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The Variation of G and Λ in Cosmology

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Abstract: The idea of varying constants of nature is very old, and has commanded a lot of attention since first mooted. The variation of the gravitational parameter G and cosmological parameter Λ is still an active area of research. Since the idea of a varying G was introduced by Dirac almost a century ago, there are even theories that have variable G such as Brans-Dicke theory and the scale covariant theory. Both these theories also have a varying Λ in their full generalisations. A varying Λ was also introduced around the same time as that of varying G. It is interesting to note that a possible solution to the cosmological constant problem can be realised from a dynamic Λ. In this work, we focus on a varying Λ and G framework. In almost all studies in the simplest framework of variable Λ and G, it is found that one of them has to increase with time. However, observations and theoretical considerations indicate that both Λ and G should decrease with time. In this paper, we propose a solution to this problem, finding theories in which both Λ and G decrease with time.

Keywords: variable Λ; variable G; conservation of energy momentum

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1. Introduction

The idea of varying constants of nature such as the fine-structure constant α, the speed of light c, Newton's gravitational constant G, Boltzmann's constant k_B, Planck's constant $\hbar$, and Fermi's constant G_F is not new. There are many reasons to believe that these constants should vary [1,2].

- From a quantum theory, string theory and other similar points of view, there are strong reasons for believing in more than three spatial dimensions. Hence, the constants from these higher dimensions need not be constant as viewed from our three dimensional point of view. Any slow change in the scale of higher dimensions, this would be revealed by measurable changes in the "constants" of our three-dimensional world.
- Spontaneous symmetry-breaking processes in the very early universe introduce irreducibly random elements into the values of constants of nature.
- The outcome of a theory of quantum gravity is expected to be probabilistic, whose probability distributions for observables may not be very sharply peaked for all possibilities. Thus, the value of the gravitation "constant", G, or its time derivative, $\dot{G}$ may vary.
- At present we have no idea why any of the constants of nature take on the numerical values that they do, and we have never successfully predicted the value of any dimensionless constant in advance of its measurement.
- Observational limits on possible variations of the constants are often very weak, although they can be made to sound strong by judicious parametrizations.
- Li et al [3] constructed a sample of 40 spectra of Ly α emitting galaxies (LAEs) and a sample of 46 spectra of QSOs at $1.09 < z < 3.73$ using the VLT/X-Shooter near-infrared spectra publicly available. They measured the wavelength ratios of the two components of the spin-orbit doublet and accordingly calculated α(z) using two methods. Analysis on all of the 86 spectra yielded $\Delta\alpha/\alpha = (-3 \pm 6) \times 10^{-5}$ with respect to the α laboratory measurements. This result is currently the subject of detailed analysis and reanalysis by the observers in order to search for possible systematic biases in the astrophysical environment or in the laboratory determinations of the spectral lines.

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- Varying G and Λ can lead to changes in the overall dynamics of the universe and affect the evolution of perturbations and hence the large-scale structure growth rate [4,5]. The apparent simplicity of the ΛCDM model belies the intricate challenges associated with the cosmological constant, including the cosmological constant problem and the coincidence problem. Consequently, alternative explanations have been sought. Their findings highlighted the differences and potential advantages of considering time-variable parameters in the cosmological model which can impact the formation of large-scale structures in the universe. While further research and observations are needed to validate their findings, valuable insights were gleaned to our understanding of the cosmos beyond the standard cosmological paradigm.

