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Fixed Point Results for α-Type F-Suzuki Contraction and α-Type F-Weak-Suzuki Contraction with Application to Non-Linear Fractional Differential Equation in b-Metric Spaces

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Abstract: In this work, we introduce a new concepts for α-type F-Suzuki contraction and α-type F-weak-Suzuki contraction in the context of b-metric spaces. Compared to the α-type F-contraction and F-Suzuki contraction mappings, these contractions are essentially weaker. For these type of contraction mappings, sufficient conditions are established for the fixed point's existence and uniqueness in b-metric spaces. As a result, the findings encompass several generalizations. To show the usability of our obtained results, we provide a supportive example and an application to a non-linear differential equation with fractional order.

Keywords: fixed points; F-Suzuki-contraction; α-type F-Suzuki contraction; α-type F-Weak-Suzuki contraction; b-metric space

1. Introduction

It is often recognized that one of the most significant and alluring findings in non-linear analysis and mathematical analysis in general is the "Banach Contraction Principle" [1]. Furthermore, fixed point theory is an important topic in many disciplines, including economics, physics, informatics, differential equations, geometry, and engineering, among others. Many researchers have extended and generalized the "Banach Contraction Principle" since its introduction, and mathematicians have been eager to advance it in various directions due to this noteworthy result (see, for instance, [2–6]).

As a generalization of the "Banach Contraction Principle", Wardowski [2] developed a new contractive mapping in 2012: the so-called F-contraction and obtained a new fixed point result. Consequently, Altun et al. [7] presented an interesting idea of multi-valued F-contraction mappings along with a few fixed point outcomes. In addition to obtaining certain fixed point findings, Wardowski and Dung [8] extended the idea of an F-contraction to an F-weak contraction and generalized the results in [2]. By presenting the idea of an F-Suzuki contraction, Piri and Kumam [10] extended the findings of Wardowski [2] in 2014 and produced some fascinating fixed point results. Also, Minak et al. [3], built on the discoveries of Wardowski [2] in 2014 by establishing the concept of an almost F-contraction and generated a few intriguing fixed point findings.

By outlining the concept of F-contraction of α-type in 2016, Gopal et al. [11] made it more general. In the same year, 2016, Budhia et al. [12] established several fixed point theorems for such contractions and presented the new ideas of an α-type almost-F-contraction and an α-type F-Suzuki contraction in metric spaces. Chandok, Huaping Huang and Stojan Radenovic [13] examined some fixed point results for the generalized F-Suzuki type contractions in the setting of b-metric Spaces in 2018, extending the work of Piri and Kumam [10]. Furthermore, in 2019, Taheri A. and Farajzadeh A.P. [14] obtained a new result on a new generalization on metric spaces of α-type F-Suzuki contractions and α-type

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almost-F-contractions. Motivated by the works discussed above, we extended on the findings of [11] and some results of [12] and [14] in this line of inquiry by presenting new type of contractions: an α-type F-Suzuki contraction and α-type F-weak-Suzuki contraction in the setting of b-metric spaces. We also proved some fixed point theorems pertaining to these contractions. Additionally, an example and an application to a non-linear differential equation of fractional order are given to illustrate the applicability of our obtained results.

2. Preliminaries

This section aims to recall various concepts as well as results used in this paper. Throughout this work, $\mathbb{N}_\circ$, $\mathbb{N}$, $\mathbb{R}_+$, $\mathbb{R}^+$, $\mathbb{R}$ and $Fix(T)$ denote the set of all non-negative integers, natural numbers, positive real numbers, non-negative real numbers, real numbers and the set of all fixed points of a self-mapping T on a non-empty set X, respectively.

Definition 2.1. [2] Suppose $F : \mathbb{R}_+ \to \mathbb{R}$ be a mapping that satisfies:

(F_1) F is non-decreasing, meaning, for all $p, q \in \mathbb{R}_+$, $p < q$ implies $F(p) < F(q)$;
(F_2) For any sequence $\{\beta_n\}$ in $\mathbb{R}_+$, $\lim_{n\to\infty} \beta_n = 0$ if and only if $\lim_{n\to\infty} F(\beta_n) = -\infty$;
(F_3) There exists a constant $k \in (0,1)$, such that $\lim_{\beta\to 0^+} \beta^k F(\beta) = 0$.

The family of all functions F satisfying the conditions (F_1)-(F_3) is denoted by $\mathcal{F}$.

Example 2.2. [12] Consider the following functions $F : \mathbb{R}_+ \to \mathbb{R}$:

(a) $F(\kappa) = \ln(\kappa^2 + 1)$,
(b) $F(\kappa) = \kappa + \ln \kappa$,
(c) $F(\kappa) = -\dfrac{1}{\sqrt{\kappa}}$.

Clearly (a), (b) and (c) pertains to $\mathcal{F}$.

