Research Series in Pure Mathematics
Topology, *Derived and Coderived Operators*,
Series 2018–2019, № 8, Pages 1–89.

— Theory of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-Derived and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-Coderived Operators —
Definitions, Essential Properties, Iterations, and Ranks

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Abstract. In a generalized topological space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$ ($\mathscr{T}_{\mathfrak{g}}$-space), the $\mathfrak{g}$-topology $\mathscr{T}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ can be characterized in the generalized sense by specifying the generalized open, generalized closed sets ($\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-open, $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-closed sets), generalized interior, generalized closure operators $\mathfrak{g}\text{-Int}_{\mathfrak{g}}$, $\mathfrak{g}\text{-Cl}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ ($\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-interior, $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-closure operators), or generalized derived, generalized coderived operators $\mathfrak{g}\text{-Der}_{\mathfrak{g}}$, $\mathfrak{g}\text{-Cod}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ ($\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-derived, $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-coderived operators), respectively. For very many $\mathscr{T}_{\mathfrak{g}}$-spaces, the δ^{th}-iterates $\mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta)}$, $\mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\delta)} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ of $\mathfrak{g}\text{-Der}_{\mathfrak{g}}$, $\mathfrak{g}\text{-Cod}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$, respectively, defined by transfinite recursion on the class of successor ordinals are also themselves $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-derived, $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-coderived operators for new $\mathfrak{g}$-topologies in the generalized sense on Ω. Thus, the use of novel definitions of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-derived, $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-coderived operators $\mathfrak{g}\text{-Der}_{\mathfrak{g}}$, $\mathfrak{g}\text{-Cod}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$, respectively, based on a very clever construction, together with their δ^{th}-iterates $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-operators $\mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta)}$, $\mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\delta)} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$, defined by transfinite recursion on the class of successor ordinals, will give rise to novel generalized $\mathfrak{g}$-topologies on Ω. The present authors have been actively engaged in the study of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-operators in $\mathscr{T}_{\mathfrak{g}}$-spaces. The study of the essential properties and the commutativity of novel definitions of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-interior and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-closure operators $\mathfrak{g}\text{-Int}_{\mathfrak{g}}$, $\mathfrak{g}\text{-Cl}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$, respectively, in $\mathfrak{T}_{\mathfrak{g}}$ has formed the first part, and the study of the essential properties and sets of consistent, independent axioms of novel definitions of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-exterior and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-frontier operators $\mathfrak{g}\text{-Ext}_{\mathfrak{g}}$, $\mathfrak{g}\text{-Fr}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$, respectively, has formed the second part. In this work, which forms the last part on the theory of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-operators in $\mathscr{T}_{\mathfrak{g}}$-spaces, the present authors propose to present novel definitions and the study of the essential properties of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-derived and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-coderived operators $\mathfrak{g}\text{-Der}_{\mathfrak{g}}$, $\mathfrak{g}\text{-Cod}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$, respectively, and their δ^{th}-iterates, and the notions of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-open and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-closed sets of ranks δ in $\mathscr{T}_{\mathfrak{g}}$-spaces.

Key words and phrases. *Generalized topological space, generalized sets, generalized derived operator, generalized coderived operator, iterations, ranks*

Contents

1. Introduction

The $\mathfrak{T}$,[1] $\mathfrak{g}$-$\mathfrak{T}$-derived operators der, $\mathfrak{g}$-Der : $\mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ (ordinary derived and generalized derived operators in ordinary topological spaces) and their duals called, respectively, $\mathfrak{T}$, $\mathfrak{g}$-$\mathfrak{T}$-coderived operators cod, $\mathfrak{g}$-Cod : $\mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ (ordinary coderived and generalized coderived operators in ordinary topological spaces) in $\mathscr{T}$-spaces as well as the $\mathfrak{T}_{\mathfrak{g}}$, $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-derived operators $\mathrm{der}_{\mathfrak{g}}$, $\mathfrak{g}$-$\mathrm{Der}_{\mathfrak{g}}$: $\mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ (ordinary derived and generalized derived operators in generalized topological spaces) and their duals called, respectively, $\mathfrak{T}_{\mathfrak{g}}$, $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-coderived operators $\mathrm{cod}_{\mathfrak{g}}$, $\mathfrak{g}$-$\mathrm{Cod}_{\mathfrak{g}}$: $\mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ (ordinary coderived and generalized coderived operators in generalized topological spaces) in $\mathscr{T}_{\mathfrak{g}}$-spaces, respectively, can all play very important roles, yielding to nice characterizations in their $\mathscr{T}$, $\mathscr{T}_{\mathfrak{g}}$-spaces.

For instance, ordinary and generalized characterizations of $\mathscr{T} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ of a $\mathscr{T}$-space $\mathfrak{T} = (\Omega, \mathscr{T})$ can be realized by specifying either the $\mathfrak{T}$, $\mathfrak{g}$-$\mathfrak{T}$-derived operators der, $\mathfrak{g}$-Der : $\mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ or the $\mathfrak{T}$, $\mathfrak{g}$-$\mathfrak{T}$-coderived operators cod, $\mathfrak{g}$-Cod : $\mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$, respectively. Likewise, ordinary and generalized characterizations of $\mathscr{T}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ of a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$ can be realized by specifying either the $\mathfrak{T}_{\mathfrak{g}}$, $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-derived operators $\mathrm{der}_{\mathfrak{g}}$, $\mathfrak{g}$-$\mathrm{Der}_{\mathfrak{g}}$: $\mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ or the $\mathfrak{T}_{\mathfrak{g}}$, $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-coderived operators $\mathrm{cod}_{\mathfrak{g}}$, $\mathfrak{g}$-$\mathrm{Cod}_{\mathfrak{g}}$: $\mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$, respectively. Moreover, if the δ^{th}-iterates $\mathrm{der}^{(\delta)}$, $\mathfrak{g}$-$\mathrm{Der}^{(\delta)}$, $\mathrm{cod}^{(\delta)}$, $\mathfrak{g}$-$\mathrm{Cod}^{(\delta)}$: $\mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ of der, $\mathfrak{g}$-Der, cod, $\mathfrak{g}$-Cod : $\mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$, respectively, defined by transfinite recursion on the class of successor ordinals are also themselves $\mathfrak{T}$, $\mathfrak{g}$-$\mathfrak{T}$-derived and $\mathfrak{T}$, $\mathfrak{g}$-$\mathfrak{T}$-coderived operators in their $\mathscr{T}$, $\mathscr{T}_{\mathfrak{g}}$-spaces, similar roles can be played, thereby realizing other ordinary and generalized characterizations of $\mathscr{T}$, $\mathscr{T}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$.

In actual fact, $\mathfrak{g}$-$\mathfrak{T}$-derivedness and $\mathfrak{g}$-$\mathfrak{T}$-coderivedness are generalizations of $\mathfrak{T}$-derivedness and $\mathfrak{T}$-coderivedness in $\mathscr{T}$-spaces, respectively; $\mathfrak{T}$-derivedness and $\mathfrak{T}$-coderivedness in $\mathscr{T}$-spaces are generalizations of $\mathbb{R}$-derivedness and $\mathbb{R}$-coderivedness in $\mathbb{R}$ (deriveness and coderiveness in the real number line $\mathbb{R}$); $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-derivedness and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-coderivedness are generalizations of $\mathfrak{T}_{\mathfrak{g}}$-derivedness and $\mathfrak{T}_{\mathfrak{g}}$-coderivedness in $\mathscr{T}_{\mathfrak{g}}$-spaces, respectively.

Since coderivedness is the dual of derivedness, the concept of $\mathbb{R}$-derived operator der : $\mathscr{P}(\mathbb{R}) \longrightarrow \mathscr{P}(\mathbb{R})$ is, amongst those primitive $\mathfrak{T}$, $\mathfrak{g}$-$\mathfrak{T}$, $\mathfrak{T}_{\mathfrak{g}}$, $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-operators and

[1]Notes to the reader: The structures $\mathfrak{T} = (\Omega, \mathscr{T})$ and $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$, respectively, are called ordinary and generalized topological spaces (briefly, $\mathscr{T}$-space and $\mathscr{T}_{\mathfrak{g}}$-space). The symbols $\mathscr{T}$ and $\mathscr{T}_{\mathfrak{g}}$, respectively, are called ordinary topology and generalized topology (briefly, topology and $\mathfrak{g}$-topology). Subsets of $\mathfrak{T}$ and $\mathfrak{T}_{\mathfrak{g}}$, respectively, are called $\mathfrak{T}$-sets and $\mathfrak{T}_{\mathfrak{g}}$-sets; subsets of $\mathscr{T}$ and $\mathscr{T}_{\mathfrak{g}}$, respectively, are called $\mathscr{T}$-open and $\mathscr{T}_{\mathfrak{g}}$-open sets, and their complements are called $\mathscr{T}$-closed and $\mathscr{T}_{\mathfrak{g}}$-closed sets. Generalizations of $\mathfrak{T}$-sets, $\mathscr{T}$-open and $\mathscr{T}$-closed sets in $\mathscr{T}$, respectively, are called $\mathfrak{g}$-$\mathfrak{T}$-sets, $\mathfrak{g}$-$\mathscr{T}$-open and $\mathfrak{g}$-$\mathscr{T}$-closed sets; generalizations of $\mathfrak{T}_{\mathfrak{g}}$-sets, $\mathscr{T}_{\mathfrak{g}}$-open and $\mathscr{T}_{\mathfrak{g}}$-closed sets in $\mathscr{T}_{\mathfrak{g}}$, respectively, are called $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-sets, $\mathfrak{g}$-$\mathscr{T}_{\mathfrak{g}}$-open and $\mathfrak{g}$-$\mathscr{T}_{\mathfrak{g}}$-closed sets. By a Λ-operator is meant an operator using Λ-sets to characterize its argument, where $\Lambda \in \{\mathscr{T}, \mathfrak{T}, \mathfrak{g}\text{-}\mathscr{T}, \mathfrak{g}\text{-}\mathfrak{T}\} \cup \{\mathscr{T}_{\mathfrak{g}}, \mathfrak{T}_{\mathfrak{g}}, \mathfrak{g}\text{-}\mathscr{T}_{\mathfrak{g}}, \mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}\}$.

their duals in their $\mathscr{T}$, $\mathscr{T}_{\mathfrak{g}}$-spaces, the oldest concept. If one year can be specified as the time when $\mathrm{der} : \mathscr{P}(\mathbb{R}) \longrightarrow \mathscr{P}(\mathbb{R})$ was first introduced and the iteration of $\mathrm{der} : \mathscr{P}(\mathbb{R}) \longrightarrow \mathscr{P}(\mathbb{R})$ considered, thereby coming to define ordinals in order to defined the notion of δ^{th} order $\mathbb{R}$-derived operator $\mathrm{der}^{(\delta)} : \mathscr{P}(\mathbb{R}) \longrightarrow \mathscr{P}(\mathbb{R})$ by transfinite recursion on the class of successor ordinals, that year should probably be 1872, the year in which Georg Cantor investigated the convergence of Fourier series [Can72, Can82]. Thereafter, various Mathematicians have studied some types of $\mathfrak{T}$, $\mathfrak{g}$-$\mathfrak{T}$, $\mathfrak{T}_{\mathfrak{g}}$, $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-operators in $\mathscr{T}$, $\mathscr{T}_{\mathfrak{g}}$-spaces and other abstract spaces, and other types left untouched [Ahm66, BBÖS19, CJK04, CM82, Gna97, Har63, Hed11, Hig83, Kow61, Lat06, LZ19, MR12, Mod17, Moo08, AON09, RT14, RTJ13, Rut43, SM15, Spi67, Ste07, Tuc67].

In the year 1911, [Hed11] has studied the properties of an arbitrary domain defined in the sense of Fréchet for which the notion of derivedness of any subcollection of which coincides with the notion of closedness [Fré06]. Later on, [Rut43] has considered the notion of closedness of grade $\delta \in \mathbb{N}^*$ in terms of the notion of δ^{th} order derivedness and investigated some properties. In his work, [Kow61] has presented an axiom system for the $\mathfrak{T}$-derived operator $\mathrm{der} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ in $\mathscr{T}$-spaces. In the work of [Har63], the author has defined the notion of the $\mathfrak{T}$-derived operator $\mathrm{der} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ in $\mathscr{T}$-spaces based on an axiom system which is equivalent to that found in the work of [Kow61] and has proved a theorem concerning a topological characterization based on the notion of $\mathfrak{T}$-derived operator as a set-valued map $\mathrm{der} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ in a $\mathscr{T}$-space. In a note on $\mathfrak{T}$-derived axiom system for $\mathscr{T}$-spaces, [Spi67] has shown that translating the $\mathfrak{T}$-closure presented by [Kur22] weakens the $\mathfrak{T}$-derived axiom system and has given a stronger $\mathfrak{T}$-derived axiom system in a sense found by [Har63]. [Ahm66] has shown that, under a slight modification, the axiom system proposed by [Kow61] becomes absolutely independent in the sense of [Har61].

In investigating the notion of periodicity of sequences of $\mathfrak{T}$-derived sets in a $\mathscr{T}$-space, [Tuc67] has considered the δ^{th}-iterate $\mathrm{der}^{(\delta)} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ of the $\mathfrak{T}$-derived operator $\mathrm{der} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$. [CM82] have discussed some properties of the $\mathfrak{T}$-derived operator $\mathrm{der} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ in a $\mathscr{T}$-space. [Hig83] has given characterizations on the δ^{th}-iterate $\mathrm{der}^{(\delta)} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ of the $\mathfrak{T}$-derived operator $\mathrm{der} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ in a $\mathscr{T}$-space.

[CJK04] has introduced a new type of $\mathfrak{g}$-$\mathfrak{T}$-derived operator in $\mathscr{T}$-spaces called θ-derived operator and characterized by $\theta\text{-D} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ and has studied some of its properties. [DMJ12] have investigated some properties of a novel type of $\mathfrak{g}$-$\mathfrak{T}$-derived operator $\mathfrak{g}\text{-Der} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ in a $\mathscr{T}$-space called p-derived operator and characterized by $\mathrm{D}_p : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$; the properties of this notion is also presented in [Gna97]. [Lat06] has introduced a new $\mathfrak{g}$-$\mathfrak{T}$-derived operator $\mathfrak{g}\text{-Der} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ in a $\mathscr{T}$-space called γ-derived operator and characterized by $\mathrm{D}_\gamma : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ and has investigated some of its properties. [AON09] has introduced a new $\mathfrak{g}$-$\mathfrak{T}$-derived operator $\mathfrak{g}\text{-Der} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ in a $\mathscr{T}$-space called b-derived operator and characterized by $\mathrm{D}_b : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ and has investigated some of its properties.

[MR12] and [Mod17] have discussed some properties of another novel type of $\mathfrak{g}$-$\mathfrak{T}$-derived operator $\mathfrak{g}\text{-Der} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ in a $\mathscr{T}$-space called λ-derived operator and characterized by $\mathrm{D}_\lambda : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$. [RTJ13] has introduced a new $\mathfrak{g}$-$\mathfrak{T}$-derived operator $\mathfrak{g}\text{-Der} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ in a $\mathscr{T}$-space called β^*-derived operator and characterized by $\mathrm{D}_{\beta^*} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ and has investigated some of its properties. [RT14] has introduced a new type of $\mathfrak{g}$-$\mathfrak{T}$-derived operator in a $\mathscr{T}$-space called g^*s^*-derived operator and characterized by $\mathrm{g}^*\mathrm{s}^*\mathrm{D} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ and has studied some basic properties of g^*s^*-derived sets. [SR16] has introduced a new type of $\mathfrak{g}$-$\mathfrak{T}$-derived operator in a $\mathscr{T}$-space called $\alpha\mathrm{g}^*\mathrm{p}$-derived operator and characterized by $\mathrm{D}_{\alpha\mathrm{g}^*\mathrm{p}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ and has studied some of its properties.

[SM15] have introduced topological semantics in terms of the $\mathfrak{T}$-derived and $\mathfrak{T}$-coderived operators der, cod : $\mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ in a $\mathscr{T}$-space. In presenting a new topological semantics for doxastic logic, [BBÖS19] have compared their semantics to older topological semantics in terms of the $\mathfrak{T}$-derived and $\mathfrak{T}$-coderived operators der, cod : $\mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ in a $\mathscr{T}$-space [Ste07]. In the work of [LZ19], alternative axiomatic definitions for the $\mathfrak{g}$-$\mathfrak{T}$-derived and $\mathfrak{g}$-$\mathfrak{T}$-coderived operators $\mathfrak{g}$-Der, $\mathfrak{g}$-Cod : $\mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ in a $\mathscr{T}$-space have been proposed, and some propositions proved on this basis.

In view of the above references of the literature of $\mathscr{T}$, $\mathscr{T}_{\mathfrak{g}}$-spaces on $\mathfrak{T}$, $\mathfrak{g}$-$\mathfrak{T}$, $\mathfrak{T}_{\mathfrak{g}}$, $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-operators, the following can be remarked:

- I. Few Mathematicians have introduced and studied the concepts of $\mathfrak{g}$-$\mathfrak{T}$-derived and $\mathfrak{g}$-$\mathfrak{T}$-coderived operators in $\mathscr{T}$-spaces.
- II. No Mathematician has introduced and studied the concepts of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-derived and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-coderived operators in $\mathscr{T}_{\mathfrak{g}}$-spaces.

In this paper titled *Theory of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-Derived and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-Coderived Operators* and subtitled *Definitions, Essential Properties, Iterations, and Ranks*, the authors attempt to add, in as unique and unified a way as possible so as to offer a unified approach to many $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-derived and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-coderived operators properties, a further contribution to the field with these three research objectives in mind:

- I. To present the definitions and the essential properties of a new class of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-derived and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-coderived operators in $\mathscr{T}_{\mathfrak{g}}$-spaces.
- II. To present the definitions and the essential properties of the concepts of δ^{th}-order derivative $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-derived and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-coderived operators defined by transfinite recursion on the class of successor ordinals in $\mathscr{T}_{\mathfrak{g}}$-spaces.
- III. To present the definitions and the essential properties of the concepts of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-open and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-closed sets of rank δ in $\mathscr{T}_{\mathfrak{g}}$-spaces.

These three research objectives form properly three separate sections, and the rest of this paper is structured in this manner: In SECT. 2, preliminary notions are described in SUBSECT. 2.1 (APPX. A contains pre-preliminary notions extracted from the pre-preliminary and preliminary sections of our sixth work titled *Theory of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-Interior and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-Closure Operators*) and the main results of the theory of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-exterior and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-frontier operators in $\mathscr{T}_{\mathfrak{g}}$-spaces are reported in SECT. 3: results associated with essential properties of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-derived and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-coderived operators in $\mathscr{T}_{\mathfrak{g}}$-spaces are given in SUBSECT. 3.1; results associated with the essential properties of δ^{th}-order derivative $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-derived and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-coderived operators defined by transfinite recursion on the class of successor ordinals in $\mathscr{T}_{\mathfrak{g}}$-spaces are given in SUBSECT. 3.2; results associated with the essential properties of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-open and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-closed sets of rank δ in $\mathscr{T}_{\mathfrak{g}}$-spaces are given in SUBSECT. 3.3. In SECT. 4, the establishment of the various relationships between these $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-operators are discussed in SECTS 4.1. To support the work, a nice application of the $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-derived and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-coderived operators in a $\mathscr{T}_{\mathfrak{g}}$-space is presented in SUBSECT. 4.2. Finally,

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Subsect. 4.3 provides concluding remarks and future directions of the theory of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-derived and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-coderived operators in $\mathscr{T}_{\mathfrak{g}}$-spaces.

2. Theory

2.1. Preliminaries. Foreign terms used here are extracted from the preliminary section of our sixth work titled *Theory of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-Interior and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-Closure Operators* and are presented in Appx. A.

The discussion commences by defining the notions of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-derived and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-coderived operators of category ν in $\mathscr{T}_{\mathfrak{g}}$-spaces.

Definition 2.1 ($\mathfrak{g}$-ν-$\mathfrak{T}_{\mathfrak{g}}$-Derived, $\mathfrak{g}$-ν-$\mathfrak{T}_{\mathfrak{g}}$-Coderived Operators). Suppose $\mathfrak{g}\text{-Int}_{\mathfrak{g},\nu}$, $\mathfrak{g}\text{-Cl}_{\mathfrak{g},\nu} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$, respectively, denote the $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-interior and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-closure operators of category ν and, $\mathfrak{g}\text{-Op}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ denote the absolute complement $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-operator in a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$. Then, the one-valued maps of the types

$$\begin{aligned}\mathfrak{g}\text{-Der}_{\mathfrak{g},\nu} : \mathscr{P}(\Omega) &\longrightarrow \mathscr{P}(\Omega) \stackrel{\text{def}}{=} \left\{\mathscr{S}_{\mathfrak{g},\mu} \subseteq \Omega : \ \mu \in I_{\infty}^{*}\right\} \\ \mathscr{S}_{\mathfrak{g}} &\longmapsto \left\{\xi \in \mathfrak{T}_{\mathfrak{g}} : \ \xi \in \mathfrak{g}\text{-Cl}_{\mathfrak{g},\nu}\big(\mathscr{S}_{\mathfrak{g}} \cap \mathfrak{g}\text{-Op}_{\mathfrak{g}}(\{\xi\})\big)\right\},\end{aligned} \tag{2.1}$$

$$\begin{aligned}\mathfrak{g}\text{-Cod}_{\mathfrak{g},\nu} : \mathscr{P}(\Omega) &\longrightarrow \mathscr{P}(\Omega) \stackrel{\text{def}}{=} \left\{\mathscr{S}_{\mathfrak{g},\mu} \subseteq \Omega : \ \mu \in I_{\infty}^{*}\right\} \\ \mathscr{S}_{\mathfrak{g}} &\longmapsto \left\{\zeta \in \mathfrak{T}_{\mathfrak{g}} : \ \zeta \in \mathfrak{g}\text{-Int}_{\mathfrak{g},\nu}\big(\mathscr{S}_{\mathfrak{g}} \cup \{\zeta\}\big)\right\}\end{aligned} \tag{2.2}$$

on $\mathscr{P}(\Omega)$ ranging in $\mathscr{P}(\Omega)$ are called, respectively, a "$\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-derived operator of category ν" and a "$\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-coderived operator of category ν." The classes $\mathfrak{g}\text{-DE}[\mathfrak{T}_{\mathfrak{g}}] \stackrel{\text{def}}{=} \left\{\mathfrak{g}\text{-Der}_{\mathfrak{g},\nu} : \ \nu \in I_3^0\right\}$ and $\mathfrak{g}\text{-CD}[\mathfrak{T}_{\mathfrak{g}}] \stackrel{\text{def}}{=} \left\{\mathfrak{g}\text{-Cod}_{\mathfrak{g},\nu} : \ \nu \in I_3^0\right\}$ are called, respectively, the class of all $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-derived operators and the class of all $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-coderived operators.

A nice remark can be given at this very first stage.

Remark 2.2. If the notations $\mathfrak{g}\text{-Der}_{\mathfrak{g}}(\xi; \mathscr{S}_{\mathfrak{g}})$ and $\mathfrak{g}\text{-Cod}_{\mathfrak{g}}(\zeta; \mathscr{S}_{\mathfrak{g}})$, respectively, designate a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-*derived point* $\xi \in \mathfrak{T}_{\mathfrak{g}}$ and a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-*coderived point* $\zeta \in \mathfrak{T}_{\mathfrak{g}}$ of some $\mathscr{S}_{\mathfrak{g}} \in \mathscr{P}(\Omega)$, then

$$\mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \stackrel{\text{def}}{=} \left\{\mathfrak{g}\text{-Der}_{\mathfrak{g}}(\xi; \mathscr{S}_{\mathfrak{g}}) : \ \xi \in \mathfrak{T}_{\mathfrak{g}}\right\}, \tag{2.3}$$

$$\mathfrak{g}\text{-Cod}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \stackrel{\text{def}}{=} \left\{\mathfrak{g}\text{-Cod}_{\mathfrak{g}}(\zeta; \mathscr{S}_{\mathfrak{g}}) : \ \xi \in \mathfrak{T}_{\mathfrak{g}}\right\}, \tag{2.4}$$

respectively, denote the $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-*derived set* and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-*coderived set* of $\mathscr{S}_{\mathfrak{g}}$ in $\mathfrak{T}_{\mathfrak{g}}$.

It is interesting to view $\mathfrak{g}\text{-Der}_{\mathfrak{g}}$, $\mathfrak{g}\text{-Cod}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ as the components of some so-called $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-vector operator, and the definition follows.

Definition 2.3 ($\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-Vector Operator). Let $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$ be a $\mathscr{T}_{\mathfrak{g}}$-space. Then, an operator of the type

$$\begin{aligned}\mathfrak{g}\text{-}\mathbf{Dc}_{\mathfrak{g},\nu} : \textstyle\times_{\alpha \in I_2^{*}} \mathscr{P}(\Omega) &\longrightarrow \displaystyle\mathop{\times}_{\alpha \in I_2^{*}} \mathscr{P}(\Omega) \\ (\mathscr{R}_{\mathfrak{g}}, \mathscr{S}_{\mathfrak{g}}) &\longmapsto \big(\mathfrak{g}\text{-Der}_{\mathfrak{g},\nu}(\mathscr{R}_{\mathfrak{g}}), \mathfrak{g}\text{-Cod}_{\mathfrak{g},\nu}(\mathscr{S}_{\mathfrak{g}})\big)\end{aligned} \tag{2.5}$$

on $\times_{\alpha \in I_2^{*}} \mathscr{P}(\Omega)$ ranging in $\times_{\alpha \in I_2^{*}} \mathscr{P}(\Omega)$ is called a "$\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-vector operator of category ν" and, $\mathfrak{g}\text{-DC}[\mathfrak{T}_{\mathfrak{g}}] \stackrel{\text{def}}{=} \left\{\mathfrak{g}\text{-}\mathbf{Dc}_{\mathfrak{g},\nu} = \big(\mathfrak{g}\text{-Der}_{\mathfrak{g},\nu}, \mathfrak{g}\text{-Cod}_{\mathfrak{g},\nu}\big) : \ \nu \in I_3^0\right\}$ is called the class of all such $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-vector operators.

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In the remark below, the passage from $\mathfrak{g}\text{-}\mathbf{Dc}_{\mathfrak{g}} : \times_{\alpha\in I_2^*} \mathscr{P}(\Omega) \longrightarrow \times_{\alpha\in I_2^*} \mathscr{P}(\Omega)$ to $\mathfrak{g}\text{-}\mathbf{dc}_{\mathfrak{g}} : \times_{\alpha\in I_2^*} \mathscr{P}(\Omega) \longrightarrow \times_{\alpha\in I_2^*} \mathscr{P}(\Omega)$, $\mathfrak{g}\text{-}\mathbf{Dc} : \times_{\alpha\in I_2^*} \mathscr{P}(\Omega) \longrightarrow \times_{\alpha\in I_2^*} \mathscr{P}(\Omega)$ and $\mathfrak{g}\text{-}\mathbf{dc} : \times_{\alpha\in I_2^*} \mathscr{P}(\Omega) \longrightarrow \times_{\alpha\in I_2^*} \mathscr{P}(\Omega)$ is explained.

Remark 2.4. Observing that, for every $\nu \in I_3^0$, the $\mathfrak{g}$-ν-$\mathfrak{T}_{\mathfrak{g}}$-derived and $\mathfrak{g}$-ν-$\mathfrak{T}_{\mathfrak{g}}$-coderived operators $\mathfrak{g}\text{-Der}_{\mathfrak{g},\nu}, \mathfrak{g}\text{-Cod}_{\mathfrak{g},\nu} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ are based on the $\mathfrak{g}$-ν-$\mathfrak{T}_{\mathfrak{g}}$-closure and $\mathfrak{g}$-ν-$\mathfrak{T}_{\mathfrak{g}}$-interior operators $\mathfrak{g}\text{-Cl}_{\mathfrak{g},\nu}, \mathfrak{g}\text{-Int}_{\mathfrak{g},\nu} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$, respectively, it follows that:

- I. $\mathfrak{g}\text{-}\mathbf{Dc}_{\mathfrak{g},\nu} = \mathbf{dc}_{\mathfrak{g}} \stackrel{\text{def}}{=} (\text{der}_{\mathfrak{g}}, \text{cod}_{\mathfrak{g}})$ if based on $(\text{cl}_{\mathfrak{g}}, \text{int}_{\mathfrak{g}})$;
- II. $\mathfrak{g}\text{-}\mathbf{Dc}_{\mathfrak{g},\nu} = \mathfrak{g}\text{-}\mathbf{Dc}_{\nu} \stackrel{\text{def}}{=} (\mathfrak{g}\text{-Der}_{\nu}, \mathfrak{g}\text{-Cod}_{\nu})$ if based on $(\mathfrak{g}\text{-Cl}_{\nu}, \mathfrak{g}\text{-Int}_{\nu})$;
- III. $\mathfrak{g}\text{-}\mathbf{Dc}_{\mathfrak{g},\nu} = \mathbf{dc} \stackrel{\text{def}}{=} (\text{der}, \text{cod})$ if based on (cl, int).

In this way, $\mathbf{dc}_{\mathfrak{g}} \stackrel{\text{def}}{=} (\text{der}_{\mathfrak{g}}, \text{cod}_{\mathfrak{g}})$ is a $\mathfrak{T}_{\mathfrak{g}}$-vector operator whose first component is a $\mathfrak{T}_{\mathfrak{g}}$-derived operator and second component is a $\mathfrak{T}_{\mathfrak{g}}$-coderived operator in a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$; $\mathfrak{g}\text{-}\mathbf{Dc}_{\nu} \stackrel{\text{def}}{=} (\mathfrak{g}\text{-Der}_{\nu}, \mathfrak{g}\text{-Cod}_{\nu})$, a $\mathfrak{g}$-$\mathfrak{T}$-vector operator whose first component is a $\mathfrak{g}$-$\mathfrak{T}$-derived operator and second component is a $\mathfrak{g}$-$\mathfrak{T}$-coderived operator of category ν in a $\mathscr{T}$-space $\mathfrak{T} = (\Omega, \mathscr{T})$ and $\mathbf{dc} \stackrel{\text{def}}{=} (\text{der}, \text{cod})$, a $\mathfrak{T}$-vector operator whose first component is a $\mathfrak{T}$-derived operator and second component is a $\mathfrak{T}$-coderived operator in a $\mathscr{T}$-space $\mathfrak{T} = (\Omega, \mathscr{T})$. Accordingly,

$$\begin{aligned} \mathfrak{g}\text{-DC}[\mathfrak{T}] &\stackrel{\text{def}}{=} \left\{\mathfrak{g}\text{-}\mathbf{Dc}_{\nu} = (\mathfrak{g}\text{-Der}_{\nu}, \mathfrak{g}\text{-Cod}_{\nu}) :\ \nu \in I_3^0\right\} \\ &\subseteq \left\{\mathfrak{g}\text{-Der}_{\nu} :\ \nu \in I_3^0\right\} \times \left\{\mathfrak{g}\text{-Cod}_{\nu} :\ \nu \in I_3^0\right\} \stackrel{\text{def}}{=} \mathfrak{g}\text{-DE}[\mathfrak{T}] \times \mathfrak{g}\text{-CD}[\mathfrak{T}]. \end{aligned} \tag{2.6}$$

Then, $\mathfrak{g}\text{-DC}[\mathfrak{T}]$ denotes the class of all $\mathfrak{g}$-$\mathfrak{T}$-vector operators in the $\mathscr{T}$-space $\mathfrak{T} = (\Omega, \mathscr{T})$; $\mathfrak{g}\text{-DE}[\mathfrak{T}]$ denotes the class of all $\mathfrak{g}$-$\mathfrak{T}$-derived operators while $\mathfrak{g}\text{-CD}[\mathfrak{T}]$ denotes the class of all $\mathfrak{g}$-$\mathfrak{T}$-coderived operators in the $\mathscr{T}$-space $\mathfrak{T} = (\Omega, \mathscr{T})$.

For any $(\mathscr{S}_{\mathfrak{g}}, \mathfrak{g}\text{-Ope}_{\mathfrak{g}}) \in \mathscr{P}(\Omega) \times \{\mathfrak{g}\text{-Der}_{\mathfrak{g}}, \mathfrak{g}\text{-Cod}_{\mathfrak{g}}\}$, consider the following description:

$$\begin{array}{rcl} 0 & \longleftrightarrow & \mathfrak{g}\text{-Ope}_{\mathfrak{g}}^{(0)}(\mathscr{S}_{\mathfrak{g}}) \stackrel{\text{def}}{=} \bigcirc_{\alpha\in I_0^0} \mathfrak{g}\text{-Ope}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}), \\ 1 & \longleftrightarrow & \mathfrak{g}\text{-Ope}_{\mathfrak{g}}^{(1)}(\mathscr{S}_{\mathfrak{g}}) \stackrel{\text{def}}{=} \bigcirc_{\alpha\in I_1^0} \mathfrak{g}\text{-Ope}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}), \\ 2 & \longleftrightarrow & \mathfrak{g}\text{-Ope}_{\mathfrak{g}}^{(2)}(\mathscr{S}_{\mathfrak{g}}) \stackrel{\text{def}}{=} \bigcirc_{\alpha\in I_2^0} \mathfrak{g}\text{-Ope}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}), \\ & \vdots & \\ \beta-1 & \longleftrightarrow & \mathfrak{g}\text{-Ope}_{\mathfrak{g}}^{(\beta-1)}(\mathscr{S}_{\mathfrak{g}}) \stackrel{\text{def}}{=} \bigcirc_{\alpha\in I_{\beta-1}^0} \mathfrak{g}\text{-Ope}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}), \\ \beta & \longleftrightarrow & \mathfrak{g}\text{-Ope}_{\mathfrak{g}}^{(\beta)}(\mathscr{S}_{\mathfrak{g}}) \stackrel{\text{def}}{=} \bigcirc_{\alpha\in I_\beta^0} \mathfrak{g}\text{-Ope}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}), \end{array} \tag{2.7}$$

where $\bigcirc_{\alpha\in I_0^0} \mathfrak{g}\text{-Ope}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \longleftrightarrow \mathscr{S}_{\mathfrak{g}}$; next, $\bigcirc_{\alpha\in I_1^0} \mathfrak{g}\text{-Ope}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \longleftrightarrow \mathfrak{g}\text{-Ope}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$ and $\bigcirc_{\alpha\in I_2^0} \mathfrak{g}\text{-Ope}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \longleftrightarrow \mathfrak{g}\text{-Ope}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Ope}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$; more generally,

$$\bigcirc_{\alpha\in I_\beta^0} \mathfrak{g}\text{-Ope}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \longleftrightarrow \mathfrak{g}\text{-Ope}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Ope}_{\mathfrak{g}} \circ \cdots \circ \mathfrak{g}\text{-Ope}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}),$$

β factors $\mathfrak{g}\text{-Ope}_{\mathfrak{g}}$. Thus, $\mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(0)}, \mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(1)}, \mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(2)}, \ldots, \mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\beta)}, \ldots : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ are the 0^{th}, 1^{st}, 2^{nd}, …, β^{th}, … *order derivative* $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-*derived operators* of $\mathfrak{g}\text{-Der}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$; $\mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(0)}, \mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(1)}, \mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(2)}, \ldots, \mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\beta)}, \ldots :$

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$\mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ are the 0^{th}, 1^{st}, 2^{nd}, …, β^{th}, … *order derivative* $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-*coderived operators* of $\mathfrak{g}\text{-Cod}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$. Then, for any pair $\left(\mathscr{S}_{\mathfrak{g}}, \mathfrak{g}\text{-Ope}_{\mathfrak{g}}\right) \in \mathscr{P}(\Omega) \times \left\{\mathfrak{g}\text{-Der}_{\mathfrak{g}}, \mathfrak{g}\text{-Cod}_{\mathfrak{g}}\right\}$, the following statement holds:

$$\left[\left(\exists \beta \in I_{\infty}^{0}\right)\left(\mathfrak{g}\text{-Ope}_{\mathfrak{g}}^{(\beta)}\left(\mathscr{S}_{\mathfrak{g}}\right) = \emptyset\right)\right] \veebar \left[\left(\forall \beta \in I_{\infty}^{0}\right)\left(\mathfrak{g}\text{-Ope}_{\mathfrak{g}}^{(\beta)}\left(\mathscr{S}_{\mathfrak{g}}\right) \neq \emptyset\right)\right]$$

Suppose the statement preceding $\veebar$ hold, then the number of iterations of the $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-operator $\mathfrak{g}\text{-Ope}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$, respectively, required to achieve *emptiness* (if this is ever achieved) is a type of *density measure* of $\mathscr{S}_{\mathfrak{g}} \in \mathscr{P}(\Omega)$. But if the statement following $\veebar$ holds, then $\mathscr{S}_{\mathfrak{g}}^{(\lambda)} \stackrel{\text{def}}{=} \bigcap_{\beta \in I_{\infty}^{*}} \mathfrak{g}\text{-Ope}_{\mathfrak{g}}^{(\beta)}\left(\mathscr{S}_{\mathfrak{g}}\right) \neq \emptyset$. Therefore, the $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-operators $\mathfrak{g}\text{-Ope}_{\mathfrak{g}}^{(1)}$, $\mathfrak{g}\text{-Ope}_{\mathfrak{g}}^{(2)}$, …, $\mathfrak{g}\text{-Ope}_{\mathfrak{g}}^{(\beta)}$, … $: \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ can again be applied on $\mathscr{S}_{\mathfrak{g}}^{(\omega)} \in \mathscr{P}(\Omega)$, yielding $\mathfrak{g}\text{-Ope}_{\mathfrak{g}}^{(\lambda+1)}\left(\mathscr{S}_{\mathfrak{g}}\right)$, $\mathfrak{g}\text{-Ope}_{\mathfrak{g}}^{(\lambda+2)}\left(\mathscr{S}_{\mathfrak{g}}\right)$, …, $\mathfrak{g}\text{-Ope}_{\mathfrak{g}}^{(\lambda+\beta)}\left(\mathscr{S}_{\mathfrak{g}}\right)$, …, with $\mathfrak{g}\text{-Ope}_{\mathfrak{g}} \in \left\{\mathfrak{g}\text{-Der}_{\mathfrak{g}}, \mathfrak{g}\text{-Cod}_{\mathfrak{g}}\right\}$.