The literature on varying cosmological and gravitational constants is very vast, and we only focus on some of the key papers. The references therein provide additional reading matter. It seems that Dirac [6,7] first proposed the idea of varying G. Dirac noticed that the ratio of the electromagnetic interaction between a proton and an electron to their gravitational interaction, $e^2/Gm_pm_e \sim 10^{40}$, is roughly the same as the ratio of the size of the Universe to the classical radius of the electron, $m_ec^3/e^2H_0 \sim 10^{40}$, where $H_0 \sim kms^{-1}Mpc^{-1}$ is the present-day Hubble parameter. Furthermore, both ratios are roughly the square root of the total number of baryons in the observable Universe, $c^3/m_pGH_0 \sim 10^{80}$. Dirac argued that the similarity of these large ratios could not be a coincidence. If they are fundamental and constants, then noting that $H_0 \sim t^{-1}$ is not constant, Dirac proposed a time variation of Newton's constant $G \sim t^{-1}$. However, the desired result could have been achieved by taking $e^2/m_e \sim t^{1/2}$. The choice simply depends on the units one is using. Dirac's choice of units fixes e, c, and m_e as constants, but one could take other choices and achieve the same results [8,9]. Cosmological limits on varying G tell us only that $\dot{G}/G \leq 10^{-2}H_0$, where H_0 is the present Hubble rate [1].

However, the early history of variable constants, and in particular, variable G is not very clear, but there were several others apart from Dirac who were also involved in studying variable constants at around the same time. One of the earliest of these to study varying constants was Lord Kelvin and Tait [10] well over a century ago in 1874. This was some 30 years before Einstein came up with his special theory of relativity in 1905 in which c is assumed constant. At the time of Kelvin, a varying speed of light was quite acceptable to the scientific community as it played no special role in physics. However, after 1905, the situation changed completely as expected. Now it was not Dirac who had first conceived the idea of the so-called "Large Numbers Hypothesis (LNH)" which is usually attributed to his name. Hermann Weyl [11,12] in 1917 and 1919 speculated that the radius of the observed universe could be the the radius of some hypothetical particle whose ratio to the electron radius was of the order of 10^{42}. The coincidence was further developed by Eddington in 1931 [13] who related it to the estimated number of charged particles in the universe, which is around 10^{42}. Milne [14] came up with an idea of two systems of units, one for atomic purposes, and the other for gravitational, which were related by a logarithmic transformation. By requiring that the mass of the "universe" be constant, he was able to derive a variation of G of the type $G \propto t$. However, it appears that Milne was motivated by his dislike of relativity rather than the LNH. The biologist Haldane [15,16] in 1935 took an interest in the theory of Milne, writing a few papers dealing with evolution of life. The argued that biochemical activation energies might appear constant on the t timescale yet increase on the τ timescale, giving rise to non-uniformity in evolution. Jordan [17,18] was also able to derive a variation of G as the inverse of time by a slightly different method. An interesting exposition of Jordan's cosmology is given by Dubois and Furza [19]. The variation $G \propto 1/t$ was shown to be unlikely by Teller [20] since it would lead to too quick a change in the temperature of the earth, and life would not be able to exist. However, the ideas of Dirac led to Jordan, Brans and Dicke to develop Brans-Dicke theory [28] in 1961 that would allow for the variation of G by the introduction of a suitable scalar field $\Phi(t)$ which had an evolution equation for $G(\Phi)$ in the theory.

Apart from that mentioned above, the motivation for varying G specifically is the following:

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- Theoretically many theories of gravity apart from Newtonian gravity and general relativity can be cast phenomenologically including space and or time dependence of G [21–23].
- Experiments and observations have been used to set limit on $\dot{G}/G$ [24] including solar evolution, lunar occultations and eclipses ($\sim 10^{-11}yr^{-1}$), paleontological evidence ($\sim 10^{-11}yr^{-1}$), white dwarf cooling and pulsations ($\sim 10^{-10}yr^{-1}$), neutron star masses and ages ($\sim 10^{-12}yr^{-1}$), star cluster evolution ($\sim 10^{-12}yr^{-1}$), big bang nucleosynthesis abundances ($\sim 10^{-12}yr^{-1}$), astroseismology ($\sim 10^{-12}yr^{-1}$), lunar laser ranging ($\sim 10^{-14}yr^{-1}$) evolution of planetary orbits ($\sim 10^{-14}yr^{-1}$), binary pulsars ($\sim 10^{-12}yr^{-1}$), high-resolution quasar spectra ($\sim 10^{-14}yr^{-1}$), gravitational wave observations of binary neutron stars ($\sim 10^{-8}yr^{-1}$) and supernovae. (See the list of references in [24]).
- An important point to note is that if one chooses a variation of G of the type $\dot{G}/G \leq 10^{-14}yr^{-1}$, then there will be no problems with the mass and size of galaxies, stars and planets, as they will not be affected.
- It is interesting to note that a variation of fundamental constants can lead to the solution of the Hubble tension problem [25–27].