Definition 2.3. [2] Let (X, d) be a metric space. A self-mapping $T : X \to X$ is called an F-contraction on (X, d) if there exist $F \in \mathcal{F}$ and $\tau > 0$ such that for all $p, q \in X$,

$$d(Tp, Tq) > 0 \Rightarrow \tau + F(d(Tp, Tq)) \leq F(d(p, q)). \tag{2.1}$$

Wardowski [2] developed a new generalization of the "Banach Contraction Principle" in 2012. It goes as follows:

Theorem 2.1. *[2] Let (X, d) be a metric space which is complete and let a self-mapping $T : X \to X$ be an F-contraction. Then T has a unique fixed point $p^* \in X$ and for all $p \in X$, the sequence $\{T^n p\}_{n\in\mathbb{N}}$ converges to p^*.*

Wardowski and Dung [8] obtained a new fixed point result in 2014 by defining the concept of an F-weak contraction on metric spaces and offered as:

Definition 2.4. [8] Let (X, d) be a metric space. A self-mapping $T : X \to X$ is said to be an F-weak contraction on (X, d) if there exist $F \in \mathcal{F}$ and $\tau > 0$ such that for all $p, q \in X$,

$$d(Tp, Tq) > 0 \Rightarrow \tau + F(d(Tp, Tq)) \leq F(M(p, q)), \tag{2.2}$$

where $M(p, q) = \max\left\{d(p, q), d(p, Tp), d(q, Tq), \dfrac{d(p, Tq) + d(q, Tp)}{2}\right\}$.

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Theorem 2.2. *[8] Let (X,d) be a complete metric space and let a self-mapping $T : X \to X$ be an F-weak contraction. If T or F is continuous, then T has a unique fixed point $p^* \in X$ and for every $p \in X$ the sequence $\{T^n p\}_{n\in\mathbb{N}}$ converges to p^*.*

Given a self-mapping T on a metric space (X,d), we know that a contraction condition for the mapping typically comprises at most five values: $d(p,q)$, $d(p,Tp)$, $d(q,Tq)$, $d(p,Tq)$ and $d(q,Tp)$ (see, for instance, [15,16]). Fixed point theorems were recently proved by Dung and Hang [9] by supplementing a contraction condition with the following four additional values: $d(T^2p,p), d(T^2p,Tp)$, $d(T^2p,q)$, and $d(T^2p,Tq)$. They provided examples to show that their findings are valid generalization of those that are already well-known in the existing literature.

Piri and Kumam [10] in 2014 replaced the condition of (F_3) in the definition of an F-contraction by the following one:

$$(F_3^{'}) : \text{F is continuous on } \mathbb{R}_+.$$

They represented, using $\mathbb{F}$, the family of all functions $F : \mathbb{R}_+ \to \mathbb{R}$ that fulfills the conditions (F_1), (F_2), and $(F_3^{'})$. As of 2013, Secelean [6] represented the family of all functions $F : \mathbb{R}_+ \to \mathbb{R}$ that satisfy the conditions (F_1), $(F_2^{'})$, and $(F_3^{'})$ by $\mathcal{G}$. Where,

$$(F_2^{'}) : \text{There exists a sequence } \{\beta_n\} in\mathbb{R}_+ \text{ such that } \lim_{n\to\infty} F(\beta_n) = -\infty, \text{ or } \inf F = -\infty.$$

Definition 2.5. [10] Let (X,d) be a metric space. A self-mapping $T : X \to X$ is said to be an F-Suzuki contraction if there exists $\tau > 0$, such that for all $p,q \in X$ with $Tp \neq Tq$,

$$\frac{1}{2}d(p,Tp) < d(p,q) \Rightarrow \tau + F(d(Tp,Tq)) \leq F(d(p,q)), \tag{2.3}$$

where $F \in \mathcal{G}$.

Theorem 2.3. *[10] Let (X,d) be a complete metric space and a self-mapping $T : X \to X$ be an F-Suzuki contraction. Then T has a unique fixed point $p^* \in X$ and for all $p \in X$, the sequence $\{T^n p\}_{n\in\mathbb{N}}$ converges to p^*.*

Definition 2.6. [5] For a non-empty set X, let $T : X \to X$ be a self-mapping and $\alpha : X \times X \to \mathbb{R}^+$ be a given mapping. Then, T is said to be an α-admissible mapping if

$$p,q \in X, \alpha(p,q) \geq 1 \Rightarrow \alpha(Tp,Tq) \geq 1.$$

Definition 2.7. [19] An α-admissible mapping T is said to have the K-property, if for each sequence $\{p_n\} \subseteq X$ with $\alpha(p_n,p_{n+1}) \geq 1$, for all $n \in \mathbb{N}_\circ$, there exists a positive integer k such that $\alpha(Tp_m,Tp_n) \geq 1$, for all $m > n \geq k$.

Definition 2.8. [12] Let (X,d) be a metric space and $\alpha : X \times X \to \mathbb{R}_+ \cup \{-\infty\}$ be a symmetric function. A self-mapping $T : X \to X$ is said to be an α-type F-Suzuki contraction if there exists $F \in \mathcal{G}$ and $\tau > 0$ such that for all $p,q \in X$ with $Tp \neq Tq$,

$$\frac{1}{2}d(p,Tp) \leq d(xp,q) \Rightarrow \tau + \alpha(p,q)F(d(Tp,Tq)) \leq F(d(p,q)). \tag{2.4}$$

Theorem 2.4. *[12] Let (X,d) be a complete metric space and a self-mapping $T : X \to X$ be an α-type F-Suzuki contraction that satisfy the following conditions:*

(i) T is an α-admissible mapping;
(ii) there exists $p_0 \in X$, such that $\alpha(p_0,Tp_0) \geq 1$;
(iii) if $\{p_n\}$ is a sequence in X, such that $p_n \to p$ as $n \to \infty$ and $\alpha(p_n,p_{n+1}) \geq 1$ for all $n \in \mathbb{N}_\circ$, then $\alpha(p_n,x) \geq 1$ for all $n \in \mathbb{N}_\circ$;
(iv) T has the K-property.

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Then, T has a fixed point in X.