In view of the above descriptions, 1, 2, …, β, … may be viewed as *successor ordinals* while λ as *limit ordinal* and, despite the absence of a *predecessor ordinal*, 0 may, for conveniency, be included in the class of successor ordinals. To define the notion of *ordinal*, the concepts of *everywhere-ordered set*, *similarity* and *order-type* in chronological order have first to be defined. The definition of the first concept (everywhere-ordered set) follows.

Definition 2.5 (Everywhere-Ordered Set). An "everywhere-ordered set" is an ordered structure of the type

$$\mathfrak{W} \stackrel{\text{def}}{=} (\mathscr{W}, \preccurlyeq) \stackrel{\text{def}}{\longleftrightarrow} \langle \alpha_0, \alpha_1, \alpha_2, \ldots, \alpha_\nu, \ldots \rangle, \tag{2.8}$$

in which $\mathscr{W} \subset \mathfrak{U}$ is an "underlying set" and,

$$\begin{aligned} \preccurlyeq : \mathscr{W} \times \mathscr{W} &\longrightarrow \mathbb{W} \stackrel{\text{def}}{=} \{\alpha \preccurlyeq \beta : (\alpha, \beta) \in \mathscr{W} \times \mathscr{W}\} \\ (\alpha, \beta) &\longmapsto \alpha \preccurlyeq \beta \end{aligned} \tag{2.9}$$

is a "2-ary rule" satisfying these "everywhere-ordering relation axioms:"

- Ax. i. $\alpha \preccurlyeq \alpha \longleftrightarrow \alpha = \alpha \quad \forall \alpha \in \mathfrak{W}$,
- Ax. ii. $(\alpha \preccurlyeq \beta) \wedge (\beta \preccurlyeq \alpha) \longrightarrow \alpha = \beta \quad \forall (\alpha, \beta) \in \mathfrak{W} \times \mathfrak{W}$,
- Ax. iii. $(\alpha \preccurlyeq \beta) \wedge (\beta \preccurlyeq \gamma) \longrightarrow \alpha \preccurlyeq \gamma \quad \forall (\alpha, \beta, \gamma) \in \mathfrak{W} \times \mathfrak{W} \times \mathfrak{W}$,
- Ax. iv. $\mathfrak{V} \stackrel{\text{def}}{=} (\mathscr{V}, \preccurlyeq) \stackrel{\text{def}}{\longleftrightarrow} \langle \beta_0, \beta_1, \beta_2, \ldots, \beta_\nu, \ldots \rangle \longrightarrow \beta_0 \preccurlyeq \beta_1 \preccurlyeq \beta_2 \preccurlyeq \cdots \preccurlyeq \beta_\nu \preccurlyeq \cdots \quad \forall \mathfrak{V} \subseteq \mathfrak{W}$.

The above definition requires some few explanations. By Ax. i., Ax. ii. and Ax. iii. are meant that $\preccurlyeq : \mathscr{W} \times \mathscr{W} \longrightarrow \mathbb{W}$ is *reflexive*, *antisymmetric* and *transitive*, respectively; by Ax. iv. is meant that any ordered structure $\mathfrak{V} = (\mathscr{V}, \preccurlyeq)$ derived from $\mathfrak{W} = (\mathscr{W}, \preccurlyeq)$ has a *first* element (*i.e.*, $\beta_0 \in \mathfrak{V} \subseteq \mathfrak{W}$). Moreover, the following statement holds true:

$$\left(\forall (\alpha, \beta) \in \mathfrak{W} \times \mathfrak{W}\right)\left[(\alpha \preccurlyeq \beta) \veebar (\beta = \alpha) \veebar (\beta \preccurlyeq \alpha)\right]. \tag{2.10}$$

Thus, given $(\alpha, \beta) \in \mathfrak{W} \times \mathfrak{W}$ then, either α *preceeds* β (*i.e.*, $\alpha \preccurlyeq \beta$), α *is of the same order as* β (*i.e.*, $\beta = \alpha$) or α *succeeds* β (*i.e.*, $\beta \preccurlyeq \alpha$). The remark below is presented in order to avoid any danger of confusing the notations of *underlying* (not ordered) and *everywhere-ordered* sets.

Remark 2.6. Instead of such *plain sets* notations as $\alpha \in \mathscr{W}$, $(\alpha, \beta) \in \mathscr{W} \times \mathscr{W}$, … which, in actual fact, are improper, the *ordered sets* notations $\alpha \in \mathfrak{W}$, $(\alpha, \beta) \in \mathfrak{W} \times \mathfrak{W}$, … are employed solely to stress that α, β, … are elements of their *ordered*

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set $\mathfrak{W}$, not of the *underlying set* $\mathscr{W}$ of the ordered set $\mathfrak{W}$. Indeed, in the present context, it does not hold that $\langle \alpha_0, \alpha_1, \alpha_2, \ldots, \alpha_\nu, \ldots \rangle \neq \{\alpha_0, \alpha_1, \alpha_2, \ldots, \alpha_\nu, \ldots\}$, though it does hold that $\{\alpha : \alpha \in \mathfrak{W}\} = \{\alpha : \alpha \in \mathscr{W}\}$.

For each $\mathscr{U} \in \{\mathscr{V}, \mathscr{W}\}$, set $\mathbb{W}_{\mathscr{U}} = \left\{\alpha \preccurlyeq_{\mathscr{U}} \beta : (\alpha, \beta) \in \mathscr{W}_{\mathscr{U}} \times \mathscr{W}_{\mathscr{U}}\right\}$. Then, the second concept (similarity) may be defined as thus.

Definition 2.7 (Similarity). The everywhere-ordered sets $\mathfrak{V} = (\mathscr{V}, \preccurlyeq_{\mathscr{V}})$ and $\mathfrak{W} = (\mathscr{W}, \preccurlyeq_{\mathscr{W}})$, where $\preccurlyeq_{\mathscr{V}} : \mathscr{V} \times \mathscr{V} \longrightarrow \mathbb{W}_{\mathscr{V}}$ and $\preccurlyeq_{\mathscr{W}} : \mathscr{W} \times \mathscr{W} \longrightarrow \mathbb{W}_{\mathscr{W}}$, respectively, are "similar," written $\mathfrak{V} \approx \mathfrak{W}$, if and only if there is an "order isomorphism" $\varphi : \mathfrak{V} \cong \mathfrak{W}$ relating the elements α_0, α_1, α_2, …, α_ν, … of $\mathfrak{V}$ to the elements β_0, β_1, β_2, …, β_ν, … of $\mathfrak{W}$ as:

$$\begin{array}{ccc} \mathfrak{V} = (\mathscr{V}, \preccurlyeq_{\mathscr{V}}) & \stackrel{\text{def}}{\longleftrightarrow} & \langle \alpha_0,\ \alpha_1,\ \alpha_2,\ \ldots,\ \alpha_\nu,\ \ldots \rangle \\ \wr\wr & & \updownarrow \varphi \\ \mathfrak{W} = (\mathscr{W}, \preccurlyeq_{\mathscr{W}}) & \stackrel{\text{def}}{\longleftrightarrow} & \langle \beta_0,\ \beta_1,\ \beta_2,\ \ldots,\ \beta_\nu,\ \ldots \rangle . \end{array} \tag{2.11}$$

From this definition, given $\mathfrak{Y} = (\mathscr{Y}, \preccurlyeq_{\mathscr{Y}}) \longleftrightarrow \langle \gamma_0,\ \gamma_1,\ \gamma_2,\ \ldots,\ \gamma_\nu,\ \ldots \rangle$ with $(\mathfrak{Y}, \mathscr{Y}, \gamma) \in \{(\mathfrak{V}, \mathscr{V}, \alpha), (\mathfrak{W}, \mathscr{W}, \beta)\}$ and $\varphi : \mathfrak{V} \cong \mathfrak{W}$, then $\alpha_0 \preccurlyeq_{\mathscr{V}} \alpha_1 \stackrel{\varphi}{\longrightarrow} \beta_0 \preccurlyeq_{\mathscr{W}} \beta_1$, $\alpha_1 \preccurlyeq_{\mathscr{V}} \alpha_2 \stackrel{\varphi}{\longrightarrow} \beta_1 \preccurlyeq_{\mathscr{W}} \beta_2$, …, $\alpha_{\nu-1} \preccurlyeq_{\mathscr{V}} \alpha_\nu \stackrel{\varphi}{\longrightarrow} \beta_{\nu-1} \preccurlyeq_{\mathscr{W}} \alpha_\nu$, …. For any $(\mathfrak{V}, \mathfrak{W}, \mathfrak{Y}) \in \times_{\mu \in I_3^*} \left\{\mathfrak{W}_\nu = (\mathscr{W}_\nu, \preccurlyeq_\nu) : \nu \in I_\infty^*\right\}$, the relations $\mathfrak{V} \approx \mathfrak{V}$, $\mathfrak{V} \approx \mathfrak{W} \longleftrightarrow \mathfrak{W} \approx \mathfrak{V}$ and $(\mathfrak{V} \approx \mathfrak{W}) \wedge (\mathfrak{W} \approx \mathfrak{Y}) \longrightarrow (\mathfrak{V} \approx \mathfrak{Y})$ hold. Therefore, the relation of similarity $\approx : (\mathfrak{V}, \mathfrak{W}) \longmapsto \mathfrak{V} \approx \mathfrak{W}$ is *reflexive*, *symmetrical* and *transitive*.

The definition of the third concept (order-type) may be stated as thus.

Definition 2.8 (Order-Type). An operator of the type

$$\mathrm{OTyp} : \mathfrak{W} \longmapsto \mathrm{OTyp}(\mathfrak{W}) \stackrel{\text{def}}{=} \tau_{\mathscr{W}} \tag{2.12}$$

assigning to any everywhere-ordered set $\mathfrak{W} = (\mathscr{W}, \preccurlyeq_{\mathscr{W}})$ a uniquely determined symbol $\tau_{\mathscr{W}}$ is called the "order-type" of $\mathfrak{W}$, provided that if $\mathfrak{V} = (\mathscr{V}, \preccurlyeq_{\mathscr{V}})$ be any other everywhere-ordered set together with its uniquely determined order-type $\mathrm{OTyp}(\mathfrak{V}) \stackrel{\text{def}}{=} \tau_{\mathscr{V}}$, the following statement holds:

$$\mathfrak{V} \approx \mathfrak{W} \longleftrightarrow \tau_{\mathscr{V}} = \tau_{\mathscr{W}} . \tag{2.13}$$

Clearly, the manner of proceeding from the relation of similarity to the concept of order-type is exactly the same as that from the relation of equivalence to the concept of cardinal number. For, given any $\mathfrak{V} = (\mathscr{V}, \preccurlyeq_{\mathscr{V}})$ and $\mathfrak{W} = (\mathscr{W}, \preccurlyeq_{\mathscr{W}})$, then $\mathfrak{V} \approx \mathfrak{W} \longleftrightarrow \mathrm{OTyp}(\mathfrak{V}) = \mathrm{OTyp}(\mathfrak{W})$ is analogous to $\mathscr{V} \sim \mathscr{W} \longleftrightarrow \mathrm{card}(\mathscr{V}) = \mathrm{card}(\mathscr{W})$.

Remark 2.9. By $\mathfrak{V} \approx \mathfrak{W} \longleftrightarrow \tau_{\mathscr{V}} = \tau_{\mathscr{W}}$ is meant that a uniquely determined symbol actually is assigned not to a single set but to a class of everywhere-ordered sets which are similar to each other.

Granted the definitions of the concepts of *everywhere-ordered set*, *similarity* and *order-type*, the definition of the concept of *ordinal* may be stated as thus.

Definition 2.10 (Ordinal). The order-type $\mathrm{OTyp}(\mathfrak{W}) = \tau_{\mathscr{W}}$ of an everywhere-ordered set $\mathfrak{W} = (\mathscr{W}, \preccurlyeq_{\mathscr{W}})$ is called "ordinal," written $\mathrm{ord}(\mathfrak{W}) \stackrel{\text{def}}{=} \delta_{\mathscr{W}}$. Moreover:

- i. $\delta_{\mathscr{W}}$ is called a "predecessor ordinal" if and only if there exists no ordinal $\mathrm{ord}(\mathfrak{W})$ such that $\delta_{\mathscr{W}} = \mathrm{ord}(\mathfrak{W}) + 1$.

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- II. $\delta_{\mathscr{W}}$ is called a "successor ordinal" if and only if there exists an ordinal $\operatorname{ord}(\mathfrak{W})$ such that $\delta_{\mathscr{W}} = \operatorname{ord}(\mathfrak{W}) + 1$.
- III. $\delta_{\mathscr{W}}$ is called a "limit ordinal," denoted as $\delta_{\mathscr{W}} \stackrel{\text{def}}{=} \lambda_{\mathscr{W}}$, if and only if it has no immediate predecessor.

Let the symbols 0, δ, and λ (instead of the symbols $0_{\mathscr{W}}$, $\delta_{\mathscr{W}}$, and $\lambda_{\mathscr{W}}$) stand for *predecessor ordinal*, *successor ordinal* and *limit ordinal*, respectively. Then, the definitions of the notions of *ordered derivative* $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$*-derived and* $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$*-coderived operators* of $\mathfrak{g}\text{-Der}_{\mathfrak{g}}, \mathfrak{g}\text{-Cod}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$, respectively, may well be stated as thus.

Definition 2.11 (δ^{th}-Iterations: $\mathfrak{g}$-ν-$\mathfrak{T}_{\mathfrak{g}}$-Derived, $\mathfrak{g}$-ν-$\mathfrak{T}_{\mathfrak{g}}$-Coderived Operators). Let $\mathfrak{g}\text{-Der}_{\mathfrak{g},\nu}, \mathfrak{g}\text{-Cod}_{\mathfrak{g},\nu} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$, respectively, be a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-derived and a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-coderived operators of category ν in a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$. Then:

- I. The "δ^{th}-iterate of $\mathfrak{g}\text{-Der}_{\mathfrak{g},\nu} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$" is a set-valued map $\mathfrak{g}\text{-Der}^{(\delta)}_{\mathfrak{g},\nu} : \mathscr{S}_{\mathfrak{g}} \in \mathscr{P}(\Omega) \longmapsto \mathfrak{g}\text{-Der}^{(\delta)}_{\mathfrak{g},\nu}(\mathscr{S}_{\mathfrak{g}})$ defined by transfinite recursion on the class of successor ordinals as,
  - (*i.*) $\mathfrak{g}\text{-Der}^{(0)}_{\mathfrak{g},\nu}(\mathscr{S}_{\mathfrak{g}}) \stackrel{\text{def}}{\longleftrightarrow} \mathscr{S}_{\mathfrak{g}}$,
  - (*ii.*) $\mathfrak{g}\text{-Der}^{(1)}_{\mathfrak{g},\nu}(\mathscr{S}_{\mathfrak{g}}) \stackrel{\text{def}}{\longleftrightarrow} \mathfrak{g}\text{-Der}_{\mathfrak{g},\nu}(\mathscr{S}_{\mathfrak{g}})$,
  - (*iii.*) $\mathfrak{g}\text{-Der}^{(\delta+1)}_{\mathfrak{g},\nu}(\mathscr{S}_{\mathfrak{g}}) \stackrel{\text{def}}{\longleftrightarrow} \mathfrak{g}\text{-Der}_{\mathfrak{g},\nu} \circ \mathfrak{g}\text{-Der}^{(\delta)}_{\mathfrak{g},\nu}(\mathscr{S}_{\mathfrak{g}})$,
  - (*iv.*) $\mathfrak{g}\text{-Der}^{(\lambda)}_{\mathfrak{g},\nu}(\mathscr{S}_{\mathfrak{g}}) \stackrel{\text{def}}{\longleftrightarrow} \bigcap_{\delta \prec \lambda} \mathfrak{g}\text{-Der}^{(\delta)}_{\mathfrak{g},\nu}(\mathscr{S}_{\mathfrak{g}})$.
- II. The "δ^{th}-iterate of $\mathfrak{g}\text{-Cod}_{\mathfrak{g},\nu} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$" is a set-valued map $\mathfrak{g}\text{-Cod}^{(\delta)}_{\mathfrak{g},\nu} : \mathscr{S}_{\mathfrak{g}} \in \mathscr{P}(\Omega) \longmapsto \mathfrak{g}\text{-Cod}^{(\delta)}_{\mathfrak{g},\nu}(\mathscr{S}_{\mathfrak{g}})$ defined by transfinite recursion on the class of successor ordinals as,
  - (*i.*) $\mathfrak{g}\text{-Cod}^{(0)}_{\mathfrak{g},\nu}(\mathscr{S}_{\mathfrak{g}}) \stackrel{\text{def}}{\longleftrightarrow} \mathscr{S}_{\mathfrak{g}}$,
  - (*ii.*) $\mathfrak{g}\text{-Cod}^{(1)}_{\mathfrak{g},\nu}(\mathscr{S}_{\mathfrak{g}}) \stackrel{\text{def}}{\longleftrightarrow} \mathfrak{g}\text{-Cod}_{\mathfrak{g},\nu}(\mathscr{S}_{\mathfrak{g}})$,
  - (*iii.*) $\mathfrak{g}\text{-Cod}^{(\delta+1)}_{\mathfrak{g},\nu}(\mathscr{S}_{\mathfrak{g}}) \stackrel{\text{def}}{\longleftrightarrow} \mathfrak{g}\text{-Cod}_{\mathfrak{g},\nu} \circ \mathfrak{g}\text{-Cod}^{(\delta)}_{\mathfrak{g},\nu}(\mathscr{S}_{\mathfrak{g}})$,
  - (*iv.*) $\mathfrak{g}\text{-Cod}^{(\lambda)}_{\mathfrak{g},\nu}(\mathscr{S}_{\mathfrak{g}}) \stackrel{\text{def}}{\longleftrightarrow} \bigcap_{\delta \prec \lambda} \mathfrak{g}\text{-Cod}^{(\delta)}_{\mathfrak{g},\nu}(\mathscr{S}_{\mathfrak{g}})$.

In the following remark, the concepts of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-derived and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-coderived sets of category ν and order δ are presented.

Remark 2.12. Suppose $(\mathscr{R}^{(\delta)}_{\mathfrak{g}}, \mathscr{S}_{\mathfrak{g}}) \in \times_{\alpha \in I^{*}_{2}} \mathscr{P}(\Omega)$ such that $\mathscr{R}^{(\delta)}_{\mathfrak{g}} = \mathfrak{g}\text{-Der}^{(\delta)}_{\mathfrak{g},\nu}(\mathscr{S}_{\mathfrak{g}})$ for some ordinal δ, then $\mathscr{R}^{(\delta)}_{\mathfrak{g}}$ may be called a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$*-derived set of* $\mathscr{S}_{\mathfrak{g}}$ *of category* ν *and order* δ. Likewise, given $(\mathscr{U}^{(\delta)}_{\mathfrak{g}}, \mathscr{V}_{\mathfrak{g}}) \in \times_{\alpha \in I^{*}_{2}} \mathscr{P}(\Omega)$ such that $\mathscr{U}^{(\delta)}_{\mathfrak{g}} = \mathfrak{g}\text{-Cod}^{(\delta)}_{\mathfrak{g},\nu}(\mathscr{V}_{\mathfrak{g}})$ for some ordinal δ, then $\mathscr{U}^{(\delta)}_{\mathfrak{g}}$ may be called a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$*-coderived set of* $\mathscr{V}_{\mathfrak{g}}$ *of category* ν *and order* δ. Hence, any $\{\xi\} \in \mathscr{P}(\Omega)$ such that $(\xi \in \mathscr{R}^{(\delta)}_{\mathfrak{g}} \in \mathscr{P}(\Omega)) \wedge (\xi \notin \mathscr{R}^{(\delta+1)}_{\mathfrak{g}} \in \mathscr{P}(\Omega))$ may be called a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$*-derived unit set of* $\mathscr{R}_{\mathfrak{g}}$ *of category* ν *and order* δ, and any $\{\zeta\} \in \mathscr{P}(\Omega)$ such that $(\zeta \in \mathscr{U}^{(\delta)}_{\mathfrak{g}} \in \mathscr{P}(\Omega)) \wedge (\zeta \notin \mathscr{U}^{(\delta+1)}_{\mathfrak{g}} \in \mathscr{P}(\Omega))$ may be called a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$*-coderived unit set of* $\mathscr{U}_{\mathfrak{g}}$ *of category* ν *and order* δ.

Evidently, the use of $\text{der}_{\mathfrak{g}}$, $\mathfrak{g}\text{-Der}_{\nu}$, $\text{der} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ instead of $\mathfrak{g}\text{-Der}_{\mathfrak{g},\nu} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ introduce the notions of $\mathfrak{T}_{\mathfrak{g}}$*-derived set of* $\mathscr{S}_{\mathfrak{g}}$ *of order* δ, $\mathfrak{g}$-$\mathfrak{T}$*-derived set of* $\mathscr{S}_{\mathfrak{g}}$ *of category* ν *and order* δ, and $\mathfrak{T}$*-derived set of* $\mathscr{S}_{\mathfrak{g}}$ *of order* δ,

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respectively; the use of $\mathrm{doc}_{\mathfrak{g}}$, $\mathfrak{g}\text{-}\mathrm{Cod}_{\nu}$, $\mathrm{cod} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ instead of $\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g},\nu} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ introduce the notions of *$\mathfrak{T}_{\mathfrak{g}}$-coderived set of $\mathscr{S}_{\mathfrak{g}}$ of order δ*, *$\mathfrak{g}$-$\mathfrak{T}$-coderived set of $\mathscr{S}_{\mathfrak{g}}$ of category ν and order δ*, and *$\mathfrak{T}$-coderived set of $\mathscr{S}_{\mathfrak{g}}$ of order δ*, respectively.

Definition 2.13 ($\mathfrak{g}$-ν-$\mathfrak{T}_{\mathfrak{g}}$-Open, $\mathfrak{g}$-ν-$\mathfrak{T}_{\mathfrak{g}}$-Closed Sets of Rank δ). Let $\mathscr{S}_{\mathfrak{g}} \subset \mathfrak{T}_{\mathfrak{g}}$ be arbitrary and let $\mathfrak{g}\text{-}\mathrm{Der}^{(\delta)}_{\mathfrak{g},\nu}$, $\mathfrak{g}\text{-}\mathrm{Cod}^{(\delta)}_{\mathfrak{g},\nu} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ be the δ^{th}-iterates of $\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g},\nu}$, $\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g},\nu} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$, respectively, in a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$. Then:

- I. $\mathscr{S}_{\mathfrak{g}}$ is said to be "$\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-closed of category ν and rank δ" if and only if:

$$\mathscr{S}_{\mathfrak{g}} \supseteq \mathfrak{g}\text{-}\mathrm{Cl}^{(\delta)}_{\mathfrak{g},\nu}(\mathscr{S}_{\mathfrak{g}}) \stackrel{\text{def}}{=} \bigcup_{\mathscr{W}_{\mathfrak{g}} \in \mathfrak{g}\text{-}\nu\text{-}\mathrm{DE}^{(\delta)}[\mathscr{S}_{\mathfrak{g}};\mathfrak{T}_{\mathfrak{g}}]} \mathscr{W}_{\mathfrak{g}}, \tag{2.14}$$

  where $\mathfrak{g}\text{-}\nu\text{-}\mathrm{DE}^{(\delta)}[\mathscr{S}_{\mathfrak{g}};\mathfrak{T}_{\mathfrak{g}}] \stackrel{\text{def}}{=} \{\mathscr{V}_{\mathfrak{g}} \subseteq \mathscr{S}_{\mathfrak{g}} : (\exists \mathscr{K}_{\mathfrak{g}})[\mathscr{V}_{\mathfrak{g}} = \mathfrak{g}\text{-}\mathrm{Der}^{(\delta)}_{\mathfrak{g},\nu}(\mathscr{K}_{\mathfrak{g}})]\}$.
- II. $\mathscr{S}_{\mathfrak{g}}$ is said to be "$\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-open of category ν and rank δ" if and only if:

$$\mathscr{S}_{\mathfrak{g}} \subseteq \mathfrak{g}\text{-}\mathrm{Int}^{(\delta)}_{\mathfrak{g},\nu}(\mathscr{S}_{\mathfrak{g}}) \stackrel{\text{def}}{=} \bigcap_{\mathscr{W}_{\mathfrak{g}} \in \mathfrak{g}\text{-}\nu\text{-}\mathrm{CD}^{(\delta)}[\mathscr{S}_{\mathfrak{g}};\mathfrak{T}_{\mathfrak{g}}]} \mathscr{W}_{\mathfrak{g}}, \tag{2.15}$$

  where $\mathfrak{g}\text{-}\nu\text{-}\mathrm{CD}^{(\delta)}[\mathscr{S}_{\mathfrak{g}};\mathfrak{T}_{\mathfrak{g}}] \stackrel{\text{def}}{=} \{\mathscr{U}_{\mathfrak{g}} \supseteq \mathscr{S}_{\mathfrak{g}} : (\exists \mathscr{O}_{\mathfrak{g}})[\mathscr{U}_{\mathfrak{g}} = \mathfrak{g}\text{-}\mathrm{Cod}^{(\delta)}_{\mathfrak{g},\nu}(\mathscr{O}_{\mathfrak{g}})]\}$.

From this definition, various notions of derivedness and coderivedness can be derived in $\mathscr{T}$, $\mathscr{T}_{\mathfrak{g}}$-spaces. To establish *$\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-set of rank δ* in $\mathfrak{T}_{\mathfrak{g}}$, set

$$\begin{aligned}
\mathfrak{g}\text{-}\mathrm{DE}^{(\delta)}[\mathscr{S}_{\mathfrak{g}};\mathfrak{T}_{\mathfrak{g}}] &= \bigcup_{\nu \in I_3^0} \{\mathscr{V}_{\mathfrak{g}} \subseteq \mathscr{S}_{\mathfrak{g}} : (\exists \mathscr{K}_{\mathfrak{g}})[\mathscr{V}_{\mathfrak{g}} = \mathfrak{g}\text{-}\mathrm{Der}^{(\delta)}_{\mathfrak{g},\nu}(\mathscr{K}_{\mathfrak{g}})]\} \\
&= \left\{\mathscr{V}_{\mathfrak{g}} \subseteq \mathscr{S}_{\mathfrak{g}} : (\exists \mathscr{K}_{\mathfrak{g}})\left[\bigvee_{\nu \in I_3^0} (\mathscr{V}_{\mathfrak{g}} = \mathfrak{g}\text{-}\mathrm{Der}^{(\delta)}_{\mathfrak{g},\nu}(\mathscr{K}_{\mathfrak{g}}))\right]\right\} \\
&= \bigcup_{\nu \in I_3^0} \mathfrak{g}\text{-}\nu\text{-}\mathrm{DE}^{(\delta)}[\mathscr{S}_{\mathfrak{g}};\mathfrak{T}_{\mathfrak{g}}].
\end{aligned}$$

Then, $\mathscr{S}_{\mathfrak{g}}$ is said to be *$\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-closed of rank δ* in $\mathfrak{T}_{\mathfrak{g}}$ if and only if:

$$\mathscr{S}_{\mathfrak{g}} \supseteq \mathfrak{g}\text{-}\mathrm{Cl}^{(\delta)}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \stackrel{\text{def}}{=} \bigcup_{\mathscr{W}_{\mathfrak{g}} \in \mathfrak{g}\text{-}\mathrm{DE}^{(\delta)}[\mathscr{S}_{\mathfrak{g}};\mathfrak{T}_{\mathfrak{g}}]} \mathscr{W}_{\mathfrak{g}}. \tag{2.16}$$

Similarly, set

$$\begin{aligned}
\mathfrak{g}\text{-}\mathrm{CD}^{(\delta)}[\mathscr{S}_{\mathfrak{g}};\mathfrak{T}_{\mathfrak{g}}] &= \bigcup_{\nu \in I_3^0} \{\mathscr{U}_{\mathfrak{g}} \supseteq \mathscr{S}_{\mathfrak{g}} : (\exists \mathscr{O}_{\mathfrak{g}})[\mathscr{U}_{\mathfrak{g}} = \mathfrak{g}\text{-}\mathrm{Cod}^{(\delta)}_{\mathfrak{g},\nu}(\mathscr{O}_{\mathfrak{g}})]\} \\
&= \left\{\mathscr{U}_{\mathfrak{g}} \supseteq \mathscr{S}_{\mathfrak{g}} : (\exists \mathscr{O}_{\mathfrak{g}})\left[\bigvee_{\nu \in I_3^0} (\mathscr{U}_{\mathfrak{g}} = \mathfrak{g}\text{-}\mathrm{Cod}^{(\delta)}_{\mathfrak{g},\nu}(\mathscr{O}_{\mathfrak{g}}))\right]\right\} \\
&= \bigcup_{\nu \in I_3^0} \mathfrak{g}\text{-}\nu\text{-}\mathrm{CD}^{(\delta)}[\mathscr{S}_{\mathfrak{g}};\mathfrak{T}_{\mathfrak{g}}].
\end{aligned}$$

Then, $\mathscr{S}_{\mathfrak{g}}$ is said to be *$\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-open of rank δ* in $\mathfrak{T}_{\mathfrak{g}}$ if and only if:

$$\mathscr{S}_{\mathfrak{g}} \subseteq \mathfrak{g}\text{-}\mathrm{Int}^{(\delta)}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \stackrel{\text{def}}{=} \bigcap_{\mathscr{W}_{\mathfrak{g}} \in \mathfrak{g}\text{-}\mathrm{CD}^{(\delta)}[\mathscr{S}_{\mathfrak{g}};\mathfrak{T}_{\mathfrak{g}}]} \mathscr{W}_{\mathfrak{g}}. \tag{2.17}$$

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To establish $\mathfrak{g}$-$\mathfrak{T}$-*set of rank* δ in $\mathfrak{T}$, set

$$\begin{aligned}\mathfrak{g}\text{-}\mathrm{DE}^{(\delta)}[\mathscr{S};\mathfrak{T}] &= \bigcup_{\nu\in I_3^0}\{\mathscr{V}\subseteq\mathscr{S}:\ (\exists\mathscr{K})[\mathscr{V}=\mathfrak{g}\text{-}\mathrm{Der}_{\nu}^{(\delta)}(\mathscr{K})]\}\\ &= \left\{\mathscr{V}\subseteq\mathscr{S}:\ (\exists\mathscr{K})\left[\bigvee_{\nu\in I_3^0}(\mathscr{V}=\mathfrak{g}\text{-}\mathrm{Der}_{\nu}^{(\delta)}(\mathscr{K}))\right]\right\}\\ &= \bigcup_{\nu\in I_3^0}\mathfrak{g}\text{-}\nu\text{-}\mathrm{DE}^{(\delta)}[\mathscr{S};\mathfrak{T}].\end{aligned}$$

Then, $\mathscr{S}$ is said to be $\mathfrak{g}$-$\mathfrak{T}$-*closed of rank* δ in $\mathfrak{T}$ if and only if:

$$\mathscr{S}\supseteq\mathfrak{g}\text{-}\mathrm{Cl}^{(\delta)}(\mathscr{S})\stackrel{\text{def}}{=}\bigcup_{\mathscr{W}\in\mathfrak{g}\text{-}\mathrm{DE}^{(\delta)}[\mathscr{S};\mathfrak{T}]}\mathscr{W}. \tag{2.18}$$

Likewise, set

$$\begin{aligned}\mathfrak{g}\text{-}\mathrm{CD}^{(\delta)}[\mathscr{S};\mathfrak{T}] &= \bigcup_{\nu\in I_3^0}\{\mathscr{U}\supseteq\mathscr{S}:\ (\exists\mathscr{O})[\mathscr{U}=\mathfrak{g}\text{-}\mathrm{Cod}_{\nu}^{(\delta)}(\mathscr{O})]\}\\ &= \left\{\mathscr{U}\supseteq\mathscr{S}:\ (\exists\mathscr{O})\left[\bigvee_{\nu\in I_3^0}(\mathscr{U}=\mathfrak{g}\text{-}\mathrm{Cod}_{\nu}^{(\delta)}(\mathscr{O}))\right]\right\}\\ &= \bigcup_{\nu\in I_3^0}\mathfrak{g}\text{-}\nu\text{-}\mathrm{CD}^{(\delta)}[\mathscr{S};\mathfrak{T}].\end{aligned}$$

Then, $\mathscr{S}$ is said to be $\mathfrak{g}$-$\mathfrak{T}$-*open of rank* δ in $\mathfrak{T}$ if and only if:

$$\mathscr{S}\subseteq\mathfrak{g}\text{-}\mathrm{Int}^{(\delta)}(\mathscr{S})\stackrel{\text{def}}{=}\bigcap_{\mathscr{W}\in\mathfrak{g}\text{-}\mathrm{CD}^{(\delta)}[\mathscr{S};\mathfrak{T}]}\mathscr{W}. \tag{2.19}$$

The passage from $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-closedness and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-openness of rank δ in $\mathfrak{T}_{\mathfrak{g}}$ to $\mathfrak{g}$-$\mathfrak{T}$-closedness and $\mathfrak{g}$-$\mathfrak{T}$-openness of rank δ in $\mathfrak{T}$ are thus established. To establish $\mathfrak{T}_{\mathfrak{g}}$-*set of rank* δ in $\mathfrak{T}_{\mathfrak{g}}$, introduce these definitions:

$$\begin{aligned}\mathrm{DE}^{(\delta)}[\mathscr{S}_{\mathfrak{g}};\mathfrak{T}_{\mathfrak{g}}] &\stackrel{\text{def}}{=} \{\mathscr{V}_{\mathfrak{g}}\subseteq\mathscr{S}_{\mathfrak{g}}:\ (\exists\mathscr{K}_{\mathfrak{g}})[\mathscr{V}_{\mathfrak{g}}=\mathrm{der}_{\mathfrak{g}}^{(\delta)}(\mathscr{K}_{\mathfrak{g}})]\},\\ \mathrm{CD}^{(\delta)}[\mathscr{S}_{\mathfrak{g}};\mathfrak{T}_{\mathfrak{g}}] &\stackrel{\text{def}}{=} \{\mathscr{U}_{\mathfrak{g}}\supseteq\mathscr{S}_{\mathfrak{g}}:\ (\exists\mathscr{O}_{\mathfrak{g}})[\mathscr{U}_{\mathfrak{g}}=\mathrm{cod}_{\mathfrak{g}}^{(\delta)}(\mathscr{O}_{\mathfrak{g}})]\}.\end{aligned}$$

Then, $\mathscr{S}_{\mathfrak{g}}$ is said to be $\mathfrak{T}_{\mathfrak{g}}$-*closed of rank* δ in $\mathfrak{T}_{\mathfrak{g}}$ if and only if:

$$\mathscr{S}_{\mathfrak{g}}\supseteq\mathrm{cl}_{\mathfrak{g}}^{(\delta)}(\mathscr{S}_{\mathfrak{g}})\stackrel{\text{def}}{=}\bigcup_{\mathscr{W}_{\mathfrak{g}}\in\mathrm{DE}^{(\delta)}[\mathscr{S}_{\mathfrak{g}};\mathfrak{T}_{\mathfrak{g}}]}\mathscr{W}_{\mathfrak{g}}. \tag{2.20}$$

It is said to be $\mathfrak{T}_{\mathfrak{g}}$-*open of rank* δ in $\mathfrak{T}_{\mathfrak{g}}$ if and only if:

$$\mathscr{S}_{\mathfrak{g}}\subseteq\mathrm{int}_{\mathfrak{g}}^{(\delta)}(\mathscr{S}_{\mathfrak{g}})\stackrel{\text{def}}{=}\bigcap_{\mathscr{W}_{\mathfrak{g}}\in\mathrm{CD}^{(\delta)}[\mathscr{S}_{\mathfrak{g}};\mathfrak{T}_{\mathfrak{g}}]}\mathscr{W}_{\mathfrak{g}}. \tag{2.21}$$

The notions of $\mathfrak{T}_{\mathfrak{g}}$-closedness and $\mathfrak{T}_{\mathfrak{g}}$-openness of rank δ in $\mathfrak{T}_{\mathfrak{g}}$ are thus established. Finally, to establish $\mathfrak{T}$-*set of rank* δ in $\mathfrak{T}$, introduce these definitions:

$$\begin{aligned}\mathrm{DE}^{(\delta)}[\mathscr{S};\mathfrak{T}] &\stackrel{\text{def}}{=} \{\mathscr{V}\subseteq\mathscr{S}:\ (\exists\mathscr{K})[\mathscr{V}=\mathrm{der}^{(\delta)}(\mathscr{K})]\},\\ \mathrm{CD}^{(\delta)}[\mathscr{S};\mathfrak{T}] &\stackrel{\text{def}}{=} \{\mathscr{U}\supseteq\mathscr{S}:\ (\exists\mathscr{O})[\mathscr{U}=\mathrm{cod}^{(\delta)}(\mathscr{O})]\}.\end{aligned}$$

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Then, $\mathscr{S}$ is said to be $\mathfrak{T}$-*closed of rank* δ in $\mathfrak{T}$ if and only if:

$$\mathscr{S} \supseteq \mathrm{cl}^{(\delta)}(\mathscr{S}) \stackrel{\text{def}}{=} \bigcup_{\mathscr{W} \in \mathrm{DE}^{(\delta)}[\mathscr{S};\mathfrak{T}]} \mathscr{W}. \tag{2.22}$$

It is said to be $\mathfrak{T}$-*open of rank* δ in $\mathfrak{T}$ if and only if:

$$\mathscr{S} \subseteq \mathrm{int}^{(\delta)}(\mathscr{S}) \stackrel{\text{def}}{=} \bigcap_{\mathscr{W} \in \mathrm{CD}^{(\delta)}[\mathscr{S};\mathfrak{T}]} \mathscr{W}. \tag{2.23}$$

The notions of $\mathfrak{T}$-closedness and $\mathfrak{T}$-openness of rank δ in $\mathfrak{T}$ are thus established.