The ΛCDM model is currently the most favoured model for explaining the current accelerated expansion of the universe, where Λ refers to the cosmological constant. Now a major problem in cosmology is the cosmological constant problem [29,30]. Now a dynamic cosmological parameter [31] can, inter alia, provide a solution to this problem. A Lagrangian description of variable Λ has been given by Poplawski [32]. The variation of the cosmological parameter seems to have first been considered by Bronstein [31,33] in 1933. Overduin and Cooperstock [34] have given a very review of all the different forms of varying cosmological parameter Λ that have been considered in the literature as of 1998. A review of varying Λ and quintessence has been given by Kragh and Overduin [35].

In general relativity, Λ is a strict constant and associated with the vacuum energy density, $\rho(vac)$. However, recent calculations involving the renormalization of quantum field theory in FLRW spacetime yield a time-varying Λ, and hence a time-varying vacuum density as well, in which Λ acquires a dynamical component through quantum effects: $\Lambda \rightarrow \Lambda + \delta\Lambda$. The GR limit can be recovered smoothly. The connection of the varying Λ with quantum field theory can be motivated from semi-qualitative renormalization group arguments [36]. However, an explicit quantum field theory calculation has appeared only very recently [37–41]. Also, recently, a running vacuum model was studied in Brans-Dicke theory, and it was found that observations favour this model over the one with constant Λ [42].

We solve the background as well as the perturbation equations for each cosmological model and test their performance against the modern wealth of cosmological data, namely a compilation of the latest SNIa+H(z)+BAO+LSS+CMB observations. We utilize the AIC and DIC statistical information criteria in order to determine if they can fit better the observations than the concordance model. The two BD extensions are tested by considering three different datasets. According to the AIC and DIC criteria, both BD extensions (i) and (ii) are competitive, but the second one (the BD-RVM scenario) is particularly favored when it is compared with the vanilla model. This fact may indicate that the current observations favor a mild dynamical evolution of the Newtonian coupling G(N) as well as of the VED. While further studies will be necessary, the results presented here suggest that the Brans & Dicke theory with running vacuum could have the potential to alleviate the two tensions at the same time.

Hence, there is a strong motivation to study the variation of the gravitational parameter G and the cosmological parameter Λ in cosmology.

In this work, in Section 2 we firstly review the simplest formulation of a Lagrangian description of general relativity with variable G and Λ. We point out that it is not possible to have both decreasing Λ and G. A decreasing G is favoured for many reason, inter alia, giving rise to late-time acceleration without the need for any exotic matter [43] as in the ΛCDM model. Then in Sections 3 and 5, we

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examine two theories, viz. the scale covariant theory and $f(R,T)$ theory. These two theories naturally have variable Λ and G. We show that it is possible to have both decreasing Λ and G.