Definition 2.9. [17] Let X be a non-empty set and $s \geq 1$ be a given real number. A mapping $d : X \times X \to \mathbb{R}^+$ is said to be a b-metric if for all $p, q, r \in X$, the following conditions are satisfied:

(D_1) $d(p,q) = 0$ if and only if $p = q$;
(D_2) $d(p,q) = d(q,p)$;
(D_3) $d(p,r) \leq s[d(p,q) + d(q,r)]$.

In this case the pair (X,d) is said to be a b-metric space (with s as a constant).

Remark. Every metric space is always a b-metric with $s = 1$, but the converse is not true.

Example 2.10. Let $X = \mathbb{R}^+$ and $\delta \geq 1$ be given. Define a mapping $d : X \times X \to \mathbb{R}^+$ by:

$$d(p,q) = \begin{cases} |p-q|, & if\ pq \neq 0, \\ \delta|p-q|, & if\ pq = 0. \end{cases} \quad \text{for any } p, q \in X.$$

Clearly, for all $p, q \in X$,

$$d(p,q) = 0 \Leftrightarrow p = q \text{ and } d(p,q) = d(q,p).$$

Hence, (D_1) and (D_2) of Definition 2.8 are satisfied.
Now, we will prove that (D_3) holds. For any $p, q, r \in X$, let us consider the following possible cases:

case 1 : Suppose that $pq \neq 0$. So $p \neq 0$ and $q \neq 0$.

- If $r = 0$, then
$$d(p,q) = |p-q| \leq |p-r| + |r-q| = \delta d(p,r) + \delta d(r,q) \leq \delta\,[d(p,r) + d(r,q)].$$
- If $r \neq 0$, then
$$d(p,q) = |p-q| \leq |p-r| + |r-q| = d(p,r) + d(r,q) \leq \delta\,[d(p,r) + d(r,q)].$$

case 2 : Suppose that $pq = 0$. Without loss of generality, assume that $p = 0$.

- If $r = 0$, then $p = r$ and (D_3) holds immediately.
- If $r \neq 0$, then
$$d(p,q) = \delta|p-q| \leq \delta|p-r| + \delta|r-q| \leq d(p,r) + \delta d(r,q) \leq \delta\,[d(p,r) + d(r,q)].$$

Therefore, by Cases 1 and case 2, we proved that (X,d) is a b-metric space, with a constant $s = \delta$. However, for $\delta > 1$, the usual triangle inequality for a metric does not hold. Thus, (X,d) is not a metric space.

Definition 2.11. [18] Let (X,d) be a b-metric space. A sequence $\{p_n\}$ in X is said to be:

(a) b-convergent if and only if there exists $p \in X$, such that $\lim_{n\to\infty} d(p_n, p) = 0$. And we write, $\lim_{n\to\infty} p_n = p$.
(b) b-Cauchy if and only if $\lim_{n,m\to\infty} d(p_n, p_m) = 0$.

Definition 2.12. [18] The b-metric space (X,d) is said to be b-complete if every b-Cauchy sequence in X converges in X.

Lemma 2.5. [10] Let $\{\beta_n\}$ be a sequence of positive real numbers and $F : \mathbb{R}_+ \to \mathbb{R}$ be an increasing function. Then the following hold:

(a) If $\lim_{n\to\infty} F(\beta_n) = -\infty$, then $\lim_{n\to\infty} \beta_n = 0$;
(b) If $\inf F = -\infty$ and $\lim_{n\to\infty} \beta_n = 0$, then $\lim_{n\to\infty} F(\beta_n) = -\infty$.

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3. Main Results

The notion of an α-type F-Suzuki contraction and an α-type F-weak-Suzuki contraction mappings in the setting of b-metric spaces are introduced. In this section some fixed point results for these class of mappings are established. Furthermore, a suitable example is provided to support the plausibility of the results drawn from our study and the validity of our generalizations. Lastly, we provide an application to a non-linear fractional differential equation.

Definition 3.1. Let (X,d) be a b-metric space and $\alpha : X \times X \to \mathbb{R}_+ \cup \{-\infty\}$ be a symmetric function. A self-mapping $T : X \to X$ is said to be an α-type F-Suzuki contraction if there exist $F \in \mathcal{G}$ and $\tau > 0$ such that for all $p,q \in X$ with $Tp \neq Tq$,

$$\frac{1}{2s}d(p,Tp) \leq d(p,q) \Rightarrow \tau + \alpha(p,q)F\left(s^3 d(Tp,Tq)\right) \leq F\left(d(p,q)\right). \tag{3.1}$$

Theorem 3.1. *Let (X,d) be a complete b-metric space and $T : X \to X$ be an α-type F-Suzuki contraction that satisfy the following conditions:*

(i) *T is α-admissible mapping;*
(ii) *there exists $p_\circ \in X$, such that $\alpha(p_\circ, Tp_\circ) \geq 1$;*
(iii) *if $\{p_n\}$ is a sequence in X, such that $p_n \to p$ as $n \to \infty$ and $\alpha(p_n, p_{n+1}) \geq 1$ for all $n \in \mathbb{N}_\circ$, then $\alpha(p_n, p) \geq 1$ for all $n \in \mathbb{N}_\circ$;*
(iv) *T has the K-property.*

Then, T has a fixed point in X.