3. Main Results

Using the foregoing definitions, the essential properties of the $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-derived and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-coderived operators, the essential properties of the concepts of δ^{th}-order derivative $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-derived and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-coderived operators, and the essential properties of the concepts of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-open and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-closed sets of rank δ in $\mathscr{T}_{\mathfrak{g}}$-spaces are presented below.

3.1. Essential Properties.

The discussion begins by giving some of the basic consequences resulting from the foregoing definitions.

In a $\mathscr{T}_{\mathfrak{g}}$-space, $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-derivedness implies $\mathfrak{T}_{\mathfrak{g}}$-derivedness and, $\mathfrak{T}_{\mathfrak{g}}$-coderivedness implies $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-coderivedness, as proved in the following theorem.

Theorem 3.1. *If* $\mathfrak{g}\text{-}\mathbf{Dc}_{\mathfrak{g}} \in \mathfrak{g}\text{-}\mathrm{DC}[\mathfrak{T}_{\mathfrak{g}}]$ *be a given pair of* $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$*-operators* $\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}$, $\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ *and* $\mathbf{dc}_{\mathfrak{g}} \in \mathrm{DC}[\mathfrak{T}_{\mathfrak{g}}]$ *be a given pair of* $\mathfrak{T}_{\mathfrak{g}}$*-operators* $\mathrm{der}_{\mathfrak{g}}$, $\mathrm{cod}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ *in a* $\mathscr{T}_{\mathfrak{g}}$*-space* $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$*, then:*

- I. $\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}) \subseteq \mathrm{der}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}) \quad \forall \mathscr{R} \in \mathscr{P}(\Omega)$,
- II. $\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \supseteq \mathrm{cod}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \quad \forall \mathscr{S} \in \mathscr{P}(\Omega)$.

Proof. Let $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$ be a $\mathscr{T}_{\mathfrak{g}}$-space. Suppose $\mathfrak{g}\text{-}\mathbf{Dc}_{\mathfrak{g}} \in \mathfrak{g}\text{-}\mathrm{DC}[\mathfrak{T}_{\mathfrak{g}}]$ and $\mathbf{dc}_{\mathfrak{g}} \in \mathrm{DC}[\mathfrak{T}_{\mathfrak{g}}]$ be given and $(\mathscr{R}_{\mathfrak{g}}, \mathscr{S}_{\mathfrak{g}}) \in \times_{\alpha \in I_2^*} \mathscr{P}(\Omega)$ be arbitrary. Then,

$$\begin{aligned}
\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}} : \mathscr{R}_{\mathfrak{g}} &\longmapsto \big\{\xi \in \mathfrak{T}_{\mathfrak{g}} :\ \xi \in \mathfrak{g}\text{-}\mathrm{Cl}_{\mathfrak{g}}\big(\mathscr{R}_{\mathfrak{g}} \cap \mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}(\{\xi\})\big)\big\} \\
&\subseteq \big\{\xi \in \mathfrak{T}_{\mathfrak{g}} :\ \xi \in \mathrm{cl}_{\mathfrak{g}}\big(\mathscr{R}_{\mathfrak{g}} \cap \mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}(\{\xi\})\big)\big\} \longleftrightarrow \mathrm{der}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}});\\
\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}} : \mathscr{S}_{\mathfrak{g}} &\longmapsto \big\{\zeta \in \mathfrak{T}_{\mathfrak{g}} :\ \zeta \in \mathfrak{g}\text{-}\mathrm{Int}_{\mathfrak{g}}\big(\mathscr{S}_{\mathfrak{g}} \cup \{\zeta\}\big)\big\} \\
&\supseteq \big\{\zeta \in \mathfrak{T}_{\mathfrak{g}} :\ \zeta \in \mathrm{int}_{\mathfrak{g}}\big(\mathscr{S}_{\mathfrak{g}} \cap \mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}(\{\zeta\})\big)\big\} \longleftrightarrow \mathrm{cod}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}).
\end{aligned}$$

Hence, the relation $\big(\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}), \mathrm{cod}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})\big) \subseteq \big(\mathrm{der}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}), \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})\big)$ holds for any $(\mathscr{R}_{\mathfrak{g}}, \mathscr{S}_{\mathfrak{g}}) \in \times_{\alpha \in I_2^*} \mathscr{P}(\Omega)$. The proof of the theorem is complete. Q.E.D.

Since $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-derivedness and $\mathfrak{T}_{\mathfrak{g}}$-coderivedness imply $\mathfrak{T}_{\mathfrak{g}}$-derivedness and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-coderivedness, respectively, the notions of *coarseness* (or, *smallness*, *weakness*), or alternatively, *finness* (or, *largeness*, *strongness*), can be introduced and is contained in the following remark.

Remark 3.2. If the relation "$\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}} \precsim \mathrm{der}_{\mathfrak{g}}$" stands for "$\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \subseteq \mathrm{der}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$" and "$\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}} \succsim \mathrm{cod}_{\mathfrak{g}}$," for "$\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \supseteq \mathrm{cod}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$," then the outstanding facts are: $\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ is *coarser* (or, *smaller*, *weaker*) than $\mathrm{der}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ or, $\mathrm{der}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ is *finer* (or, *larger*, *stronger*) than $\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$; $\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ is *finer* (or, *larger*, *stronger*)

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than $\mathrm{cod}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ or, $\mathrm{cod}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ is *coarser* (or, *smaller*, *weaker*) than $\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$.

In a $\mathscr{T}_{\mathfrak{g}}$-space, that the $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-derived operator is *dual* to the $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-coderived operator or equivalently, the $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-coderived operator is dual to the $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-derived operator, is proved in the following proposition.

PROPOSITION 3.3. *If* $\mathfrak{g}\text{-}\mathbf{Dc}_{\mathfrak{g}} \in \mathfrak{g}\text{-}\mathrm{DC}[\mathfrak{T}_{\mathfrak{g}}]$ *be a given pair of* $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$*-operators* $\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}$, $\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ *and* $\mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ *be the natural complement* $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$*-operator of their components in a* $\mathscr{T}_{\mathfrak{g}}$*-space* $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$*, then:*

$$\begin{aligned} &\big(\forall \mathscr{S}_{\mathfrak{g}} \in \mathscr{P}(\Omega)\big)\big[\big(\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \longleftrightarrow \mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}} \circ \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}} \circ \mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})\big) \\ &\wedge\big(\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \longleftrightarrow \mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}} \circ \mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}} \circ \mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})\big)\big]. \end{aligned} \tag{3.1}$$

PROOF. Let $\mathfrak{g}\text{-}\mathbf{Dc}_{\mathfrak{g}} \in \mathfrak{g}\text{-}\mathrm{DC}[\mathfrak{T}_{\mathfrak{g}}]$ be a given pair of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-operators $\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}$, $\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$, respectively, and let $\mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ be the natural complement $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-operator of their components in a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$. Then, for a $\mathscr{R}_{\mathfrak{g}} \in \mathscr{P}(\Omega)$ taken arbitrarily, it follows that

$$\begin{gathered} \mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}} \circ \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}} \circ \mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}) \\ \Big\updownarrow \\ \mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}\big(\{\xi \in \mathfrak{T}_{\mathfrak{g}} :\ \xi \in \mathfrak{g}\text{-}\mathrm{Int}_{\mathfrak{g}}\big(\mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}) \cup \{\xi\}\big)\}\big) \\ \Big\updownarrow \\ \big\{\xi \in \mathfrak{T}_{\mathfrak{g}} :\ \xi \in \mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}} \circ \mathfrak{g}\text{-}\mathrm{Int}_{\mathfrak{g}} \circ \mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}\big(\mathscr{R}_{\mathfrak{g}} \cap \mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}(\{\xi\})\big)\big\} \\ \Big\updownarrow \\ \big\{\xi \in \mathfrak{T}_{\mathfrak{g}} :\ \xi \in \mathfrak{g}\text{-}\mathrm{Cl}_{\mathfrak{g}}\big(\mathscr{R}_{\mathfrak{g}} \cap \mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}(\{\xi\})\big)\big\} \\ \Big\updownarrow \\ \mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}) \end{gathered}$$

Thus, $\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}) \longleftrightarrow \mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}} \circ \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}} \circ \mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}})$ for every $\mathscr{R}_{\mathfrak{g}} \in \mathscr{P}(\Omega)$. For a $\mathscr{S}_{\mathfrak{g}} \in \mathscr{P}(\Omega)$ taken arbitrarily, it follows that

$$\begin{gathered} \mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}} \circ \mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}} \circ \mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \\ \Big\updownarrow \\ \mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}\big(\{\xi \in \mathfrak{T}_{\mathfrak{g}} :\ \xi \in \mathfrak{g}\text{-}\mathrm{Cl}_{\mathfrak{g}}\big(\mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \cap \mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}(\{\xi\})\big)\}\big) \\ \Big\updownarrow \\ \big\{\xi \in \mathfrak{T}_{\mathfrak{g}} :\ \xi \in \mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}} \circ \mathfrak{g}\text{-}\mathrm{Cl}_{\mathfrak{g}} \circ \mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}\big(\mathscr{S}_{\mathfrak{g}} \cup \{\xi\}\big)\big\} \\ \Big\updownarrow \\ \big\{\xi \in \mathfrak{T}_{\mathfrak{g}} :\ \xi \in \mathfrak{g}\text{-}\mathrm{Int}_{\mathfrak{g}}\big(\mathscr{S}_{\mathfrak{g}} \cup \{\xi\}\big)\big\} \\ \Big\updownarrow \\ \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}). \end{gathered}$$

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Hence, $\mathfrak{g}\text{-Cod}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \longleftrightarrow \mathfrak{g}\text{-Op}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Der}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$ for every $\mathscr{S}_{\mathfrak{g}} \in \mathscr{P}(\Omega)$. The proof of the proposition is complete. Q.E.D.

The lemma below, in which it is proved that the $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-derived and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-coderived operators are both *monotone* or equivalently, *isotonic*, will be useful in the proof of the theorem following it.

Lemma 3.4. *If* $(\mathfrak{g}\text{-Der}_{\mathfrak{g}}, \mathfrak{g}\text{-Cod}_{\mathfrak{g}}) \in \mathfrak{g}\text{-DE}[\mathfrak{T}_{\mathfrak{g}}] \times \mathfrak{g}\text{-CD}[\mathfrak{T}_{\mathfrak{g}}]$ *be a pair of* $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$*-derived and* $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$*-coderived operators* $\mathfrak{g}\text{-Der}_{\mathfrak{g}}$, $\mathfrak{g}\text{-Cod}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$*, respectively, in a* $\mathscr{T}_{\mathfrak{g}}$*-space* $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$ *then:*

- I. $\mathscr{R}_{\mathfrak{g}} \subseteq \mathscr{S}_{\mathfrak{g}} \in \mathscr{P}(\Omega) \longrightarrow \mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}) \subseteq \mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$,
- II. $\mathscr{R}_{\mathfrak{g}} \subseteq \mathscr{S}_{\mathfrak{g}} \in \mathscr{P}(\Omega) \longrightarrow \mathfrak{g}\text{-Cod}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}) \subseteq \mathfrak{g}\text{-Cod}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$.

Proof. Let $(\mathfrak{g}\text{-Der}_{\mathfrak{g}}, \mathfrak{g}\text{-Cod}_{\mathfrak{g}}) \in \mathfrak{g}\text{-DE}[\mathfrak{T}_{\mathfrak{g}}] \times \mathfrak{g}\text{-CD}[\mathfrak{T}_{\mathfrak{g}}]$ be a pair of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-derived and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-coderived operators $\mathfrak{g}\text{-Der}_{\mathfrak{g}}$, $\mathfrak{g}\text{-Cod}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$, respectively, and let $(\mathscr{R}_{\mathfrak{g}}, \mathscr{S}_{\mathfrak{g}}) \in \times_{\alpha \in I_2^*} \mathscr{P}(\Omega)$ be an arbitrary pair such that $\mathscr{R}_{\mathfrak{g}} \subseteq \mathscr{S}_{\mathfrak{g}}$ in a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$. Then, since $\mathscr{R}_{\mathfrak{g}} \subseteq \mathscr{S}_{\mathfrak{g}}$, it results that

$$\begin{aligned}
\mathfrak{g}\text{-Der}_{\mathfrak{g}} : \mathscr{R}_{\mathfrak{g}} &\longmapsto \left\{\xi \in \mathfrak{T}_{\mathfrak{g}} : \ \xi \in \mathfrak{g}\text{-Cl}_{\mathfrak{g}}\big(\mathscr{R}_{\mathfrak{g}} \cap \mathfrak{g}\text{-Op}_{\mathfrak{g}}(\{\xi\})\big)\right\} \\
&\subseteq \left\{\xi \in \mathfrak{T}_{\mathfrak{g}} : \ \xi \in \mathfrak{g}\text{-Cl}_{\mathfrak{g}}\big(\mathscr{S}_{\mathfrak{g}} \cap \mathfrak{g}\text{-Op}_{\mathfrak{g}}(\{\xi\})\big)\right\} \longleftrightarrow \mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})\,; \\
\mathfrak{g}\text{-Cod}_{\mathfrak{g}} : \mathscr{R}_{\mathfrak{g}} &\longmapsto \left\{\zeta \in \mathfrak{T}_{\mathfrak{g}} : \ \zeta \in \mathfrak{g}\text{-Int}_{\mathfrak{g}}\big(\mathscr{R}_{\mathfrak{g}} \cup \{\zeta\}\big)\right\} \\
&\subseteq \left\{\zeta \in \mathfrak{T}_{\mathfrak{g}} : \ \zeta \in \mathfrak{g}\text{-Int}_{\mathfrak{g}}\big(\mathscr{S}_{\mathfrak{g}} \cup \{\zeta\}\big)\right\} \longleftrightarrow \mathfrak{g}\text{-Cod}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})\,.
\end{aligned}$$

Hence, for every $(\mathscr{R}_{\mathfrak{g}}, \mathscr{S}_{\mathfrak{g}}) \in \times_{\alpha \in I_2^*} \mathscr{P}(\Omega)$ such that $\mathscr{R}_{\mathfrak{g}} \subseteq \mathscr{S}_{\mathfrak{g}}$, $\mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}) \subseteq \mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$ and $\mathfrak{g}\text{-Cod}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}) \subseteq \mathfrak{g}\text{-Cod}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$. The proof of the lemma is complete. Q.E.D.

Equivalently stated, $\mathfrak{g}\text{-}\mathbf{Dc}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}, \mathscr{R}_{\mathfrak{g}}) \subseteq \mathfrak{g}\text{-}\mathbf{Dc}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}, \mathscr{S}_{\mathfrak{g}})$ for every $(\mathscr{R}_{\mathfrak{g}}, \mathscr{S}_{\mathfrak{g}}) \in \times_{\alpha \in I_2^*} \mathscr{P}(\Omega)$ such that $\mathscr{R}_{\mathfrak{g}} \subseteq \mathscr{S}_{\mathfrak{g}}$. In a $\mathscr{T}_{\mathfrak{g}}$-space, every element in the class of pairs of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-derived and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-coderived operators is paired with exactly one element in the class of pairs of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-interior and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-closure operators, and conversely. The theorem follows.

Theorem 3.5. *If* $\mathfrak{g}\text{-}\mathbf{Dc}_{\mathfrak{g}} \in \mathfrak{g}\text{-DC}[\mathfrak{T}_{\mathfrak{g}}]$ *and* $\mathfrak{g}\text{-}\mathbf{Ic}_{\mathfrak{g}} \in \mathfrak{g}\text{-IC}[\mathfrak{T}_{\mathfrak{g}}]$ *be given pairs of* $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$*-operators* $\mathfrak{g}\text{-Der}_{\mathfrak{g}}$, $\mathfrak{g}\text{-Cod}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ *and* $\mathfrak{g}\text{-Int}_{\mathfrak{g}}$, $\mathfrak{g}\text{-Cl}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$*, respectively, in a* $\mathscr{T}_{\mathfrak{g}}$*-space* $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$*, then:*

- I. $\mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \longleftrightarrow \mathscr{S}_{\mathfrak{g}} \cup \mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \quad \forall \mathscr{S}_{\mathfrak{g}} \in \mathscr{P}(\Omega)$,
- II. $\mathfrak{g}\text{-Int}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \longleftrightarrow \mathscr{S}_{\mathfrak{g}} \cap \mathfrak{g}\text{-Cod}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \quad \forall \mathscr{S}_{\mathfrak{g}} \in \mathscr{P}(\Omega)$.

Proof. Let $\mathfrak{g}\text{-}\mathbf{Dc}_{\mathfrak{g}} \in \mathfrak{g}\text{-DC}[\mathfrak{T}_{\mathfrak{g}}]$ and $\mathfrak{g}\text{-}\mathbf{Ic}_{\mathfrak{g}} \in \mathfrak{g}\text{-IC}[\mathfrak{T}_{\mathfrak{g}}]$ be given pairs of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-operators $\mathfrak{g}\text{-Der}_{\mathfrak{g}}$, $\mathfrak{g}\text{-Cod}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ and $\mathfrak{g}\text{-Int}_{\mathfrak{g}}$, $\mathfrak{g}\text{-Cl}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$, respectively, and suppose $\mathscr{S}_{\mathfrak{g}} \in \mathscr{P}(\Omega)$ be arbitrary in a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$. Then:

I. The relation $\mathscr{S}_{\mathfrak{g}} \subseteq \mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \in \mathfrak{g}\text{-K}[\mathfrak{T}_{\mathfrak{g}}]$ holds. Consequently, $\mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \subseteq \mathfrak{g}\text{-Der}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \subseteq \mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$ implying, $\mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \supseteq \mathscr{S}_{\mathfrak{g}} \cup \mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$. But, $\mathscr{S}_{\mathfrak{g}} \subseteq \mathscr{S}_{\mathfrak{g}} \cup \mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \in \mathfrak{g}\text{-K}[\mathfrak{T}_{\mathfrak{g}}]$ holds and consequently, $\mathscr{S}_{\mathfrak{g}} \subseteq \mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \subseteq \mathscr{S}_{\mathfrak{g}} \cup \mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$. Therefore, $\mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \subseteq \mathscr{S}_{\mathfrak{g}} \cup \mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$ and hence, $\mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \longleftrightarrow \mathscr{S}_{\mathfrak{g}} \cup \mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$ for all $\mathscr{S}_{\mathfrak{g}} \in \mathscr{P}(\Omega)$.

II. The relation $\mathscr{S}_{\mathfrak{g}} \supseteq \mathfrak{g}\text{-Int}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \in \mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}]$ holds. Therefore, $\mathfrak{g}\text{-Cod}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \supseteq \mathfrak{g}\text{-Cod}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Int}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \supseteq \mathfrak{g}\text{-Int}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$ implying, $\mathfrak{g}\text{-Int}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \subseteq \mathscr{S}_{\mathfrak{g}} \cap \mathfrak{g}\text{-Cod}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$. But,

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$\mathscr{S}_{\mathfrak{g}} \supseteq \mathscr{S}_{\mathfrak{g}} \cap \mathfrak{g}\text{-Cod}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \in \mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}]$ holds and consequently, $\mathscr{S}_{\mathfrak{g}} \supseteq \mathfrak{g}\text{-Int}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \supseteq \mathscr{S}_{\mathfrak{g}} \cap \mathfrak{g}\text{-Cod}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$. Therefore, $\mathfrak{g}\text{-Int}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \supseteq \mathscr{S}_{\mathfrak{g}} \cap \mathfrak{g}\text{-Cod}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$ holds and thus, $\mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \longleftrightarrow \mathscr{S}_{\mathfrak{g}} \cup \mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$ for all $\mathscr{S}_{\mathfrak{g}} \in \mathscr{P}(\Omega)$. The proof of the theorem is complete. Q.E.D.

In a strong $\mathscr{T}_{\mathfrak{g}}$-space, the $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-derived operator is $\emptyset$-*preserving* and the $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-coderived operator is Ω-*preserving*, as shown in the following proposition.

PROPOSITION 3.6. *If $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$ be a strong $\mathscr{T}_{\mathfrak{g}}$-space, then:*

$$(3.2)\qquad \left(\forall\, \mathfrak{g}\text{-}\mathbf{Dc}_{\mathfrak{g}} \in \mathfrak{g}\text{-DC}[\mathfrak{T}_{\mathfrak{g}}]\right)\left[\mathfrak{g}\text{-}\mathbf{Dc}_{\mathfrak{g}} : (\emptyset, \Omega) \longmapsto (\emptyset, \Omega)\right].$$

PROOF. Let $\mathfrak{g}\text{-}\mathbf{Dc}_{\mathfrak{g}} \in \mathfrak{g}\text{-DC}[\mathfrak{T}_{\mathfrak{g}}]$ be a pair of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-operators $\mathfrak{g}\text{-Der}_{\mathfrak{g}}$, $\mathfrak{g}\text{-Cod}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ in a strong $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$. Then, since $\mathfrak{T}_{\mathfrak{g}}$ is a strong $\mathscr{T}_{\mathfrak{g}}$-space, $(\emptyset, \Omega) \longleftrightarrow \left(\mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\emptyset), \mathfrak{g}\text{-Int}_{\mathfrak{g}}(\Omega)\right) \in \mathfrak{g}\text{-K}[\mathfrak{T}_{\mathfrak{g}}] \times \mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}]$. Consequently,

$$\left(\emptyset \cup \mathfrak{g}\text{-Der}_{\mathfrak{g}}(\emptyset), \Omega \cap \mathfrak{g}\text{-Cod}_{\mathfrak{g}}(\Omega)\right) \longleftrightarrow \left(\mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\emptyset), \mathfrak{g}\text{-Int}_{\mathfrak{g}}(\Omega)\right) \longleftrightarrow (\emptyset, \Omega),$$

Hence, $\mathfrak{g}\text{-}\mathbf{Dc}_{\mathfrak{g}} : (\emptyset, \Omega) \longmapsto (\emptyset, \Omega)$ for any $\mathfrak{g}\text{-}\mathbf{Dc}_{\mathfrak{g}} \in \mathfrak{g}\text{-DC}[\mathfrak{T}_{\mathfrak{g}}]$. The proof of the proposition is complete. Q.E.D.

In view of the theorem preceding the preceding proposition, the following remark presents itself.

REMARK 3.7. Relative to the $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$, $\{\xi\} \in \mathscr{P}(\Omega)$ is a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-*closure unit set* of $\mathscr{S}_{\mathfrak{g}} \in \mathscr{P}(\Omega)$ if and only if $\{\xi\}$ is a $\mathfrak{T}_{\mathfrak{g}}$-*unit set* or a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-*derived unit set* of $\mathscr{S}_{\mathfrak{g}}$; $\{\xi\} \in \mathscr{P}(\Omega)$ is a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-*interior unit set* of $\mathscr{S}_{\mathfrak{g}} \in \mathscr{P}(\Omega)$ if and only if, relative to the $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$, $\{\xi\}$ is a $\mathfrak{T}_{\mathfrak{g}}$-*unit set* and a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-*coderived unit set* of $\mathscr{S}_{\mathfrak{g}}$. Relative to the $\mathscr{T}$-space $\mathfrak{T} = (\Omega, \mathscr{T})$, $\{\xi\} \in \mathscr{P}(\Omega)$ is a $\mathfrak{g}$-$\mathfrak{T}$-*closure unit set* of $\mathscr{S} \in \mathscr{P}(\Omega)$ if and only if $\{\xi\}$ is a $\mathfrak{T}$-*unit set* or a $\mathfrak{g}$-$\mathfrak{T}$-*derived unit set* of $\mathscr{S}$; $\{\xi\} \in \mathscr{P}(\Omega)$ is a $\mathfrak{g}$-$\mathfrak{T}$-*interior unit set* of $\mathscr{S}_{\mathfrak{g}} \in \mathscr{P}(\Omega)$ if and only if $\{\xi\}$ is a $\mathfrak{T}$-*unit set* and a $\mathfrak{g}$-$\mathfrak{T}$-*coderived unit set* of $\mathscr{S}$.

Taking REMS. 2.2, 3.7 into account, an immediate consequence of THM. 3.5 is the following corollary.

COROLLARY 3.8. *If $\mathfrak{g}\text{-}\mathbf{Dc}_{\mathfrak{g}} \in \mathfrak{g}\text{-DC}[\mathfrak{T}_{\mathfrak{g}}]$ be a pair of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-operators $\mathfrak{g}\text{-Der}_{\mathfrak{g}}$, $\mathfrak{g}\text{-Cod}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ and, $\left\{\mathfrak{g}\text{-Der}_{\mathfrak{g}}(\xi; \mathscr{R}_{\mathfrak{g}}) : \xi \in \mathfrak{T}_{\mathfrak{g}}\right\}$ and $\left\{\mathfrak{g}\text{-Cod}_{\mathfrak{g}}(\zeta; \mathscr{R}_{\mathfrak{g}}) : \zeta \in \mathfrak{T}_{\mathfrak{g}}\right\}$, respectively, be the corresponding collections of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-derived and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-coderived points of some $\mathscr{R}_{\mathfrak{g}} \in \mathscr{P}(\Omega)$ in a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$, then:*

- I. $\left(\exists\, \mathfrak{g}\text{-Der}_{\mathfrak{g}}(\xi; \mathscr{R}_{\mathfrak{g}}) \in \mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}})\right)\left[\mathfrak{g}\text{-Der}_{\mathfrak{g}}(\xi; \mathscr{R}_{\mathfrak{g}}) \notin \mathscr{R}_{\mathfrak{g}}\right]$,
- II. $\left(\forall\, \mathfrak{g}\text{-Cod}_{\mathfrak{g}}(\zeta; \mathscr{R}_{\mathfrak{g}}) \in \mathfrak{g}\text{-Cod}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}})\right)\left[\mathfrak{g}\text{-Cod}_{\mathfrak{g}}(\zeta; \mathscr{R}_{\mathfrak{g}}) \in \mathscr{R}_{\mathfrak{g}}\right]$.

For any $\mathfrak{T}_{\mathfrak{g}}$-set of a $\mathscr{T}_{\mathfrak{g}}$-space, the intersection of its $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-coderived set with itself is contained in the $\mathfrak{T}_{\mathfrak{g}}$-set and, the union of its $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-derived set with itself contains the $\mathfrak{T}_{\mathfrak{g}}$-set, as proved in the following proposition.

PROPOSITION 3.9. *If $\mathfrak{g}\text{-}\mathbf{Dc}_{\mathfrak{g}} \in \mathfrak{g}\text{-DC}[\mathfrak{T}_{\mathfrak{g}}]$ be a given pair of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-operators $\mathfrak{g}\text{-Der}_{\mathfrak{g}}$, $\mathfrak{g}\text{-Cod}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ in a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$, then:*

$$(3.3)\qquad \left(\forall \mathscr{S} \in \mathscr{P}(\Omega)\right)\left[\mathscr{S}_{\mathfrak{g}} \cap \mathfrak{g}\text{-Cod}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \subseteq \mathscr{S}_{\mathfrak{g}} \subseteq \mathscr{S}_{\mathfrak{g}} \cup \mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})\right].$$

PROOF. Let $\mathfrak{g}\text{-}\mathbf{Dc}_{\mathfrak{g}} \in \mathfrak{g}\text{-DC}[\mathfrak{T}_{\mathfrak{g}}]$ be a given pair of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-operators $\mathfrak{g}\text{-Der}_{\mathfrak{g}}$, $\mathfrak{g}\text{-Cod}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ in a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$. Then, for all $\mathscr{S} \in \mathscr{P}(\Omega)$, $\mathscr{S}_{\mathfrak{g}} \cup \mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \longleftrightarrow \mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$ and $\mathscr{S}_{\mathfrak{g}} \cap \mathfrak{g}\text{-Cod}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \longleftrightarrow \mathfrak{g}\text{-Int}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$. But, for all

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$\mathscr{S} \in \mathscr{P}(\Omega)$, $\mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \supseteq \mathscr{S}_{\mathfrak{g}}$ and $\mathfrak{g}\text{-Int}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \subseteq \mathscr{S}_{\mathfrak{g}}$. Hence, for all $\mathscr{S} \in \mathscr{P}(\Omega)$, $\mathscr{S}_{\mathfrak{g}} \cap \mathfrak{g}\text{-Cod}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \subseteq \mathscr{S}_{\mathfrak{g}} \subseteq \mathscr{S}_{\mathfrak{g}} \cup \mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$. The proof of the proposition is complete. Q.E.D.

Since the relation $\mathfrak{g}\text{-Int}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \subseteq \mathscr{S}_{\mathfrak{g}} \subseteq \mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$ and the relations

$$\mathscr{S}_{\mathfrak{g}} \cup \text{der}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \longleftrightarrow \text{cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \supseteq \mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \longleftrightarrow \mathscr{S}_{\mathfrak{g}} \cup \mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}),$$
$$\mathscr{S}_{\mathfrak{g}} \cap \text{cod}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \longleftrightarrow \text{int}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \subseteq \mathfrak{g}\text{-Int}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \longleftrightarrow \mathscr{S}_{\mathfrak{g}} \cap \mathfrak{g}\text{-Cod}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$$

hold for any $\mathscr{S} \in \mathscr{P}(\Omega)$, an immediate consequence of the preceding proposition is the following corollary.

Corollary 3.10. *If* $\mathfrak{g}\text{-}\mathbf{Dc}_{\mathfrak{g}} \in \mathfrak{g}\text{-DC}[\mathfrak{T}_{\mathfrak{g}}]$ *and* $\mathbf{dc}_{\mathfrak{g}} \in \text{DC}[\mathfrak{T}_{\mathfrak{g}}]$ *be given pairs of* $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-*operators* $\mathfrak{g}\text{-Der}_{\mathfrak{g}}$, $\mathfrak{g}\text{-Cod}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ *and* $\text{der}_{\mathfrak{g}}$, $\text{cod}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$, *respectively, in a* $\mathscr{T}_{\mathfrak{g}}$-*space* $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$, *then the following logical implication holds for any* $\mathscr{S}_{\mathfrak{g}} \in \mathscr{P}(\Omega)$:

$$\begin{array}{c} \mathscr{S}_{\mathfrak{g}} \cap \mathfrak{g}\text{-Cod}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \subseteq \mathscr{S}_{\mathfrak{g}} \subseteq \mathscr{S}_{\mathfrak{g}} \cup \mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \\ \downarrow \\ \mathscr{S}_{\mathfrak{g}} \cap \text{cod}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \subseteq \mathscr{S}_{\mathfrak{g}} \subseteq \mathscr{S}_{\mathfrak{g}} \cup \text{der}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}). \end{array} \tag{3.4}$$

In a $\mathscr{T}_{\mathfrak{g}}$-space, the $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-derived operator is $\subseteq$-*preserving relative to* $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-*open sets* and the $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-coderived operator is $\supseteq$-*preserving relative to* $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-*closed sets*, as shown in the following proposition.

Proposition 3.11. *If* $\mathfrak{g}\text{-}\mathbf{Dc}_{\mathfrak{g}} \in \mathfrak{g}\text{-DC}[\mathfrak{T}_{\mathfrak{g}}]$ *be a pair of* $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-*operators* $\mathfrak{g}\text{-Der}_{\mathfrak{g}}$, $\mathfrak{g}\text{-Cod}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$, *and* $(\mathscr{U}_{\mathfrak{g}}, \mathscr{V}_{\mathfrak{g}}) \in \mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}] \times \mathfrak{g}\text{-K}[\mathfrak{T}_{\mathfrak{g}}]$ *be a pair of* $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-*open and* $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-*closed sets, respectively, in a* $\mathscr{T}_{\mathfrak{g}}$-*space* $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$, *then:*

- I. $\mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \subseteq \mathscr{V}_{\mathfrak{g}} \longleftarrow \mathscr{S}_{\mathfrak{g}} \subseteq \mathscr{V}_{\mathfrak{g}} \quad \forall \mathscr{S}_{\mathfrak{g}} \in \mathscr{P}(\Omega)$,
- II. $\mathfrak{g}\text{-Cod}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}) \supseteq \mathscr{U}_{\mathfrak{g}} \longleftarrow \mathscr{R}_{\mathfrak{g}} \supseteq \mathscr{U}_{\mathfrak{g}} \quad \forall \mathscr{R}_{\mathfrak{g}} \in \mathscr{P}(\Omega)$.

Proof. Let $\mathfrak{g}\text{-}\mathbf{Dc}_{\mathfrak{g}} \in \mathfrak{g}\text{-DC}[\mathfrak{T}_{\mathfrak{g}}]$ be a pair of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-operators $\mathfrak{g}\text{-Der}_{\mathfrak{g}}$, $\mathfrak{g}\text{-Cod}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$, and $(\mathscr{U}_{\mathfrak{g}}, \mathscr{V}_{\mathfrak{g}}) \in \mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}] \times \mathfrak{g}\text{-K}[\mathfrak{T}_{\mathfrak{g}}]$ be a pair of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-open and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-closed sets, respectively, and let $(\mathscr{R}_{\mathfrak{g}}, \mathscr{S}_{\mathfrak{g}}) \in \times_{\alpha \in I_2^*} \mathscr{P}(\Omega)$ be arbitrary in a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$. Suppose $(\mathscr{V}_{\mathfrak{g}}, \mathscr{R}_{\mathfrak{g}}) \supseteq (\mathscr{S}_{\mathfrak{g}}, \mathscr{U}_{\mathfrak{g}})$, then

$$\left(\mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{V}_{\mathfrak{g}}), \mathfrak{g}\text{-Cod}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}})\right) \supseteq \left(\mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}), \mathfrak{g}\text{-Cod}_{\mathfrak{g}}(\mathscr{U}_{\mathfrak{g}})\right).$$

But $(\mathscr{U}_{\mathfrak{g}}, \mathscr{V}_{\mathfrak{g}}) \in \mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}] \times \mathfrak{g}\text{-K}[\mathfrak{T}_{\mathfrak{g}}]$ implies $\left(\mathscr{V}_{\mathfrak{g}}, \mathfrak{g}\text{-Cod}_{\mathfrak{g}}(\mathscr{U}_{\mathfrak{g}})\right) \supseteq \left(\mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{V}_{\mathfrak{g}}), \mathscr{U}_{\mathfrak{g}}\right)$ and consequently,

$$\left(\mathscr{V}_{\mathfrak{g}}, \mathfrak{g}\text{-Cod}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}})\right) \supseteq \left(\mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{V}_{\mathfrak{g}}), \mathfrak{g}\text{-Cod}_{\mathfrak{g}}(\mathscr{U}_{\mathfrak{g}})\right) \supseteq \left(\mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}), \mathscr{U}_{\mathfrak{g}}\right).$$

Hence, $\mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \subseteq \mathscr{V}_{\mathfrak{g}}$ and $\mathfrak{g}\text{-Cod}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}) \supseteq \mathscr{U}_{\mathfrak{g}}$. The proof of the proposition is complete. Q.E.D.

In view of the above proposition, it follows, then, that, in a $\mathscr{T}_{\mathfrak{g}}$-space, $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-derived sets and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-coderived sets can also be characterized in terms of their $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-closed sets and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-open sets, respectively. The theorem follows.

Theorem 3.12. *If* $\mathfrak{g}\text{-}\mathbf{Dc}_{\mathfrak{g}} \in \mathfrak{g}\text{-DC}[\mathfrak{T}_{\mathfrak{g}}]$ *be a given pair of* $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-*operators* $\mathfrak{g}\text{-Der}_{\mathfrak{g}}$, $\mathfrak{g}\text{-Cod}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$, *respectively, in a* $\mathscr{T}_{\mathfrak{g}}$-*space* $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$, *then:*

- I. $\mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}) \subseteq \mathscr{R}_{\mathfrak{g}} \longleftrightarrow \mathscr{R}_{\mathfrak{g}} \in \mathfrak{g}\text{-K}[\mathfrak{T}_{\mathfrak{g}}] \quad \forall \mathscr{R}_{\mathfrak{g}} \in \mathscr{P}(\Omega)$,
- II. $\mathfrak{g}\text{-Cod}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \supseteq \mathscr{S}_{\mathfrak{g}} \longleftrightarrow \mathscr{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}] \quad \forall \mathscr{S}_{\mathfrak{g}} \in \mathscr{P}(\Omega)$.

