-\Delta\le r\le T_2+\Delta$,

$$J(r)\doteq\left(\frac{r}{2\pi}\right)^{\alpha} r^{1/2}\sigma^{1/2}e^{\pi i/4}. \tag{2.7}$$

And if $r<T_1-\Delta$, *or* $r>T_2+\Delta$, *and* $\Delta\ge\Delta_0 T^{\epsilon}$, *then*

$$J(r)=O(1). \tag{2.8}$$

Proof. Denote by

$$F(u)=\int_{u-\Delta}^{u+\Delta}\left(\frac{t}{2\pi}\right)^{\alpha} e^{-(t-u)^2/2\sigma}\exp\left(it\log\left(\frac{t}{re}\right)\right)dt$$

And let $t=u(1+x)$,then

$$F(u)=\left(\frac{u}{2\pi}\right)^{\alpha} ue^{iu\rho}F_1(u),$$

where $\rho=\log(u/r)-1$,

$$F_1(u)=\int_{-\Delta/u}^{\Delta/u}\exp(A(x)+B(x)i)dx,$$

$$A(x)=\alpha\log(1+x)-(ux)^2/2\sigma,\quad B(x)=u(1+x)\log(1+x)+ux\rho.$$

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By Gauss's integration, it is easy to follow

$$F_1(u) \doteq \frac{1}{u}\frac{2\pi\sigma}{\sqrt{1+\alpha\sigma/u^2-i\sigma/u}}\exp\left(-\frac{(1+\rho-i\alpha/u)^2u}{2(u/\sigma+\alpha/u-i)}\right)$$

and

$$F(u) \doteq \left(\frac{u}{2\pi}\right)^{\alpha}e^{iu\rho}\frac{\sqrt{2\pi\sigma}}{\sqrt{1+\alpha\sigma/u^2-i\sigma/u}}\exp\left(-\frac{(\log(u/r)-i\alpha/u)^2\sigma}{2(1+\alpha\sigma/u^2-i\sigma/u)}\right)$$

If $r \in [T_1-\Delta, T_2+\Delta]$, let $u = r(1+x)$, then

$$F(u) \doteq \left(\frac{r}{2\pi}\right)^{\alpha}\sqrt{2\pi\sigma}\exp\left(-\frac{x^2(\sigma-ri)}{2}\right)(1+x)^{\alpha}$$

and

$$\begin{aligned}\frac{1}{2\pi}\int_{T_1/r-1}^{T_2/r-1}F(u)rdx &\doteq \frac{r}{2\pi}\left(\frac{r}{2\pi}\right)^{\alpha}\frac{\sqrt{2\pi\sigma}\cdot\sqrt{\pi}}{\sqrt{(\sigma+\alpha-ri)/2}}\exp\left(\frac{\alpha^2(\sigma+\alpha+ri)}{2(\sigma+\alpha)^2+r^2)}\right)\\ &\doteq \left(\frac{r}{2\pi}\right)^{\alpha}r^{1/2}\sqrt{\sigma}e^{\pi i/4}\end{aligned}$$

If $r < T_1-\Delta$, or $r > T_2+\Delta$, and if $\Delta \geq \Delta_0 T^{\epsilon}$, then

$$\begin{aligned}|F(u)| &\ll T^{\alpha+1/2}\exp\left(-\left(\frac{\Delta}{T}\right)^2\frac{\sigma}{2}\right)\\ &\ll T^{\alpha+1/2-\beta} \ll O(1/T).\end{aligned}$$

□

3. The Proof of Theorem 1.1

Proof. Let $h(s) = \pi^{-s/2}\Gamma(s/2)$, then the functional equation of $\zeta(s)$ can be written as

$$h(s)\zeta(s) = h(1-s)\zeta(1-s) \tag{3.1}$$

By Stirling's formula, it has

$$\log h(s) = \frac{1}{2}(s-1)\log\frac{s}{2\pi} - \frac{s}{2} + C_0 + O\left(\frac{1}{s}\right) \tag{3.2}$$