2. Lagrangian Formulation

Let us consider Einstein's field equations in suitable units with variable G and Λ:

$$R_{ab} - \frac{1}{2}RG_{ab} + \Lambda(x^d)G_{ab} = G(x^d)T_{ab} \tag{1}$$

where the symbols have their usual meanings, but Λ and G are allowed to be variable. It should be noted that one has to have a variable Λ together with variable G in order for the usual conservation law to hold. This form of the field equations can be shown to arise from several different approaches. Firstly from the following Lagrangian [44] as follows. We assume that G and Λ are related, and that the action is:

$$A = \int d^4x\mathcal{L} = \int d^4x\{\sqrt{-g}[R/G - V(G)] + L_m\} + A'... \tag{2}$$

where G is a variable, $V(G)$ is a function of G and L_m is the matter Lagrangian. Here A' is introduced to cancel terms involving the second derivatives of the metric so as to obtain simple equations in the variables G and Λ [45]. The Euler-Lagrange equations used here are:

$$\frac{\partial\mathcal{L}}{\partial G} = \nabla_a \frac{\partial\mathcal{L}}{\partial(\partial_a G)} \tag{3}$$

This yields:

$$V'(G) = \frac{R}{G^2} \tag{4}$$

Varying the action (2) with respect to g_{ab}, we obtain:

$$R_{ab} - \frac{1}{2}Rg_{ab} = GTab - (\frac{1}{2}GV(G))g_{ab} \tag{5}$$

where the matter tensor T_{ab} arises from the matter Lagrangian L_m. If we now let

$$\frac{1}{2}GV(G) = \Lambda \tag{6}$$

then we get finally

$$R_{ab} - \frac{1}{2}Rg_{ab} + \Lambda gab = GT_{ab} \tag{7}$$

where G and Λ are variable.

From Equation (7), we can derive the field equations as:

$$-3H^2 + \Lambda(t) = -G(t)\rho \tag{8}$$

$$-2\dot{H} - 3H^2 + \Lambda(t) = G(t)p \tag{9}$$

From the above two equations, we can derive a modified energy conservation equation with variable G and Λ:

$$\dot{\rho} + 3H(\rho + p) = -\rho\left(\frac{\dot{G}}{G}\right) - \frac{\dot{\Lambda}}{G}. \tag{10}$$

We notice that in this simplest formalism with variable G and Λ, using a Lagrangian formulation, we can still retain the usual energy conservation law in general relativity, and get a separate equation for the variation of G and Λ:

$$\dot{\rho} + 3H(\rho + p) = 0, \tag{11}$$

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$$\rho\left(\frac{\dot{G}}{G}\right)+\frac{\dot{\Lambda}}{G}=0. \tag{12}$$

Another Lagrangian derivation has been given by Lau [46] and Lau and Prokhovnik [47] in terms of a scalar formulation for Λ.

Since ρ is non-negative, it follows immediately from Equation (12) that if $\dot{G}$ is positive (negative), then $\dot{\Lambda}$ is negative (positive). So, a decreasing cosmological parameter implies an increasing gravitational parameter and vice versa. Usually, a decreasing cosmological parameter is favoured, and this implies an increasing gravitational parameter. A variation of the type (12) seems to have been first considered by DerSarkissian [50] and since then, has been considered by many authors. A partial list of them is [51–58]. Further references can be found in these citations.

It is interesting to investigate what is the situation in other theories of gravity.

3. Scale Covariant Theory

Canuto et al. [48] came up with an alternative theory, the scale-covariant theory of gravitation. In this theory, there are two systems of units. One of them is gravitational units in which Einstein's field equations are valid, and the other is atomic units in which physical quantities are expressed in atomic units. A conformal transformation relates the two systems of units:

$$\bar{g}_{ab}=\phi^2 g_{ab}, \tag{13}$$

where indices a,b take their values $0,1,2,3$, the bar indicates gravitational units, and unbarred quantities refer to atomic units. The scalar function ϕ satisfies $0<\phi<\infty$. The action for the theory is:

$$I=\int\left(A\phi^4-\phi^2R+6\phi^a\phi_a+2G(\phi)L\right)\sqrt{-g}d^4x, \tag{14}$$

where R is the Ricci scalar, L is the Lagrangian of the matter and ϕ_a denotes the ordinary derivative. We note that the gravitational parameter $G(\phi)$ is no longer a constant, but a function of ϕ. Performing a variation of (14), we find the field equations for the scale covariant theory as [48]:

$$R_{ab}-\frac{1}{2}Rg_{ab}+f_{ab}(\phi)+\Lambda(\phi)g_{ab}=GT_{ab}, \tag{15}$$

where

$$f_{ab}(\phi)=\frac{1}{\phi^2}\left[2\phi\phi_{a;b}-4\phi_a\phi_b-g_{ab}(2\phi\phi^d{}_{;d}-\phi^d\phi_d)\right]. \tag{16}$$