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PROOF. Let $\mathfrak{g}\text{-}\mathbf{Dc}_{\mathfrak{g}} \in \mathfrak{g}\text{-}\mathrm{DC}[\mathfrak{T}_{\mathfrak{g}}]$ be a given pair of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-operators $\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}$, $\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$, respectively, and let $(\mathscr{R}_{\mathfrak{g}}, \mathscr{S}_{\mathfrak{g}}) \in \times_{\alpha \in I_2^*} \mathscr{P}(\Omega)$ be arbitrary in a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$. Then:

Necessity. Suppose $\big(\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}), \mathscr{S}_{\mathfrak{g}}\big) \subseteq \big(\mathscr{R}_{\mathfrak{g}}, \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})\big)$ holds. Then, $\mathscr{R}_{\mathfrak{g}} \cup \mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}) \longleftrightarrow \mathscr{R}_{\mathfrak{g}}$ and $\mathscr{S}_{\mathfrak{g}} \cap \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \longleftrightarrow \mathscr{S}_{\mathfrak{g}}$. But, $\mathscr{R}_{\mathfrak{g}} \cup \mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}) \longleftrightarrow \mathfrak{g}\text{-}\mathrm{Cl}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}) \in \mathfrak{g}\text{-}\mathrm{K}[\mathfrak{T}_{\mathfrak{g}}]$ and $\mathscr{S}_{\mathfrak{g}} \cap \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \longleftrightarrow \mathfrak{g}\text{-}\mathrm{Int}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \in \mathfrak{g}\text{-}\mathrm{O}[\mathfrak{T}_{\mathfrak{g}}]$. Hence, $(\mathscr{R}_{\mathfrak{g}}, \mathscr{S}_{\mathfrak{g}}) \in \mathfrak{g}\text{-}\mathrm{K}[\mathfrak{T}_{\mathfrak{g}}] \times \mathfrak{g}\text{-}\mathrm{O}[\mathfrak{T}_{\mathfrak{g}}]$. The conditions of ITEMS I., II. are, therefore, necessary.

Sufficiency. Conversely, suppose $(\mathscr{R}_{\mathfrak{g}}, \mathscr{S}_{\mathfrak{g}}) \in \mathfrak{g}\text{-}\mathrm{K}[\mathfrak{T}_{\mathfrak{g}}] \times \mathfrak{g}\text{-}\mathrm{O}[\mathfrak{T}_{\mathfrak{g}}]$ holds. Then, $\mathscr{R}_{\mathfrak{g}} \longleftrightarrow \mathfrak{g}\text{-}\mathrm{Cl}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}})$ and $\mathscr{S}_{\mathfrak{g}} \longleftrightarrow \mathfrak{g}\text{-}\mathrm{Int}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$. But, $\mathfrak{g}\text{-}\mathrm{Cl}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}) \longleftrightarrow \mathscr{R}_{\mathfrak{g}} \cup \mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}})$ and $\mathfrak{g}\text{-}\mathrm{Int}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \longleftrightarrow \mathscr{S}_{\mathfrak{g}} \cap \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$. Consequently, $\mathscr{R}_{\mathfrak{g}} \longleftrightarrow \mathscr{R}_{\mathfrak{g}} \cup \mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}})$ and $\mathscr{S}_{\mathfrak{g}} \longleftrightarrow \mathscr{S}_{\mathfrak{g}} \cap \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$. Thus, $\big(\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}), \mathscr{S}_{\mathfrak{g}}\big) \subseteq \big(\mathscr{R}_{\mathfrak{g}}, \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})\big)$. The conditions of ITEMS I., II. are, therefore, sufficient. The proof of the theorem is complete. Q.E.D.

In a $\mathscr{T}_{\mathfrak{g}}$-space, the statement that a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-derived set is $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-open is equivalent to the statement that the $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-coderived operation on it is *extensive* and, the statement that a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-coderived set is $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-closed is equivalent to the statement that the $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-derived operation on it is *intensive*. The proposition follows.

PROPOSITION 3.13. *Let* $\mathfrak{g}\text{-}\mathbf{Dc}_{\mathfrak{g}} \in \mathfrak{g}\text{-}\mathrm{DC}[\mathfrak{T}_{\mathfrak{g}}]$ *be a pair of* $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-*operators* $\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}$, $\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ *in a* $\mathscr{T}_{\mathfrak{g}}$-*space* $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$, *and let* $\mathfrak{g}\text{-}\mathbf{Dc}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}, \mathscr{S}_{\mathfrak{g}}) \in \mathfrak{g}\text{-}\mathrm{O}[\mathfrak{T}_{\mathfrak{g}}] \times \mathfrak{g}\text{-}\mathrm{K}[\mathfrak{T}_{\mathfrak{g}}]$ *holds for some* $(\mathscr{R}_{\mathfrak{g}}, \mathscr{S}_{\mathfrak{g}}) \in \times_{\alpha \in I_2^*} \mathscr{P}(\Omega)$. *Then:*

- I. $\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}) \in \mathfrak{g}\text{-}\mathrm{O}[\mathfrak{T}_{\mathfrak{g}}] \;\longleftrightarrow\; \mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}) \subseteq \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}} \circ \mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}})$,
- II. $\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \in \mathfrak{g}\text{-}\mathrm{K}[\mathfrak{T}_{\mathfrak{g}}] \;\longleftrightarrow\; \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \supseteq \mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}} \circ \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$.

PROOF. Let $\mathfrak{g}\text{-}\mathbf{Dc}_{\mathfrak{g}} \in \mathfrak{g}\text{-}\mathrm{DC}[\mathfrak{T}_{\mathfrak{g}}]$ be a pair of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-operators $\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}$, $\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ in a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$ and, let it be supposed that the condition $\mathfrak{g}\text{-}\mathbf{Dc}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}, \mathscr{S}_{\mathfrak{g}}) \in \mathfrak{g}\text{-}\mathrm{O}[\mathfrak{T}_{\mathfrak{g}}] \times \mathfrak{g}\text{-}\mathrm{K}[\mathfrak{T}_{\mathfrak{g}}]$ holds for some $(\mathscr{R}_{\mathfrak{g}}, \mathscr{S}_{\mathfrak{g}}) \in \times_{\alpha \in I_2^*} \mathscr{P}(\Omega)$. Then:

I. Since $\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}) \in \mathfrak{g}\text{-}\mathrm{O}[\mathfrak{T}_{\mathfrak{g}}]$, $\mathfrak{g}\text{-}\mathrm{Int}_{\mathfrak{g}} \circ \mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}) \supseteq \mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}})$, where $\mathfrak{g}\text{-}\mathrm{Int}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ is the $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-interior operator in $\mathfrak{T}_{\mathfrak{g}}$. But,

$$\begin{aligned} \mathfrak{g}\text{-}\mathrm{Int}_{\mathfrak{g}} \circ \mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}) &\longleftrightarrow \mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}) \cap \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}} \circ \mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}) \\ &\subseteq \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}} \circ \mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}). \end{aligned}$$

Thus, $\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}) \in \mathfrak{g}\text{-}\mathrm{O}[\mathfrak{T}_{\mathfrak{g}}] \longleftrightarrow \mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}) \subseteq \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}} \circ \mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}})$.

II. Since $\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \in \mathfrak{g}\text{-}\mathrm{K}[\mathfrak{T}_{\mathfrak{g}}]$, $\mathfrak{g}\text{-}\mathrm{Cl}_{\mathfrak{g}} \circ \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \subseteq \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$, where $\mathfrak{g}\text{-}\mathrm{Cl}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ is the $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-closure operator in $\mathfrak{T}_{\mathfrak{g}}$. But,

$$\begin{aligned} \mathfrak{g}\text{-}\mathrm{Cl}_{\mathfrak{g}} \circ \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) &\longleftrightarrow \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \cup \mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}} \circ \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \\ &\supseteq \mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}} \circ \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}). \end{aligned}$$

Hence, $\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \in \mathfrak{g}\text{-}\mathrm{K}[\mathfrak{T}_{\mathfrak{g}}] \longleftrightarrow \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \supseteq \mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}} \circ \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$. The proof of the proposition is complete. Q.E.D.

In a $\mathscr{T}_{\mathfrak{g}}$-space, the $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-derived operator is $\cup$-*preserving* and the $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-coderived operator is $\cap$-*preserving*, as shown in the following theorem.

THEOREM 3.14. *If* $\mathfrak{g}\text{-}\mathbf{Dc}_{\mathfrak{g}} \in \mathfrak{g}\text{-}\mathrm{DC}[\mathfrak{T}_{\mathfrak{g}}]$ *be a pair of* $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-*operators* $\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}$, $\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ *in a* $\mathscr{T}_{\mathfrak{g}}$-*space* $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$, *then:*

- I. $\mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}\cup\mathscr{S}_{\mathfrak{g}}) \longleftrightarrow \bigcup_{\mathscr{U}_{\mathfrak{g}}=\mathscr{R}_{\mathfrak{g}},\mathscr{S}_{\mathfrak{g}}} \mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{U}_{\mathfrak{g}})$,
- II. $\mathfrak{g}\text{-Cod}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}\cap\mathscr{S}_{\mathfrak{g}}) \longleftrightarrow \bigcap_{\mathscr{V}_{\mathfrak{g}}=\mathscr{R}_{\mathfrak{g}},\mathscr{S}_{\mathfrak{g}}} \mathfrak{g}\text{-Cod}_{\mathfrak{g}}(\mathscr{U}_{\mathfrak{g}})$,

for every $(\mathscr{R}_{\mathfrak{g}},\mathscr{S}_{\mathfrak{g}})\in\times_{\alpha\in I_2^*}\mathscr{P}(\Omega)$.

Proof. Let $\mathfrak{g}\text{-}\mathbf{Dc}_{\mathfrak{g}}\in\mathfrak{g}\text{-DC}[\mathfrak{T}_{\mathfrak{g}}]$ be a pair of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-operators $\mathfrak{g}\text{-Der}_{\mathfrak{g}}$, $\mathfrak{g}\text{-Cod}_{\mathfrak{g}}$: $\mathscr{P}(\Omega)\longrightarrow\mathscr{P}(\Omega)$, and let $(\mathscr{R}_{\mathfrak{g}},\mathscr{S}_{\mathfrak{g}})\in\times_{\alpha\in I_2^*}\mathscr{P}(\Omega)$ be arbitrary in a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}}=(\Omega,\mathscr{T}_{\mathfrak{g}})$. Then, since $(\mathscr{R}_{\mathfrak{g}}\cup\mathscr{S}_{\mathfrak{g}})\cap\mathfrak{g}\text{-Op}_{\mathfrak{g}}(\{\xi\})\longleftrightarrow\bigcup_{\mathscr{U}_{\mathfrak{g}}=\mathscr{R}_{\mathfrak{g}},\mathscr{S}_{\mathfrak{g}}}(\mathscr{U}_{\mathfrak{g}}\cap\mathfrak{g}\text{-Op}_{\mathfrak{g}}(\{\xi\}))$ and $(\mathscr{R}_{\mathfrak{g}}\cap\mathscr{S}_{\mathfrak{g}})\cup\{\xi\}\longleftrightarrow\bigcap_{\mathscr{V}_{\mathfrak{g}}=\mathscr{R}_{\mathfrak{g}},\mathscr{S}_{\mathfrak{g}}}(\mathscr{V}_{\mathfrak{g}}\cup\{\xi\})$, it results that

$$\mathfrak{g}\text{-Cl}_{\mathfrak{g}}:\bigcup_{\mathscr{U}_{\mathfrak{g}}=\mathscr{R}_{\mathfrak{g}},\mathscr{S}_{\mathfrak{g}}}(\mathscr{U}_{\mathfrak{g}}\cap\mathfrak{g}\text{-Op}_{\mathfrak{g}}(\{\xi\})) \longmapsto \bigcup_{\mathscr{U}_{\mathfrak{g}}=\mathscr{R}_{\mathfrak{g}},\mathscr{S}_{\mathfrak{g}}}\mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{U}_{\mathfrak{g}}\cap\mathfrak{g}\text{-Op}_{\mathfrak{g}}(\{\xi\})),$$

$$\mathfrak{g}\text{-Int}_{\mathfrak{g}}:\bigcap_{\mathscr{V}_{\mathfrak{g}}=\mathscr{R}_{\mathfrak{g}},\mathscr{S}_{\mathfrak{g}}}(\mathscr{V}_{\mathfrak{g}}\cup\{\xi\}) \longmapsto \bigcap_{\mathscr{V}_{\mathfrak{g}}=\mathscr{R}_{\mathfrak{g}},\mathscr{S}_{\mathfrak{g}}}\mathfrak{g}\text{-Int}_{\mathfrak{g}}(\mathscr{V}_{\mathfrak{g}}\cup\{\xi\}).$$

Consequently,

$$\begin{aligned}\mathfrak{g}\text{-Der}_{\mathfrak{g}}:\mathscr{R}_{\mathfrak{g}}\cup\mathscr{S}_{\mathfrak{g}} &\longmapsto \left\{\xi\in\mathfrak{T}_{\mathfrak{g}}:\ \xi\in\mathfrak{g}\text{-Cl}_{\mathfrak{g}}\left(\left(\bigcup_{\mathscr{U}_{\mathfrak{g}}=\mathscr{R}_{\mathfrak{g}},\mathscr{S}_{\mathfrak{g}}}\mathscr{U}_{\mathfrak{g}}\right)\cap\mathfrak{g}\text{-Op}_{\mathfrak{g}}(\{\xi\})\right)\right\}\\ &\longleftrightarrow \bigcup_{\mathscr{U}_{\mathfrak{g}}=\mathscr{R}_{\mathfrak{g}},\mathscr{S}_{\mathfrak{g}}}\{\xi\in\mathfrak{T}_{\mathfrak{g}}:\ \xi\in\mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{U}_{\mathfrak{g}}\cap\mathfrak{g}\text{-Op}_{\mathfrak{g}}(\{\xi\}))\}\\ &\longleftrightarrow \bigcup_{\mathscr{U}_{\mathfrak{g}}=\mathscr{R}_{\mathfrak{g}},\mathscr{S}_{\mathfrak{g}}}\mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{U}_{\mathfrak{g}}),\end{aligned}$$

and

$$\begin{aligned}\mathfrak{g}\text{-Cod}_{\mathfrak{g}}:\mathscr{R}_{\mathfrak{g}}\cap\mathscr{S}_{\mathfrak{g}} &\longmapsto \left\{\xi\in\mathfrak{T}_{\mathfrak{g}}:\ \xi\in\mathfrak{g}\text{-Int}_{\mathfrak{g}}\left(\left(\bigcap_{\mathscr{V}_{\mathfrak{g}}=\mathscr{R}_{\mathfrak{g}},\mathscr{S}_{\mathfrak{g}}}\mathscr{U}_{\mathfrak{g}}\right)\cup\{\xi\}\right)\right\}\\ &\longleftrightarrow \bigcap_{\mathscr{V}_{\mathfrak{g}}=\mathscr{R}_{\mathfrak{g}},\mathscr{S}_{\mathfrak{g}}}\{\xi\in\mathfrak{T}_{\mathfrak{g}}:\ \xi\in\mathfrak{g}\text{-Int}_{\mathfrak{g}}(\mathscr{V}_{\mathfrak{g}}\cup\{\xi\})\}\\ &\longleftrightarrow \bigcap_{\mathscr{V}_{\mathfrak{g}}=\mathscr{R}_{\mathfrak{g}},\mathscr{S}_{\mathfrak{g}}}\mathfrak{g}\text{-Cod}_{\mathfrak{g}}(\mathscr{V}_{\mathfrak{g}}).\end{aligned}$$

Hence, it follows that $\mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}\cup\mathscr{S}_{\mathfrak{g}})$ and $\mathfrak{g}\text{-Cod}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}\cap\mathscr{S}_{\mathfrak{g}})$ are equivalent to $\bigcup_{\mathscr{U}_{\mathfrak{g}}=\mathscr{R}_{\mathfrak{g}},\mathscr{S}_{\mathfrak{g}}}\mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{U}_{\mathfrak{g}})$ and $\bigcap_{\mathscr{V}_{\mathfrak{g}}=\mathscr{R}_{\mathfrak{g}},\mathscr{S}_{\mathfrak{g}}}\mathfrak{g}\text{-Cod}_{\mathfrak{g}}(\mathscr{U}_{\mathfrak{g}})$, respectively. The proof of the theorem is complete. Q.E.D.

Since the relation $\mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}\cup\mathscr{S}_{\mathfrak{g}})\subseteq\bigcup_{\mathscr{U}_{\mathfrak{g}}=\mathscr{R}_{\mathfrak{g}},\mathscr{S}_{\mathfrak{g}}}\mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{U}_{\mathfrak{g}})$ and the relation $\mathfrak{g}\text{-Int}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}\cap\mathscr{S}_{\mathfrak{g}})\supseteq\bigcap_{\mathscr{V}_{\mathfrak{g}}=\mathscr{R}_{\mathfrak{g}},\mathscr{S}_{\mathfrak{g}}}\mathfrak{g}\text{-Int}_{\mathfrak{g}}(\mathscr{U}_{\mathfrak{g}})$ hold for every $(\mathscr{R}_{\mathfrak{g}},\mathscr{S}_{\mathfrak{g}})\in\times_{\alpha\in I_2^*}\mathscr{P}(\Omega)$, Items I., II. of Thm. 3.14 can be rewritten, giving their *weaker* forms which read, the $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-derived operator is $\cup$-*subadditive* and the $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-coderived operator is $\cap$-*supadditive*. Accordingly, the corollary stated below can be viewed as a consequence of Thm. 3.14.

Corollary 3.15. *If* $\mathfrak{g}\text{-}\mathbf{Dc}_{\mathfrak{g}}\in\mathfrak{g}\text{-DC}[\mathfrak{T}_{\mathfrak{g}}]$ *be a pair of* $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-*operators* $\mathfrak{g}\text{-Der}_{\mathfrak{g}}$, $\mathfrak{g}\text{-Cod}_{\mathfrak{g}}$: $\mathscr{P}(\Omega)\longrightarrow\mathscr{P}(\Omega)$ *in a* $\mathscr{T}_{\mathfrak{g}}$-*space* $\mathfrak{T}_{\mathfrak{g}}=(\Omega,\mathscr{T}_{\mathfrak{g}})$, *then:*

- I. $\mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}\cup\mathscr{S}_{\mathfrak{g}})\subseteq\bigcup_{\mathscr{U}_{\mathfrak{g}}=\mathscr{R}_{\mathfrak{g}},\mathscr{S}_{\mathfrak{g}}}\mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{U}_{\mathfrak{g}})$,

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- II. $\mathfrak{g}\text{-Cod}_{\mathfrak{g}}\left(\mathscr{R}_{\mathfrak{g}} \cap \mathscr{S}_{\mathfrak{g}}\right) \supseteq \bigcap_{\mathscr{V}_{\mathfrak{g}}=\mathscr{R}_{\mathfrak{g}},\mathscr{S}_{\mathfrak{g}}} \mathfrak{g}\text{-Cod}_{\mathfrak{g}}\left(\mathscr{U}_{\mathfrak{g}}\right)$,

for every $(\mathscr{R}_{\mathfrak{g}}, \mathscr{S}_{\mathfrak{g}}) \in \times_{\alpha \in I_2^*} \mathscr{P}(\Omega)$.

In a $\mathscr{T}_{\mathfrak{g}}$-space, the image of a $\mathfrak{T}_{\mathfrak{g}}$-set under a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-derived operator is equivalent to the image of the relative complement of any $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-derived unit set in the $\mathfrak{T}_{\mathfrak{g}}$-set under the $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-derived operator; the image of the $\mathfrak{T}_{\mathfrak{g}}$-set under a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-coderived operator is equivalent to the image of the union of the $\mathfrak{T}_{\mathfrak{g}}$-set and any $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-coderived unit set under the $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-coderived operator. The theorem follows.

THEOREM 3.16. *If* $\mathfrak{g}\text{-}\mathbf{Dc}_{\mathfrak{g}} \in \mathfrak{g}\text{-DC}[\mathfrak{T}_{\mathfrak{g}}]$ *be a pair of* $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$*-operators* $\mathfrak{g}\text{-Der}_{\mathfrak{g}}$, $\mathfrak{g}\text{-Cod}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ *in a* $\mathscr{T}_{\mathfrak{g}}$*-space* $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$*, then:*

$$\begin{gathered}\left(\{\xi\},\{\zeta\}\right) \subset \mathfrak{g}\text{-}\mathbf{Dc}_{\mathfrak{g}}\left(\mathscr{R}_{\mathfrak{g}},\mathscr{S}_{\mathfrak{g}}\right)\\ \Big\updownarrow \\ \left(\{\xi\},\{\zeta\}\right) \subset \mathfrak{g}\text{-}\mathbf{Dc}_{\mathfrak{g}}\left(\mathscr{R}_{\mathfrak{g}} \cap \mathfrak{g}\text{-Op}_{\mathfrak{g}}\left(\{\xi\}\right), \mathscr{S}_{\mathfrak{g}} \cup \{\zeta\}\right)\end{gathered} \tag{3.5}$$

for any $(\mathscr{R}_{\mathfrak{g}}, \mathscr{S}_{\mathfrak{g}}) \in \times_{\alpha \in I_2^*} \mathscr{P}(\Omega)$.

PROOF. Let $\mathfrak{g}\text{-}\mathbf{Dc}_{\mathfrak{g}} \in \mathfrak{g}\text{-DC}[\mathfrak{T}_{\mathfrak{g}}]$ be a pair of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-operators $\mathfrak{g}\text{-Der}_{\mathfrak{g}}$, $\mathfrak{g}\text{-Cod}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$, and let $(\mathscr{R}_{\mathfrak{g}}, \mathscr{S}_{\mathfrak{g}}) \in \times_{\alpha \in I_2^*} \mathscr{P}(\Omega)$ be arbitrary in a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$. Then, since $\mathscr{S}_{\mathfrak{g}} \cap \mathfrak{g}\text{-Op}_{\mathfrak{g}}(\{\xi\}) \longleftrightarrow \left(\mathscr{S}_{\mathfrak{g}} \cap \mathfrak{g}\text{-Op}_{\mathfrak{g}}(\{\xi\})\right) \cap \mathfrak{g}\text{-Op}_{\mathfrak{g}}(\{\xi\})$ and $\mathscr{S}_{\mathfrak{g}} \cup \{\xi\} \longleftrightarrow \left(\mathscr{S}_{\mathfrak{g}} \cup \{\xi\}\right) \cup \{\xi\}$, it results that

$$\begin{gathered}\left(\{\xi\},\{\zeta\}\right) \subset \mathfrak{g}\text{-}\mathbf{Dc}_{\mathfrak{g}}\left(\mathscr{R}_{\mathfrak{g}},\mathscr{S}_{\mathfrak{g}}\right)\\ \Big\updownarrow \\ \left(\{\xi\},\{\zeta\}\right) \subset \mathfrak{g}\text{-Der}_{\mathfrak{g}}\left(\mathscr{R}_{\mathfrak{g}} \cap \mathfrak{g}\text{-Op}_{\mathfrak{g}}\left(\{\xi\}\right)\right) \times \mathfrak{g}\text{-Cod}_{\mathfrak{g}}\left(\mathscr{S}_{\mathfrak{g}} \cup \{\zeta\}\right)\\ \Big\updownarrow \\ \left(\{\xi\},\{\zeta\}\right) \subset \mathfrak{g}\text{-}\mathbf{Dc}_{\mathfrak{g}}\left(\mathscr{R}_{\mathfrak{g}} \cap \mathfrak{g}\text{-Op}_{\mathfrak{g}}\left(\{\xi\}\right), \mathscr{S}_{\mathfrak{g}} \cup \{\zeta\}\right)\end{gathered}$$

holds for any $(\mathscr{R}_{\mathfrak{g}}, \mathscr{S}_{\mathfrak{g}}) \in \times_{\alpha \in I_2^*} \mathscr{P}(\Omega)$. Hence, $\left(\{\xi\},\{\zeta\}\right) \subset \mathfrak{g}\text{-}\mathbf{Dc}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}, \mathscr{S}_{\mathfrak{g}}) \longleftrightarrow \left(\{\xi\},\{\zeta\}\right) \subset \mathfrak{g}\text{-}\mathbf{Dc}_{\mathfrak{g}}\left(\mathscr{R}_{\mathfrak{g}} \cap \mathfrak{g}\text{-Op}_{\mathfrak{g}}(\{\xi\}), \mathscr{S}_{\mathfrak{g}} \cup \{\zeta\}\right)$. The proof of the theorem is complete. Q.E.D.

An immediate consequence of the above theorem is the corollary stated below.

COROLLARY 3.17. *If* $\mathfrak{g}\text{-}\mathbf{Dc}_{\mathfrak{g}} \in \mathfrak{g}\text{-DC}[\mathfrak{T}_{\mathfrak{g}}]$ *be a pair of* $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$*-operators* $\mathfrak{g}\text{-Der}_{\mathfrak{g}}$, $\mathfrak{g}\text{-Cod}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ *in a* $\mathscr{T}_{\mathfrak{g}}$*-space* $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$*, then:*

$$\begin{gathered}\left(\{\xi\},\{\zeta\}\right) \not\subset \mathfrak{g}\text{-}\mathbf{Dc}_{\mathfrak{g}}\left(\mathscr{R}_{\mathfrak{g}},\mathscr{S}_{\mathfrak{g}}\right)\\ \Big\updownarrow \\ \left(\{\xi\},\{\zeta\}\right) \subset \left(\mathfrak{g}\text{-Cod}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Op}_{\mathfrak{g}}\left(\mathscr{R}_{\mathfrak{g}}\right), \mathfrak{g}\text{-Der}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Op}_{\mathfrak{g}}\left(\mathscr{S}_{\mathfrak{g}}\right)\right)\end{gathered} \tag{3.6}$$

for any $(\mathscr{R}_{\mathfrak{g}}, \mathscr{S}_{\mathfrak{g}}) \in \times_{\alpha \in I_2^*} \mathscr{P}(\Omega)$.

The proposition given below is a further consequence of the above theorem and corollary.

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Proposition 3.18. *If* $\mathfrak{g}\text{-}\mathbf{Dc}_{\mathfrak{g}} \in \mathfrak{g}\text{-}\mathrm{DC}[\mathfrak{T}_{\mathfrak{g}}]$ *be a pair of* $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}$*-operators* $\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}$, $\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ *in a* $\mathscr{T}_{\mathfrak{g}}$*-space* $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$*, then:*

$$\mathfrak{g}\text{-}\mathbf{Dc}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}, \mathscr{S}_{\mathfrak{g}}) \longleftrightarrow \mathfrak{g}\text{-}\mathbf{Dc}_{\mathfrak{g}}\big(\mathscr{R}_{\mathfrak{g}} \cup \{\xi\}, \mathscr{S}_{\mathfrak{g}} \cap \mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}(\{\zeta\})\big) \tag{3.7}$$

for any $(\mathscr{R}_{\mathfrak{g}}, \mathscr{S}_{\mathfrak{g}}) \in \times_{\alpha\in I_2^*} \mathscr{P}(\Omega)$.

Proof. Let $\mathfrak{g}\text{-}\mathbf{Dc}_{\mathfrak{g}} \in \mathfrak{g}\text{-}\mathrm{DC}[\mathfrak{T}_{\mathfrak{g}}]$ be a pair of $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}$-operators $\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}$, $\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$, and let $(\mathscr{R}_{\mathfrak{g}}, \mathscr{S}_{\mathfrak{g}}) \in \times_{\alpha\in I_2^*} \mathscr{P}(\Omega)$ be arbitrary in a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$. Then, since

$$\begin{aligned}\mathfrak{g}\text{-}\mathrm{Cl}_{\mathfrak{g}}\big(\mathscr{R}_{\mathfrak{g}} \cap \mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}(\{\xi\})\big) &\longleftrightarrow \mathfrak{g}\text{-}\mathrm{Cl}_{\mathfrak{g}}\big((\mathscr{R}_{\mathfrak{g}} \cap \mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}(\{\xi\})) \cup (\{\xi\} \cap \mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}(\{\xi\}))\big)\\ &\longleftrightarrow \mathfrak{g}\text{-}\mathrm{Cl}_{\mathfrak{g}}\big((\mathscr{R}_{\mathfrak{g}} \cup \{\xi\}) \cap \mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}(\{\xi\})\big),\end{aligned}$$

it results that

$$\begin{aligned}\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}) &\longleftrightarrow \big\{\xi \in \mathfrak{T}_{\mathfrak{g}} :\ \xi \in \mathfrak{g}\text{-}\mathrm{Cl}_{\mathfrak{g}}\big(\mathscr{S}_{\mathfrak{g}} \cap \mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}(\{\xi\})\big)\big\}\\ &\longleftrightarrow \big\{\xi \in \mathfrak{T}_{\mathfrak{g}} :\ \xi \in \mathfrak{g}\text{-}\mathrm{Cl}_{\mathfrak{g}}\big((\mathscr{R}_{\mathfrak{g}} \cup \{\xi\}) \cap \mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}(\{\xi\})\big)\big\}\\ &\longleftrightarrow \mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}\big(\mathscr{R}_{\mathfrak{g}} \cup \{\xi\}\big).\end{aligned}$$

Therefore, $\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}) \longleftrightarrow \mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}\big(\mathscr{R}_{\mathfrak{g}} \cup \{\xi\}\big)$. Since

$$\begin{aligned}\mathfrak{g}\text{-}\mathrm{Int}_{\mathfrak{g}}\big(\mathscr{S}_{\mathfrak{g}} \cup \{\zeta\}\big) &\longleftrightarrow \mathfrak{g}\text{-}\mathrm{Int}_{\mathfrak{g}}\big((\mathscr{S}_{\mathfrak{g}} \cup \{\zeta\}) \cap (\{\zeta\} \cap \mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}(\{\zeta\}))\big)\\ &\longleftrightarrow \mathfrak{g}\text{-}\mathrm{Int}_{\mathfrak{g}}\big((\mathscr{S}_{\mathfrak{g}} \cap \mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}(\{\zeta\})) \cup \{\zeta\}\big),\end{aligned}$$

it follows that

$$\begin{aligned}\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) &\longleftrightarrow \big\{\zeta \in \mathfrak{T}_{\mathfrak{g}} :\ \zeta \in \mathfrak{g}\text{-}\mathrm{Int}_{\mathfrak{g}}\big(\mathscr{S}_{\mathfrak{g}} \cup \{\zeta\}\big)\big\}\\ &\longleftrightarrow \big\{\zeta \in \mathfrak{T}_{\mathfrak{g}} :\ \zeta \in \mathfrak{g}\text{-}\mathrm{Int}_{\mathfrak{g}}\big((\mathscr{S}_{\mathfrak{g}} \cap \mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}(\{\zeta\})) \cup \{\zeta\}\big)\big\}\\ &\longleftrightarrow \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}\big(\mathscr{S}_{\mathfrak{g}} \cap \mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}(\{\zeta\})\big).\end{aligned}$$

Therefore, $\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \longleftrightarrow \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}\big(\mathscr{S}_{\mathfrak{g}} \cap \mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}(\{\zeta\})\big)$. Hence, it follows that $\mathfrak{g}\text{-}\mathbf{Dc}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}, \mathscr{S}_{\mathfrak{g}}) \longleftrightarrow \mathfrak{g}\text{-}\mathbf{Dc}_{\mathfrak{g}}\big(\mathscr{R}_{\mathfrak{g}} \cup \{\xi\}, \mathscr{S}_{\mathfrak{g}} \cap \mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}(\{\zeta\})\big)$. The proof of the proposition is complete. Q.E.D.

In a $\mathscr{T}_{\mathfrak{g}}$-space, the notions of $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}$-derived and $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}$-coderived points can also be characterized in terms of the notions of $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}$-unit sets. These are embodied in the proposition that follows.

Proposition 3.19. *If* $\mathfrak{g}\text{-}\mathbf{Dc}_{\mathfrak{g}} \in \mathfrak{g}\text{-}\mathrm{DC}[\mathfrak{T}_{\mathfrak{g}}]$ *be a pair of* $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}$*-operators* $\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}$, $\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$*, and* $\big(\{\xi\}, \{\zeta\}\big) \subset \mathfrak{T}_{\mathfrak{g}} \times \mathfrak{T}_{\mathfrak{g}}$ *be a pair of* $\mathfrak{T}_{\mathfrak{g}}$*-unit sets in a* $\mathscr{T}_{\mathfrak{g}}$*-space* $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$*, then:*

$$\big(\{\xi\}, \{\zeta\}\big) \not\subset \mathfrak{g}\text{-}\mathbf{Dc}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}, \mathscr{S}_{\mathfrak{g}}) \longleftrightarrow \big(\{\xi\}, \{\zeta\}\big) \in \mathfrak{g}\text{-}\mathrm{O}[\mathfrak{T}_{\mathfrak{g}}] \times \mathfrak{g}\text{-}\mathrm{K}[\mathfrak{T}_{\mathfrak{g}}] \tag{3.8}$$

for any $(\mathscr{R}_{\mathfrak{g}}, \mathscr{S}_{\mathfrak{g}}) \in \times_{\alpha\in I_2^*} \mathscr{P}(\Omega)$.

Proof. Let $\mathfrak{g}\text{-}\mathbf{Dc}_{\mathfrak{g}} \in \mathfrak{g}\text{-}\mathrm{DC}[\mathfrak{T}_{\mathfrak{g}}]$ be a pair of $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}$-operators $\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}$, $\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$, and let $\big(\{\xi\}, \{\zeta\}\big) \subset \mathfrak{T}_{\mathfrak{g}} \times \mathfrak{T}_{\mathfrak{g}}$ be a pair of $\mathfrak{T}_{\mathfrak{g}}$-unit sets in a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$. Suppose $\big(\{\xi\}, \{\zeta\}\big) \not\subset \mathfrak{g}\text{-}\mathbf{Dc}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}, \mathscr{S}_{\mathfrak{g}})$ for an arbitrary $(\mathscr{R}_{\mathfrak{g}}, \mathscr{S}_{\mathfrak{g}}) \in \times_{\alpha\in I_2^*} \mathscr{P}(\Omega)$. Then, $\big(\{\xi\}, \{\zeta\}\big) \not\subset \mathfrak{g}\text{-}\mathbf{Dc}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}, \mathscr{S}_{\mathfrak{g}})$ implies $\big(\{\xi\}, \{\zeta\}\big) \not\subset \mathfrak{g}\text{-}\mathrm{Cl}_{\mathfrak{g}}\big(\mathscr{R}_{\mathfrak{g}} \cap \mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}(\{\xi\})\big) \times \mathfrak{g}\text{-}\mathrm{Int}_{\mathfrak{g}}\big(\mathscr{S}_{\mathfrak{g}} \cup \{\xi\}\big)$. Consequently, it follows that the relation $\big(\{\xi\}, \{\zeta\}\big) \subset \mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}} \circ \mathfrak{g}\text{-}\mathrm{Cl}_{\mathfrak{g}}\big(\mathscr{R}_{\mathfrak{g}} \cap \mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}(\{\xi\})\big) \times \mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}} \circ \mathfrak{g}\text{-}\mathrm{Int}_{\mathfrak{g}}\big(\mathscr{S}_{\mathfrak{g}} \cup \{\xi\}\big)$ holds. But, $\mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}} \circ \mathfrak{g}\text{-}\mathrm{Cl}_{\mathfrak{g}}\big(\mathscr{R}_{\mathfrak{g}} \cap \mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}(\{\xi\})\big) \longleftrightarrow \mathfrak{g}\text{-}\mathrm{Int}_{\mathfrak{g}}\big(\mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}) \cup \{\xi\}\big)$ and

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$\mathfrak{g}\text{-Op}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Int}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}} \cup \{\xi\}) \longleftrightarrow \mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \cap \mathfrak{g}\text{-Op}_{\mathfrak{g}}(\{\xi\}))$ and on the other hand, $\mathfrak{g}\text{-Int}_{\mathfrak{g}}(\mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}) \cup \{\xi\}) \in \mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}]$ and $\mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \cap \mathfrak{g}\text{-Op}_{\mathfrak{g}}(\{\xi\})) \in \mathfrak{g}\text{-K}[\mathfrak{T}_{\mathfrak{g}}]$. Therefore, $(\{\xi\}, \{\zeta\}) \subset (\mathscr{U}_{\mathfrak{g}}, \mathscr{V}_{\mathfrak{g}})$ holds for some $(\mathscr{U}_{\mathfrak{g}}, \mathscr{V}_{\mathfrak{g}}) \in \mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}] \times \mathfrak{g}\text{-K}[\mathfrak{T}_{\mathfrak{g}}]$ and consequently, $(\{\xi\}, \{\zeta\}) \in \mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}] \times \mathfrak{g}\text{-K}[\mathfrak{T}_{\mathfrak{g}}]$. Hence, $(\{\xi\}, \{\zeta\}) \not\subset \mathfrak{g}\text{-}\mathbf{Dc}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}, \mathscr{S}_{\mathfrak{g}}) \longleftrightarrow (\{\xi\}, \{\zeta\}) \in \mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}] \times \mathfrak{g}\text{-K}[\mathfrak{T}_{\mathfrak{g}}]$. The proof of the proposition is complete. Q.E.D.