Let $f(s) = \log h(s)$, then

$$f'(s) = \frac{h'(s)}{h(s)} = \frac{1}{2}\log\frac{s}{2\pi} + O\left(\frac{1}{s}\right) \tag{3.3}$$

and for larger t

$$f'(s) + f'(1-s) = \log\frac{t}{2\pi} + O\left(\frac{1}{s}\right) \tag{3.4}$$

Taking logarithm of equation (3.1), and then derivative, it follows

$$h(s)\zeta(s)(f'(s)+f'(1-s)) = -h(s)\zeta'(s) - h(1-s)\zeta'(1-s) \tag{3.5}$$

We note that the right side of (3.5) is a sum of two conjugative complex numbers as $s = 1/2+it$, so the zeros of the right side of (3.5) occur if and only if

$$\arg(h(s)\zeta'(s)) \equiv \pi/2 \mod \pi \tag{3.6}$$

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On the left side of (3.5), clearly, $h(s)$ is never zero, and by (3.4), so these zeros are just the zeros of $\zeta(1/2+it)$.
Moreover, let $\chi(s) = h(1-s)/h(s)$, then $\zeta(s) = \chi(s)\zeta(1-s)$, and

$$\zeta'(s) = -\chi(s)\{(f'(s) + f'(1-s))\zeta(1-s) + \zeta'(1-s)\} \tag{3.7}$$

By (3.6), the zeros of $\zeta(1/2+it)$ are the ones

$$\arg(h(1-s)\{(f'(s) + f'(1-s))\zeta(1-s) + \zeta'(1-s)\}) \equiv \pi/2 \mod \pi$$

on $\sigma = 1/2$, equivalently,

$$\arg(h(s)\{(f'(s) + f'(1-s))\zeta(s) + \zeta'(s)\}) \equiv \pi/2 \mod \pi \tag{3.8}$$

on $\sigma = 1/2$. Write $\mathcal{L}(s) = f'(s) + f'(1-s)$, and denote by

$$G(s) = \zeta(s) + \zeta'(s)/\mathcal{L}(s) \tag{3.9}$$

The investigation above means

$$N_0(T) = \frac{1}{\pi}\Delta_0^T \arg(hG(1/2+it)) \tag{3.10}$$

By(3.2), it can be known that

$$\Delta_0^T \arg(h(1/2+it)) = \frac{T}{2}\log\frac{T}{2\pi} - \frac{T}{2} + O(\log T) \tag{3.11}$$

So, the main task to determine $N_0(T)$ is to calculate $\Delta_0^T \arg(G(1/2+it))$.

Let $L = \log(T/2\pi)$, $U \le T$, and let D be the rectangle with the vertices $1/2+iT$, $c+iT$, $c+i(T+U)$, $1/2+i(T+U)$,$(c \ge 3)$. First of all, we might as well assume there are no zeros of $G(s)$ on the boundary of D, then by the principle of argument, the change of $\arg G(s)$ around D is equal to 2π times $N_G(D)$, the number of zeros of $G(s)$ in D.

On the right side of D

$$|G(c+it) - 1| \le \sum_{n\ge 2} n^{-c} + O(1/L) \le 1/3$$

so, $\arg G(s)$ change less than π. On the lower side and the upper side of D, by a known result [9, §9.4], a extension of Jessen's theorem, taking account on the order of $G(s)$, we can know that $\arg G(s) = O(L)$ as $0 < \sigma \le 3$, and $\arg G(\sigma+it) = O(2^{-\sigma})$ as $\sigma \ge 3$, hence, for any $0 \le b \le c$, it has

$$\int_b^c \arg G(\sigma+iT)d\sigma, \int_b^c \arg G(\sigma+i(T+U))d\sigma \ll O(L) \tag{3.12}$$

So,

$$\Delta_T^{T+U} \arg(G(1/2+it)) = -2\pi N_G(D) + O(\log T) \tag{3.13}$$

Now the work is turned into to evaluate $N_G(D)$.