Here R_{ab} and T_{ab} stand for the Ricci tensor and energy-momentum tensor, respectively. Again, we notice that the cosmological parameter $\Lambda(\phi)$ is no longer a constant, but a function of the scalar ϕ. The scale-covariant theory involves a non-minimal coupling between the gauge function ϕ and the Ricci scalar R.

For a flat FLRW space-time, the metric is:

$$ds^2=-a^2(t)(dx^2+dy^2+dz^2)+dt^2. \tag{17}$$

where a is the scale factor. Also, the energy-momentum tensor for a perfect fluid can be written as

$$T_{ab}=(\rho+p)\,u_au_b+pg_{ab}, \tag{18}$$

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where ρ, p, and u^a represent the energy density, pressure, and the four-velocity vector, respectively. In a co-moving coordinate system, $u^a u_a = -1$ and $u^a u_b = 0$. The field equations in the scale covariant theory for a flat FLRW space-time can be found by expanding the tensor Equations (15) and (16):

$$2\dot{H} + 3H^2 + 6H\frac{\dot{\phi}}{\phi} + 2\frac{\ddot{\phi}}{\phi} - \frac{\dot{\phi}^2}{\phi^2} = -8\pi Gp + \Lambda, \tag{19}$$

$$3H^2 + 6H\frac{\dot{\phi}}{\phi} + 3\frac{\dot{\phi}^2}{\phi^2} = 8\pi G\rho + \Lambda, \tag{20}$$

The continuity equation which is a consequence of the field Equations (19) and (20) is given by [49,59,60]:

$$\dot{\rho} + 3H\left[\rho + p\left(1 + \frac{\dot{\phi}}{H\phi}\right)\right] = -\rho\left(\frac{\dot{G}}{G} + \frac{\dot{\phi}}{\phi}\right) - \frac{\dot{\Lambda}}{G}. \tag{21}$$

4. Models in SCT with Decreasing Parameters

The modified energy conservation Equation (21) can be split up in the following way:

$$\dot{\rho} + 3H(\rho + p) + 3p\frac{\dot{\phi}}{\phi} = 0 \tag{22}$$

$$\rho\left(\frac{\dot{G}}{G} + \frac{\dot{\phi}}{\phi}\right) + \frac{\dot{\Lambda}}{G} = 0 \tag{23}$$

The reason for splitting up the modified conservation equation in this is the following. The term $3p\frac{\dot{\phi}}{\phi}$ involves the term p, and hence it needs to be added to the term $3Hp$. Also, we note that when $\phi = 1$ in Equation (22), then we recover general relativity. Secondly, putting $\phi = 1$ in Equation (23), we get the same equation as Equation (11) [50]. We can write Equation (23) in the following way:

$$\frac{\dot{\Lambda}}{G} = -\rho\left(\frac{\dot{G}}{G} + \frac{\dot{\phi}}{\phi}\right) \tag{24}$$