In a $\mathscr{T}_{\mathfrak{g}}$-space, the $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-derived operator is *intensive* and the $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-coderived operator is *extensive*, as shown in the following theorem.

THEOREM 3.20. *If* $\mathfrak{g}\text{-}\mathbf{Dc}_{\mathfrak{g}} \in \mathfrak{g}\text{-DC}[\mathfrak{T}_{\mathfrak{g}}]$ *be a pair of* $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$*-operators* $\mathfrak{g}\text{-Der}_{\mathfrak{g}}, \mathfrak{g}\text{-Cod}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ *in a* $\mathscr{T}_{\mathfrak{g}}$*-space* $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$*, then:*

- I. $\mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}} \cup \mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}})) \subseteq \mathscr{R}_{\mathfrak{g}} \cup \mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}) \quad \forall \mathscr{R}_{\mathfrak{g}} \in \mathscr{P}(\Omega)$,
- II. $\mathfrak{g}\text{-Cod}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}} \cap \mathfrak{g}\text{-Cod}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})) \supseteq \mathscr{S}_{\mathfrak{g}} \cap \mathfrak{g}\text{-Cod}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \quad \forall \mathscr{S}_{\mathfrak{g}} \in \mathscr{P}(\Omega)$.

PROOF. Let $\mathfrak{g}\text{-}\mathbf{Dc}_{\mathfrak{g}} \in \mathfrak{g}\text{-DC}[\mathfrak{T}_{\mathfrak{g}}]$ be a pair of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-operators $\mathfrak{g}\text{-Der}_{\mathfrak{g}}, \mathfrak{g}\text{-Cod}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$, and let $(\mathscr{R}_{\mathfrak{g}}, \mathscr{S}_{\mathfrak{g}}) \in \times_{\alpha \in I_2^*} \mathscr{P}(\Omega)$ be arbitrary in a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$. Then:

I. Set $\mathscr{U}_{\mathfrak{g}} = \mathscr{R}_{\mathfrak{g}} \cup \mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}})$. Then,

$$\begin{aligned} \mathscr{U}_{\mathfrak{g}} \cup \mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{U}_{\mathfrak{g}}) \longleftrightarrow \mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{U}_{\mathfrak{g}}) &\longleftrightarrow \mathfrak{g}\text{-Cl}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}) \\ &\longleftrightarrow \mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}) \longleftrightarrow \mathscr{R}_{\mathfrak{g}} \cup \mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}). \end{aligned}$$

Hence, $\mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}} \cup \mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}})) \subseteq \mathscr{R}_{\mathfrak{g}} \cup \mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}})$.

II. Set $\mathscr{V}_{\mathfrak{g}} = \mathscr{S}_{\mathfrak{g}} \cap \mathfrak{g}\text{-Cod}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$. Then,

$$\begin{aligned} \mathscr{V}_{\mathfrak{g}} \cap \mathfrak{g}\text{-Cod}_{\mathfrak{g}}(\mathscr{V}_{\mathfrak{g}}) \longleftrightarrow \mathfrak{g}\text{-Int}_{\mathfrak{g}}(\mathscr{V}_{\mathfrak{g}}) &\longleftrightarrow \mathfrak{g}\text{-Int}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Int}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \\ &\longleftrightarrow \mathfrak{g}\text{-Int}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \longleftrightarrow \mathscr{S}_{\mathfrak{g}} \cap \mathfrak{g}\text{-Cod}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}). \end{aligned}$$

Thus, $\mathfrak{g}\text{-Cod}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}} \cap \mathfrak{g}\text{-Cod}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})) \supseteq \mathscr{S}_{\mathfrak{g}} \cap \mathfrak{g}\text{-Cod}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$. The proof of the theorem is complete. Q.E.D.

In a $\mathscr{T}_{\mathfrak{g}}$-space, the image of the union of a $\mathfrak{T}_{\mathfrak{g}}$-set and its $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-derived set under that $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-derived set operator by means of which the $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-derived set is established is equivalent to the image of the $\mathfrak{T}_{\mathfrak{g}}$-set under the $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-derived set operator composition with itself; the image of the intersection of a $\mathfrak{T}_{\mathfrak{g}}$-set and its $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-coderived set under that $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-coderived set operator by means of which the $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-coderived set is established is equivalent to the image of the $\mathfrak{T}_{\mathfrak{g}}$-set under the $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-coderived set operator composition with itself. These are embodied in the following proposition.

PROPOSITION 3.21. *If* $\mathfrak{g}\text{-}\mathbf{Dc}_{\mathfrak{g}} \in \mathfrak{g}\text{-DC}[\mathfrak{T}_{\mathfrak{g}}]$ *be a pair of* $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$*-operators* $\mathfrak{g}\text{-Der}_{\mathfrak{g}}, \mathfrak{g}\text{-Cod}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ *in a* $\mathscr{T}_{\mathfrak{g}}$*-space* $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$*, then:*

- I. $\mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}} \cup \mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}})) \longleftrightarrow \mathfrak{g}\text{-Der}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}) \quad \forall \mathscr{R}_{\mathfrak{g}} \in \mathscr{P}(\Omega)$,
- II. $\mathfrak{g}\text{-Cod}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}} \cap \mathfrak{g}\text{-Cod}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})) \longleftrightarrow \mathfrak{g}\text{-Cod}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Cod}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \quad \forall \mathscr{S}_{\mathfrak{g}} \in \mathscr{P}(\Omega)$.

PROOF. Let $\mathfrak{g}\text{-}\mathbf{Dc}_{\mathfrak{g}} \in \mathfrak{g}\text{-DC}[\mathfrak{T}_{\mathfrak{g}}]$ be a pair of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-operators $\mathfrak{g}\text{-Der}_{\mathfrak{g}}, \mathfrak{g}\text{-Cod}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$, and let $(\mathscr{R}_{\mathfrak{g}}, \mathscr{S}_{\mathfrak{g}}) \in \times_{\alpha \in I_2^*} \mathscr{P}(\Omega)$ be arbitrary in a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$. Then:

I. Since $\mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{U}_{\mathfrak{g}}) \longleftrightarrow \mathscr{U}_{\mathfrak{g}} \cup \mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{U}_{\mathfrak{g}})$ holds for any $\mathscr{U}_{\mathfrak{g}} \in \mathscr{P}(\Omega)$, setting $\mathscr{U}_{\mathfrak{g}} = \mathscr{R}_{\mathfrak{g}} \cup \mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}})$ yields

$$\mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}} \cup \mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}})) \longleftrightarrow (\mathscr{R}_{\mathfrak{g}} \cup \mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}})) \cup \mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}} \cup \mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}})).$$

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But,

$$\begin{aligned}\mathfrak{g}\text{-Cl}_{\mathfrak{g}}\big(\mathscr{R}_{\mathfrak{g}}\cup\mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}})\big) &\longleftrightarrow \mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}})\cup\mathfrak{g}\text{-Cl}_{\mathfrak{g}}\circ\mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}})\\ &\longleftrightarrow \big(\mathscr{R}_{\mathfrak{g}}\cup\mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}})\big)\\ &\quad\cup \big(\mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}})\cup\mathfrak{g}\text{-Der}_{\mathfrak{g}}\circ\mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}})\big)\\ &\longleftrightarrow \big(\mathscr{R}_{\mathfrak{g}}\cup\mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}})\big)\cup\mathfrak{g}\text{-Der}_{\mathfrak{g}}\circ\mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}).\end{aligned}$$

Thus, $\mathfrak{g}\text{-Der}_{\mathfrak{g}}\big(\mathscr{R}_{\mathfrak{g}}\cup\mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}})\big)\longleftrightarrow\mathfrak{g}\text{-Der}_{\mathfrak{g}}\circ\mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}})$.

II. Since $\mathfrak{g}\text{-Int}_{\mathfrak{g}}(\mathscr{V}_{\mathfrak{g}})\longleftrightarrow\mathscr{V}_{\mathfrak{g}}\cap\mathfrak{g}\text{-Cod}_{\mathfrak{g}}(\mathscr{V}_{\mathfrak{g}})$ holds for any $\mathscr{V}_{\mathfrak{g}}\in\mathscr{P}(\Omega)$, setting $\mathscr{V}_{\mathfrak{g}}=\mathscr{S}_{\mathfrak{g}}\cap\mathfrak{g}\text{-Cod}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$ yields

$$\mathfrak{g}\text{-Int}_{\mathfrak{g}}\big(\mathscr{S}_{\mathfrak{g}}\cap\mathfrak{g}\text{-Cod}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})\big)\longleftrightarrow\big(\mathscr{S}_{\mathfrak{g}}\cap\mathfrak{g}\text{-Cod}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})\big)\cap\mathfrak{g}\text{-Cod}_{\mathfrak{g}}\big(\mathscr{S}_{\mathfrak{g}}\cap\mathfrak{g}\text{-Cod}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})\big).$$

But,

$$\begin{aligned}\mathfrak{g}\text{-Int}_{\mathfrak{g}}\big(\mathscr{S}_{\mathfrak{g}}\cap\mathfrak{g}\text{-Cod}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})\big) &\longleftrightarrow \mathfrak{g}\text{-Int}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})\cap\mathfrak{g}\text{-Int}_{\mathfrak{g}}\circ\mathfrak{g}\text{-Cod}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})\\ &\longleftrightarrow \big(\mathscr{S}_{\mathfrak{g}}\cap\mathfrak{g}\text{-Cod}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})\big)\\ &\quad\cap \big(\mathfrak{g}\text{-Cod}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})\cap\mathfrak{g}\text{-Cod}_{\mathfrak{g}}\circ\mathfrak{g}\text{-Cod}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})\big)\\ &\longleftrightarrow \big(\mathscr{S}_{\mathfrak{g}}\cap\mathfrak{g}\text{-Cod}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})\big)\cap\mathfrak{g}\text{-Cod}_{\mathfrak{g}}\circ\mathfrak{g}\text{-Cod}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}).\end{aligned}$$

Hence, $\mathfrak{g}\text{-Cod}_{\mathfrak{g}}\big(\mathscr{S}_{\mathfrak{g}}\cap\mathfrak{g}\text{-Cod}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})\big)\longleftrightarrow\mathfrak{g}\text{-Cod}_{\mathfrak{g}}\circ\mathfrak{g}\text{-Cod}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$. The proof of the proposition is complete. Q.E.D.

The corollary stated below is a consequence of the above proposition and the theorem preceding it.

Corollary 3.22. *If* $\mathfrak{g}\text{-}\mathbf{Dc}_{\mathfrak{g}}\in\mathfrak{g}\text{-DC}[\mathfrak{T}_{\mathfrak{g}}]$ *be a pair of* $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$*-operators* $\mathfrak{g}\text{-Der}_{\mathfrak{g}},\mathfrak{g}\text{-Cod}_{\mathfrak{g}}:\mathscr{P}(\Omega)\longrightarrow\mathscr{P}(\Omega)$ *in a* $\mathscr{T}_{\mathfrak{g}}$*-space* $\mathfrak{T}_{\mathfrak{g}}=(\Omega,\mathscr{T}_{\mathfrak{g}})$*, then:*

- I. $\mathfrak{g}\text{-Der}_{\mathfrak{g}}\circ\mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}})\subseteq\mathscr{R}_{\mathfrak{g}}\cup\mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}})\quad\forall\mathscr{R}_{\mathfrak{g}}\in\mathscr{P}(\Omega)$,
- II. $\mathfrak{g}\text{-Cod}_{\mathfrak{g}}\circ\mathfrak{g}\text{-Cod}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})\supseteq\mathscr{S}_{\mathfrak{g}}\cap\mathfrak{g}\text{-Cod}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})\quad\forall\mathscr{S}_{\mathfrak{g}}\in\mathscr{P}(\Omega)$.

In other words, the above corollary states that, in a $\mathscr{T}_{\mathfrak{g}}$-space, the union of a $\mathfrak{T}_{\mathfrak{g}}$-set and its $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-derived set includes the image of the $\mathfrak{T}_{\mathfrak{g}}$-set under that $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-derived set operator composition with itself; the intersection of a $\mathfrak{T}_{\mathfrak{g}}$-set and its $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-coderived set is included in the image of the $\mathfrak{T}_{\mathfrak{g}}$-set under that $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-coderived set operator composition with itself. The $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-derived set and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-coderived set operators have other properties which are equivalent to those presented in the above corollary, and are embodied in the following proposition.

Proposition 3.23. *If* $\mathfrak{g}\text{-}\mathbf{Dc}_{\mathfrak{g}}\in\mathfrak{g}\text{-DC}[\mathfrak{T}_{\mathfrak{g}}]$ *be a pair of* $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$*-operators* $\mathfrak{g}\text{-Der}_{\mathfrak{g}}$, $\mathfrak{g}\text{-Cod}_{\mathfrak{g}}:\mathscr{P}(\Omega)\longrightarrow\mathscr{P}(\Omega)$ *in a* $\mathscr{T}_{\mathfrak{g}}$*-space* $\mathfrak{T}_{\mathfrak{g}}=(\Omega,\mathscr{T}_{\mathfrak{g}})$*, then the following logical implications holds:*

- I. *For any* $(\mathscr{R}_{\mathfrak{g}},\mathscr{S}_{\mathfrak{g}})\in\times_{\alpha\in I_2^*}\mathscr{P}(\Omega)$,

$$\begin{gathered}\big(\mathfrak{g}\text{-Der}_{\mathfrak{g}}\big(\mathscr{R}_{\mathfrak{g}}\cup\mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}})\big)\subseteq\mathscr{R}_{\mathfrak{g}}\cup\mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}})\big)\\ \wedge\big(\mathfrak{g}\text{-Der}_{\mathfrak{g}}\big(\mathscr{R}_{\mathfrak{g}}\cup\mathscr{S}_{\mathfrak{g}}\big)=\mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}})\cup\mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})\big)\\ \updownarrow\\ \mathfrak{g}\text{-Der}_{\mathfrak{g}}\circ\mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}})\subseteq\mathscr{R}_{\mathfrak{g}}\cup\mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}).\end{gathered}\tag{3.9}$$

- II. *For any* $(\mathscr{R}_{\mathfrak{g}}, \mathscr{S}_{\mathfrak{g}}) \in \times_{\alpha\in I_2^*} \mathscr{P}(\Omega)$,

$$\begin{array}{c}
\big(\mathfrak{g}\text{-Cod}_{\mathfrak{g}}\big(\mathscr{S}_{\mathfrak{g}} \cap \mathfrak{g}\text{-Cod}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})\big) \supseteq \mathscr{S}_{\mathfrak{g}} \cap \mathfrak{g}\text{-Cod}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})\big) \\
\wedge\big(\mathfrak{g}\text{-Cod}_{\mathfrak{g}}\big(\mathscr{R}_{\mathfrak{g}} \cap \mathscr{S}_{\mathfrak{g}}\big) = \mathfrak{g}\text{-Cod}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}) \cap \mathfrak{g}\text{-Cod}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})\big) \\
\updownarrow \\
\mathfrak{g}\text{-Cod}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Cod}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \supseteq \mathscr{S}_{\mathfrak{g}} \cap \mathfrak{g}\text{-Cod}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}).
\end{array} \tag{3.10}$$

PROOF. Let $\mathfrak{g}\text{-}\mathbf{Dc}_{\mathfrak{g}} \in \mathfrak{g}\text{-DC}[\mathfrak{T}_{\mathfrak{g}}]$ be a pair of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-operators $\mathfrak{g}\text{-Der}_{\mathfrak{g}}$, $\mathfrak{g}\text{-Cod}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$, and let $(\mathscr{R}_{\mathfrak{g}}, \mathscr{S}_{\mathfrak{g}}) \in \times_{\alpha\in I_2^*} \mathscr{P}(\Omega)$ be arbitrary in a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$. Then:

I. Substitute $\mathscr{S}_{\mathfrak{g}} = \mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}})$ in $\mathfrak{g}\text{-Der}_{\mathfrak{g}}\big(\mathscr{R}_{\mathfrak{g}} \cup \mathscr{S}_{\mathfrak{g}}\big) = \bigcup_{\mathscr{U}_{\mathfrak{g}}=\mathscr{R}_{\mathfrak{g}},\mathscr{S}_{\mathfrak{g}}} \mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{U}_{\mathfrak{g}})$ and then take $\mathfrak{g}\text{-Der}_{\mathfrak{g}}\big(\mathscr{R}_{\mathfrak{g}} \cup \mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}})\big) \subseteq \mathscr{R}_{\mathfrak{g}} \cup \mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}})$ into account. Consequently,

$$\mathfrak{g}\text{-Der}_{\mathfrak{g}}\big(\mathscr{R}_{\mathfrak{g}} \cup \mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}})\big) = \bigcup_{\mathscr{U}_{\mathfrak{g}}=\mathscr{R}_{\mathfrak{g}},\mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}})} \mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{U}_{\mathfrak{g}}) \subseteq \mathscr{R}_{\mathfrak{g}} \cup \mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}).$$

Thus, $\mathfrak{g}\text{-Der}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}) \subseteq \mathscr{R}_{\mathfrak{g}} \cup \mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}})$.

II. Substitute $\mathscr{R}_{\mathfrak{g}} = \mathfrak{g}\text{-Cod}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$ in $\mathfrak{g}\text{-Cod}_{\mathfrak{g}}\big(\mathscr{R}_{\mathfrak{g}} \cap \mathscr{S}_{\mathfrak{g}}\big) = \bigcap_{\mathscr{V}_{\mathfrak{g}}=\mathscr{R}_{\mathfrak{g}},\mathscr{S}_{\mathfrak{g}}} \mathfrak{g}\text{-Cod}_{\mathfrak{g}}(\mathscr{V}_{\mathfrak{g}})$ and then take $\mathfrak{g}\text{-Cod}_{\mathfrak{g}}\big(\mathscr{S}_{\mathfrak{g}} \cap \mathfrak{g}\text{-Cod}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})\big) \supseteq \mathscr{S}_{\mathfrak{g}} \cap \mathfrak{g}\text{-Cod}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$ into account. Consequently,

$$\mathfrak{g}\text{-Cod}_{\mathfrak{g}}\big(\mathscr{S}_{\mathfrak{g}} \cap \mathfrak{g}\text{-Cod}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})\big) = \bigcap_{\mathscr{V}_{\mathfrak{g}}=\mathscr{S}_{\mathfrak{g}},\mathfrak{g}\text{-Cod}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})} \mathfrak{g}\text{-Cod}_{\mathfrak{g}}(\mathscr{V}_{\mathfrak{g}}) \supseteq \mathscr{S}_{\mathfrak{g}} \cap \mathfrak{g}\text{-Cod}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}).$$

Hence, $\mathfrak{g}\text{-Cod}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Cod}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \supseteq \mathscr{S}_{\mathfrak{g}} \cap \mathfrak{g}\text{-Cod}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$. The proof of the proposition is complete. Q.E.D.

In a $\mathscr{T}_{\mathfrak{g}}$-space, just as the $\mathfrak{T}_{\mathfrak{g}}$-derived set and $\mathfrak{T}_{\mathfrak{g}}$-coderived set operators are both *monotone*, so are both the $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-derived set and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-derived set operators, as shown in the following proposition.

PROPOSITION 3.24. *If* $\mathfrak{g}\text{-}\mathbf{Dc}_{\mathfrak{g}} \in \mathfrak{g}\text{-DC}[\mathfrak{T}_{\mathfrak{g}}]$ *be a given pair of* $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$*-operators* $\mathfrak{g}\text{-Der}_{\mathfrak{g}}$, $\mathfrak{g}\text{-Cod}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ *and* $\mathbf{dc}_{\mathfrak{g}} \in \text{DC}[\mathfrak{T}_{\mathfrak{g}}]$ *be a given pair of* $\mathfrak{T}_{\mathfrak{g}}$*-operators* $\text{der}_{\mathfrak{g}}$, $\text{cod}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ *in a* $\mathscr{T}_{\mathfrak{g}}$*-space* $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$, *then:*

- I. *For any* $(\mathscr{R}_{\mathfrak{g}}, \mathscr{S}_{\mathfrak{g}}) \in \times_{\alpha\in I_2^*} \mathscr{P}(\Omega)$,

$$\begin{array}{ccc}
\mathscr{R}_{\mathfrak{g}} \subseteq \mathscr{S}_{\mathfrak{g}} & \longrightarrow & \mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}) \subseteq \mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \\
 & \searrow & \downarrow \\
 & & \text{der}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}) \subseteq \text{der}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}).
\end{array} \tag{3.11}$$

- II. *For any* $(\mathscr{U}_{\mathfrak{g}}, \mathscr{V}_{\mathfrak{g}}) \in \times_{\alpha\in I_2^*} \mathscr{P}(\Omega)$,

$$\begin{array}{ccc}
\mathscr{U}_{\mathfrak{g}} \supseteq \mathscr{V}_{\mathfrak{g}} & \longrightarrow & \mathfrak{g}\text{-Cod}_{\mathfrak{g}}(\mathscr{U}_{\mathfrak{g}}) \supseteq \mathfrak{g}\text{-Cod}_{\mathfrak{g}}(\mathscr{V}_{\mathfrak{g}}) \\
 & \searrow & \uparrow \\
 & & \text{cod}_{\mathfrak{g}}(\mathscr{U}_{\mathfrak{g}}) \supseteq \text{cod}_{\mathfrak{g}}(\mathscr{V}_{\mathfrak{g}}).
\end{array} \tag{3.12}$$

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PROOF. Let $\mathfrak{g}\text{-}\mathbf{Dc}_{\mathfrak{g}} \in \mathfrak{g}\text{-}\mathrm{DC}[\mathfrak{T}_{\mathfrak{g}}]$ be a given pair of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-operators $\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}$, $\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ and $\mathbf{dc}_{\mathfrak{g}} \in \mathrm{DC}[\mathfrak{T}_{\mathfrak{g}}]$ be a given pair of $\mathfrak{T}_{\mathfrak{g}}$-operators $\mathrm{der}_{\mathfrak{g}}$, $\mathrm{cod}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ in a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$. Then:

I. Since $\mathscr{R}_{\mathfrak{g}} \subseteq \mathscr{S}_{\mathfrak{g}}$ implies $\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}) \subseteq \mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$ and $\mathrm{der}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}) \subseteq \mathrm{der}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$, to prove the diagram it suffices to prove that, for any $(\mathscr{R}_{\mathfrak{g}}, \mathscr{S}_{\mathfrak{g}}) \in \times_{\alpha\in I_2^*} \mathscr{P}(\Omega)$, $\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}) \subseteq \mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$ implies $\mathrm{der}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}) \subseteq \mathrm{der}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$. Suppose $\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}) \subseteq \mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$, then

$$\begin{aligned} \mathrm{der}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}) &\longleftrightarrow \mathrm{der}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}) \cup \mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}) \\ &\subseteq \mathrm{der}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}) \cup \mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \\ &\subseteq \mathrm{der}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}) \cup \mathrm{der}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \longleftrightarrow \mathrm{der}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \end{aligned}$$

Thus, $\mathrm{der}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}) \subseteq \mathrm{der}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$ for any $(\mathscr{R}_{\mathfrak{g}}, \mathscr{S}_{\mathfrak{g}}) \in \times_{\alpha\in I_2^*} \mathscr{P}(\Omega)$.

II. Since $\mathscr{U}_{\mathfrak{g}} \supseteq \mathscr{V}_{\mathfrak{g}}$ implies $\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}(\mathscr{U}_{\mathfrak{g}}) \supseteq \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}(\mathscr{V}_{\mathfrak{g}})$ and $\mathrm{cod}_{\mathfrak{g}}(\mathscr{U}_{\mathfrak{g}}) \supseteq \mathrm{cod}_{\mathfrak{g}}(\mathscr{V}_{\mathfrak{g}})$, to prove the diagram it suffices to prove that, for any $(\mathscr{U}_{\mathfrak{g}}, \mathscr{V}_{\mathfrak{g}}) \in \times_{\alpha\in I_2^*} \mathscr{P}(\Omega)$, $\mathrm{cod}_{\mathfrak{g}}(\mathscr{U}_{\mathfrak{g}}) \supseteq \mathrm{cod}_{\mathfrak{g}}(\mathscr{V}_{\mathfrak{g}})$ implies $\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}(\mathscr{U}_{\mathfrak{g}}) \supseteq \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}(\mathscr{V}_{\mathfrak{g}})$. Suppose $\mathrm{cod}_{\mathfrak{g}}(\mathscr{U}_{\mathfrak{g}}) \supseteq \mathrm{cod}_{\mathfrak{g}}(\mathscr{V}_{\mathfrak{g}})$, then

$$\begin{aligned} \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}(\mathscr{U}_{\mathfrak{g}}) &\longleftrightarrow \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}(\mathscr{U}_{\mathfrak{g}}) \cup \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}(\mathscr{V}_{\mathfrak{g}}) \\ &\supseteq \mathrm{cod}_{\mathfrak{g}}(\mathscr{U}_{\mathfrak{g}}) \cup \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}(\mathscr{V}_{\mathfrak{g}}) \\ &\supseteq \mathrm{cod}_{\mathfrak{g}}(\mathscr{V}_{\mathfrak{g}}) \cup \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}(\mathscr{V}_{\mathfrak{g}}) \longleftrightarrow \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}(\mathscr{V}_{\mathfrak{g}}). \end{aligned}$$

Hence, $\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}(\mathscr{U}_{\mathfrak{g}}) \supseteq \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}(\mathscr{V}_{\mathfrak{g}})$ for any $(\mathscr{U}_{\mathfrak{g}}, \mathscr{V}_{\mathfrak{g}}) \in \times_{\alpha\in I_2^*} \mathscr{P}(\Omega)$. The proof of the proposition is complete. Q.E.D.

Our first research objective concerning the definitions and the essential properties of a new class of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-derived and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-coderived operators in $\mathscr{T}_{\mathfrak{g}}$-spaces is now complete. Of the notions of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-derived and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-coderived operators in $\mathscr{T}_{\mathfrak{g}}$-spaces, we conclude the present section with two corollaries and two axiomatic definitions derived from these two corollaries which will be useful in the section following it.

The first corollary stated below contains the necessary and sufficient condition for a set-valued map to be a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-derived operator in a strong $\mathscr{T}_{\mathfrak{g}}$-space.

COROLLARY 3.25. *A necessary and sufficient condition for the set-valued map* $\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ *to be a* $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$*-derived operator in a strong* $\mathscr{T}_{\mathfrak{g}}$*-space* $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$ *is that, for every* $(\{\xi\}, \mathscr{R}_{\mathfrak{g}}, \mathscr{S}_{\mathfrak{g}}) \in \times_{\alpha\in I_3^*} \mathscr{P}(\Omega)$ *such that* $\{\xi\} \subset \mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}})$*, it satisfies:*

- I. $\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}(\emptyset) = \emptyset$,
- II. $\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}) = \mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}} \cap \mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}(\{\xi\}))$,
- III. $\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}} \circ \mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}) \subseteq \mathscr{R}_{\mathfrak{g}} \cup \mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}})$,
- IV. $\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}} \cup \mathscr{S}_{\mathfrak{g}}) = \bigcup_{\mathscr{U}_{\mathfrak{g}} = \mathscr{R}_{\mathfrak{g}}, \mathscr{S}_{\mathfrak{g}}} \mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}(\mathscr{U}_{\mathfrak{g}})$.

The second corollary stated below contains the necessary and sufficient condition for a set-valued map to be a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-coderived operator in a strong $\mathscr{T}_{\mathfrak{g}}$-space.

COROLLARY 3.26. *A necessary and sufficient condition for the set-valued map* $\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ *to be a* $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$*-coderived operator in a strong* $\mathscr{T}_{\mathfrak{g}}$*-space* $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$ *is that, for each* $(\{\zeta\}, \mathscr{U}_{\mathfrak{g}}, \mathscr{V}_{\mathfrak{g}}) \in \times_{\alpha\in I_3^*} \mathscr{P}(\Omega)$ *such that* $\{\zeta\} \subset \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}(\mathscr{U}_{\mathfrak{g}})$*, it satisfies:*

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- I. $\mathfrak{g}\text{-Cod}_{\mathfrak{g}}(\Omega) = \Omega$,
- II. $\mathfrak{g}\text{-Cod}_{\mathfrak{g}}(\mathscr{U}_{\mathfrak{g}}) = \mathfrak{g}\text{-Cod}_{\mathfrak{g}}(\mathscr{U}_{\mathfrak{g}} \cup \{\zeta\})$,
- III. $\mathfrak{g}\text{-Cod}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Cod}_{\mathfrak{g}}(\mathscr{U}_{\mathfrak{g}}) \supseteq \mathscr{U}_{\mathfrak{g}} \cap \mathfrak{g}\text{-Cod}_{\mathfrak{g}}(\mathscr{U}_{\mathfrak{g}})$,
- IV. $\mathfrak{g}\text{-Cod}_{\mathfrak{g}}(\mathscr{U}_{\mathfrak{g}} \cap \mathscr{V}_{\mathfrak{g}}) = \bigcap_{\mathscr{W}_{\mathfrak{g}} = \mathscr{U}_{\mathfrak{g}}, \mathscr{V}_{\mathfrak{g}}} \mathfrak{g}\text{-Cod}_{\mathfrak{g}}(\mathscr{W}_{\mathfrak{g}})$.

Hence, in a strong $\mathscr{T}_{\mathfrak{g}}$-space, for a set-valued map $\mathfrak{g}\text{-Der}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ on $\mathscr{P}(\Omega)$ ranging in $\mathscr{P}(\Omega)$ to be characterized as a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-derived operator it must necessarily and sufficiently satisfy a list of *derived set* $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$*-derived operator conditions* (ITEMS I.–IV. of COR. 3.25), and similarly, for a set-valued map $\mathfrak{g}\text{-Der}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ on $\mathscr{P}(\Omega)$ ranging in $\mathscr{P}(\Omega)$ to be characterized as a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-coderived operator it must necessarily and sufficiently satisfy a list of *derived set* $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$*-coderived operator conditions* (ITEMS V.–VIII. of COR. 3.26).

Some nice Mathematical vocabulary follow. In COR. 3.25, ITEMS I., II., III. and IV. may well be taken as stating that the $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-derived operator $\mathfrak{g}\text{-Der}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ is $\emptyset$*-grounded* (alternatively, $\emptyset$*-preserving*), ξ*-invariant* (alternatively, ξ*-unaffected*), $\mathfrak{g}\text{-Cl}_{\mathfrak{g}}$*-intensive* and $\cup$*-additive* (alternatively, $\cup$*-distributive*), respectively. On the other hand, ITEMS I., II., III. and IV. of COR. 3.26, may well be taken as stating that the $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-coderived operator $\mathfrak{g}\text{-Cod}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ is Ω*-grounded* (alternatively, Ω*-preserving*), ζ*-invariant* (alternatively, ζ*-unaffected*), $\mathfrak{g}\text{-Int}_{\mathfrak{g}}$*-extensive* and $\cap$*-additive* (alternatively, $\cap$*-distributive*), respectively.

Viewing the derived set $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-derived operator conditions (ITEMS I.–IV. of COR. 3.25 above) as $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$*-derived operator axioms*, the axiomatic definition of the concept of a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-derived operator, then, can be defined as a set-valued map $\mathfrak{g}\text{-Der}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ on $\mathscr{P}(\Omega)$ ranging in $\mathscr{P}(\Omega)$ satisfying a list of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$*-derived operator axioms.* The axiomatic definition of the concept of a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-derived operator in $\mathscr{T}_{\mathfrak{g}}$-spaces follows.

DEFINITION 3.27 (Axiomatic Definition: $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-Derived Operator). A set-valued map of the type $\mathfrak{g}\text{-Der}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ in a strong $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$ is called a "$\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-derived operator" on $\mathscr{P}(\Omega)$ ranging in $\mathscr{P}(\Omega)$ if and only if, for any $(\{\xi\}, \mathscr{R}_{\mathfrak{g}}, \mathscr{S}_{\mathfrak{g}}) \in \times_{\alpha \in I_3^*} \mathscr{P}(\Omega)$ such that $\{\xi\} \subset \mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}})$, it satisfies each "$\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-derived operator axiom" in $\mathrm{AX}[\mathfrak{g}\text{-DE}[\mathfrak{T}_{\mathfrak{g}}]; \mathbb{B}] \stackrel{\text{def}}{=} \{\mathrm{Ax}_{\mathrm{DE},\nu}(\mathfrak{g}\text{-Der}_{\mathfrak{g}}) : \nu \in I_4^*\}$, where $\mathrm{Ax}_{\mathrm{DE},\nu} : \mathfrak{g}\text{-DE}[\mathfrak{T}_{\mathfrak{g}}] \longrightarrow \mathbb{B} \stackrel{\text{def}}{=} \{0,1\}$, $\nu \in I_4^*$, is defined as thus:

- $\mathrm{Ax}_{\mathrm{DE},1}(\mathfrak{g}\text{-Der}_{\mathfrak{g}}) \stackrel{\text{def}}{\longleftrightarrow} \mathfrak{g}\text{-Der}_{\mathfrak{g}}(\emptyset) = \emptyset$,
- $\mathrm{Ax}_{\mathrm{DE},2}(\mathfrak{g}\text{-Der}_{\mathfrak{g}}) \stackrel{\text{def}}{\longleftrightarrow} \mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}) = \mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}} \cap \mathfrak{g}\text{-Op}_{\mathfrak{g}}(\{\xi\}))$,
- $\mathrm{Ax}_{\mathrm{DE},3}(\mathfrak{g}\text{-Der}_{\mathfrak{g}}) \stackrel{\text{def}}{\longleftrightarrow} \mathfrak{g}\text{-Der}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}) \subseteq \mathscr{R}_{\mathfrak{g}} \cup \mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}})$,
- $\mathrm{Ax}_{\mathrm{DE},4}(\mathfrak{g}\text{-Der}_{\mathfrak{g}}) \stackrel{\text{def}}{\longleftrightarrow} \mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}} \cup \mathscr{S}_{\mathfrak{g}}) = \bigcup_{\mathscr{U}_{\mathfrak{g}} = \mathscr{R}_{\mathfrak{g}}, \mathscr{S}_{\mathfrak{g}}} \mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{U}_{\mathfrak{g}})$.

Thus, a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-derived operator $\mathfrak{g}\text{-Der}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ in a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$ is a $\emptyset$*-grounded* ($\mathrm{Ax}_{\mathrm{DE},1}$), ξ*-invariant* ($\mathrm{Ax}_{\mathrm{DE},2}$), $\mathfrak{g}\text{-Cl}_{\mathfrak{g}}$*-intensive* ($\mathrm{Ax}_{\mathrm{DE},3}$) and $\cup$*-additive* ($\mathrm{Ax}_{\mathrm{DE},4}$) $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-set-valued set map forming a generalization of the

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$\mathfrak{T}_{\mathfrak{g}}$-set-valued set map $\operatorname{der}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ in the strong $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}}$, provided

$$\begin{gathered}\left(\mathfrak{g}\text{-Der}_{\mathfrak{g}}\left(\mathscr{R}_{\mathfrak{g}} \cup \mathfrak{g}\text{-Der}_{\mathfrak{g}}\left(\mathscr{R}_{\mathfrak{g}}\right)\right) \subseteq \mathscr{R}_{\mathfrak{g}} \cup \mathfrak{g}\text{-Der}_{\mathfrak{g}}\left(\mathscr{R}_{\mathfrak{g}}\right)\right) \\ \wedge\left(\mathfrak{g}\text{-Der}_{\mathfrak{g}}\left(\mathscr{R}_{\mathfrak{g}} \cup \mathscr{S}_{\mathfrak{g}}\right) = \mathfrak{g}\text{-Der}_{\mathfrak{g}}\left(\mathscr{R}_{\mathfrak{g}}\right) \cup \mathfrak{g}\text{-Der}_{\mathfrak{g}}\left(\mathscr{S}_{\mathfrak{g}}\right)\right) \\ \updownarrow \\ \mathfrak{g}\text{-Der}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Der}_{\mathfrak{g}}\left(\mathscr{R}_{\mathfrak{g}}\right) \subseteq \mathscr{R}_{\mathfrak{g}} \cup \mathfrak{g}\text{-Der}_{\mathfrak{g}}\left(\mathscr{R}_{\mathfrak{g}}\right)\end{gathered}$$

holds for any $(\mathscr{R}_{\mathfrak{g}}, \mathscr{S}_{\mathfrak{g}}) \in \times_{\alpha \in I_2^*} \mathscr{P}(\Omega)$.

Having introduced an alternative definition defining the notion of a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-derived operator in a strong $\mathscr{T}_{\mathfrak{g}}$-space axiomatically, it may not be without interest to prove some further propositions based on such axiomatic definition. The theorem follows.