Let $1/2 - a = O(1/L)$, and $\mathscr{C}$ be the rectangle with vertices $a+iT$, $c+iT$, $c+i(T+U)$, $a+i(T+U)$. Taking the integral $\int_{\mathscr{C}} \log G(s)ds$, by the Littlewood's Lemma [9, §9.9], it has

$$\int_T^{T+U} \log|G(a+it)|dt - \int_T^{T+U} \log|G(c+it)|dt + \int_a^c \arg G(\sigma+i(T+U))d\sigma$$
$$-\int_a^c \arg G(\sigma+iT)d\sigma = 2\pi\sum \text{dist} \tag{3.14}$$

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where $\sum$ dist is the sum of the distances of the zeros of $G(s)$ from the left.
By (3.9), it is easy to know

$$\int_T^{T+U} \log G(c+it)dt = \int_T^{T+U} \log \zeta(c+it)dt + O(1/L)$$

and it is familiar that

$$\log \zeta(s) = \sum_n \frac{-\Lambda(n)}{n^s \log n}$$

So

$$\int_T^{T+U} \log |G(c+it)|dt \ll 1.$$

With (3.12), the rest is to calculate the first integral of (3.14).

By the concavity of logarithm, it has

$$\begin{aligned} \int_T^{T+U} \log |G(a+it)|dt &= \frac{1}{2} \int_T^{T+U} \log |G(a+it)|^2 dt \\ &\leq \frac{1}{2} U \log \left(\frac{1}{U} \int_T^{T+U} |G(a+it)|^2 dt \right) \end{aligned} \tag{3.15}$$

At first, we simplify $G(s)$ as

$$G_0(s) = \zeta(s) + \frac{\zeta'(s)}{L}.$$

Then

$$G(s) = G_0(s) + E(s).$$

$$E(s) = \left(\frac{1}{\mathcal{L}(s)} - \frac{1}{L} \right) \zeta'(s) \ll \frac{1}{L^3} \zeta'(s).$$

And

$$\begin{aligned} \int_{T_1}^{T_2} |G(a+it)|^2 dt = \int_{T_1}^{T_2} |G_0(a+it)|^2 dt + 2\mathrm{Re} \int_{T_1}^{T_2} G_0(a+it)E(a-it)dt \\ + \int_{T_1}^{T_2} |E(a+it)|^2 dt \end{aligned} \tag{3.16}$$

By Cauchy's inequality

$$\int_{T_1}^{T_2} G_0(a+it)E(a-it)dt \leq \left(\int_{T_1}^{T_2} |G_0(a+it)|^2 dt \int_{T_1}^{T_2} |E(a+it)|^2 dt \right)^{1/2}$$

The third integral in the right side of (3.16) is much smaller than the first one, which will be actually calculated later, hence

$$\int_{T_1}^{T_2} |G(a+it)|^2 dt = (1+\epsilon) \int_{T_1}^{T_2} |G_0(a+it)|^2 dt.$$

Let

$$\omega_1(s) = \frac{1}{2\pi i} \int_{|u-t| \leq \Delta} g(s_1 - s) ds_1. \tag{3.17}$$

From Lemma 2.1, it is known that $\omega_1(s)$ is the domination of $\omega(s, T_1, T_2)$, and which is a positive real number apart from a small error term.

Moreover, let

$$\phi(s) = \omega_1^{1/2}(s). \tag{3.18}$$

Then it has

$$\arg\phi(v+T_1 i) = o(1), \quad \arg\phi(v+T_2 i) = o(1).$$

And

$$\int_a^c \arg\phi(v+T_1 i)dv \le o(c), \quad \int_a^c \arg\phi(v+T_2 i)dv \le o(c). \tag{3.19}$$