Proof. By (ii), there exists $p_\circ \in X$ such that, $\alpha(p_\circ, Tp_\circ) \geq 1$. For any $n \in \mathbb{N}_\circ$, we define a sequence $\{p_n\}$ by:

$$p_{n+1} = Tp_n.$$

So, we have $\alpha(p_\circ, p_1) = \alpha(p_\circ, Tp_\circ) \geq 1$ and also $\alpha(p_1, p_2) = \alpha(Tp_\circ, Tp_1) \geq 1$, since T is α-admissible. By following this procedure, one can simply get that

$$\alpha(p_n, p_{n+1}) \geq 1, \text{ for all } n \in \mathbb{N}_\circ. \tag{3.2}$$

If $p_{k_\circ+1} = p_{k_\circ}$ for some $k_\circ \in \mathbb{N}_\circ$, then $p_{k_\circ}$ is a fixed point of T.
Let us assume that $p_{n+1} \neq p_n$ for all $n \in \mathbb{N}_\circ$, that is, $d(p_n, p_{n+1}) > 0$ and so for all $n \in \mathbb{N}_\circ$,

$$\frac{1}{2s}d(p_n, Tp_n) \leq \frac{1}{2}d(p_n, Tp_n) = \frac{1}{2}d(p_n, p_{n+1}) < d(p_n, p_{n+1}). \tag{3.3}$$

Now, since T is an α-type F-Suzuki-contraction, by (3.1) and (3.3), we obtain

$$\tau + F\left(d(Tp_n, Tp_{n+1})\right) \leq \tau + \alpha(p_n, p_{n+1})F\left(s^3 d(Tp_n, Tp_{n+1})\right) \leq F\left(d(p_n, p_{n+1})\right). \tag{3.4}$$

From this we have,

$$\tau + F\left(d(p_{n+1}, p_{n+2})\right) \leq F\left(d(p_n, p_{n+1})\right).$$

That is,

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$$F\left(d(p_{n+1}, p_{n+2})\right) \leq F\left(d(p_n, p_{n+1})\right) - \tau. \tag{3.5}$$

In general, repeating this process one can get

$$F\left(d(p_{n+1}, p_{n+2})\right) \leq F\left(d(p_\circ, p_1)\right) - n\tau. \tag{3.6}$$

Hence, $\lim_{n\to\infty} F\left(d(p_{n+1}, p_{n+2})\right) = -\infty$, as $\tau > 0$. This, in conjunction with $(F_2^{'})$ and using Lemma 2.5, yields

$$\lim_{n\to\infty} d(p_{n+1}, p_{n+2}) = 0.$$

Now, we assert that the sequence $\{p_n\}$ is a b-Cauchy one. In the contrary, there exists $\varepsilon > 0$ and two sequences of positive integers, $\{m_k\}$ and $\{n_k\}$, where $m_k > n_k > k$, and $d(p_{m_k}, p_{n_k}) \geq \varepsilon$ and $d(p_{m_k-1}, p_{n_k}) < \varepsilon$.
From this, one can observe that

$$\varepsilon \leq d(p_{m_k}, p_{n_k}) \leq sd(p_{m_k}, p_{m_k-1}) + sd(p_{m_k-1}, p_{n_k}) \leq sd(p_{m_k}, p_{m_k-1}) + s\varepsilon.$$

Therefore,

$$\varepsilon \leq \lim_{k\to\infty} \sup d(p_{m_k}, p_{n_k}) \leq s\varepsilon. \tag{3.7}$$

From the triangle inequality, we obtain

$$\varepsilon \leq d(p_{m_k}, p_{n_k}) \leq s[d(p_{m_k}, p_{n_k+1}) + d(p_{n_k+1}, p_{n_k})]$$

and

$$d(p_{m_k}, p_{n_k+1}) \leq s[d(p_{m_k}, p_{n_k}) + d(p_{n_k}, p_{n_k+1})].$$

So as $k \to \infty$, from the above two inequalities and using (3.7), we have

$$\frac{\varepsilon}{s} \leq \lim_{k\to\infty} \sup d(p_{m_k}, p_{n_k+1}) \leq s^2\varepsilon. \tag{3.8}$$

Repeating the above process, we get

$$\frac{\varepsilon}{s} \leq \lim_{k\to\infty} \sup d(p_{m_k+1}, p_{n_k}) \leq s^2\varepsilon. \tag{3.9}$$

From (3.8) and the triangle inequality

$$d(p_{m_k}, p_{n_k+1}) \leq s[d(p_{m_k}, p_{m_k+1}) + d(p_{m_k+1}, p_{n_k+1})],$$

we have

$$\frac{\varepsilon}{s^2} \leq \lim_{k\to\infty} \sup d(p_{m_k+1}, p_{n_k+1}). \tag{3.10}$$

From (3.7) and the inequality

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$$
\begin{aligned}
d(p_{m_k+1}, p_{n_k+1}) &\le s[d(p_{m_k+1}, p_{n_k}) + d(p_{n_k}, p_{n_k+1})] \\
&\le s^2[d(p_{m_k+1}, p_{m_k}) + d(p_{m_k}, p_{n_k})] + sd(p_{n_k}, p_{n_k+1}),
\end{aligned}
$$

we get

$$\lim_{k\to\infty} \sup d(p_{m_k+1}, p_{n_k+1}) \le s^3\varepsilon. \tag{3.11}$$

It follows from (3.10) and (3.11) that

$$\frac{\varepsilon}{s^2} \le \lim_{k\to\infty} \sup d(p_{m_k+1}, p_{n_k+1}) \le s^3\varepsilon. \tag{3.12}$$

Consequently, we can select a positive integer $n_1 \in \mathbb{N}$ so that

$$\frac{1}{2s} d(p_{m_k}, Tp_{m_k}) < \frac{\varepsilon}{2s} < d(p_{m_k}, p_{n_k}), \text{ for all } k \ge n_1.$$