THEOREM 3.28. *Let* $\mathrm{AX}\left[\mathfrak{g}\text{-DE}\left[\mathfrak{T}_{\mathfrak{g}}\right]; \mathbb{B}\right] \stackrel{\text{def}}{=} \left\{\mathrm{Ax}_{\mathrm{DE},\nu} : \nu \in I_4^*\right\}$ *be the class of* $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$*-derived operator axioms in a strong* $\mathscr{T}_{\mathfrak{g}}$*-space* $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$ *and, let* $\mathrm{Ax}_{\mathrm{DE,I}} : \mathfrak{g}\text{-DE}\left[\mathfrak{T}_{\mathfrak{g}}\right] \longrightarrow \mathbb{B}$ *such that, for any* $(\mathscr{R}_{\mathfrak{g}}, \mathscr{S}_{\mathfrak{g}}) \in \times_{\alpha \in I_2^*} \mathscr{P}(\Omega)$,

$$\begin{aligned}\mathrm{Ax}_{\mathrm{DE,I}}\left(\mathfrak{g}\text{-Der}_{\mathfrak{g}}\right) \stackrel{\text{def}}{\Longleftrightarrow}\ & \mathfrak{g}\text{-Der}_{\mathfrak{g}}\left(\mathscr{S}_{\mathfrak{g}} \cup \mathfrak{g}\text{-Der}_{\mathfrak{g}}\left(\mathscr{S}_{\mathfrak{g}}\right)\right) \\ & \cup\left(\bigcup_{\mathscr{U}_{\mathfrak{g}} = \mathscr{R}_{\mathfrak{g}}, \mathscr{S}_{\mathfrak{g}}}\left(\mathscr{U}_{\mathfrak{g}} \cup \mathfrak{g}\text{-Der}_{\mathfrak{g}}\left(\mathscr{U}_{\mathfrak{g}}\right)\right)\right) \\ & = \left(\mathscr{R}_{\mathfrak{g}} \cup \mathscr{S}_{\mathfrak{g}} \cup \mathfrak{g}\text{-Der}_{\mathfrak{g}}\left(\mathscr{R}_{\mathfrak{g}} \cup \mathscr{S}_{\mathfrak{g}}\right)\right) \setminus \mathfrak{g}\text{-Der}_{\mathfrak{g}}(\emptyset).\end{aligned} \tag{3.13}$$

Then, $\mathrm{Ax}_{\mathrm{DE,I}}\left(\mathfrak{g}\text{-Der}_{\mathfrak{g}}\right) = 1 \longrightarrow \bigwedge_{\nu \in I_4^*} \mathrm{Ax}_{\mathrm{DE},\nu}\left(\mathfrak{g}\text{-Der}_{\mathfrak{g}}\right) = 1.$

PROOF. Let $\mathrm{AX}\left[\mathfrak{g}\text{-DE}\left[\mathfrak{T}_{\mathfrak{g}}\right]; \mathbb{B}\right] = \left\{\mathrm{Ax}_{\mathrm{DE},\nu} : \nu \in I_4^*\right\}$ be the class of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-derived operator axioms in a strong $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$ and, let $\mathrm{Ax}_{\mathrm{DE,I}} : \mathfrak{g}\text{-DE}\left[\mathfrak{T}_{\mathfrak{g}}\right] \longrightarrow \mathbb{B}$ such that, for any $(\mathscr{R}_{\mathfrak{g}}, \mathscr{S}_{\mathfrak{g}}) \in \times_{\alpha \in I_2^*} \mathscr{P}(\Omega)$,

$$\begin{aligned}\mathrm{Ax}_{\mathrm{DE,I}}\left(\mathfrak{g}\text{-Der}_{\mathfrak{g}}\right) \stackrel{\text{def}}{\Longleftrightarrow}\ & \mathfrak{g}\text{-Der}_{\mathfrak{g}}\left(\mathscr{S}_{\mathfrak{g}} \cup \mathfrak{g}\text{-Der}_{\mathfrak{g}}\left(\mathscr{S}_{\mathfrak{g}}\right)\right) \\ & \cup\left(\bigcup_{\mathscr{U}_{\mathfrak{g}} = \mathscr{R}_{\mathfrak{g}}, \mathscr{S}_{\mathfrak{g}}}\left(\mathscr{U}_{\mathfrak{g}} \cup \mathfrak{g}\text{-Der}_{\mathfrak{g}}\left(\mathscr{U}_{\mathfrak{g}}\right)\right)\right) \\ & = \left(\mathscr{R}_{\mathfrak{g}} \cup \mathscr{S}_{\mathfrak{g}} \cup \mathfrak{g}\text{-Der}_{\mathfrak{g}}\left(\mathscr{R}_{\mathfrak{g}} \cup \mathscr{S}_{\mathfrak{g}}\right)\right) \setminus \mathfrak{g}\text{-Der}_{\mathfrak{g}}(\emptyset).\end{aligned}$$

Suppose $\mathrm{Ax}_{\mathrm{DE,I}}\left(\mathfrak{g}\text{-Der}_{\mathfrak{g}}\right) = 1$ holds. Then:

If $(\mathscr{R}_{\mathfrak{g}}, \mathscr{S}_{\mathfrak{g}}) = (\emptyset, \emptyset)$, then

$$\begin{aligned}\mathrm{Ax}_{\mathrm{DE,I}}\left(\mathfrak{g}\text{-Der}_{\mathfrak{g}}\right) \stackrel{\text{def}}{\Longleftrightarrow}\ & \mathfrak{g}\text{-Der}_{\mathfrak{g}}\left(\emptyset \cup \mathfrak{g}\text{-Der}_{\mathfrak{g}}(\emptyset)\right) \\ & \cup\left(\bigcup_{\mathscr{U}_{\mathfrak{g}} = \emptyset, \emptyset}\left(\mathscr{U}_{\mathfrak{g}} \cup \mathfrak{g}\text{-Der}_{\mathfrak{g}}\left(\mathscr{U}_{\mathfrak{g}}\right)\right)\right) \\ & = \left(\emptyset \cup \emptyset \cup \mathfrak{g}\text{-Der}_{\mathfrak{g}}(\emptyset \cup \emptyset)\right) \setminus \mathfrak{g}\text{-Der}_{\mathfrak{g}}(\emptyset).\end{aligned}$$

Consequently, $\mathfrak{g}\text{-Der}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Der}_{\mathfrak{g}}(\emptyset)$, $\mathfrak{g}\text{-Der}_{\mathfrak{g}}(\emptyset) = \emptyset$. Therefore, $\mathfrak{g}\text{-Der}_{\mathfrak{g}}(\emptyset) = \emptyset \stackrel{\text{def}}{\Longleftrightarrow} \mathrm{Ax}_{\mathrm{DE},1}\left(\mathfrak{g}\text{-Der}_{\mathfrak{g}}\right)$ and thus, $\mathrm{Ax}_{\mathrm{DE,I}} \longrightarrow \mathrm{Ax}_{\mathrm{DE},1}$.

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If $(\mathscr{R}_{\mathfrak{g}},\mathscr{S}_{\mathfrak{g}})\in\times_{\alpha\in I_2^*}\mathscr{P}(\Omega)$ be arbitrary such that $\mathscr{S}_{\mathfrak{g}}=\mathscr{R}_{\mathfrak{g}}\cap\mathfrak{g}\text{-Op}_{\mathfrak{g}}(\{\xi\})$ and $\{\xi\}\subset\mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}})$, then

$$\begin{aligned}\text{Ax}_{\text{DE,I}}\left(\mathfrak{g}\text{-Der}_{\mathfrak{g}}\right) \stackrel{\text{def}}{\longleftrightarrow}\; & \mathfrak{g}\text{-Der}_{\mathfrak{g}}\big((\mathscr{R}_{\mathfrak{g}}\cap\mathfrak{g}\text{-Op}_{\mathfrak{g}}(\{\xi\}))\cup\mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}\cap\mathfrak{g}\text{-Op}_{\mathfrak{g}}(\{\xi\}))\big)\\ & \cup\Bigg(\bigcup_{\mathscr{U}_{\mathfrak{g}}=\mathscr{R}_{\mathfrak{g}},\mathscr{R}_{\mathfrak{g}}\cap\mathfrak{g}\text{-Op}_{\mathfrak{g}}(\{\xi\})}\left(\mathscr{U}_{\mathfrak{g}}\cup\mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{U}_{\mathfrak{g}})\right)\Bigg)\\ & =\big(\mathscr{R}_{\mathfrak{g}}\cup(\mathscr{R}_{\mathfrak{g}}\cap\mathfrak{g}\text{-Op}_{\mathfrak{g}}(\{\xi\}))\\ & \cup\mathfrak{g}\text{-Der}_{\mathfrak{g}}\big(\mathscr{R}_{\mathfrak{g}}\cup(\mathscr{R}_{\mathfrak{g}}\cap\mathfrak{g}\text{-Op}_{\mathfrak{g}}(\{\xi\}))\big)\big)\setminus\mathfrak{g}\text{-Der}_{\mathfrak{g}}(\emptyset).\end{aligned}$$

Since the relation $\mathfrak{g}\text{-Der}_{\mathfrak{g}}\left(\mathscr{S}_{\mathfrak{g}}\cup\mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})\right)\subseteq\mathscr{S}_{\mathfrak{g}}\cup\mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$ holds, implying $\mathfrak{g}\text{-Der}_{\mathfrak{g}}\left(\mathscr{S}_{\mathfrak{g}}\cup\mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})\right)\cup\left(\mathscr{S}_{\mathfrak{g}}\cup\mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})\right)=\mathscr{S}_{\mathfrak{g}}\cup\mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$, and $\mathfrak{g}\text{-Der}_{\mathfrak{g}}(\emptyset)=\emptyset$ by virtue of $\text{Ax}_{\text{DE},1}$, the above expression reduces to

$$\mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}})\cup\mathfrak{g}\text{-Der}_{\mathfrak{g}}\left(\mathscr{R}_{\mathfrak{g}}\cap\mathfrak{g}\text{-Op}_{\mathfrak{g}}(\{\xi\})\right)=\mathfrak{g}\text{-Der}_{\mathfrak{g}}\left(\mathscr{R}_{\mathfrak{g}}\cup(\mathscr{R}_{\mathfrak{g}}\cap\mathfrak{g}\text{-Op}_{\mathfrak{g}}(\{\xi\}))\right).$$

Clearly, $\mathfrak{g}\text{-Der}_{\mathfrak{g}}\left(\mathscr{R}_{\mathfrak{g}}\cup(\mathscr{R}_{\mathfrak{g}}\cap\mathfrak{g}\text{-Op}_{\mathfrak{g}}(\{\xi\}))\right)\longleftrightarrow\mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}})$. Consequently, it results that $\mathfrak{g}\text{-Der}_{\mathfrak{g}}\left(\mathscr{R}_{\mathfrak{g}}\cap\mathfrak{g}\text{-Op}_{\mathfrak{g}}(\{\xi\})\right)\subseteq\mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}})$. But,

$$\begin{aligned}\mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}) &\subseteq \left\{\xi\in\mathfrak{T}_{\mathfrak{g}}:\ \xi\in\mathfrak{g}\text{-Cl}_{\mathfrak{g}}\left(\mathscr{R}_{\mathfrak{g}}\cap\mathfrak{g}\text{-Op}_{\mathfrak{g}}(\{\xi\})\right)\right\}\\ &\longleftrightarrow \left\{\xi\in\mathfrak{T}_{\mathfrak{g}}:\ \xi\in\mathfrak{g}\text{-Cl}_{\mathfrak{g}}\left((\mathscr{R}_{\mathfrak{g}}\cap\mathfrak{g}\text{-Op}_{\mathfrak{g}}(\{\xi\}))\cap\mathfrak{g}\text{-Op}_{\mathfrak{g}}(\{\xi\})\right)\right\}\\ &\longleftrightarrow \mathfrak{g}\text{-Der}_{\mathfrak{g}}\left(\mathscr{R}_{\mathfrak{g}}\cap\mathfrak{g}\text{-Op}_{\mathfrak{g}}(\{\xi\})\right).\end{aligned}$$

Consequently, $\mathfrak{g}\text{-Der}_{\mathfrak{g}}\left(\mathscr{R}_{\mathfrak{g}}\cap\mathfrak{g}\text{-Op}_{\mathfrak{g}}(\{\xi\})\right)\supseteq\mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}})$. Therefore, it results that $\mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}})=\mathfrak{g}\text{-Der}_{\mathfrak{g}}\left(\mathscr{R}_{\mathfrak{g}}\cap\mathfrak{g}\text{-Op}_{\mathfrak{g}}(\{\xi\})\right)\stackrel{\text{def}}{\longleftrightarrow}\text{Ax}_{\text{DE},2}\left(\mathfrak{g}\text{-Der}_{\mathfrak{g}}\right)$ holds and hence, $\text{Ax}_{\text{DE,I}}\longrightarrow\text{Ax}_{\text{DE},2}$.

If $(\mathscr{R}_{\mathfrak{g}},\mathscr{S}_{\mathfrak{g}})\in\times_{\alpha\in I_2^*}\mathscr{P}(\Omega)$ such that $\mathscr{S}_{\mathfrak{g}}=\mathscr{R}_{\mathfrak{g}}$ be arbitrary, then

$$\begin{aligned}\text{Ax}_{\text{DE,I}}\left(\mathfrak{g}\text{-Der}_{\mathfrak{g}}\right) \stackrel{\text{def}}{\longleftrightarrow}\; & \mathfrak{g}\text{-Der}_{\mathfrak{g}}\left(\mathscr{R}_{\mathfrak{g}}\cup\mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}})\right)\\ & \cup\Bigg(\bigcup_{\mathscr{U}_{\mathfrak{g}}=\mathscr{R}_{\mathfrak{g}},\mathscr{R}_{\mathfrak{g}}}\left(\mathscr{U}_{\mathfrak{g}}\cup\mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{U}_{\mathfrak{g}})\right)\Bigg)\\ & =\left(\mathscr{R}_{\mathfrak{g}}\cup\mathscr{R}_{\mathfrak{g}}\cup\mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}\cup\mathscr{R}_{\mathfrak{g}})\right)\setminus\mathfrak{g}\text{-Der}_{\mathfrak{g}}(\emptyset).\end{aligned}$$

Consequently, $\mathfrak{g}\text{-Der}_{\mathfrak{g}}\left(\mathscr{R}_{\mathfrak{g}}\cup\mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}})\right)\subseteq\mathscr{R}_{\mathfrak{g}}\cup\mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}})$, since $\mathfrak{g}\text{-Der}_{\mathfrak{g}}(\emptyset)=\emptyset$ by virtue of $\text{Ax}_{\text{DE},1}$. But, $\mathfrak{g}\text{-Der}_{\mathfrak{g}}\left(\mathscr{R}_{\mathfrak{g}}\cup\mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}})\right)\supseteq\mathfrak{g}\text{-Der}_{\mathfrak{g}}\circ\mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}})$. Therefore, $\mathfrak{g}\text{-Der}_{\mathfrak{g}}\circ\mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}})\subseteq\mathscr{R}_{\mathfrak{g}}\cup\mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}})\stackrel{\text{def}}{\longleftrightarrow}\text{Ax}_{\text{DE},3}\left(\mathfrak{g}\text{-Der}_{\mathfrak{g}}\right)$ and thus, $\text{Ax}_{\text{DE,I}}\longrightarrow\text{Ax}_{\text{DE},3}$.

If $(\mathscr{R}_{\mathfrak{g}},\mathscr{S}_{\mathfrak{g}})\in\times_{\alpha\in I_2^*}\mathscr{P}(\Omega)$ be arbitrary, then

$$\begin{aligned}\text{Ax}_{\text{DE,I}}\left(\mathfrak{g}\text{-Der}_{\mathfrak{g}}\right) \stackrel{\text{def}}{\longleftrightarrow}\; & \mathfrak{g}\text{-Der}_{\mathfrak{g}}\left(\mathscr{S}_{\mathfrak{g}}\cup\mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})\right)\\ & \cup\Bigg(\bigcup_{\mathscr{U}_{\mathfrak{g}}=\mathscr{R}_{\mathfrak{g}},\mathscr{S}_{\mathfrak{g}}}\left(\mathscr{U}_{\mathfrak{g}}\cup\mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{U}_{\mathfrak{g}})\right)\Bigg)\\ & =\left(\mathscr{R}_{\mathfrak{g}}\cup\mathscr{S}_{\mathfrak{g}}\cup\mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}\cup\mathscr{S}_{\mathfrak{g}})\right)\setminus\mathfrak{g}\text{-Der}_{\mathfrak{g}}(\emptyset).\end{aligned}$$

By virtue of the relation $\mathfrak{g}\text{-Der}_{\mathfrak{g}}\left(\mathscr{S}_{\mathfrak{g}}\cup\mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})\right)\subseteq\mathscr{S}_{\mathfrak{g}}\cup\mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$ or equivalently, $\mathfrak{g}\text{-Der}_{\mathfrak{g}}\left(\mathscr{S}_{\mathfrak{g}}\cup\mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})\right)\cup\left(\mathscr{S}_{\mathfrak{g}}\cup\mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})\right)=\mathscr{S}_{\mathfrak{g}}\cup\mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$, together

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with $\mathrm{Ax}_{\mathrm{DE},1}$, $\mathrm{Ax}_{\mathrm{DE,I}}$ reduces to $\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}\cup\mathscr{S}_{\mathfrak{g}}) = \bigcup_{\mathscr{U}_{\mathfrak{g}}=\mathscr{R}_{\mathfrak{g}},\mathscr{S}_{\mathfrak{g}}} \mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}(\mathscr{U}_{\mathfrak{g}}) \stackrel{\mathrm{def}}{\longleftrightarrow} \mathrm{Ax}_{\mathrm{DE},4}(\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}})$ and hence, $\mathrm{Ax}_{\mathrm{DE,I}} \longrightarrow \mathrm{Ax}_{\mathrm{DE},4}$.

Hence, $\mathrm{Ax}_{\mathrm{DE,I}}(\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}) = 1 \longrightarrow \bigwedge_{\nu\in I_4^*} \mathrm{Ax}_{\mathrm{DE},\nu}(\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}) = 1$ and the proof of the theorem is complete. Q.E.D.

In the proposition given below there is contained further properties.

Proposition 3.29. *Let* $\mathrm{AX}[\mathfrak{g}\text{-}\mathrm{DE}[\mathfrak{T}_{\mathfrak{g}}];\mathbb{B}] \stackrel{\mathrm{def}}{=} \{\mathrm{Ax}_{\mathrm{DE},\nu} : \ \nu\in I_4^*\}$ *be the class of* $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$*-derived operator axioms in a strong* $\mathscr{T}_{\mathfrak{g}}$*-space* $\mathfrak{T}_{\mathfrak{g}} = (\Omega,\mathscr{T}_{\mathfrak{g}})$ *and, let* $\mathrm{Ax}_{\mathrm{DE,II}} : \mathfrak{g}\text{-}\mathrm{DE}[\mathfrak{T}_{\mathfrak{g}}] \longrightarrow \mathbb{B}$ *such that, for any* $(\mathscr{R}_{\mathfrak{g}},\mathscr{S}_{\mathfrak{g}}) \in \bigtimes_{\alpha\in I_2^*}\mathscr{P}(\Omega)$,

$$\begin{aligned}\mathrm{Ax}_{\mathrm{DE,II}}(\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}) \stackrel{\mathrm{def}}{\longleftrightarrow} & \Big(\bigcup_{\mathscr{U}_{\mathfrak{g}}=\mathscr{R}_{\mathfrak{g}},\mathscr{S}_{\mathfrak{g}}} \mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}(\mathscr{U}_{\mathfrak{g}}\cup\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}(\mathscr{U}_{\mathfrak{g}}))\Big)\\ & \cup\Big(\bigcup_{\mathscr{U}_{\mathfrak{g}}=\mathscr{R}_{\mathfrak{g}},\mathscr{S}_{\mathfrak{g}}} (\mathscr{U}_{\mathfrak{g}}\cup\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}(\mathscr{U}_{\mathfrak{g}}))\Big)\\ & = (\mathscr{R}_{\mathfrak{g}}\cup\mathscr{S}_{\mathfrak{g}})\cup\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}\cup\mathscr{S}_{\mathfrak{g}}).\end{aligned}\tag{3.14}$$

Then, $\mathrm{Ax}_{\mathrm{DE,II}}(\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}) = 1 \longrightarrow \bigwedge_{\nu\in I_4^*} \mathrm{Ax}_{\mathrm{DE},\nu}(\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}) = 1$.

Proof. Let $\mathrm{AX}[\mathfrak{g}\text{-}\mathrm{DE}[\mathfrak{T}_{\mathfrak{g}}];\mathbb{B}] \stackrel{\mathrm{def}}{=} \{\mathrm{Ax}_{\mathrm{DE},\nu} : \ \nu\in I_4^*\}$ be the class of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-derived operator axioms in a strong $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega,\mathscr{T}_{\mathfrak{g}})$ and, let $\mathrm{Ax}_{\mathrm{DE,II}} : \mathfrak{g}\text{-}\mathrm{DE}[\mathfrak{T}_{\mathfrak{g}}] \longrightarrow \mathbb{B}$ such that, for any $(\mathscr{R}_{\mathfrak{g}},\mathscr{S}_{\mathfrak{g}}) \in \bigtimes_{\alpha\in I_2^*}\mathscr{P}(\Omega)$,

$$\begin{aligned}\mathrm{Ax}_{\mathrm{DE,II}}(\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}) \stackrel{\mathrm{def}}{\longleftrightarrow} & \Big(\bigcup_{\mathscr{U}_{\mathfrak{g}}=\mathscr{R}_{\mathfrak{g}},\mathscr{S}_{\mathfrak{g}}} \mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}(\mathscr{U}_{\mathfrak{g}}\cup\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}(\mathscr{U}_{\mathfrak{g}}))\Big)\\ & \cup\Big(\bigcup_{\mathscr{U}_{\mathfrak{g}}=\mathscr{R}_{\mathfrak{g}},\mathscr{S}_{\mathfrak{g}}} (\mathscr{U}_{\mathfrak{g}}\cup\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}(\mathscr{U}_{\mathfrak{g}}))\Big)\\ & = (\mathscr{R}_{\mathfrak{g}}\cup\mathscr{S}_{\mathfrak{g}})\cup\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}\cup\mathscr{S}_{\mathfrak{g}}).\end{aligned}$$

Suppose $\mathrm{Ax}_{\mathrm{DE,II}}(\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}) = 1$ holds. Then:

If $(\mathscr{R}_{\mathfrak{g}},\mathscr{S}_{\mathfrak{g}}) = (\emptyset,\emptyset)$, then

$$\begin{aligned}\mathrm{Ax}_{\mathrm{DE,II}}(\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}) \stackrel{\mathrm{def}}{\longleftrightarrow} & \Big(\bigcup_{\mathscr{U}_{\mathfrak{g}}=\emptyset,\emptyset} \mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}(\mathscr{U}_{\mathfrak{g}}\cup\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}(\mathscr{U}_{\mathfrak{g}}))\Big)\\ & \cup\Big(\bigcup_{\mathscr{U}_{\mathfrak{g}}=\emptyset,\emptyset} (\mathscr{U}_{\mathfrak{g}}\cup\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}(\mathscr{U}_{\mathfrak{g}}))\Big)\\ & = (\emptyset\cup\emptyset)\cup\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}(\emptyset\cup\emptyset).\end{aligned}$$

Consequently, $\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}\circ\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}(\emptyset)\cup\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}(\emptyset) = \mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}(\emptyset)$. But, $\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}(\emptyset) \longleftrightarrow \emptyset\cup\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}(\emptyset) \longleftrightarrow \mathfrak{g}\text{-}\mathrm{Cl}_{\mathfrak{g}}(\emptyset) = \emptyset$. Therefore, $\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}(\emptyset) = \emptyset \stackrel{\mathrm{def}}{\longleftrightarrow} \mathrm{Ax}_{\mathrm{DE},1}(\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}})$ and thus, $\mathrm{Ax}_{\mathrm{DE,II}} \longrightarrow \mathrm{Ax}_{\mathrm{DE},1}$.

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If $(\mathscr{R}_{\mathfrak{g}}, \mathscr{S}_{\mathfrak{g}}) \in \times_{\alpha\in I_2^*} \mathscr{P}(\Omega)$ be arbitrary such that $\mathscr{S}_{\mathfrak{g}} = \mathscr{R}_{\mathfrak{g}} \cap \mathfrak{g}\text{-Op}_{\mathfrak{g}}(\{\xi\})$ and $\{\xi\} \subset \mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}})$, then

$$\begin{aligned}
\mathrm{Ax}_{\mathrm{DE,II}}(\mathfrak{g}\text{-Der}_{\mathfrak{g}}) \overset{\text{def}}{\longleftrightarrow} & \Big(\bigcup_{\mathscr{U}_{\mathfrak{g}}=\mathscr{R}_{\mathfrak{g}},\mathscr{R}_{\mathfrak{g}}\cap\mathfrak{g}\text{-Op}_{\mathfrak{g}}(\{\xi\})} \mathfrak{g}\text{-Der}_{\mathfrak{g}}\big(\mathscr{U}_{\mathfrak{g}} \cup \mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{U}_{\mathfrak{g}})\big)\Big) \\
& \cup \Big(\bigcup_{\mathscr{U}_{\mathfrak{g}}=\mathscr{R}_{\mathfrak{g}},\mathscr{R}_{\mathfrak{g}}\cap\mathfrak{g}\text{-Op}_{\mathfrak{g}}(\{\xi\})} \big(\mathscr{U}_{\mathfrak{g}} \cup \mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{U}_{\mathfrak{g}})\big)\Big) \\
& = \big(\mathscr{R}_{\mathfrak{g}} \cup (\mathscr{R}_{\mathfrak{g}} \cap \mathfrak{g}\text{-Op}_{\mathfrak{g}}(\{\xi\}))\big) \\
& \cup \mathfrak{g}\text{-Der}_{\mathfrak{g}}\big(\mathscr{R}_{\mathfrak{g}} \cup (\mathscr{R}_{\mathfrak{g}} \cap \mathfrak{g}\text{-Op}_{\mathfrak{g}}(\{\xi\}))\big).
\end{aligned}$$

Since the relation $\mathfrak{g}\text{-Der}_{\mathfrak{g}}\big(\mathscr{U}_{\mathfrak{g}} \cup \mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{U}_{\mathfrak{g}})\big) \subseteq \mathscr{U}_{\mathfrak{g}} \cup \mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{U}_{\mathfrak{g}})$ holds, implying $\mathfrak{g}\text{-Der}_{\mathfrak{g}}\big(\mathscr{U}_{\mathfrak{g}} \cup \mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{U}_{\mathfrak{g}})\big) \cup \big(\mathscr{U}_{\mathfrak{g}} \cup \mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{U}_{\mathfrak{g}})\big) = \mathscr{U}_{\mathfrak{g}} \cup \mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{U}_{\mathfrak{g}})$ for any $\mathscr{U}_{\mathfrak{g}} \in \{\mathscr{R}_{\mathfrak{g}},\ \mathscr{R}_{\mathfrak{g}} \cap \mathfrak{g}\text{-Op}_{\mathfrak{g}}(\{\xi\})\}$, $\mathrm{Ax}_{\mathrm{DE,II}}$ reduces to

$$\bigcup_{\mathscr{U}_{\mathfrak{g}}=\mathscr{R}_{\mathfrak{g}},\mathscr{R}_{\mathfrak{g}}\cap\mathfrak{g}\text{-Op}_{\mathfrak{g}}(\{\xi\})} \mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{U}_{\mathfrak{g}}) = \mathfrak{g}\text{-Der}_{\mathfrak{g}}\big(\mathscr{R}_{\mathfrak{g}} \cup (\mathscr{R}_{\mathfrak{g}} \cap \mathfrak{g}\text{-Op}_{\mathfrak{g}}(\{\xi\}))\big).$$

Because $\mathscr{R}_{\mathfrak{g}} \supseteq \mathscr{R}_{\mathfrak{g}} \cap \mathfrak{g}\text{-Op}_{\mathfrak{g}}(\{\xi\})$ holds, $\mathfrak{g}\text{-Der}_{\mathfrak{g}}\big(\mathscr{R}_{\mathfrak{g}} \cup (\mathscr{R}_{\mathfrak{g}} \cap \mathfrak{g}\text{-Op}_{\mathfrak{g}}(\{\xi\}))\big) \longleftrightarrow \mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}})$. Consequently, $\mathfrak{g}\text{-Der}_{\mathfrak{g}}\big(\mathscr{R}_{\mathfrak{g}} \cap \mathfrak{g}\text{-Op}_{\mathfrak{g}}(\{\xi\})\big) \subseteq \mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}})$. But,

$$\begin{aligned}
\mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}) &\subseteq \big\{\xi \in \mathfrak{T}_{\mathfrak{g}} :\ \xi \in \mathfrak{g}\text{-Cl}_{\mathfrak{g}}\big(\mathscr{R}_{\mathfrak{g}} \cap \mathfrak{g}\text{-Op}_{\mathfrak{g}}(\{\xi\})\big)\big\} \\
&\longleftrightarrow \big\{\xi \in \mathfrak{T}_{\mathfrak{g}} :\ \xi \in \mathfrak{g}\text{-Cl}_{\mathfrak{g}}\big((\mathscr{R}_{\mathfrak{g}} \cap \mathfrak{g}\text{-Op}_{\mathfrak{g}}(\{\xi\})) \cap \mathfrak{g}\text{-Op}_{\mathfrak{g}}(\{\xi\})\big)\big\} \\
&\longleftrightarrow \mathfrak{g}\text{-Der}_{\mathfrak{g}}\big(\mathscr{R}_{\mathfrak{g}} \cap \mathfrak{g}\text{-Op}_{\mathfrak{g}}(\{\xi\})\big),
\end{aligned}$$

implying $\mathfrak{g}\text{-Der}_{\mathfrak{g}}\big(\mathscr{R}_{\mathfrak{g}} \cap \mathfrak{g}\text{-Op}_{\mathfrak{g}}(\{\xi\})\big) \supseteq \mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}})$. Therefore, $\mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}) = \mathfrak{g}\text{-Der}_{\mathfrak{g}}\big(\mathscr{R}_{\mathfrak{g}} \cap \mathfrak{g}\text{-Op}_{\mathfrak{g}}(\{\xi\})\big) \overset{\text{def}}{\longleftrightarrow} \mathrm{Ax}_{\mathrm{DE},2}(\mathfrak{g}\text{-Der}_{\mathfrak{g}})$ and hence, $\mathrm{Ax}_{\mathrm{DE,I}} \longrightarrow \mathrm{Ax}_{\mathrm{DE},2}$.

If $(\mathscr{R}_{\mathfrak{g}}, \mathscr{S}_{\mathfrak{g}}) \in \times_{\alpha\in I_2^*} \mathscr{P}(\Omega)$ such that $\mathscr{S}_{\mathfrak{g}} = \mathscr{R}_{\mathfrak{g}}$ be arbitrary, then

$$\begin{aligned}
\mathrm{Ax}_{\mathrm{DE,II}}(\mathfrak{g}\text{-Der}_{\mathfrak{g}}) \overset{\text{def}}{\longleftrightarrow} & \Big(\bigcup_{\mathscr{U}_{\mathfrak{g}}=\mathscr{R}_{\mathfrak{g}},\mathscr{R}_{\mathfrak{g}}} \mathfrak{g}\text{-Der}_{\mathfrak{g}}\big(\mathscr{U}_{\mathfrak{g}} \cup \mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{U}_{\mathfrak{g}})\big)\Big) \\
& \cup \Big(\bigcup_{\mathscr{U}_{\mathfrak{g}}=\mathscr{R}_{\mathfrak{g}},\mathscr{R}_{\mathfrak{g}}} \mathscr{U}_{\mathfrak{g}} \cup \mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{U}_{\mathfrak{g}})\Big) \\
& = (\mathscr{R}_{\mathfrak{g}} \cup \mathscr{R}_{\mathfrak{g}}) \cup \mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}} \cup \mathscr{R}_{\mathfrak{g}}).
\end{aligned}$$

Consequently, $\mathfrak{g}\text{-Der}_{\mathfrak{g}}\big(\mathscr{R}_{\mathfrak{g}} \cup \mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}})\big) \cup \big(\mathscr{R}_{\mathfrak{g}} \cup \mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}})\big) = \mathscr{R}_{\mathfrak{g}} \cup \mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}})$, implying $\mathfrak{g}\text{-Der}_{\mathfrak{g}}\big(\mathscr{R}_{\mathfrak{g}} \cup \mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}})\big) \subseteq \mathscr{R}_{\mathfrak{g}} \cup \mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}})$. But, because the relation $\mathfrak{g}\text{-Der}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}) \subseteq \mathfrak{g}\text{-Der}_{\mathfrak{g}}\big(\mathscr{R}_{\mathfrak{g}} \cup \mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}})\big)$ holds, it results, therefore, that $\mathfrak{g}\text{-Der}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}) \subseteq \mathscr{R}_{\mathfrak{g}} \cup \mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}) \overset{\text{def}}{\longleftrightarrow} \mathrm{Ax}_{\mathrm{DE},3}(\mathfrak{g}\text{-Der}_{\mathfrak{g}})$ and thus, $\mathrm{Ax}_{\mathrm{DE,II}} \longrightarrow \mathrm{Ax}_{\mathrm{DE},3}$.

If $(\mathscr{R}_{\mathfrak{g}}, \mathscr{S}_{\mathfrak{g}}) \in \times_{\alpha \in I_2^*} \mathscr{P}(\Omega)$ be arbitrary, then

$$\begin{aligned} \mathrm{Ax}_{\mathrm{DE,II}}\left(\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}\right) \stackrel{\text{def}}{\longleftrightarrow} & \left(\bigcup_{\mathscr{U}_{\mathfrak{g}}=\mathscr{R}_{\mathfrak{g}},\mathscr{S}_{\mathfrak{g}}} \mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}\left(\mathscr{U}_{\mathfrak{g}} \cup \mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}\left(\mathscr{U}_{\mathfrak{g}}\right)\right)\right) \\ & \cup \left(\bigcup_{\mathscr{U}_{\mathfrak{g}}=\mathscr{R}_{\mathfrak{g}},\mathscr{S}_{\mathfrak{g}}} \mathscr{U}_{\mathfrak{g}} \cup \mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}\left(\mathscr{U}_{\mathfrak{g}}\right)\right) \\ & = \left(\mathscr{R}_{\mathfrak{g}} \cup \mathscr{S}_{\mathfrak{g}}\right) \cup \mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}\left(\mathscr{R}_{\mathfrak{g}} \cup \mathscr{S}_{\mathfrak{g}}\right). \end{aligned}$$

Since $\bigcup_{\mathscr{U}_{\mathfrak{g}}=\mathscr{R}_{\mathfrak{g}},\mathscr{S}_{\mathfrak{g}}} \mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}\left(\mathscr{U}_{\mathfrak{g}} \cup \mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}(\mathscr{U}_{\mathfrak{g}})\right) \subseteq \bigcup_{\mathscr{U}_{\mathfrak{g}}=\mathscr{R}_{\mathfrak{g}},\mathscr{S}_{\mathfrak{g}}} \left(\mathscr{U}_{\mathfrak{g}} \cup \mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}(\mathscr{U}_{\mathfrak{g}})\right)$ holds, $\mathrm{Ax}_{\mathrm{DE,II}}$, evidently, reduces to $\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}} \cup \mathscr{S}_{\mathfrak{g}}) = \bigcup_{\mathscr{U}_{\mathfrak{g}}=\mathscr{R}_{\mathfrak{g}},\mathscr{S}_{\mathfrak{g}}} \mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}(\mathscr{U}_{\mathfrak{g}}) \stackrel{\text{def}}{\longleftrightarrow}$ $\mathrm{Ax}_{\mathrm{DE},4}\left(\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}\right)$ and hence, $\mathrm{Ax}_{\mathrm{DE,II}} \longrightarrow \mathrm{Ax}_{\mathrm{DE},4}$.

Thus, $\mathrm{Ax}_{\mathrm{DE,II}}\left(\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}\right) = 1 \longrightarrow \bigwedge_{\nu \in I_4^*} \mathrm{Ax}_{\mathrm{DE},\nu}\left(\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}\right) = 1$ and the proof of the proposition is complete. Q.E.D.

The corollary stated below is an immediate consequence of the foregoing theorem and proposition.

COROLLARY 3.30. *If $\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ be a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-derived operator on $\mathscr{P}(\Omega)$ ranging in $\mathscr{P}(\Omega)$ in a strong $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$, then it satisfies the following "$\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-derived operator axiomatic diagram:"*

$$\begin{array}{ccc} \mathrm{Ax}_{\mathrm{DE,I}}\left(\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}\right) = 1 & \longrightarrow & \bigwedge_{\nu \in I_4^*} \mathrm{Ax}_{\mathrm{DE},\nu}\left(\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}\right) = 1 \\ & \nwarrow\!\!\searrow & \uparrow \\ & & \mathrm{Ax}_{\mathrm{DE,II}}\left(\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}\right) = 1. \end{array} \tag{3.15}$$

Likewise, viewing the derived set $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-coderived operator conditions (ITEMS I.–IV. of COR. 3.26 above) as $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-*coderived operator axioms*, the axiomatic definition of the concept of a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-coderived operator, then, can be defined as a set-valued map $\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ on $\mathscr{P}(\Omega)$ ranging in $\mathscr{P}(\Omega)$ satisfying a list of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-*coderived operator axioms*. The axiomatic definition of the concept of a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-coderived operator in $\mathscr{T}_{\mathfrak{g}}$-spaces follows.