Moreover, by Lemma 2.1, there is

$$\int_{T_1}^{T_2} \log|\phi(c+it)|dt \ll T^{-b}. \tag{3.20}$$

(3.19) and (3.20) indicate that function $\phi(s)$ may be used as a mollifier.
Let

$$\mathcal{G}(s) = G_0(s)\phi(s). \tag{3.21}$$

In the following we will replace $G(s)$ by $\mathcal{G}(s)$, and let

$$\mathcal{I} = \int_{T_1}^{T_2} |\mathcal{G}(s)|^2 dt \tag{3.22}$$

Then,

$$\begin{aligned}
\mathcal{I} &= \int_{T_1}^{T_2} |G_0(s)|^2 \omega_1(s) dt \\
&= \frac{1}{2\pi}\int_{T_1}^{T_2}\int_{t-\Delta}^{t+\Delta} g(s_1-s)du|G_0(a+it)|^2 dt \\
&= \frac{1}{2\pi}\int_{T_1}^{T_2}\int_{u-\Delta}^{u+\Delta} g(s_1-s)|G_0(a+it)|^2 dtdu + R_1 + R_2 - R_3 - R_4
\end{aligned}$$

where

$$\begin{aligned}
R_1 =& \frac{1}{2\pi}\int_{T_1}^{T_1+\Delta}\int_{u-\Delta}^{T_1} g(s_1-s)du|G_0(a+it)|^2 dt, \\
R_2 =& \frac{1}{2\pi}\int_{T_2-\Delta}^{T_2}\int_{u+\Delta}^{T_1} g(s_1-s)du|G_0(a+it)|^2 dt, \\
R_3 =& \frac{1}{2\pi}\int_{T_1-\Delta}^{T_1}\int_{T_1}^{u+\Delta} g(s_1-s)du|G_0(a+it)|^2 dt, \\
R_4 =& \frac{1}{2\pi}\int_{T_2}^{T_2+\Delta}\int_{u-\Delta}^{T_2} g(s_1-s)du|G_0(a+it)|^2 dt.
\end{aligned}$$

By the mean-value theorems (cf. [9, Ch.7]), it has

$$R_1 \ll \int_{T_1}^{T_1+\Delta} |G_0(a+it)|^2 dt \ll \Delta L,$$

similarly,

$$R_i \ll \Delta L, \quad i = 2,3,4.$$

Then, by Lemma 2.2, it has

$$\mathcal{I} = \frac{1}{2\pi i}\int_{T_1}^{T_2}\int_{b+(u-\Delta)i}^{b+(u+\Delta)i} g(s_1-s)G_0(s)G_0(2a-s)ds + O(\Delta L)$$

Moreover, by the functional equation of $\zeta(z)$, it has

$$\zeta(2a-(b+it)) = \chi(2a-(b+it))\zeta(1+b-2a-it),$$

$$\zeta'(2a-(b+it)) = -\chi(2a-(b+it))(L\zeta(1+b-2a+it)+\zeta'(1+b-2a+it))$$

and

$$G_0(2a-(b+it)) = -\chi(2a-(b+it))\frac{\zeta'(1+b-2a+it)}{L}$$

and it is easy to know

$$\chi(2a-(b+it)) = t^{1/2+b-2a}\exp\left(-\frac{\pi i}{4}+it\log\left(\frac{t}{2\pi e}\right)\right)$$

Hence,

$$\mathcal{I} = I_1 + I_2 + O(\Delta L).$$

where

$$I_1 = -\frac{1}{2\pi}\int_{T_1}^{T_2}\int_{u-\Delta}^{u+\Delta} g(s_1-s)\chi(2a-(b+it))\zeta(b+it)\frac{\zeta'(1+b-2a+it)}{L}dtdu$$

$$I_2 = -\frac{1}{2\pi}\int_{T_1}^{T_2}\int_{u-\Delta}^{u+\Delta} g(s_1-s)\chi(2a-(b+it))\frac{\zeta'(b+it)}{L}\frac{\zeta'(1+b-2a+it)}{L}dtdu$$