Therefore, using the K-property of T, for every $k \ge n_1$, we have

$$\tau + F\left(d(Tp_{m_k}, Tp_{n_k})\right) \le \tau + \alpha(p_{m_k}, p_{n_k}) F\left(s^3 d(Tp_{m_k}, Tp_{n_k})\right) \le F\left(d(p_{m_k}, p_{n_k})\right). \tag{3.13}$$

Taking the limit supremum as $k \to \infty$ of (3.13), applying condition (F_3') and using (3.7) and (3.12), we get

$$\tau + F(s\varepsilon) \le F(s\varepsilon),$$

which is a contradiction, as $\tau > 0$. Hence $\{p_n\}$ is a b-Cauchy sequence in the complete b-metric space X and so it converges to some point $p^* \in X$.
To complete the proof, we show that p^* is a fixed point of T. First, we claim that, for all $n \in \mathbb{N}_\circ$,

$$\frac{1}{2s} d(p_n, p_{n+1}) < d(p_n, p^*)$$

or

$$\frac{1}{2s} d(p_{n+1}, p_{n+2}) < d(p_{n+1}, p^*).$$

Assume on the contrary that there exists $m \in \mathbb{N}_\circ$, such that

$$\frac{1}{2s} d(p_m, p_{m+1}) \ge d(p_m, p^*) \text{ and } \frac{1}{2s} d(p_{m+1}, p_{m+2}) \ge d(p_{m+1}, p^*). \tag{3.14}$$

It follows that,

$$2sd(p_m, p^*) \le d(p_m, p_{m+1}) \le sd(p_m, p^*) + sd(p^*, p_{m+1}), \tag{3.15}$$

this implies

$$d(p_m, p^*) \le d(p_m, p_{m+1}).$$

Following from (3.14) and (3.15), we get that

$$d(p_m, p^*) \le d(p^*, p_{m+1}) \le \frac{1}{2s} d(p_{m+1}, p_{m+2}). \tag{3.16}$$

Since, $\frac{1}{2s} d(p_m, p_{m+1}) < d(p_m, p_{m+1})$, we have that

$$
\begin{aligned}
\tau + F\left(d(p_{m+1}, p_{m+2})\right) &\le \tau + \alpha(p_m, p_{m+1}) F\left(s^3 d(Tp_m, Tp_{m+1})\right) \\
&\le F\left(d(p_m, p_{m+1})\right).
\end{aligned}
$$

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Since $\tau > 0$, it follows

$$F\left(d(p_{m+1}, p_{m+2})\right) < F\left(d(p_m, p_{m+1})\right).$$

Using the fact that F is non-decreasing, we obtain that

$$d(p_{m+1}, p_{m+2}) < d(p_m, p_{m+1}).$$

So, using this together with (3.14) and (3.16), we have

$$\begin{aligned} d\left(p_{m+1}, p_{m+2}\right) &< d\left(p_m, p_{m+1}\right) \\ &\leq sd(p_m, p^*) + d(p^*, p_{m+1}) \\ &\leq \frac{1}{2}d(p_{m+1}, p_{m+2}) + \frac{1}{2}d(p_{m+1}, p_{m+2}) \\ &= d(p_{m+1}, p_{m+2}), \end{aligned}$$

which is a contradiction. So, we must have that

$$\frac{1}{2s}d(p_n, p_{n+1}) < d(p_n, p^*)$$

or

$$\frac{1}{2s}d(p_{n+1}, p_{n+2}) < d(p_{n+1}, p^*).$$

Hence,

$$\begin{aligned} \tau + F\left(d(p_{n+1}, Tp^*)\right) &= \tau + F\left(d(Tp_n, Tp^*)\right) \\ &\leq \tau + \alpha(p_n, p^*)F\left(s^3 d(Tp_n, Tp^*)\right) \\ &\leq F\left(d(p_n, p^*)\right). \end{aligned}$$

Using the fact that $F \in \mathcal{G}$ and by Lemma 2.5, we get that

$$\lim_{n\to\infty} F\left(d(Tp_n, Tp^*)\right) = -\infty,$$

and so

$$\lim_{n\to\infty} d\left(Tp_n, Tp^*\right) = 0.$$

similarly,

$$\lim_{n\to\infty} F\left(d(T^2p_n, Tp^*)\right) = -\infty \text{ and } \lim_{n\to\infty} d\left(T^2p_n, Tp^*\right) = 0.$$

Now, observe

$$0 \leq d(p^*, Tp^*) \leq s[d(p^*, Tp_n) + d(Tp_n, Tp^*)].$$

Taking the limit as $n \to \infty$, it follows that

$$d(p^*, Tp^*) = 0.$$

Therefore, T has a fixed point. □

Theorem 3.2. *Let (X, d) be a complete b-metric space and $T : X \to X$ be an α-type F-Suzuki contraction that satisfy all the conditions of Theorem 3.1. Additionally, suppose that $\alpha(p, q) \geq 1$, for all $p, q \in Fix(T)$, then T has a unique fixed point.*

Proof. Suppose that p^* and q^* be two fixed points of T.
If $p^* \neq q^*$, then $d(Tp^*, Tq^*) > 0$. Since $p, q \in Fix(T)$, then $\alpha(p^*, q^*) \geq 1$.
Also,

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$$\frac{1}{2s}d(p^*,Tp^*) = 0 < d(p^*,q^*).$$