DEFINITION 3.31 (Axiomatic Definition: $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-Coderived Operator). A one-valued map of the type $\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ in a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$ is called a "$\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-coderived operator" on $\mathscr{P}(\Omega)$ ranging in $\mathscr{P}(\Omega)$ if and only if, for any $\left(\{\zeta\}, \mathscr{U}_{\mathfrak{g}}, \mathscr{V}_{\mathfrak{g}}\right) \in \times_{\alpha \in I_3^*} \mathscr{P}(\Omega)$, it satisfies each "$\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-coderived operator axiom" in $\mathrm{AX}\left[\mathfrak{g}\text{-}\mathrm{CD}\left[\mathfrak{T}_{\mathfrak{g}}\right]; \mathbb{B}\right] \stackrel{\text{def}}{=} \left\{\mathrm{Ax}_{\mathrm{CD},\nu}\left(\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}\right) : \nu \in I_4^*\right\}$, where $\mathrm{Ax}_{\mathrm{CD},\nu} : \mathfrak{g}\text{-}\mathrm{CD}\left[\mathfrak{T}_{\mathfrak{g}}\right] \longrightarrow \mathbb{B} \stackrel{\text{def}}{=} \{0,1\}$, $\nu \in I_4^*$, is defined as thus:

- $\mathrm{Ax}_{\mathrm{CD},1}\left(\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}\right) \stackrel{\text{def}}{\longleftrightarrow} \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}(\Omega) = \Omega$,
- $\mathrm{Ax}_{\mathrm{CD},2}\left(\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}\right) \stackrel{\text{def}}{\longleftrightarrow} \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}\left(\mathscr{U}_{\mathfrak{g}}\right) = \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}\left(\mathscr{U}_{\mathfrak{g}} \cup \{\zeta\}\right)$,
- $\mathrm{Ax}_{\mathrm{CD},3}\left(\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}\right) \stackrel{\text{def}}{\longleftrightarrow} \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}} \circ \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}(\mathscr{U}_{\mathfrak{g}}) \supseteq \mathscr{U}_{\mathfrak{g}} \cap \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}(\mathscr{U}_{\mathfrak{g}})$,
- $\mathrm{Ax}_{\mathrm{CD},4}\left(\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}\right) \stackrel{\text{def}}{\longleftrightarrow} \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}(\mathscr{U}_{\mathfrak{g}} \cap \mathscr{V}_{\mathfrak{g}}) = \bigcap_{\mathscr{W}_{\mathfrak{g}}=\mathscr{U}_{\mathfrak{g}},\mathscr{V}_{\mathfrak{g}}} \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}(\mathscr{W}_{\mathfrak{g}})$.

Hence, a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-coderived operator $\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ in a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$ is a Ω-*grounded* ($\mathrm{Ax}_{\mathrm{CD},1}$), ζ-*invariant* ($\mathrm{Ax}_{\mathrm{CD},2}$), $\mathfrak{g}\text{-}\mathrm{Int}_{\mathfrak{g}}$-*extensive*

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$(\mathrm{Ax}_{\mathrm{CD},3})$ and $\cap$-*additive* $(\mathrm{Ax}_{\mathrm{CD},4})$ $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-set-valued set map forming a generalization of the $\mathfrak{T}_{\mathfrak{g}}$-set-valued set map $\mathrm{cod}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ in the $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}}$, provided

$$\begin{aligned}\big(\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}\big(\mathscr{S}_{\mathfrak{g}} \cap \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})\big) \supseteq \mathscr{S}_{\mathfrak{g}} \cap \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})\big)&\\ \wedge\big(\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}\big(\mathscr{R}_{\mathfrak{g}} \cap \mathscr{S}_{\mathfrak{g}}\big) = \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}) \cap \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})\big)&\\ \Updownarrow\qquad&\\ \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}} \circ \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \supseteq \mathscr{S}_{\mathfrak{g}} \cap \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})&\end{aligned}$$

holds for any $(\mathscr{R}_{\mathfrak{g}}, \mathscr{S}_{\mathfrak{g}}) \in \times_{\alpha \in I_2^*} \mathscr{P}(\Omega)$.

As above, having introduced an alternative definition defining the notion of a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-coderived operator in a $\mathscr{T}_{\mathfrak{g}}$-space axiomatically, it may not be without interest to prove some further propositions based on such axiomatic definition. The theorem follows.

Theorem 3.32. *Let* $\mathrm{AX}\big[\mathfrak{g}\text{-}\mathrm{CD}[\mathfrak{T}_{\mathfrak{g}}];\mathbb{B}\big] \stackrel{\text{def}}{=} \{\mathrm{Ax}_{\mathrm{CD},\nu} : \nu \in I_4^*\}$ *be the class of* $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$*-coderived operator axioms in a* $\mathscr{T}_{\mathfrak{g}}$*-space* $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$ *and, let* $\mathrm{Ax}_{\mathrm{CD,I}} : \mathfrak{g}\text{-}\mathrm{CD}[\mathfrak{T}_{\mathfrak{g}}] \longrightarrow \mathbb{B}$ *such that, for any* $(\mathscr{U}_{\mathfrak{g}}, \mathscr{V}_{\mathfrak{g}}) \in \times_{\alpha \in I_2^*} \mathscr{P}(\Omega)$*,*

$$\begin{aligned}\mathrm{Ax}_{\mathrm{CD,I}}\big(\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}\big) \quad &\stackrel{\text{def}}{\longleftrightarrow} \quad \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}\big(\mathscr{U}_{\mathfrak{g}} \cap \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}(\mathscr{U}_{\mathfrak{g}})\big)\\ &\qquad \cap\Big(\bigcap_{\mathscr{W}_{\mathfrak{g}} = \mathscr{U}_{\mathfrak{g}}, \mathscr{V}_{\mathfrak{g}}} \big(\mathscr{W}_{\mathfrak{g}} \cap \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}(\mathscr{W}_{\mathfrak{g}})\big)\Big)\\ &\qquad = \big(\mathscr{U}_{\mathfrak{g}} \cap \mathscr{V}_{\mathfrak{g}} \cap \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}(\mathscr{U}_{\mathfrak{g}} \cap \mathscr{V}_{\mathfrak{g}})\big) \setminus \mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}} \circ \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}(\Omega).\end{aligned} \tag{3.16}$$

Then, $\mathrm{Ax}_{\mathrm{CD,I}}\big(\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}\big) = 1 \longrightarrow \bigwedge_{\nu \in I_4^*} \mathrm{Ax}_{\mathrm{CD},\nu}\big(\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}\big) = 1$.

Proof. Let $\mathrm{AX}\big[\mathfrak{g}\text{-}\mathrm{CD}[\mathfrak{T}_{\mathfrak{g}}];\mathbb{B}\big] = \{\mathrm{Ax}_{\mathrm{CD},\nu} : \nu \in I_4^*\}$ be the class of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-coderived operator axioms in a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$ and, let $\mathrm{Ax}_{\mathrm{CD,I}} : \mathfrak{g}\text{-}\mathrm{CD}[\mathfrak{T}_{\mathfrak{g}}] \longrightarrow \mathbb{B}$ such that, for any $(\mathscr{U}_{\mathfrak{g}}, \mathscr{V}_{\mathfrak{g}}) \in \times_{\alpha \in I_2^*} \mathscr{P}(\Omega)$,

$$\begin{aligned}\mathrm{Ax}_{\mathrm{CD,I}}\big(\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}\big) \quad &\stackrel{\text{def}}{\longleftrightarrow} \quad \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}\big(\mathscr{U}_{\mathfrak{g}} \cap \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}(\mathscr{U}_{\mathfrak{g}})\big)\\ &\qquad \cap\Big(\bigcap_{\mathscr{W}_{\mathfrak{g}} = \mathscr{U}_{\mathfrak{g}}, \mathscr{V}_{\mathfrak{g}}} \big(\mathscr{W}_{\mathfrak{g}} \cap \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}(\mathscr{W}_{\mathfrak{g}})\big)\Big)\\ &\qquad = \big(\mathscr{U}_{\mathfrak{g}} \cap \mathscr{V}_{\mathfrak{g}} \cap \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}(\mathscr{U}_{\mathfrak{g}} \cap \mathscr{V}_{\mathfrak{g}})\big) \setminus \mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}} \circ \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}(\Omega).\end{aligned}$$

Suppose $\mathrm{Ax}_{\mathrm{CD,I}}\big(\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}\big) = 1$ holds. Then:

If $(\mathscr{U}_{\mathfrak{g}}, \mathscr{V}_{\mathfrak{g}}) = (\Omega, \Omega)$, then

$$\begin{aligned}\mathrm{Ax}_{\mathrm{CD,I}}\big(\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}\big) \quad &\stackrel{\text{def}}{\longleftrightarrow} \quad \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}\big(\Omega \cap \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}(\Omega)\big)\\ &\qquad \cap\Big(\bigcap_{\mathscr{W}_{\mathfrak{g}} = \Omega, \Omega} \big(\mathscr{W}_{\mathfrak{g}} \cap \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}(\mathscr{W}_{\mathfrak{g}})\big)\Big)\\ &\qquad = \big(\Omega \cap \Omega \cap \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}(\Omega \cap \Omega)\big) \setminus \mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}} \circ \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}(\Omega).\end{aligned}$$

Consequently, $\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}} \circ \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}(\Omega)$, $\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}(\Omega) = \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}(\Omega)$. Thus, $\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}(\Omega) = \Omega \stackrel{\text{def}}{\longleftrightarrow} \mathrm{Ax}_{\mathrm{CD},1}\big(\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}\big)$ and hence, $\mathrm{Ax}_{\mathrm{CD,I}} \longrightarrow \mathrm{Ax}_{\mathrm{CD},1}$.

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If $(\mathscr{U}_{\mathfrak{g}},\mathscr{V}_{\mathfrak{g}})\in\times_{\alpha\in I_2^*}\mathscr{P}(\Omega)$ be arbitrary such that $\mathscr{V}_{\mathfrak{g}}=\mathscr{U}_{\mathfrak{g}}\cup\{\zeta\}$ and $\{\zeta\}\subset\mathfrak{g}\text{-Cod}_{\mathfrak{g}}(\mathscr{U}_{\mathfrak{g}})$, then

$$\begin{aligned}\mathrm{Ax}_{\mathrm{CD,I}}\left(\mathfrak{g}\text{-Cod}_{\mathfrak{g}}\right) \overset{\text{def}}{\longleftrightarrow}\ & \mathfrak{g}\text{-Cod}_{\mathfrak{g}}\left(\mathscr{U}_{\mathfrak{g}}\cap\mathfrak{g}\text{-Cod}_{\mathfrak{g}}(\mathscr{U}_{\mathfrak{g}})\right)\\ &\cap\Bigg(\bigcap_{\mathscr{W}_{\mathfrak{g}}=\mathscr{U}_{\mathfrak{g}},\mathscr{U}_{\mathfrak{g}}\cup\{\zeta\}}\left(\mathscr{W}_{\mathfrak{g}}\cap\mathfrak{g}\text{-Cod}_{\mathfrak{g}}(\mathscr{W}_{\mathfrak{g}})\right)\Bigg)\\ &=\left(\mathscr{U}_{\mathfrak{g}}\cap(\mathscr{U}_{\mathfrak{g}}\cup\{\zeta\})\cap\mathfrak{g}\text{-Cod}_{\mathfrak{g}}\left(\mathscr{U}_{\mathfrak{g}}\cap(\mathscr{U}_{\mathfrak{g}}\cup\{\zeta\})\right)\right)\\ &\setminus\mathfrak{g}\text{-Op}_{\mathfrak{g}}\circ\mathfrak{g}\text{-Cod}_{\mathfrak{g}}(\Omega).\end{aligned}$$

Since the relation $\mathfrak{g}\text{-Cod}_{\mathfrak{g}}\left(\mathscr{U}_{\mathfrak{g}}\cap\mathfrak{g}\text{-Cod}_{\mathfrak{g}}(\mathscr{U}_{\mathfrak{g}})\right)\supseteq\mathscr{U}_{\mathfrak{g}}\cap\mathfrak{g}\text{-Cod}_{\mathfrak{g}}(\mathscr{U}_{\mathfrak{g}})$ holds, implying $\mathfrak{g}\text{-Cod}_{\mathfrak{g}}\left(\mathscr{U}_{\mathfrak{g}}\cap\mathfrak{g}\text{-Cod}_{\mathfrak{g}}(\mathscr{U}_{\mathfrak{g}})\right)\cap\left(\mathscr{U}_{\mathfrak{g}}\cap\mathfrak{g}\text{-Cod}_{\mathfrak{g}}(\mathscr{U}_{\mathfrak{g}})\right)=\mathscr{U}_{\mathfrak{g}}\cap\mathfrak{g}\text{-Cod}_{\mathfrak{g}}(\mathscr{U}_{\mathfrak{g}})$, and $\mathrm{Ax}_{\mathrm{CD},1}$ implies $\mathfrak{g}\text{-Op}_{\mathfrak{g}}\circ\mathfrak{g}\text{-Cod}_{\mathfrak{g}}(\Omega)=\emptyset$, $\mathrm{Ax}_{\mathrm{CD,I}}$ reduces to

$$\mathfrak{g}\text{-Cod}_{\mathfrak{g}}(\mathscr{U}_{\mathfrak{g}})\cap\mathfrak{g}\text{-Cod}_{\mathfrak{g}}\left(\mathscr{U}_{\mathfrak{g}}\cup\{\zeta\}\right)=\mathfrak{g}\text{-Cod}_{\mathfrak{g}}\left(\mathscr{U}_{\mathfrak{g}}\cap(\mathscr{U}_{\mathfrak{g}}\cup\{\zeta\})\right).$$

Clearly, $\mathfrak{g}\text{-Cod}_{\mathfrak{g}}\left(\mathscr{U}_{\mathfrak{g}}\cap(\mathscr{U}_{\mathfrak{g}}\cup\{\zeta\})\right)\longleftrightarrow\mathfrak{g}\text{-Cod}_{\mathfrak{g}}(\mathscr{U}_{\mathfrak{g}})$. Consequently, it results that $\mathfrak{g}\text{-Cod}_{\mathfrak{g}}\left(\mathscr{U}_{\mathfrak{g}}\cup\{\zeta\}\right)\supseteq\mathfrak{g}\text{-Cod}_{\mathfrak{g}}(\mathscr{U}_{\mathfrak{g}})$. But,

$$\begin{aligned}\mathfrak{g}\text{-Cod}_{\mathfrak{g}}(\mathscr{U}_{\mathfrak{g}}) &\supseteq \left\{\zeta\in\mathfrak{T}_{\mathfrak{g}}:\ \zeta\in\mathfrak{g}\text{-Int}_{\mathfrak{g}}\left(\mathscr{U}_{\mathfrak{g}}\cup\{\zeta\}\right)\right\}\\ &\longleftrightarrow \left\{\zeta\in\mathfrak{T}_{\mathfrak{g}}:\ \zeta\in\mathfrak{g}\text{-Int}_{\mathfrak{g}}\left((\mathscr{U}_{\mathfrak{g}}\cup\{\zeta\})\cup\{\zeta\}\right)\right\}\\ &\longleftrightarrow \mathfrak{g}\text{-Cod}_{\mathfrak{g}}(\mathscr{U}_{\mathfrak{g}}\cup\{\zeta\}).\end{aligned}$$

Consequently, $\mathfrak{g}\text{-Cod}_{\mathfrak{g}}\left(\mathscr{U}_{\mathfrak{g}}\cup\{\zeta\}\right)\subseteq\mathfrak{g}\text{-Cod}_{\mathfrak{g}}(\mathscr{U}_{\mathfrak{g}})$. Therefore, it follows that the relation $\mathfrak{g}\text{-Cod}_{\mathfrak{g}}(\mathscr{U}_{\mathfrak{g}})=\mathfrak{g}\text{-Cod}_{\mathfrak{g}}\left(\mathscr{U}_{\mathfrak{g}}\cup\{\zeta\}\right)\overset{\text{def}}{\longleftrightarrow}\mathrm{Ax}_{\mathrm{CD},2}\left(\mathfrak{g}\text{-Cod}_{\mathfrak{g}}\right)$ holds and thus, $\mathrm{Ax}_{\mathrm{CD,I}}\longrightarrow\mathrm{Ax}_{\mathrm{CD},2}$.

If $(\mathscr{U}_{\mathfrak{g}},\mathscr{V}_{\mathfrak{g}})\in\times_{\alpha\in I_2^*}\mathscr{P}(\Omega)$ such that $\mathscr{V}_{\mathfrak{g}}=\mathscr{U}_{\mathfrak{g}}$ be arbitrary, then

$$\begin{aligned}\mathrm{Ax}_{\mathrm{CD,I}}\left(\mathfrak{g}\text{-Cod}_{\mathfrak{g}}\right) \overset{\text{def}}{\longleftrightarrow}\ & \mathfrak{g}\text{-Cod}_{\mathfrak{g}}\left(\mathscr{U}_{\mathfrak{g}}\cap\mathfrak{g}\text{-Cod}_{\mathfrak{g}}(\mathscr{U}_{\mathfrak{g}})\right)\\ &\cap\Bigg(\bigcap_{\mathscr{W}_{\mathfrak{g}}=\mathscr{U}_{\mathfrak{g}},\mathscr{U}_{\mathfrak{g}}}\left(\mathscr{W}_{\mathfrak{g}}\cap\mathfrak{g}\text{-Cod}_{\mathfrak{g}}(\mathscr{W}_{\mathfrak{g}})\right)\Bigg)\\ &=\left(\mathscr{U}_{\mathfrak{g}}\cap\mathscr{U}_{\mathfrak{g}}\cap\mathfrak{g}\text{-Cod}_{\mathfrak{g}}(\mathscr{U}_{\mathfrak{g}}\cap\mathscr{U}_{\mathfrak{g}})\right)\setminus\mathfrak{g}\text{-Op}_{\mathfrak{g}}\circ\mathfrak{g}\text{-Cod}_{\mathfrak{g}}(\Omega).\end{aligned}$$

Consequently, $\mathfrak{g}\text{-Cod}_{\mathfrak{g}}\left(\mathscr{U}_{\mathfrak{g}}\cap\mathfrak{g}\text{-Cod}_{\mathfrak{g}}(\mathscr{U}_{\mathfrak{g}})\right)\supseteq\mathscr{U}_{\mathfrak{g}}\cap\mathfrak{g}\text{-Cod}_{\mathfrak{g}}(\mathscr{U}_{\mathfrak{g}})$, since $\mathrm{Ax}_{\mathrm{CD},1}$ implies $\mathfrak{g}\text{-Op}_{\mathfrak{g}}\circ\mathfrak{g}\text{-Cod}_{\mathfrak{g}}(\Omega)=\emptyset$. But, $\mathfrak{g}\text{-Cod}_{\mathfrak{g}}\left(\mathscr{U}_{\mathfrak{g}}\cap\mathfrak{g}\text{-Cod}_{\mathfrak{g}}(\mathscr{U}_{\mathfrak{g}})\right)\subseteq\mathfrak{g}\text{-Cod}_{\mathfrak{g}}\circ\mathfrak{g}\text{-Cod}_{\mathfrak{g}}(\mathscr{U}_{\mathfrak{g}})$. Therefore, $\mathfrak{g}\text{-Cod}_{\mathfrak{g}}\circ\mathfrak{g}\text{-Cod}_{\mathfrak{g}}(\mathscr{U}_{\mathfrak{g}})\supseteq\mathscr{U}_{\mathfrak{g}}\cap\mathfrak{g}\text{-Cod}_{\mathfrak{g}}(\mathscr{U}_{\mathfrak{g}})\overset{\text{def}}{\longleftrightarrow}\mathrm{Ax}_{\mathrm{CD},3}\left(\mathfrak{g}\text{-Cod}_{\mathfrak{g}}\right)$ and thus, $\mathrm{Ax}_{\mathrm{CD,I}}\longrightarrow\mathrm{Ax}_{\mathrm{CD},3}$.

If $(\mathscr{U}_{\mathfrak{g}},\mathscr{V}_{\mathfrak{g}})\in\times_{\alpha\in I_2^*}\mathscr{P}(\Omega)$ be arbitrary, then

$$\begin{aligned}\mathrm{Ax}_{\mathrm{CD,I}}\left(\mathfrak{g}\text{-Cod}_{\mathfrak{g}}\right) \overset{\text{def}}{\longleftrightarrow}\ & \mathfrak{g}\text{-Cod}_{\mathfrak{g}}\left(\mathscr{U}_{\mathfrak{g}}\cap\mathfrak{g}\text{-Cod}_{\mathfrak{g}}(\mathscr{U}_{\mathfrak{g}})\right)\\ &\cap\Bigg(\bigcap_{\mathscr{W}_{\mathfrak{g}}=\mathscr{U}_{\mathfrak{g}},\mathscr{V}_{\mathfrak{g}}}\left(\mathscr{W}_{\mathfrak{g}}\cap\mathfrak{g}\text{-Cod}_{\mathfrak{g}}(\mathscr{W}_{\mathfrak{g}})\right)\Bigg)\\ &=\left(\mathscr{U}_{\mathfrak{g}}\cap\mathscr{V}_{\mathfrak{g}}\cap\mathfrak{g}\text{-Cod}_{\mathfrak{g}}(\mathscr{U}_{\mathfrak{g}}\cap\mathscr{V}_{\mathfrak{g}})\right)\setminus\mathfrak{g}\text{-Op}_{\mathfrak{g}}\circ\mathfrak{g}\text{-Cod}_{\mathfrak{g}}(\Omega).\end{aligned}$$

By virtue of the relation $\mathfrak{g}\text{-Cod}_{\mathfrak{g}}\left(\mathscr{U}_{\mathfrak{g}}\cap\mathfrak{g}\text{-Cod}_{\mathfrak{g}}(\mathscr{U}_{\mathfrak{g}})\right)\supseteq\mathscr{U}_{\mathfrak{g}}\cap\mathfrak{g}\text{-Cod}_{\mathfrak{g}}(\mathscr{U}_{\mathfrak{g}})$ or equivalently, $\mathfrak{g}\text{-Cod}_{\mathfrak{g}}\left(\mathscr{U}_{\mathfrak{g}}\cap\mathfrak{g}\text{-Cod}_{\mathfrak{g}}(\mathscr{U}_{\mathfrak{g}})\right)\cap\left(\mathscr{U}_{\mathfrak{g}}\cap\mathfrak{g}\text{-Cod}_{\mathfrak{g}}(\mathscr{U}_{\mathfrak{g}})\right)=\mathscr{U}_{\mathfrak{g}}\cap\mathfrak{g}\text{-Cod}_{\mathfrak{g}}(\mathscr{U}_{\mathfrak{g}})$, together

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with $\mathrm{Ax}_{\mathrm{CD},1}$, $\mathrm{Ax}_{\mathrm{CD},\mathrm{I}}$ reduces to $\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}(\mathscr{U}_{\mathfrak{g}} \cap \mathscr{V}_{\mathfrak{g}}) = \bigcap_{\mathscr{W}_{\mathfrak{g}}=\mathscr{U}_{\mathfrak{g}},\mathscr{V}_{\mathfrak{g}}} \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}(\mathscr{W}_{\mathfrak{g}}) \stackrel{\mathrm{def}}{\longleftrightarrow} \mathrm{Ax}_{\mathrm{CD},4}(\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}})$ and hence, $\mathrm{Ax}_{\mathrm{CD},\mathrm{I}} \longrightarrow \mathrm{Ax}_{\mathrm{CD},4}$.

Thus, $\mathrm{Ax}_{\mathrm{CD},\mathrm{I}}(\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}) = 1 \longrightarrow \bigwedge_{\nu\in I_4^*} \mathrm{Ax}_{\mathrm{CD},\nu}(\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}) = 1$ and the proof of the theorem is complete. Q.E.D.

The proposition given below contains further properties.

Proposition 3.33. *Let* $\mathrm{AX}[\mathfrak{g}\text{-}\mathrm{CD}[\mathfrak{T}_{\mathfrak{g}}];\mathbb{B}] \stackrel{\mathrm{def}}{=} \{\mathrm{Ax}_{\mathrm{CD},\nu} : \nu \in I_4^*\}$ *be the class of* $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$*-coderived operator axioms in a* $\mathscr{T}_{\mathfrak{g}}$*-space* $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$ *and, let* $\mathrm{Ax}_{\mathrm{CD},\mathrm{II}} : \mathfrak{g}\text{-}\mathrm{CD}[\mathfrak{T}_{\mathfrak{g}}] \longrightarrow \mathbb{B}$ *such that, for any* $(\mathscr{U}_{\mathfrak{g}}, \mathscr{V}_{\mathfrak{g}}) \in \times_{\alpha\in I_2^*} \mathscr{P}(\Omega)$,

$$\begin{aligned}\mathrm{Ax}_{\mathrm{CD},\mathrm{II}}(\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}) \stackrel{\mathrm{def}}{\longleftrightarrow}\; & \Big(\bigcap_{\mathscr{W}_{\mathfrak{g}}=\mathscr{U}_{\mathfrak{g}},\mathscr{V}_{\mathfrak{g}}} (\mathscr{W}_{\mathfrak{g}} \cap \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}(\mathscr{W}_{\mathfrak{g}}))\Big) \\ & \cap \Big(\bigcap_{\mathscr{W}_{\mathfrak{g}}=\mathscr{U}_{\mathfrak{g}},\mathscr{V}_{\mathfrak{g}}} \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}(\mathscr{W}_{\mathfrak{g}} \cap \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}(\mathscr{W}_{\mathfrak{g}}))\Big) \\ & = (\mathscr{U}_{\mathfrak{g}} \cap \mathscr{V}_{\mathfrak{g}}) \cap \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}(\mathscr{U}_{\mathfrak{g}} \cap \mathscr{V}_{\mathfrak{g}}).\end{aligned} \tag{3.17}$$

Then, $\mathrm{Ax}_{\mathrm{CD},\mathrm{II}}(\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}) = 1 \longrightarrow \bigwedge_{\nu\in I_4^*} \mathrm{Ax}_{\mathrm{CD},\nu}(\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}) = 1$.

Proof. Let $\mathrm{AX}[\mathfrak{g}\text{-}\mathrm{CD}[\mathfrak{T}_{\mathfrak{g}}];\mathbb{B}] \stackrel{\mathrm{def}}{=} \{\mathrm{Ax}_{\mathrm{CD},\nu} : \nu \in I_4^*\}$ be the class of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-coderived operator axioms in a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$ and further, let $\mathrm{Ax}_{\mathrm{CD},\mathrm{II}} : \mathfrak{g}\text{-}\mathrm{CD}[\mathfrak{T}_{\mathfrak{g}}] \longrightarrow \mathbb{B}$ such that, for any $(\mathscr{U}_{\mathfrak{g}}, \mathscr{V}_{\mathfrak{g}}) \in \times_{\alpha\in I_2^*} \mathscr{P}(\Omega)$,

$$\begin{aligned}\mathrm{Ax}_{\mathrm{CD},\mathrm{II}}(\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}) \stackrel{\mathrm{def}}{\longleftrightarrow}\; & \Big(\bigcap_{\mathscr{W}_{\mathfrak{g}}=\mathscr{U}_{\mathfrak{g}},\mathscr{V}_{\mathfrak{g}}} (\mathscr{W}_{\mathfrak{g}} \cap \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}(\mathscr{W}_{\mathfrak{g}}))\Big) \\ & \cap \Big(\bigcap_{\mathscr{W}_{\mathfrak{g}}=\mathscr{U}_{\mathfrak{g}},\mathscr{V}_{\mathfrak{g}}} \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}(\mathscr{W}_{\mathfrak{g}} \cap \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}(\mathscr{W}_{\mathfrak{g}}))\Big) \\ & = (\mathscr{U}_{\mathfrak{g}} \cap \mathscr{V}_{\mathfrak{g}}) \cap \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}(\mathscr{U}_{\mathfrak{g}} \cap \mathscr{V}_{\mathfrak{g}}).\end{aligned}$$

Suppose $\mathrm{Ax}_{\mathrm{CD},\mathrm{II}}(\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}) = 1$ holds. Then:

If $(\mathscr{U}_{\mathfrak{g}}, \mathscr{V}_{\mathfrak{g}}) = (\Omega, \Omega)$, then

$$\begin{aligned}\mathrm{Ax}_{\mathrm{CD},\mathrm{II}}(\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}) \stackrel{\mathrm{def}}{\longleftrightarrow}\; & \Big(\bigcap_{\mathscr{W}_{\mathfrak{g}}=\Omega,\Omega} (\mathscr{W}_{\mathfrak{g}} \cap \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}(\mathscr{W}_{\mathfrak{g}}))\Big) \\ & \cap \Big(\bigcap_{\mathscr{W}_{\mathfrak{g}}=\Omega,\Omega} \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}(\mathscr{W}_{\mathfrak{g}} \cap \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}(\mathscr{W}_{\mathfrak{g}}))\Big) \\ & = (\Omega \cap \Omega) \cap \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}(\Omega \cap \Omega).\end{aligned}$$

Consequently, $\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}(\Omega) \cap \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}} \circ \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}(\Omega) = \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}(\Omega)$. But, the relation $\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}(\Omega) \longleftrightarrow \Omega \cap \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}(\Omega) \longleftrightarrow \mathfrak{g}\text{-}\mathrm{Int}_{\mathfrak{g}}(\Omega) = \Omega$ holds. Therefore, $\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}(\Omega) = \Omega \stackrel{\mathrm{def}}{\longleftrightarrow} \mathrm{Ax}_{\mathrm{CD},1}(\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}})$ and hence, $\mathrm{Ax}_{\mathrm{CD},\mathrm{II}} \longrightarrow \mathrm{Ax}_{\mathrm{CD},1}$.

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If $(\mathscr{U}_{\mathfrak{g}},\mathscr{V}_{\mathfrak{g}})\in\times_{\alpha\in I_2^*}\mathscr{P}(\Omega)$ be arbitrary such that $\mathscr{V}_{\mathfrak{g}}=\mathscr{U}_{\mathfrak{g}}\cup\{\zeta\}$ and $\{\zeta\}\subset\mathfrak{g}\text{-Cod}_{\mathfrak{g}}(\mathscr{U}_{\mathfrak{g}})$, then

$$\begin{aligned}\mathrm{Ax}_{\mathrm{CD,II}}\left(\mathfrak{g}\text{-Cod}_{\mathfrak{g}}\right) \overset{\text{def}}{\longleftrightarrow} & \left(\bigcap_{\mathscr{W}_{\mathfrak{g}}=\mathscr{U}_{\mathfrak{g}},\mathscr{U}_{\mathfrak{g}}\cup\{\zeta\}}\left(\mathscr{W}_{\mathfrak{g}}\cap\mathfrak{g}\text{-Cod}_{\mathfrak{g}}(\mathscr{W}_{\mathfrak{g}})\right)\right)\\ &\cap\left(\bigcap_{\mathscr{W}_{\mathfrak{g}}=\mathscr{U}_{\mathfrak{g}},\mathscr{U}_{\mathfrak{g}}\cup\{\zeta\}}\mathfrak{g}\text{-Cod}_{\mathfrak{g}}\left(\mathscr{W}_{\mathfrak{g}}\cap\mathfrak{g}\text{-Cod}_{\mathfrak{g}}(\mathscr{W}_{\mathfrak{g}})\right)\right)\\ &=\left(\mathscr{U}_{\mathfrak{g}}\cap(\mathscr{U}_{\mathfrak{g}}\cup\{\zeta\})\right)\cap\mathfrak{g}\text{-Cod}_{\mathfrak{g}}\left(\mathscr{U}_{\mathfrak{g}}\cap(\mathscr{U}_{\mathfrak{g}}\cup\{\zeta\})\right).\end{aligned}$$

Since the relation $\mathfrak{g}\text{-Cod}_{\mathfrak{g}}\left(\mathscr{W}_{\mathfrak{g}}\cap\mathfrak{g}\text{-Cod}_{\mathfrak{g}}(\mathscr{W}_{\mathfrak{g}})\right)\supseteq\mathscr{W}_{\mathfrak{g}}\cap\mathfrak{g}\text{-Cod}_{\mathfrak{g}}(\mathscr{W}_{\mathfrak{g}})$ holds, implying $\mathfrak{g}\text{-Cod}_{\mathfrak{g}}\left(\mathscr{W}_{\mathfrak{g}}\cap\mathfrak{g}\text{-Cod}_{\mathfrak{g}}(\mathscr{W}_{\mathfrak{g}})\right)\cap\left(\mathscr{W}_{\mathfrak{g}}\cap\mathfrak{g}\text{-Cod}_{\mathfrak{g}}(\mathscr{W}_{\mathfrak{g}})\right)=\mathscr{W}_{\mathfrak{g}}\cap\mathfrak{g}\text{-Cod}_{\mathfrak{g}}(\mathscr{W}_{\mathfrak{g}})$ for any $\mathscr{W}_{\mathfrak{g}}\in\{\mathscr{U}_{\mathfrak{g}},\ \mathscr{U}_{\mathfrak{g}}\cup\{\zeta\}\}$, $\mathrm{Ax}_{\mathrm{CD,II}}$ reduces to

$$\bigcap_{\mathscr{W}_{\mathfrak{g}}=\mathscr{U}_{\mathfrak{g}},\mathscr{U}_{\mathfrak{g}}\cup\{\zeta\}}\mathfrak{g}\text{-Cod}_{\mathfrak{g}}(\mathscr{W}_{\mathfrak{g}})=\mathfrak{g}\text{-Cod}_{\mathfrak{g}}\left(\mathscr{U}_{\mathfrak{g}}\cap\left(\mathscr{U}_{\mathfrak{g}}\cup\{\zeta\}\right)\right).$$

Because $\mathscr{U}_{\mathfrak{g}}\subseteq\mathscr{U}_{\mathfrak{g}}\cup\{\zeta\}$ holds, $\mathfrak{g}\text{-Cod}_{\mathfrak{g}}\left(\mathscr{U}_{\mathfrak{g}}\cap(\mathscr{U}_{\mathfrak{g}}\cup\{\zeta\})\right)\longleftrightarrow\mathfrak{g}\text{-Cod}_{\mathfrak{g}}(\mathscr{U}_{\mathfrak{g}})$. Consequently, $\mathfrak{g}\text{-Cod}_{\mathfrak{g}}\left(\mathscr{U}_{\mathfrak{g}}\cup\{\zeta\}\right)\supseteq\mathfrak{g}\text{-Cod}_{\mathfrak{g}}(\mathscr{U}_{\mathfrak{g}})$. But,

$$\begin{aligned}\mathfrak{g}\text{-Cod}_{\mathfrak{g}}(\mathscr{U}_{\mathfrak{g}}) &\supseteq \left\{\zeta\in\mathfrak{T}_{\mathfrak{g}}:\ \zeta\in\mathfrak{g}\text{-Int}_{\mathfrak{g}}\left(\mathscr{U}_{\mathfrak{g}}\cup\{\zeta\}\right)\right\}\\ &\longleftrightarrow \left\{\zeta\in\mathfrak{T}_{\mathfrak{g}}:\ \zeta\in\mathfrak{g}\text{-Int}_{\mathfrak{g}}\left((\mathscr{U}_{\mathfrak{g}}\cup\{\zeta\})\cup\{\zeta\}\right)\right\}\\ &\longleftrightarrow \mathfrak{g}\text{-Cod}_{\mathfrak{g}}(\mathscr{U}_{\mathfrak{g}}\cup\{\zeta\}),\end{aligned}$$

implying the relation $\mathfrak{g}\text{-Cod}_{\mathfrak{g}}\left(\mathscr{U}_{\mathfrak{g}}\cup\{\zeta\}\right)\subseteq\mathfrak{g}\text{-Cod}_{\mathfrak{g}}(\mathscr{U}_{\mathfrak{g}})$. Therefore, $\mathfrak{g}\text{-Cod}_{\mathfrak{g}}(\mathscr{U}_{\mathfrak{g}})=\mathfrak{g}\text{-Cod}_{\mathfrak{g}}\left(\mathscr{U}_{\mathfrak{g}}\cup\{\zeta\}\right)\overset{\text{def}}{\longleftrightarrow}\mathrm{Ax}_{\mathrm{CD,2}}\left(\mathfrak{g}\text{-Cod}_{\mathfrak{g}}\right)$ and hence, $\mathrm{Ax}_{\mathrm{CD,I}}\longrightarrow\mathrm{Ax}_{\mathrm{CD,2}}$.