Expanding $\zeta(s)$ and $\zeta'(s)$ as Dirichlet's series, and dividing I_1 and I_2 into three parts respectively,

$$I_1 = I_{1,1} + I_{1,2} + I_{1,3} = \sum_{2\pi xy<T_1-\Delta} + \sum_{T_1-\Delta\le 2\pi xy\le T_2+\Delta} + \sum_{2\pi xy>T_2+\Delta},$$

$$I_2 = I_{2,1} + I_{2,2} + I_{2,3} = \sum_{2\pi xy<T_1-\Delta} + \sum_{T_1-\Delta\le 2\pi xy\le T_2+\Delta} + \sum_{2\pi xy>T_2+\Delta}.$$

Then by Lemmas 2.1,∼,3, there are

$$I_{1,1}, I_{1,3}, I_{2,1}, I_{2,3} \ll o(1).$$

and

$$I_{1,2} = \frac{2\pi}{L}\sum_{T_1-\Delta\le 2\pi xy\le T_2+\Delta}\frac{(xy)^{b'}\log y}{x^b y^{b'}}$$

$$I_{2,2} = \frac{-2\pi}{L^2}\sum_{T_1-\Delta\le 2\pi xy\le T_2+\Delta}\frac{(xy)^{b'}\log x\log y}{x^b y^{b'}}.$$

where $b' = b+1-2a$, and $b = c(>1)$.
i.e.

$$I_{1,2} = \frac{2\pi}{L}\sum_{T_1-\Delta\le 2\pi xy\le T_2+\Delta} x^{1-2a}\log y,$$

$$I_{2,2} = \frac{-2\pi}{L^2}\sum_{T_1-\Delta\le 2\pi xy\le T_2+\Delta} x^{1-2a}\log x\log y.$$

Let

$$H(n) = \frac{1}{L}\sum_{1\le xy\le n} x^{1-2a}\log y - \frac{1}{L^2}\sum_{1\le xy\le n} x^{1-2a}\log x\log y.$$

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It is easy to follow that

$$H(n) \doteq \frac{n}{L^2(1-a)}\left(\frac{n^{1-2a}}{(1-2a)^3} - \frac{\log^2 n}{2(1-2a)} - \frac{\log n}{(1-2a)^2} - \frac{1}{(1-2a)^3}\right)$$

Let $x = (1-2a)L$, then

$$H(n) \doteq \frac{2nL}{x^3}(e^x - 1 - x - x^2/2) \doteq nL/3 \tag{3.23}$$

Obviously, estimation (3.23) is not sufficient for the proof of Theorem 1.1. Nevertheless, there is an alternative way to improve it.

Actually, the argument of Levinson [6] can be extended to differentiate the functional equation of $\zeta(s)$ to higher order k, and similarly to obtain the functions $G(s,k)$, $k = 1, 2, ...$, and similar results as (3.23), and more sharper.

For examples, for $k = 1$,

$$G(s,1) = \zeta(s) + \zeta'(s)/L,$$

and

$$H(n,1) = \frac{nL}{x^3}\{a_1(x)e^x - b_1(x)\} \doteq nL/3. \tag{3.24}$$

where

$$a_1(x) = 2,$$
$$b_1(x) = 2 + 2x + x^2.$$

For $k = 2$,

$$G(s,2) = \zeta(s) + 4\zeta'(s)/L + 4\zeta''(s)/L^2.$$

and

$$H(n,2) \doteq \frac{nL}{x^5}\{a_2(x)e^x - b_2(x)\} \doteq nL/5 \tag{3.25}$$

where

$$a_2(x) = 384 - 192x + 48x^2 - 8x^3 + x^4,$$
$$b_2(x) = 384 + 192x + 48x^2 + 8x^3 + x^4.$$

And for $k = 3$,

$$G(s,3) = \zeta(s) + 6\zeta'(s)/L + 12\zeta''(s)/L^2 + 8\zeta'''(s)/L^3.$$

and

$$H(n,3) \doteq \frac{nL}{x^7}\{a_3(x)e^x - b_3(x)\} \doteq nL/7. \tag{3.26}$$

where

$$a_3(x) = 46080 - 23040x + 5760x^2 - 960x^3 + 120x^4 - 12x^5 + x^6,$$
$$b_3(x) = 46080 + 23040x + 5760x^2 + 960x^3 + 120x^4 + 12x^5 + x^6.$$

So, it is predictable that for $k \geq L/2$, there will be

$$H(n,k) \doteq n.$$

A rough proof for this is added in the appendix.