Hence, by (3.4) we obtain

$$\tau + F\left(d(p^*,q^*)\right) = \tau + F\left(d(Tp^*,Tq^*)\right) \leq \tau + \alpha(p^*,q^*)F\left(s^3 d(Tp^*,Tq^*)\right) \leq F\left(d(p^*,q^*)\right),$$

which is a contradiction, since $\tau > 0$. So, $p^* = q^*$. □

Definition 3.2. Let (X,d) be a b-metric space and $\alpha : X \times X \to \mathbb{R}_+ \cup \{-\infty\}$ be a symmetric function. The mapping $T : X \to X$ is called an α-type F-weak-Suzuki contraction if there exists $F \in \mathcal{G}$ and $\tau > 0$, such that for all $p,q \in X$ with $Tp \neq Tq$,

$$\frac{1}{2s}d(p,Tp) \leq d(p,q) \Rightarrow \tau + \alpha(p,q)F\left(s^3 d(Tp,Tq)\right) \leq F\left(M(p,q)\right). \tag{3.17}$$

Where,

$$M(p,q) = \max\left\{d(p,q), d(p,Tp), d(q,Tq), \frac{d(p,Tq)+d(q,Tp)}{2s}\right\}.$$

Remark. Every α-type F-Suzuki contraction is an α-type F-weak-Suzuki contraction, but the converse is not necessarily true.

Theorem 3.3. *Let (X,d) be a complete b-metric space and $T : X \to X$ be an α-type F-weak-Suzuki contraction that satisfy all the conditions in Theorem 3.1, then T has a fixed point in X. In addition, if $\alpha(p,q) \geq 1$ for all $p,q \in Fix(T)$, then T has a unique fixed point.*

Proof. Using similar steps as in Theorem 3.1 one can show that T has a fixed point and its uniqueness follows immediately after Theorem 3.2. □

4. Example and Application

In this section, we establish an example that supports Theorem 3.1. Also we provide an application for this theorem to determine the existence of solutions for a non-linear differential equation of fractional order. First, we consider the following example:

Example 4.1. Let $X = \mathbb{R}^+$ and $d : X \times X \to \mathbb{R}^+$ be defined as $d(p,q) = |p-q|^2$ for all $p,q \in X$. It is clear that (X,d) is a b-metric space with $s = 2$. Let us define $T : X \to X$ by:

$$Tp = \begin{cases} \frac{p}{16}, & if\ p \in [0,1], \\ 4p, & if\ p \in (1,\infty). \end{cases}$$

Define $\alpha : X \times X \to \mathbb{R}_+ \cup \{-\infty\}$ by

$$\alpha(p,q) = \begin{cases} 8, & if\ p,q \in [0,1], \\ -\infty, & if\ p,q \in (1,\infty), \end{cases}$$

and let

$$F(t) = -\frac{1}{t} + t.$$

Then for $\tau = 1$, T is an α-type F-Suzuki contraction mapping and also T satisfies all the hypothesis of Theorem 3.1 with $p = 0$ as its fixed point.

Proof. We know that $\alpha(p,q) = 8$, $Tp = \dfrac{p}{16}$ and $\alpha(Tp,Tq) = 8$, for any $p \in [0,1]$.
Thus, T is an α-admissible mapping.

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Also, for any $p_\circ \in [0,1]$, we have that, $\alpha(p_\circ, Tp_\circ) = 8$.
Now, let $\{p_n\}$ be a sequence in X with $\alpha(p_n, p_{n+1}) \geq 8$ for all $n \in \mathbb{N}_\circ$ and $p_n \to p$ as $n \to \infty$.
So, using the definition of α, we must have that $\{p_n\} \subset [0,1]$ and hence $p \in [0,1]$.
Following this, we have that $\alpha(p_n, p) = 8$.
To show that T is an α-type F-Suzuki contraction for any $p, q \in [0,1]$ with $\frac{1}{2s} d(p, Tp) \leq d(p,q)$, without loss of generality suppose that $p \leq q$. So, We have

$$\frac{1}{2s} d(p, Tp) = \frac{1}{4}|p - \frac{p}{16}|^2 = \left(\frac{15p}{32}\right)^2.$$

Thus, as $s = 2$, for $\frac{1}{4} d(p, Tp) \leq d(p,q)$, we must have that $\frac{47p}{32} \leq q$.
From this, since $\tau = 1$, one can observe that,

$$\begin{aligned}
\tau + \alpha(p,q) F\left(s^3 d(Tp, Tq)\right) &= 1 + 8F\left(8|\frac{q}{16} - \frac{p}{16}|^2\right) \\
&= 1 + 8F\left(\frac{1}{32}|q-p|^2\right) = 1 + \frac{|q-p|^2}{4} - \frac{256}{|q-p|^2} \\
&= |q-p|^2 + \left[1 - \frac{3|q-p|^2}{4} - \frac{256}{|q-p|^2}\right] \\
&\leq |q-p|^2 = F\left(d(p,q)\right).
\end{aligned}$$

Therefore, T is an α-type F-Suzuki contraction mapping and T satisfies all the hypothesis of Theorem 3.1 with $p = 0$ as its fixed point. □

In the last part of this section, we provide an application for our main theorem 3.1 for non-linear fractional differential equation. Our application of this section is devoted to the existence of solution for such differential equation with two boundary conditions. Finally, we will investigate if the following non-linear fractional differential problem has any solution:

$$^{C}D^{\delta} p(t) = h(t, p(t)), (0 < t < 1, 1 < \delta \leq 2). \tag{4.1}$$

via the boundary conditions

$$p(0) = 0, \quad p(1) = \int_0^k p(z)\, \mathrm{d}z, (o < k < 1).$$

Where, $^{C}D^{\delta}$ denotes the Caputo fractional derivative with order δ and $h : [0,1] \times \mathbb{R} \to \mathbb{R}$ is a continuous function. Let $X = C([0,1], \mathbb{R})$. Here, $(X, \|\cdot\|_\infty)$ is the Banach space of continuous functions mapped from $[0,1]$ into $\mathbb{R}$ endowed with the supremum norm $\|p\|_\infty = \sup_{t \in [0,1]} |p(t)|$.