From this equation, it is possible to deduce that there is a large class of solutions that permit both decreasing Λ and G. The variable Λ is positive [62] and expected to decrease with time for many reasons, including a possible solution to the cosmological constant problem [63]. Therefore, since G is positive, the left side of Equation (24) is negative. This means that the right side must also be negative, i.e.,

$$-\rho\left(\frac{\dot{G}}{G} + \frac{\dot{\phi}}{\phi}\right) < 0 \tag{25}$$

or, since the energy density ρ is assumed positive, we must have

$$\frac{\dot{G}}{G} + \frac{\dot{\phi}}{\phi} > 0 \tag{26}$$

or,

$$\frac{\dot{G}}{G} > -\frac{\dot{\phi}}{\phi} \tag{27}$$

For decreasing G, we must have $\dot{G}/G < 0$. Unfortunately in the scale covariant theory, the specific choice of the gauge function ϕ is not dictated to by the theory itself, but has to be put in by hand. Then one has to check how well this fits in with observational constraints. Canuto et al. [48,61] have resricted ϕ as follows:

$$\phi \sim t^n, \ -1 \le n \le +1 \tag{28}$$

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Then:

$$\frac{\dot{\phi}}{\phi} = \frac{n}{t} \tag{29}$$

Let us choose $n = 1$, and for G let us choose

$$G = \frac{1}{logt} \tag{30}$$

Then, $\dot{G}/G < 0$, and we find that the requirement (27) is satisfied.

Hence we have demonstrated that there are solutions in the scale covariant theory for which both the cosmological parameter Λ and gravitational parameter G decrease with time. This is not possible in the equivalent formulation in general relativity.

5. f(R,T) Theory of Gravity

Another very interesting modified gravity theory is $f(R,T)$ gravity [64], which has garnered the attention of researchers recently. In this theory, the gravitational Lagrangian is given by an arbitrary function of the Ricci scalar (R) and the trace (T) of the energy momentum tensor. It also depends on a source term representing the variation of the matter stress energy tensor with respect to the metric. The source term is obtained as a function of the matter Lagrangian (L_m). As a result, for each choice of L_m, a specific set of field equations is generated. The field equations of $f(R,T)$ gravity are obtained [64] from the action:

$$S = \int \sqrt{-g}\left(\frac{1}{G}f(R,T) + L_m\right) d^4x \tag{31}$$

where $f(R,T)$ is an arbitrary function of the Ricci scalar (R) and trace (T) of the energy momentum tensor T_{ab} of the matter, and L_m represents the matter Lagrangian density. The field equations of the theory can be derived by varying the action S of the gravitational field with respect to the metric tensor components g_{ab}. Consider a perfect fluid given by (18), and matter Lagrangian density $L_m = -p$. In order to analyse whether we could get both Λ and G decreasing, we choose a relatively simple form of $f(R,T)$, viz., $f(R,T) = R + 2\eta T$, where η is a constant which indicates the departure from general relativity. For the above choices, the action (31) leads to the field equations:

$$R_{ab} - \frac{1}{2}Rg_{ab} + \Lambda g_{ab} = (G + 2\eta)T_{ab} + \eta(\rho - p)g_{ab} \tag{32}$$

For $\eta = 0$, we recover the field equations in general relativity with cosmological constant Λ.

Then, the FLRW metric (17) in conjunction with the field Equations (32) lead to the Raychaudhuri type equation:

$$-2\dot{H} - 3H^2 + \frac{k}{S^2} + \Lambda = -\eta G\rho + (1+\eta)Gp \tag{33}$$

and the Friedmann equation:

$$-3H^2 + \Lambda = -(1+3\eta)G\rho + \eta Gp \tag{34}$$

We note that for $\eta = 0$, we recover the corresponding equations in general relativity with cosmological constant Λ. By differentiating Equation (34) and substituting into Equation (33), we are able to derive an energy conservation type equation in $f(R,T)$ gravity

$$\dot{\rho} + 3H(\rho + p) = -\left(\frac{\dot{G}}{G}\right) - \frac{\dot{\Lambda}}{G} - \frac{3\eta}{G}\dot{\rho} - \frac{6\eta}{G}\rho H - \frac{6\eta}{G}pH + \frac{\eta}{G}\dot{p}. \tag{35}$$