So,

$$\mathcal{I} = U + T^{\epsilon},$$

where ϵ is a arbitrary small positive number.
And

$$\int_T^{T+U} |\mathcal{G}(a+it)|^2 dt = U + T^\epsilon.$$

Let $U = T$, by (3.15), it follows

$$\begin{aligned}\int_T^{T+U} \log|\mathcal{G}(a+it)|dt &\leq \frac{T}{2}\log\left(1+T^{-1+\epsilon}\right) \\ &\ll T^\epsilon. \end{aligned} \tag{3.27}$$

With (3.12), (3.14), (3.19),(3.20) and (3.27), and $(1-2a)L = 1$, it follows

$$2\pi N_G(D) \leq \frac{T^\epsilon + O(\Delta L)}{1/2 - a} \ll T^{1/2}L^3.$$

i.e.

$$\Delta_{2T}^T \arg G(1/2+it) \leq O(T^{1/2}L^3).$$

and

$$(N(2T) - N(T)) - (N_0(2T) - N_0(T)) \leq O(T^{1/2}L^3).$$

Then let T be $T/2^k, 1 \leq k \leq \log_2(T)$, and summing. This proves Theorem 1.1 in the case that there are no zeros of $G(s)$ on the boundary of D.

For the rest case, let N_1 and N_2 be the numbers of zeros of $G(s)$ on the left side of D, $\sigma = 1/2$, and in D with $\sigma > 1/2$, respectively. Indent the left side of D with small semicircles with centers at the zeros and lying in $\sigma \geq 1/2$. Let N_1' be the number of distinct zeros in the N_1 zeros. Let V_j be the variation in $\arg G$ in the jth interval between the successive semicircles. Then by the principle of argument, it has

$$\sum_j V_j - \pi N_1 = 2\pi N_2 + O(L), \tag{3.28}$$

Let W_j be the variation of argument of

$$h(s)(f'(s) + f'(1-s))G(s)$$

in the jth interval, where W_j is taken for increasing t, while V_j is taken for decreasing t. With (3.2) and (3.28), it has

$$\begin{aligned}\sum_j W_j &= \mathrm{Im}(f)|_T^{T+U} - \sum_j V_j \\ &= \mathrm{Im}(f)|_T^{T+U} - (2\pi N_2 + \pi N_1) + O(L) \end{aligned} \tag{3.29}$$

By (3.8), in the jth open interval, the number of zeros of $\zeta(1/2+it)$ is at least

$$(W_j/\pi) - 1.$$

and in all the open intervals, the number of zeros is at least

$$\begin{aligned}\frac{1}{\pi}\sum_j W_j - N_1' - 1 &= \frac{1}{\pi}\mathrm{Im}(f)|_T^{T+U} - (2N_2 + N_1) - N_1' - 1 + O(L) \\ &= \frac{1}{\pi}\mathrm{Im}(f)|_T^{T+U} - 2N_G(D) + N_1 - N_1' + O(L) \end{aligned} \tag{3.30}$$

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Moreover, by (3.7), we can know that on the side $\sigma = 1/2$, a zero of $G(s)$ is also a zero of $\zeta'(s)$, and so a zero of $\zeta(s)$, with multiplicity one greater, so there are $N_1 + N_1'$ such zeros of $\zeta(1/2 + it)$, adding to (3.30), in total, it has

$$N_0(T+U) - N_0(T) \geq \frac{1}{\pi}\mathrm{Im}(f)|_T^{T+U} - 2N_G(D) + 2N_1 + O(L).$$

By (3.11), we can know

$$\frac{1}{\pi}\mathrm{Im}(f)|_T^{T+U} = N(T+U) - N(T) + O(L).$$

i.e.