In [20], for a continuous function $f : \mathbb{R}^+ \to \mathbb{R}$, the Caputo derivative with fractional order $\delta > 0$, is defined as follows:

$$^{C}D^{\delta} f(t) = \frac{1}{\Gamma(n-\delta)} \int_0^t (t-z)^{n-\delta-1} f^n(z)\, \mathrm{d}z, (n - 1 < \delta < n, n = [\delta] + 1).$$

Where, $[\delta]$ denotes the integer part of the positive real number δ, and Γ is a gamma function. Also for a continuous function $g : \mathbb{R}^+ \to \mathbb{R}$, the Riemann-Liouville fractional derivatives of order δ is defined by:

$$D^{\delta} f(t) = \frac{1}{\Gamma(n-\delta)} \frac{d^n}{dt^n} \int_0^t \frac{f(z)}{(t-z)^{\delta-n+1}}\, \mathrm{d}z, (n = [\delta] + 1),$$

where the right-hand sided part is point-wise defined on $(0, \infty)$.

At last, we prove the following existence theorem that supports our Theorem 3.1:

Theorem 4.1. *Assume that*

(i) *for $s \geq 1$ and $\tau > 0$, there exist a function $\beta : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ such that*

$$|h(t,p) - h(t,q)| \leq \frac{\Gamma(\delta+1)}{5s^3} e^{-\tau} |p-q|,$$

for each $t \in [0,1]$ and $p, q \in \mathbb{R}$ such that $\beta(p,q) > 0$;

(ii) *for all $t \in [0,1]$, there exists $p_\circ \in X$, such that $\beta(p_\circ(t), Tp_\circ(t)) > 0$, where the mapping $T : X \to X$ is defined by*

$$\begin{aligned} Tp(t) = {} & \frac{1}{\Gamma(\delta)} \int_0^t (t-z)^{\delta-1} h(z, p(z))\, \mathrm{d}z - \frac{2t}{(2-k^2)\Gamma(\delta)} \int_0^1 (1-z)^{\delta-1} h(z, p(z))\, \mathrm{d}z \\ & + \frac{2t}{(2-k^2)\Gamma(\delta)} \int_0^k \left(\int_0^s (z-m)^{\delta-1} h(m, p(m) dm) \right) \mathrm{d}z \quad (t \in [0,1]); \end{aligned}$$

(iii) *for each $p, q \in X$, $\beta(p(t), q(t)) > 0$ implies $\beta(Tp(t), Tq(t)) > 0$, for all $t \in [0,1]$;*

(iv) *for all $n \in \mathbb{N}$, if $\{p_n\}$ is a sequence in X, such that $p_n \to p$ in X and $\beta(p_n, p_{n+1}) > 0$, then $\beta(p_n, p) > 0$ for all $n \in \mathbb{N}$.*

Then, there is at least one solution for problem (4.1).

Proof. Since h is continuous, we know that, $p \in X$ is a solution of (4.1) if and only if it is a solution of the integral equation

$$\begin{aligned} p(t) = {} & \frac{1}{\Gamma(\delta)} \int_0^t (t-z)^{\delta-1} h(z, p(z))\, \mathrm{d}z - \frac{2t}{(2-k^2)\Gamma(\delta)} \int_0^1 (1-z)^{\delta-1} h(z, p(z))\, \mathrm{d}z \\ & + \frac{2t}{(2-k^2)\Gamma(\delta)} \int_0^k \left(\int_0^z (z-m)^{\delta-1} h(m, p(m) dm) \right) \mathrm{d}z, \quad t \in [0,1], \end{aligned}$$

which is the inverted form of equation (4.1). Then, solving problem (4.1) is equivalent to finding $p^* \in X$, which is a fixed point of T.

Following this, let $p, q \in X$ such that $\beta(p(t), q(t)) > 0$ for all $t \in [0,1]$. Taking the supremum over $z \in [0,1]$ and by (i), we obtain

$$\begin{aligned} s^3 \Big| [Tp(t) - Tq(t)] \Big| = {} & s^3 \Bigg| \frac{1}{\Gamma(\delta)} \int_0^t (t-z)^{\delta-1} h(z, p(z))\, \mathrm{d}z - \frac{2t}{(2-k^2)\Gamma(\delta)} \int_0^1 (1-z)^{\delta-1} h(z, p(z))\, \mathrm{d}z \\ & + \frac{2t}{(2-k^2)\Gamma(\delta)} \int_0^k \left(\int_0^z (z-m)^{\delta-1} h(m, p(m) dm) \right) \mathrm{d}z \\ & - \frac{1}{\Gamma(\delta)} \int_0^t (t-z)^{\delta-1} h(z, q(z))\, \mathrm{d}z - \frac{2t}{(2-k^2)\Gamma(\delta)} \int_0^1 (1-z)^{\delta-1} h(z, q(z))\, \mathrm{d}z \\ & - \frac{2t}{(2-k^2)\Gamma(\delta)} \int_0^k \left(\int_0^z (z-m)^{\delta-1} h(m, q(m) dm) \right) \mathrm{d}z \Bigg| \\ \leq {} & \frac{s^3}{\Gamma(\delta)} \int_0^t |t-z|^{\delta-1} |h(z, p(z)) - h(z, q(z))|\, \mathrm{d}z \\ & + \frac{2ts^3}{(2-k^2)\Gamma(\delta)} \int_0^1 |t-z|^{\delta-1} |h(z, p(z)) - h(z, q(z))|\, \mathrm{d}z \\ & + \frac{2ts^3}{(2-k^2)\Gamma(\delta)} \int_0^k \left| \int_0^z (z-m)^{\delta-1} \left(h(m, q(m)) - h(m, p(m)) \right) dm \right| \mathrm{d}z \end{aligned}$$