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Again, for $\eta = 0$, we see that we get the same equation as Equation (11). To have the minimum departure from general relativity, we assume the usual energy conservation law by splitting the modified energy conservation law (35) as follows:

$$\dot{\rho} + 3H(\rho + p) = 0 \tag{36}$$

and the following equation which includes the evolution of G and Λ:

$$-\left(\frac{\dot{G}}{G}\right)\rho - \frac{\dot{\Lambda}}{G} - \frac{3\eta}{G}\dot{\rho} - \frac{6\eta}{G}\rho H - \frac{6\eta}{G}pH + \frac{\eta}{G}\dot{p} = 0 \tag{37}$$

This equation can be written as

$$-\left(\frac{\dot{G}}{G}\right)\rho - \frac{\dot{\Lambda}}{G} - \frac{2\eta}{G}\left[\dot{\rho} + 3H(\rho + p)\right] - \frac{\eta}{G}\dot{\rho} + \frac{\eta}{G}\dot{p} = 0 \tag{38}$$

By Equation (36), we get finally

$$\dot{\Lambda} = -\dot{G}\rho - \eta\dot{\rho} + \eta\dot{p} \tag{39}$$

Hence here we can have both G and Λ decreasing by choosing $\dot{G}\rho > -\eta\dot{\rho} + \eta\dot{p}$. In fact, we have the possibility of all nine combinations of decreasing Λ, constant Λ and increasing Λ with decreasing G, constant G and even increasing G. Hence, we have demonstrated that in $f(R,T)$ theory of gravity, it is possible to have both G and Λ decreasing.

6. Results

We have analysed cosmological models in which the gravitational parameter G and the cosmological parameter Λ are allowed to be variable in the simplest formalism, as considered by many authors. In general relativity, one parameter has to be increasing, and the other decreasing. The situation in several modified theories of gravity is shown to be more flexible, and we can have both parameters decreasing, which is preferred by observations.

We comment on the splitting the full energy conservation law (35) into Equations (36) and (37). Firstly, we wish to regain the usual energy conservation law as in general relativity in the appropriate limit. Secondly, it enables us to get one more equation if we are seeking a full solution to the equations. Several authors have used this splitting, such as [65–67] and references therein.

7. Discussion

The idea of variable constants of nature is not new and goes back over a century. In this work, we focus on variable cosmological and gravitational parameters. We firstly consider the simplest extension of general relativity allowing for variable cosmological and gravitational parameters. Such a similar formalism can also arise in special forms of other theories of gravity such as Brans-Dicke theory (or the scalar tensor theory based on it), the scale covariant theory and $f(R,T)$ gravity. In general relativity, one parameter can increase, but the other has to decrease. The preferred variation of these parameters is a decrease of both. We find that in the scale covariant theory and $f(R,T)$ gravity, we can have both parameters decreasing. Harko and Mak [68] also found a similar result with particle creation in general relativity with variable G and Λ. We also noted that a variation of fundamental constants of nature can solve the familiar Hubble tension problem. Variation of Λ provides a better fit to data than does the standard ΛCDM model [69,70]. Sola et al [71] have carried out a very detailed analysis of several models with a running vacuum parameter (basically a varying Λ), and found a better fit to observations than the ΛCDM model. The transition redshift was found to be approximately the same as that of the ΛCDM model. Perico et al [72] have studied a large class of time-dependent vacuum energy density (variable Λ) models in the form of power series of the Hubble rate. The proposed class of such models provide:

- a new mechanism for inflation (different from the usual inflaton models);
- a natural mechanism for a graceful exit; which is universal for the whole class of models
- the radiation and matter eras;
- the currently accelerated expansion of the universe;
- a mild dynamical dark energy at present;
- a final de Sitter stage.

Remarkably, the late-time cosmic expansion history of the class of models is very close to the concordance ΛCDM model, but above all it furnishes the necessary smooth link between the initial and final de Sitter stages through the radiation- and matter-dominated epochs. As far the future is concerned, better observations will be able to constrain the fundamental constants and their variation to better accuracy.

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