$$(N(T+U) - N(T)) - (N_0(T+U) - N_0(T)) \leq O(T^{1/2}L^3).$$

□

Besides, we know that on the critical line a zero of $G(s)$ is also a zero of $\zeta'(s)$, and so a zero of $\zeta(s)$, with multiplicity one greater. Hence

$$\sum(m-1) \leq N_G(D).$$

where sum is over the distinct zeros of $\zeta(s)$ on the left side of D, m is the multiplicity of a zero. And so,

$$\sum_{m \geq 2} m \leq 2N_G(D) \leq O(T^{1/2}L^3). \tag{3.31}$$

This means that the non-trivial zeros of $\zeta(s)$ are all on the critical line, and all are simple, with at most $O(T^{1/2}L^3)$ ones excepted.

Appendix A. Some Phased Results

It will be used the formula

$$\int_1^n \frac{\log^r y}{y^{2-2a}} dy = \frac{r!}{(1-2a)^{r+1}} - \frac{1}{n^{1-2a}} \sum_{0 \leq d \leq r} \frac{[r,d]}{(1-2a)^{d+1}} \log^{r-d} n. \tag{1}$$

where $[r,d] = r \cdots (r-d+1), [r,0] = 1$. which is easy to be followed by the integration by parts. By deriving the functional equation of $\zeta(s)$ successively, as in [6], it will be obtained the functions similar to $G(s)$

$$G(s,k) = \sum_i C_k^i (2/L)^i \zeta^{(i)}(s) = \left(1 + \frac{2}{L}\frac{d}{ds}\right)^k \zeta(s). \quad (k > 1) \tag{2}$$

By the induction, it is easy to deduce that

$$\zeta^{(k)}(s) = (-1)^k \chi(s) \sum_{0 \leq v \leq k} L^{k-v} C_k^v \zeta^{(v)}(1-s). \tag{3}$$

With (2),(3) and the functional equation, it has

$$G(s,m) = (-1)^m \chi(s) \sum_{0 \leq i \leq m} C_m^i (2/L)^i \zeta^{(i)}(1-s). \tag{4}$$

Lemma 1. Denote by

$$\tau_0 = \sum_{0 \leq j,k \leq m} C_m^k C_m^j 2^{k+j} (-1)^{k+j} \sum_{0 \leq l \leq k} (-1)^l C_k^l \frac{1}{j+l+1}.$$

Then

$$\tau_0 = \frac{(-1)^m}{2m+1}. \tag{5}$$

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Proof. Let

$$g(x) = (1-2(1-x))^m(1-2x)^m.$$

Expanding the binormals, it has

$$g(x) = \sum_{0\le j,k\le m} C_m^k C_m^j 2^{k+j}(-1)^{k+j} \sum_{0\le l\le k} (-1)^l C_k^l x^{j+l}$$

and

$$\tau_0 = \int_0^1 g(x)dx = \int_0^1 (-1)^m(1-2x)^{2m}dx = \frac{(-1)^m}{2m+1}.$$

□

Lemma 2. Let

$$\Phi_0 = (-1)^m \frac{n^{2-2a}}{2-2a} \sum_{0\le j,k\le m} C_m^k C_m^j (2/L)^{j+k}(-1)^{j+k} \int_1^n \frac{\log^j y \log^k(n/y)}{y^{2-2a}} dy.$$

Then

$$\Phi_0 = \frac{1}{(2-2a)} \frac{nL}{(2m+1)}. \tag{6}$$

Proof. With formula (1), it has

$$\Phi_0 = \frac{(-1)^m n}{2-2a} \sum_{0\le j,k\le m} C_m^k C_m^j (2/L)^{j+k}(-1)^{j+k}(A+B),$$

where

$$\begin{aligned} A =& n^{1-2a} \sum_{0\le l\le k} (-1)^l C_k^l \frac{(j+l)!}{(1-2a)^{j+l+1}} \log^{k-l} n, \\ B =& \sum_{0\le l\le k} (-1)^l C_k^l \sum_{0\le d\le j+l} \frac{[j+l,d]}{(1-2a)^{d+1}} \log^{j+k-d} n. \end{aligned}$$