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$$
\begin{aligned}
&\leq \frac{s^3}{\Gamma(\delta)} \int_0^t |t-z|^{\delta-1} \frac{\Gamma(\delta+1)}{5s^3} e^{-\tau} |q(t)-p(t)| \, \mathrm{d}z \\
&+ \frac{2ts^3}{(2-k^2)\,\Gamma(\delta)} \int_0^1 |1-z|^{\delta-1} \frac{\Gamma(\delta+1)}{5s^3} e^{-\tau} |q(z)-p(z)| \, \mathrm{d}z \\
&+ \frac{2ts^3}{(2-k^2)\,\Gamma(\delta)} \int_0^k \left(\int_0^z |z-m|^{\delta-1} \frac{\Gamma(\delta+1)}{5s^3} e^{-\tau} |q(m)-p(m)| dm \right) \mathrm{d}z \\
&\leq \frac{s^3\Gamma(\delta+1)}{5s^3} e^{-\tau} \|p-q\|_\infty \sup_{t\in(0,1)} \left(\frac{1}{\Gamma(\delta)} \int_0^1 |t-z|^{\delta-1}| \, \mathrm{d}z \right. \\
&\left. + \frac{2t}{(2-k^2)\,\Gamma(\delta)} \int_0^1 |1-z|^{\delta-1} \, \mathrm{d}z + \frac{2t}{(2-k^2)\,\Gamma(\delta)} \int_0^k \int_0^z |z-m|^{\delta-1} dm \, \mathrm{d}z \right) \\
&\leq e^{-\tau} \|p-q\|_\infty.
\end{aligned}
$$

Hence, for each $p, q \in X$ with $\beta(p(t), q(t)) > 0$, for each $t \in [0,1]$, we have

$$s^3\|Tp-Tq\|_\infty \leq e^{-\tau}\|p-q\|_\infty, \text{ which means } s^3 d(Tp,Tq) \leq e^{-\tau} d(p,q).$$

Where,

$$d(p,q) = \|p-q\|_\infty = \lim_{t\in[0,1]} |p(t)-q(t)|.$$

Taking a logarithm both sides of the inequality $s^3 d(Tp,Tq) \leq e^{-\tau} d(p,q)$, we have

$$\ln\left(s^3 d(Tp,Tq)\right) \leq \ln\left(e^{-\tau} d(p,q)\right),$$

which implies

$$\tau + \ln\left(s^3 d(Tp,Tq)\right) \leq \ln\left(e^{-\tau} d(p,q)\right).$$

Next, consider the function $F : \mathbb{R}_+ \to \mathbb{R}$ defined by:

$$F(v) = \ln v.$$

Clearly, F satisfies conditions (F_1) and $(F_3^{'})$.
Now, we define $\alpha : X \times X \to \mathbb{R}_+ \cup \{-\infty\}$ by:

$$\alpha(p,q) = \begin{cases} 1, & if\ \beta(p(t),q(t)) > 0,\ \ t \in [0,1], \\ -\infty, & otherwise, \end{cases} \quad \text{for all} p, q \in X.$$

Therefore,

$$\frac{1}{2s} d(p,Tp) \leq d(p,q) \Rightarrow \tau + \alpha(p,q) F\left(s^3 d(Tp,Tq)\right) \leq F\left(d(p,q)\right),$$

which implies that, T is an α-type F-Suzuki-contraction.
Also, by condition (iii), we obtain

$$\alpha(p,q) > 0 \Rightarrow \beta(p(t),q(t)) > 0 \Rightarrow \beta(Tp(t),Tq(t)) > 0 \Rightarrow \alpha(Tp,Tq) \geq 1,$$

for all $p, q \in X$.
Thus, T is α-admissible.
Again by condition (ii), there exists $p_\circ \in X$ such that $\alpha(p_\circ, Tp_\circ) \geq 1$ and also T satisfies the K-property.
Additionally, from (iv), F is continuous.
This implies that, all the conditions of Theorem 3.1 are satisfied.
So following from this, there exist $p^* \in X$, such that $Tp^* = p^*$.
Therefore, p^* is a solution of problem (4.1). □

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5. Conclusion

The last few decades have seen a significant increase in interest in the search for fixed points with F-contractive type conditions due to its intriguing applications. Our paper's primary goal is to introduce novel ideas for α-type F-Suzuki-contraction and α-type F-weak-Suzuki-contraction in the setting of b-metric spaces. These concept are fundamentally weaker than the class of F-contraction mappings described in [2,4,6,11,12,14]. The existence and uniqueness of some fixed point results in this spaces are established for these kind of contractions. Additionally, an example and an application to a non-linear differential equation of fractional order are provided to show the applicability of our obtained results.

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