Let $x = (1-2a)L$, then

$$\begin{aligned} \Phi_0 =& \frac{(-1)^m nL}{2-2a} \sum_{0\le j,k\le m} C_m^k C_m^j 2^{j+k}(-1)^{j+k} \\ & \cdot \left\{ e^x \sum_{0\le l\le k} (-1)^l C_k^l \frac{(j+l)!}{x^{j+l+1}} - \sum_{0\le l\le k} (-1)^l C_k^l \sum_{0\le d\le j+l} \frac{[j+l,d]}{x^{d+1}} \right\} \\ =& \frac{(-1)^m nL}{2-2a} \sum_{0\le j,k\le m} C_m^k C_m^j 2^{j+k}(-1)^{j+k} \\ & \cdot \left\{ \sum_{0\le l\le k} \sum_{r\ge 0} (-1)^l C_k^l \frac{(j+l)!}{x^{j+l-r+1} r!} - \sum_{0\le l\le k} (-1)^l C_k^l \sum_{0\le d\le j+l} \frac{[j+l,d]}{x^{d+1}} \right\} \\ =& \frac{(-1)^m nL}{2-2a} \sum_{0\le j,k\le m} C_m^k C_m^j 2^{j+k}(-1)^{j+k} \sum_{r>j+l} (-1)^l C_k^l \frac{x^{r-j-l-1}}{[r, r-(j+l]} \\ \doteq& \frac{(-1)^m nL}{2-2a} \tau_0 \\ \doteq& \frac{nL}{(2-2a)(2m+1)}. \end{aligned}$$

□

Let

$$\mathfrak{I}(m) = \int_{T_1}^{T_2} \omega_1(s) G(s,m) G(2a-s) ds. \tag{7}$$

Then by Lemmas 2.1∼ 2.3 and formules (2),(3) and (4), there is

$$\mathfrak{I}(m) = \mathfrak{H}(m) + O(\Delta L).$$

where

$$\mathfrak{H}(m) = (-1)^m \sum_{0\le j,k\le m} C_m^k C_m^j (2/L)^{k+j} (-1)^{k+j} \sum_{T_1-\Delta\le 2\pi xy\le T_2+\Delta} \frac{(xy)^{b'} \log^k x \log^j y}{x^b y^{b'}}.$$

Correspondingly, define

$$H(n,m) = (-1)^m \sum_{0\le j,k\le m} C_m^k C_m^j (2/L)^{k+j} (-1)^{k+j} \sum_{1\le xy\le n} \frac{(xy)^{b'} \log^k x \log^j y}{x^b y^{b'}}.$$

Then, it has

$$\begin{aligned} H(n,m) =& (-1)^m \sum_{0\le j,k\le m} C_m^k C_m^j (2/L)^{k+j} (-1)^{k+j} \sum_{1\le y\le n} \log^j y \sum_{1}^{n/y} x^{1-2a} \log^k x \\ =& (-1)^m \sum_{0\le j,k\le m} C_m^k C_m^j (2/L)^{k+j} (-1)^{k+j} \int_1^n \log^j y \int_1^{n/y} x^{1-2a} \log^k x dx dy \\ =& \sum_{0\le d\le m} (-1)^d \Phi_d. \end{aligned}$$

where

$$\Phi_d = \frac{(-1)^m n^{2-2a}}{(2-2a)^{d+1}} \sum_{0\le j,k\le m} C_m^k C_m^j (2/L)^{k+j} (-1)^{k+j} [k,d] \int_1^n \frac{\log^j y \log^{k-d} x}{y^{2-2a}} dy.$$

Φ_0 is calculated above, and it is easy to know Φ_0 is the dominant of $H(n,m)$ when $m = o(L)$. The calculations of $\Phi_d, d > 0$ are slightly more complex, a preliminlary calculation shows

$$\Phi_d = \frac{[m,d] 2^d n L}{(2m-d+1) L^d}. \quad (d \ge 0) \tag{8}$$

So, if the calculations above are correct, then $m = [L/2]$ will be sufficient for the proof of Theorem 1.1.

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