If $(\mathscr{U}_{\mathfrak{g}},\mathscr{V}_{\mathfrak{g}})\in\times_{\alpha\in I_2^*}\mathscr{P}(\Omega)$ such that $\mathscr{V}_{\mathfrak{g}}=\mathscr{U}_{\mathfrak{g}}$ be arbitrary, then

$$\begin{aligned}\mathrm{Ax}_{\mathrm{CD,II}}\left(\mathfrak{g}\text{-Cod}_{\mathfrak{g}}\right) \overset{\text{def}}{\longleftrightarrow} & \left(\bigcap_{\mathscr{W}_{\mathfrak{g}}=\mathscr{U}_{\mathfrak{g}},\mathscr{U}_{\mathfrak{g}}}\left(\mathscr{W}_{\mathfrak{g}}\cap\mathfrak{g}\text{-Cod}_{\mathfrak{g}}(\mathscr{W}_{\mathfrak{g}})\right)\right)\\ &\cap\left(\bigcap_{\mathscr{W}_{\mathfrak{g}}=\mathscr{U}_{\mathfrak{g}},\mathscr{U}_{\mathfrak{g}}}\mathfrak{g}\text{-Cod}_{\mathfrak{g}}\left(\mathscr{W}_{\mathfrak{g}}\cap\mathfrak{g}\text{-Cod}_{\mathfrak{g}}(\mathscr{W}_{\mathfrak{g}})\right)\right)\\ &=\left(\mathscr{U}_{\mathfrak{g}}\cap\mathscr{U}_{\mathfrak{g}}\right)\cap\mathfrak{g}\text{-Cod}_{\mathfrak{g}}\left(\mathscr{U}_{\mathfrak{g}}\cap\mathscr{U}_{\mathfrak{g}}\right).\end{aligned}$$

Consequently, $\mathfrak{g}\text{-Cod}_{\mathfrak{g}}\left(\mathscr{U}_{\mathfrak{g}}\cap\mathfrak{g}\text{-Cod}_{\mathfrak{g}}(\mathscr{U}_{\mathfrak{g}})\right)\cap\left(\mathscr{U}_{\mathfrak{g}}\cap\mathfrak{g}\text{-Cod}_{\mathfrak{g}}(\mathscr{U}_{\mathfrak{g}})\right)=\mathscr{U}_{\mathfrak{g}}\cap\mathfrak{g}\text{-Cod}_{\mathfrak{g}}(\mathscr{U}_{\mathfrak{g}})$, implying $\mathfrak{g}\text{-Cod}_{\mathfrak{g}}\left(\mathscr{U}_{\mathfrak{g}}\cap\mathfrak{g}\text{-Cod}_{\mathfrak{g}}(\mathscr{U}_{\mathfrak{g}})\right)\supseteq\mathscr{U}_{\mathfrak{g}}\cap\mathfrak{g}\text{-Cod}_{\mathfrak{g}}(\mathscr{U}_{\mathfrak{g}})$. But, because the relation $\mathfrak{g}\text{-Cod}_{\mathfrak{g}}\circ\mathfrak{g}\text{-Cod}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}})\supseteq\mathfrak{g}\text{-Cod}_{\mathfrak{g}}\left(\mathscr{U}_{\mathfrak{g}}\cap\mathfrak{g}\text{-Cod}_{\mathfrak{g}}(\mathscr{U}_{\mathfrak{g}})\right)$ holds, it results, therefore, that $\mathfrak{g}\text{-Cod}_{\mathfrak{g}}\circ\mathfrak{g}\text{-Cod}_{\mathfrak{g}}(\mathscr{U}_{\mathfrak{g}})\supseteq\mathscr{U}_{\mathfrak{g}}\cap\mathfrak{g}\text{-Cod}_{\mathfrak{g}}(\mathscr{U}_{\mathfrak{g}})\overset{\text{def}}{\longleftrightarrow}\mathrm{Ax}_{\mathrm{CD,3}}\left(\mathfrak{g}\text{-Cod}_{\mathfrak{g}}\right)$ and thus, $\mathrm{Ax}_{\mathrm{CD,II}}\longrightarrow\mathrm{Ax}_{\mathrm{CD,3}}$.

If $(\mathscr{U}_{\mathfrak{g}},\mathscr{V}_{\mathfrak{g}})\in\times_{\alpha\in I_2^*}\mathscr{P}(\Omega)$ be arbitrary, then

$$\begin{aligned}\mathrm{Ax}_{\mathrm{CD,II}}\left(\mathfrak{g}\text{-Cod}_{\mathfrak{g}}\right) \overset{\text{def}}{\longleftrightarrow} & \left(\bigcap_{\mathscr{W}_{\mathfrak{g}}=\mathscr{U}_{\mathfrak{g}},\mathscr{V}_{\mathfrak{g}}}\left(\mathscr{W}_{\mathfrak{g}}\cap\mathfrak{g}\text{-Cod}_{\mathfrak{g}}(\mathscr{W}_{\mathfrak{g}})\right)\right)\\ &\cap\left(\bigcap_{\mathscr{W}_{\mathfrak{g}}=\mathscr{U}_{\mathfrak{g}},\mathscr{V}_{\mathfrak{g}}}\mathfrak{g}\text{-Cod}_{\mathfrak{g}}\left(\mathscr{W}_{\mathfrak{g}}\cap\mathfrak{g}\text{-Cod}_{\mathfrak{g}}(\mathscr{W}_{\mathfrak{g}})\right)\right)\\ &=\left(\mathscr{U}_{\mathfrak{g}}\cap\mathscr{V}_{\mathfrak{g}}\right)\cap\mathfrak{g}\text{-Cod}_{\mathfrak{g}}\left(\mathscr{U}_{\mathfrak{g}}\cap\mathscr{V}_{\mathfrak{g}}\right).\end{aligned}$$

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Since $\bigcap_{\mathscr{W}_{\mathfrak{g}}=\mathscr{U}_{\mathfrak{g}},\mathscr{V}_{\mathfrak{g}}} \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}\left(\mathscr{W}_{\mathfrak{g}} \cap \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}\left(\mathscr{W}_{\mathfrak{g}}\right)\right) \supseteq \bigcap_{\mathscr{W}_{\mathfrak{g}}=\mathscr{U}_{\mathfrak{g}},\mathscr{V}_{\mathfrak{g}}}\left(\mathscr{W}_{\mathfrak{g}} \cap \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}\left(\mathscr{W}_{\mathfrak{g}}\right)\right)$ holds, $\mathrm{Ax}_{\mathrm{CD,II}}$, evidently, reduces to $\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}\left(\mathscr{U}_{\mathfrak{g}} \cap \mathscr{V}_{\mathfrak{g}}\right) = \bigcap_{\mathscr{W}_{\mathfrak{g}}=\mathscr{U}_{\mathfrak{g}},\mathscr{V}_{\mathfrak{g}}} \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}\left(\mathscr{W}_{\mathfrak{g}}\right) \stackrel{\mathrm{def}}{\longleftrightarrow} \mathrm{Ax}_{\mathrm{CD},4}\left(\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}\right)$ and hence, $\mathrm{Ax}_{\mathrm{CD,II}} \longrightarrow \mathrm{Ax}_{\mathrm{CD},4}$.

Thus, $\mathrm{Ax}_{\mathrm{CD,II}}\left(\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}\right) = 1 \longrightarrow \bigwedge_{\nu \in I_4^*} \mathrm{Ax}_{\mathrm{CD},\nu}\left(\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}\right) = 1$ and the proof of the proposition is complete. Q.E.D.

The corollary stated below is an immediate consequence of the foregoing theorem and proposition.

Corollary 3.34. *If $\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ be a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-coderived operator on $\mathscr{P}(\Omega)$ ranging in $\mathscr{P}(\Omega)$ in a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$, then it satisfies the following "$\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-coderived operator axiomatic diagram:"*

$$
\begin{array}{ccc}
\mathrm{Ax}_{\mathrm{CD,I}}\left(\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}\right) = 1 & \longrightarrow & \bigwedge_{\nu \in I_4^*} \mathrm{Ax}_{\mathrm{CD},\nu}\left(\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}\right) = 1 \\
 & \nwarrow\!\!\searrow & \uparrow \\
 & & \mathrm{Ax}_{\mathrm{CD,II}}\left(\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}\right) = 1.
\end{array}
\tag{3.18}
$$

The proven lemma presented below will be helpful in proving the theorem following it.

Lemma 3.35. *Let $\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}$, $\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ be a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-derived and a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-coderived operators, respectively, in a strong $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$. Then:*

- I. $\mathscr{T}_{\mathfrak{g},\mathrm{Der}}(\Omega) \stackrel{\mathrm{def}}{=} \left\{\mathscr{K}_{\mathfrak{g}} \in \mathscr{P}(\Omega) : \mathscr{K}_{\mathfrak{g}} \supseteq \mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}\left(\mathscr{K}_{\mathfrak{g}}\right)\right\}$ *satisfies the $\mathscr{T}_{\mathfrak{g}}$-closed set axioms for the strong $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}}$,*
- II. $\mathscr{T}_{\mathfrak{g},\mathrm{Cod}}(\Omega) \stackrel{\mathrm{def}}{=} \left\{\mathscr{O}_{\mathfrak{g}} \in \mathscr{P}(\Omega) : \mathscr{O}_{\mathfrak{g}} \subseteq \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}\left(\mathscr{O}_{\mathfrak{g}}\right)\right\}$ *satisfies the $\mathscr{T}_{\mathfrak{g}}$-open set axioms for the strong $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}}$.*

Proof. Let $\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}$, $\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ be a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-derived and a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-coderived operators, respectively, in a strong $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$. Since $\mathfrak{T}_{\mathfrak{g}}$ is a strong $\mathscr{T}_{\mathfrak{g}}$-space, it satisfies the $\mathscr{T}_{\mathfrak{g}}$-open set axioms $\mathscr{T}_{\mathfrak{g}}(\emptyset) = \emptyset$, $\mathscr{T}_{\mathfrak{g}}(\Omega) = \Omega$, $\mathscr{T}_{\mathfrak{g}}(\mathscr{O}_{\mathfrak{g}}) \subseteq \mathscr{O}_{\mathfrak{g}}$, and $\mathscr{T}_{\mathfrak{g}}\left(\bigcup_{\nu \in I_\infty^*} \mathscr{O}_{\mathfrak{g},\nu}\right) = \bigcup_{\nu \in I_\infty^*} \mathscr{T}_{\mathfrak{g}}(\mathscr{O}_{\mathfrak{g},\nu})$, and it also satisfies the $\mathscr{T}_{\mathfrak{g}}$-closed set axioms $\neg\mathscr{T}_{\mathfrak{g}}(\Omega) = \Omega$, $\neg\mathscr{T}_{\mathfrak{g}}(\emptyset) = \emptyset$, $\neg\mathscr{T}_{\mathfrak{g}}(\mathscr{K}_{\mathfrak{g}}) \supseteq \mathscr{K}_{\mathfrak{g}}$, and $\neg\mathscr{T}_{\mathfrak{g}}\left(\bigcap_{\nu \in I_\infty^*} \mathscr{K}_{\mathfrak{g},\nu}\right) = \bigcap_{\nu \in I_\infty^*} \neg\mathscr{T}_{\mathfrak{g}}(\mathscr{K}_{\mathfrak{g},\nu})$. Therefore, to prove Item I. and Item II., it suffices to show that $\mathscr{T}_{\mathfrak{g},\mathrm{Der}} \longleftrightarrow \neg\mathscr{T}_{\mathfrak{g}}$ and $\mathscr{T}_{\mathfrak{g},\mathrm{Cod}} \longleftrightarrow \mathscr{T}_{\mathfrak{g}}$, respectively. Then:

I. By the definition of $\mathscr{T}_{\mathfrak{g},\mathrm{Der}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$, $\Omega \supseteq \mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}(\Omega)$. Thus, $\mathscr{T}_{\mathfrak{g},\mathrm{Der}}(\Omega) = \Omega$. By virtue of $\mathrm{Ax}_{\mathrm{DE},1}$, $\emptyset = \mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}(\emptyset) \longleftrightarrow \emptyset \supseteq \mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}(\emptyset)$. Hence, $\mathscr{T}_{\mathfrak{g},\mathrm{Der}}(\emptyset) = \emptyset$. Since $\mathscr{T}_{\mathfrak{g},\mathrm{Der}}(\Omega) \supseteq \left\{\mathscr{K}_{\mathfrak{g}} \in \mathscr{P}(\Omega) : \mathscr{K}_{\mathfrak{g}} \supseteq \mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}(\mathscr{K}_{\mathfrak{g}})\right\}$, it results that, for every $\left(\mathscr{K}_{\mathfrak{g}}, \mathscr{T}_{\mathfrak{g},\mathrm{Der}}(\mathscr{K}_{\mathfrak{g}})\right) \in \mathscr{P}(\Omega) \times \mathscr{T}_{\mathfrak{g},\mathrm{Der}}(\Omega)$, the relation $\mathscr{T}_{\mathfrak{g},\mathrm{Der}}(\mathscr{K}_{\mathfrak{g}}) \supseteq \mathscr{K}_{\mathfrak{g}}$ holds. Suppose $(\mathscr{K}_{\mathfrak{g},\nu}, \mathscr{K}_{\mathfrak{g},\mu}) \in \bigtimes_{\alpha \in I_2^*} \mathscr{P}(\Omega)$ such that, for each $\eta \in \{\nu, \mu\}$, $\mathscr{K}_{\mathfrak{g},\mu} \supseteq \mathscr{K}_{\mathfrak{g},\nu}$ and, for all $\sigma \in I_\infty^*$, the relation $\mathscr{K}_{\mathfrak{g},\sigma} \supseteq \mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}(\mathscr{K}_{\mathfrak{g},\sigma})$ holds. Then, $\mathscr{K}_{\mathfrak{g},\mu} \supseteq \mathscr{K}_{\mathfrak{g},\nu}$ implies $\mathscr{K}_{\mathfrak{g},\mu} \longleftrightarrow \mathscr{K}_{\mathfrak{g},\mu} \cup \mathscr{K}_{\mathfrak{g},\nu} \longleftrightarrow \mathscr{K}_{\mathfrak{g},\mu} \cup (\mathscr{K}_{\mathfrak{g},\mu} \cap \mathscr{K}_{\mathfrak{g},\nu})$. By virtue of $\mathrm{Ax}_{\mathrm{DE},4}$, it follows that the relation $\bigcap_{\eta=\nu,\mu} \mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}(\mathscr{K}_{\mathfrak{g},\eta}) \supseteq \mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}(\mathscr{K}_{\mathfrak{g},\nu} \cap \mathscr{K}_{\mathfrak{g},\mu}) \cap \mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}(\mathscr{K}_{\mathfrak{g},\nu}) \longleftrightarrow \mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}(\mathscr{K}_{\mathfrak{g},\nu} \cap \mathscr{K}_{\mathfrak{g},\mu})$ holds, implying $\bigcap_{\eta=\nu,\mu} \mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}(\mathscr{K}_{\mathfrak{g},\eta}) \supseteq \mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}(\mathscr{K}_{\mathfrak{g},\nu} \cap \mathscr{K}_{\mathfrak{g},\mu})$. But $\mathscr{K}_{\mathfrak{g},\eta} \supseteq \mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}(\mathscr{K}_{\mathfrak{g},\eta})$ holds for each $\eta \in \{\nu, \mu\}$ implies $\bigcap_{\eta=\nu,\mu} \mathscr{K}_{\mathfrak{g},\eta} \supseteq \bigcap_{\eta=\nu,\mu} \mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}(\mathscr{K}_{\mathfrak{g},\eta})$. Thus, $\bigcap_{\eta=\nu,\mu} \mathscr{K}_{\mathfrak{g},\eta} \supseteq \mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}(\mathscr{K}_{\mathfrak{g},\nu} \cap \mathscr{K}_{\mathfrak{g},\mu})$.

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The condition $\mathscr{T}_{\mathfrak{g},\mathrm{Der}} \longleftrightarrow \neg\mathscr{T}_{\mathfrak{g}}$ is proved and hence, $\mathscr{T}_{\mathfrak{g},\mathrm{Der}}\mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ satisfies the $\mathscr{T}_{\mathfrak{g}}$-closed set axioms for the strong $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}}$.

II. By virtue of $\mathrm{Ax}_{\mathrm{CD},1}$, $\Omega = \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}(\Omega) \longleftrightarrow \Omega \subseteq \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}(\Omega)$. Thus, $\mathscr{T}_{\mathfrak{g},\mathrm{Cod}}(\Omega) = \Omega$. By the definition of $\mathscr{T}_{\mathfrak{g},\mathrm{Cod}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$, $\emptyset \subseteq \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}(\emptyset)$. Hence, $\mathscr{T}_{\mathfrak{g},\mathrm{Cod}}(\emptyset) = \emptyset$. Since $\mathscr{T}_{\mathfrak{g},\mathrm{Cod}}(\Omega) \subseteq \{\mathscr{O}_{\mathfrak{g}} \in \mathscr{P}(\Omega) : \ \mathscr{O}_{\mathfrak{g}} \subseteq \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}(\mathscr{O}_{\mathfrak{g}})\}$, it follows that, for every $(\mathscr{O}_{\mathfrak{g}}, \mathscr{T}_{\mathfrak{g},\mathrm{Cod}}(\mathscr{O}_{\mathfrak{g}})) \in \mathscr{P}(\Omega) \times \mathscr{T}_{\mathfrak{g},\mathrm{Cod}}(\Omega)$, the relation $\mathscr{T}_{\mathfrak{g},\mathrm{Cod}}(\mathscr{O}_{\mathfrak{g}}) \subseteq \mathscr{O}_{\mathfrak{g}}$ holds. Let $(\mathscr{O}_{\mathfrak{g},\nu}, \mathscr{O}_{\mathfrak{g},\mu}) \in \times_{\alpha\in I_2^*}\mathscr{P}(\Omega)$ such that, for each $\eta \in \{\nu,\mu\}$, $\mathscr{O}_{\mathfrak{g},\mu} \subseteq \mathscr{O}_{\mathfrak{g},\nu}$ and, for all $\sigma \in I_{\infty}^*$, the relation $\mathscr{O}_{\mathfrak{g},\sigma} \subseteq \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}(\mathscr{O}_{\mathfrak{g},\sigma})$ holds. Then, $\mathscr{O}_{\mathfrak{g},\nu} \subseteq \mathscr{O}_{\mathfrak{g},\nu}$ implies $\mathscr{O}_{\mathfrak{g},\mu} \longleftrightarrow \mathscr{O}_{\mathfrak{g},\mu} \cap \mathscr{O}_{\mathfrak{g},\nu} \longleftrightarrow (\mathscr{O}_{\mathfrak{g},\mu} \cup \mathscr{O}_{\mathfrak{g},\nu}) \cap \mathscr{O}_{\mathfrak{g},\nu}$. By virtue of $\mathrm{Ax}_{\mathrm{CD},4}$, it results that the relation $\bigcup_{\eta=\nu,\mu}\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}(\mathscr{O}_{\mathfrak{g},\eta}) \subseteq \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}(\mathscr{O}_{\mathfrak{g},\nu} \cup \mathscr{O}_{\mathfrak{g},\mu}) \cup \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}(\mathscr{O}_{\mathfrak{g},\nu}) \longleftrightarrow \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}(\mathscr{O}_{\mathfrak{g},\nu} \cup \mathscr{O}_{\mathfrak{g},\mu})$ holds which, in turn, implies $\bigcup_{\eta=\nu,\mu}\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}(\mathscr{O}_{\mathfrak{g},\eta}) \subseteq \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}(\mathscr{O}_{\mathfrak{g},\nu} \cup \mathscr{O}_{\mathfrak{g},\mu})$. But the relation $\mathscr{O}_{\mathfrak{g},\eta} \subseteq \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}(\mathscr{O}_{\mathfrak{g},\eta})$ holding true for each $\eta \in \{\nu,\mu\}$ implies, in turn, $\bigcup_{\eta=\nu,\mu}\mathscr{O}_{\mathfrak{g},\eta} \subseteq \bigcap_{\eta=\nu,\mu}\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}(\mathscr{O}_{\mathfrak{g},\eta})$. Thus, $\bigcup_{\eta=\nu,\mu}\mathscr{O}_{\mathfrak{g},\eta} \subseteq \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}(\mathscr{O}_{\mathfrak{g},\nu} \cup \mathscr{O}_{\mathfrak{g},\mu})$. The condition $\mathscr{T}_{\mathfrak{g},\mathrm{Cod}} \longleftrightarrow \mathscr{T}_{\mathfrak{g}}$ is proved and hence, $\mathscr{T}_{\mathfrak{g},\mathrm{Cod}}\mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ satisfies the $\mathscr{T}_{\mathfrak{g}}$-open set axioms for the strong $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}}$. The proof of the lemma is complete. Q.E.D.

The theorem is now stated and proved by the aid of the above lemma.

THEOREM 3.36. *Let* $\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}$, $\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ *be a* $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}$*-derived and a* $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}$*-coderived operators, respectively, in a unique strong* $\mathscr{T}_{\mathfrak{g}}$*-space* $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$. *Then:*

- I. $\mathscr{T}_{\mathfrak{g},\mathrm{Der}}(\Omega) \stackrel{\mathrm{def}}{=} \{\mathscr{K}_{\mathfrak{g}} \in \mathscr{P}(\Omega) : \ \mathscr{K}_{\mathfrak{g}} \supseteq \mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}(\mathscr{K}_{\mathfrak{g}})\}$ *forms the* $\mathscr{T}_{\mathfrak{g}}$*-closed sets for the unique strong* $\mathscr{T}_{\mathfrak{g}}$*-space* $\mathfrak{T}_{\mathfrak{g}}$,
- II. $\mathscr{T}_{\mathfrak{g},\mathrm{Cod}}(\Omega) \stackrel{\mathrm{def}}{=} \{\mathscr{O}_{\mathfrak{g}} \in \mathscr{P}(\Omega) : \ \mathscr{O}_{\mathfrak{g}} \subseteq \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}(\mathscr{O}_{\mathfrak{g}})\}$ *forms the* $\mathscr{T}_{\mathfrak{g}}$*-open set axioms for the unique strong* $\mathscr{T}_{\mathfrak{g}}$*-space* $\mathfrak{T}_{\mathfrak{g}}$.

PROOF. Let $\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}$, $\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ be a $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}$-derived and a $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}$-coderived operators, respectively, in a strong $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$. Then:

I. $\mathscr{T}_{\mathfrak{g},\mathrm{Der}}(\Omega) \stackrel{\mathrm{def}}{=} \{\mathscr{K}_{\mathfrak{g}} \in \mathscr{P}(\Omega) : \ \mathscr{K}_{\mathfrak{g}} \supseteq \mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}(\mathscr{K}_{\mathfrak{g}})\}$ forms the collection of $\mathscr{T}_{\mathfrak{g}}$-closed set in the strong $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}}$. Suppose $\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g},\mathrm{ind}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ be the induced $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}$-derived operator, then to show uniqueness it only suffices to prove that $\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g},\mathrm{ind}}(\mathscr{R}_{\mathfrak{g}}) \longleftrightarrow \mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}})$ holds true for any $\mathscr{R}_{\mathfrak{g}} \subseteq \mathfrak{T}_{\mathfrak{g}}$. Let $\mathscr{R}_{\mathfrak{g}} \subseteq \mathfrak{T}_{\mathfrak{g}}$ be arbitrary and by hypothesis, let $(\xi \in \mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}})) \wedge (\xi \notin \mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g},\mathrm{ind}}(\mathscr{R}_{\mathfrak{g}}))$ hold true. Then, uniqueness is shown by proving that such hypothesis is a contradiction. Thus, the following cases present themselves:

Case (i.) Suppose $\xi \notin \mathscr{R}_{\mathfrak{g}}$. Then, $(\xi \notin \mathscr{R}_{\mathfrak{g}}) \wedge (\xi \notin \mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g},\mathrm{ind}}(\mathscr{R}_{\mathfrak{g}}))$ by virtue of the supposition and the hypothesis. Consequently, $\xi \notin \mathfrak{g}\text{-}\mathrm{Cl}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}} \cap \mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}(\{\xi\}))$. Therefore, a $\mathscr{T}_{\mathfrak{g}}$-open set $\mathscr{O}_{\mathfrak{g}} \in \mathscr{T}_{\mathfrak{g}}$ can be found, satisfying $\xi \in \mathscr{O}_{\mathfrak{g}}$, such that $\mathscr{O}_{\mathfrak{g}} \cap (\mathscr{R}_{\mathfrak{g}} \cap \mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}(\{\xi\})) = \mathscr{O}_{\mathfrak{g}} \cap \mathscr{R}_{\mathfrak{g}} = \emptyset$. Clearly, $\mathscr{K}_{\mathfrak{g}} = \mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}(\mathscr{O}_{\mathfrak{g}})$ is a $\mathscr{T}_{\mathfrak{g}}$-closed set and therefore, it satisfies $\mathscr{K}_{\mathfrak{g}} \supseteq \mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}(\mathscr{K}_{\mathfrak{g}})$, implying $(\mathscr{K}_{\mathfrak{g}} \in \mathscr{T}_{\mathfrak{g},\mathrm{Der}}) \wedge (\mathscr{K}_{\mathfrak{g}} \supseteq \mathscr{R}_{\mathfrak{g}})$ holds true. Consequently, it follows that $\mathscr{K}_{\mathfrak{g}} \supseteq \mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}(\mathscr{K}_{\mathfrak{g}}) \supseteq \mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}})$. But, $\xi \in \mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}})$ by hypothesis. Hence, $(\xi \in \mathscr{O}_{\mathfrak{g}}) \wedge (\xi \in \mathscr{K}_{\mathfrak{g}} = \mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}(\mathscr{O}_{\mathfrak{g}}))$, a contradiction. The hypothesis is therefore a contradiction.

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Case (ii.) Suppose $\xi \in \mathscr{R}_{\mathfrak{g}}$. Then, $(\xi \in \mathscr{R}_{\mathfrak{g}}) \wedge (\xi \in \mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}))$ by virtue of the supposition and the hypothesis. Consequently, $\xi \in \mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}} \cap \mathfrak{g}\text{-Op}_{\mathfrak{g}}(\{\xi\}))$. Therefore, a $\mathscr{T}_{\mathfrak{g}}$-closed set $\mathscr{K}_{\mathfrak{g}} \in \neg\mathscr{T}_{\mathfrak{g}}$ can be found, satisfying $\xi \in \mathscr{K}_{\mathfrak{g}}$, such that $\mathscr{K}_{\mathfrak{g}} \cap (\mathscr{R}_{\mathfrak{g}} \cap \mathfrak{g}\text{-Op}_{\mathfrak{g}}(\{\xi\})) = \mathscr{R}_{\mathfrak{g}} \cap \mathfrak{g}\text{-Op}_{\mathfrak{g}}(\{\xi\}) \neq \emptyset$. Then, $\mathscr{K}_{\mathfrak{g}} \supseteq \mathscr{R}_{\mathfrak{g}} \cap \mathfrak{g}\text{-Op}_{\mathfrak{g}}(\{\xi\})$ and consequently, $\mathscr{K}_{\mathfrak{g}} \supseteq \mathfrak{g}\text{-Der}_{\mathfrak{g},\text{ind}}(\mathscr{K}_{\mathfrak{g}}) \supseteq \mathfrak{g}\text{-Der}_{\mathfrak{g},\text{ind}}(\mathscr{R}_{\mathfrak{g}} \cap \mathfrak{g}\text{-Op}_{\mathfrak{g}}(\{\xi\})) = \mathfrak{g}\text{-Der}_{\mathfrak{g},\text{ind}}(\mathscr{R}_{\mathfrak{g}})$. But, $\xi \notin \mathfrak{g}\text{-Der}_{\mathfrak{g},\text{ind}}(\mathscr{R}_{\mathfrak{g}})$ by hypothesis. Thus, $(\xi \in \mathscr{K}_{\mathfrak{g}}) \wedge (\xi \notin \mathscr{K}_{\mathfrak{g}})$, a contradiction. The hypothesis is therefore a contradiction. Hence, $\mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}) \subseteq \mathfrak{g}\text{-Der}_{\mathfrak{g},\text{ind}}(\mathscr{R}_{\mathfrak{g}})$.

The relation $\mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}) \supseteq \mathfrak{g}\text{-Der}_{\mathfrak{g},\text{ind}}(\mathscr{R}_{\mathfrak{g}})$ is now proved. By hypothesis, let $(\xi \notin \mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}})) \wedge (\xi \in \mathfrak{g}\text{-Der}_{\mathfrak{g},\text{ind}}(\mathscr{R}_{\mathfrak{g}}))$ hold true. Then, uniqueness is again shown by proving that such hypothesis is a contradiction. Thus, the following cases present themselves:

Case (i.) Suppose $\xi \notin \mathscr{R}_{\mathfrak{g}}$. Clearly, a $\mathscr{T}_{\mathfrak{g}}$-closed set $\mathscr{K}_{\mathfrak{g}} \in \neg\mathscr{T}_{\mathfrak{g}}$ can be found such that $\mathscr{K}_{\mathfrak{g}} = \mathscr{R}_{\mathfrak{g}} \cup \mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}})$, and consequently, $\mathscr{K}_{\mathfrak{g}} \supseteq \mathfrak{g}\text{-Der}_{\mathfrak{g},\text{ind}}(\mathscr{R}_{\mathfrak{g}})$. But, by virtue of the supposition and the hypothesis, $(\xi \notin \mathscr{R}_{\mathfrak{g}}) \wedge (\xi \notin \mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}))$, implying $\xi \notin \mathfrak{g}\text{-Der}_{\mathfrak{g},\text{ind}}(\mathscr{R}_{\mathfrak{g}})$, a contradiction. The hypothesis is therefore a contradiction.

Case (ii.) Suppose $\xi \in \mathscr{R}_{\mathfrak{g}}$. Then, $(\xi \in \mathscr{R}_{\mathfrak{g}}) \wedge (\xi \in \mathfrak{g}\text{-Der}_{\mathfrak{g},\text{ind}}(\mathscr{R}_{\mathfrak{g}}))$ by virtue of the supposition and the hypothesis. Consequently, $\xi \in \mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}} \cap \mathfrak{g}\text{-Op}_{\mathfrak{g}}(\{\xi\}))$. Since $\xi \in \mathfrak{g}\text{-Der}_{\mathfrak{g},\text{ind}}(\mathscr{R}_{\mathfrak{g}})$ is equivalent to $\xi \in \mathfrak{g}\text{-Der}_{\mathfrak{g},\text{ind}}(\mathscr{R}_{\mathfrak{g}} \cap \mathfrak{g}\text{-Op}_{\mathfrak{g}}(\{\xi\}))$ and, on the other hand, $\mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}} \cap \mathfrak{g}\text{-Op}_{\mathfrak{g}}(\{\xi\}))$ is equivalent to $(\mathscr{R}_{\mathfrak{g}} \cap \mathfrak{g}\text{-Op}_{\mathfrak{g}}(\{\xi\})) \cup \mathfrak{g}\text{-Der}_{\mathfrak{g},\text{ind}}(\mathscr{R}_{\mathfrak{g}} \cap \mathfrak{g}\text{-Op}_{\mathfrak{g}}(\{\xi\})) = \mathscr{K}_{\mathfrak{g}}$ for some $\mathscr{T}_{\mathfrak{g}}$-closed set $\mathscr{K}_{\mathfrak{g}} \in \neg\mathscr{T}_{\mathfrak{g}}$, it follows that $\xi \in \mathfrak{g}\text{-Der}_{\mathfrak{g},\text{ind}}(\mathscr{R}_{\mathfrak{g}} \cap \mathfrak{g}\text{-Op}_{\mathfrak{g}}(\{\xi\})) \subseteq \mathfrak{g}\text{-Der}_{\mathfrak{g},\text{ind}}(\mathscr{R}_{\mathfrak{g}})$, implying $\xi \in \mathfrak{g}\text{-Der}_{\mathfrak{g},\text{ind}}(\mathscr{R}_{\mathfrak{g}})$. But, by virtue of the supposition and the hypothesis, $(\xi \in \mathscr{R}_{\mathfrak{g}}) \wedge (\xi \in \mathfrak{g}\text{-Der}_{\mathfrak{g},\text{ind}}(\mathscr{R}_{\mathfrak{g}}))$, implying $\xi \in \mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}})$, a contradiction. The hypothesis is therefore a contradiction. Thus, $\mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}) \supseteq \mathfrak{g}\text{-Der}_{\mathfrak{g},\text{ind}}(\mathscr{R}_{\mathfrak{g}})$

II. $\mathscr{T}_{\mathfrak{g},\text{Cod}}(\Omega) \stackrel{\text{def}}{=} \{\mathscr{O}_{\mathfrak{g}} \in \mathscr{P}(\Omega) : \mathscr{O}_{\mathfrak{g}} \subseteq \mathfrak{g}\text{-Cod}_{\mathfrak{g}}(\mathscr{O}_{\mathfrak{g}})\}$ forms the collection of $\mathscr{T}_{\mathfrak{g}}$-open set in the strong $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}}$. Suppose $\mathfrak{g}\text{-Cod}_{\mathfrak{g},\text{ind}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ be the induced $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-coderived operator, then to show uniqueness it only suffices to prove that $\mathfrak{g}\text{-Cod}_{\mathfrak{g},\text{ind}}(\mathscr{S}_{\mathfrak{g}}) \longleftrightarrow \mathfrak{g}\text{-Cod}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$ holds true for any $\mathscr{S}_{\mathfrak{g}} \subseteq \mathfrak{T}_{\mathfrak{g}}$. Let $\mathscr{S}_{\mathfrak{g}} \subseteq \mathfrak{T}_{\mathfrak{g}}$ be arbitrary and by hypothesis, let $(\zeta \in \mathfrak{g}\text{-Cod}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})) \wedge (\zeta \notin \mathfrak{g}\text{-Cod}_{\mathfrak{g},\text{ind}}(\mathscr{S}_{\mathfrak{g}}))$ hold true. Then, uniqueness is shown by proving that such hypothesis is a contradiction. Thus, the following cases present themselves:

Case (i.) Suppose $\zeta \notin \mathscr{S}_{\mathfrak{g}}$. By virtue of the supposition and the hypothesis, the relation $(\zeta \notin \mathscr{S}_{\mathfrak{g}}) \wedge (\zeta \notin \mathfrak{g}\text{-Cod}_{\mathfrak{g},\text{ind}}(\mathscr{S}_{\mathfrak{g}}))$, then, holds true. Consequently, $\zeta \notin \mathfrak{g}\text{-Int}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}} \cup \{\zeta\})$, implying $\zeta \in \mathfrak{g}\text{-Cl}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}} \cup \{\zeta\})$. Therefore, a $\mathscr{T}_{\mathfrak{g}}$-closed set $\mathscr{K}_{\mathfrak{g}} \in \neg\mathscr{T}_{\mathfrak{g}}$ can be found, satisfying $\zeta \in \mathscr{K}_{\mathfrak{g}}$, such that $\mathscr{K}_{\mathfrak{g}} \cap \mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}} \cup \{\zeta\}) = \mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}} \cup \{\zeta\})$. Clearly, $\mathscr{O}_{\mathfrak{g}} = \mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{K}_{\mathfrak{g}})$ is a $\mathscr{T}_{\mathfrak{g}}$-open set and therefore, it satisfies $\mathscr{O}_{\mathfrak{g}} \subseteq \mathfrak{g}\text{-Cod}_{\mathfrak{g}}(\mathscr{O}_{\mathfrak{g}})$, implying $(\mathscr{O}_{\mathfrak{g}} \in \mathscr{T}_{\mathfrak{g},\text{Cod}}) \wedge (\mathscr{O}_{\mathfrak{g}} \subseteq \mathscr{S}_{\mathfrak{g}})$ holds true. Consequently, it follows that $\mathscr{O}_{\mathfrak{g}} \subseteq \mathfrak{g}\text{-Cod}_{\mathfrak{g}}(\mathscr{O}_{\mathfrak{g}}) \subseteq \mathfrak{g}\text{-Cod}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$. But, $\zeta \in \mathfrak{g}\text{-Cod}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$ by hypothesis. Hence, $(\zeta \notin \mathscr{O}_{\mathfrak{g}} = \mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{K}_{\mathfrak{g}})) \wedge (\zeta \notin \mathscr{K}_{\mathfrak{g}})$, a contradiction. The hypothesis is therefore a contradiction.

Case (ii.) Suppose $\zeta \in \mathscr{S}_{\mathfrak{g}}$. By virtue of the supposition and the hypothesis, the relation $\left(\zeta \in \mathscr{S}_{\mathfrak{g}}\right) \wedge \left(\zeta \in \mathfrak{g}\text{-Cod}_{\mathfrak{g}}\left(\mathscr{S}_{\mathfrak{g}}\right)\right)$, then, holds true. Consequently, $\zeta \in \mathfrak{g}\text{-Int}_{\mathfrak{g}}\left(\mathscr{S}_{\mathfrak{g}} \cup \{\zeta\}\right)$, implying $\zeta \notin \mathfrak{g}\text{-Cl}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Op}_{\mathfrak{g}}\left(\mathscr{S}_{\mathfrak{g}} \cup \{\zeta\}\right)$. Therefore, a $\mathscr{T}_{\mathfrak{g}}$-open set $\mathscr{O}_{\mathfrak{g}} \in \mathscr{T}_{\mathfrak{g}}$ can be found, satisfying $\zeta \in \mathscr{O}_{\mathfrak{g}}$, such that $\mathscr{O}_{\mathfrak{g}} \cup (\mathscr{S}_{\mathfrak{g}} \cup \{\xi\}) = \mathscr{S}_{\mathfrak{g}} \cup \{\xi\}$. Then, $\mathscr{O}_{\mathfrak{g}} \subseteq \mathscr{S}_{\mathfrak{g}} \cup \{\xi\}$ and consequently, $\mathscr{O}_{\mathfrak{g}} \subseteq \mathfrak{g}\text{-Cod}_{\mathfrak{g},\mathrm{ind}}\left(\mathscr{O}_{\mathfrak{g}}\right) \subseteq \mathfrak{g}\text{-Cod}_{\mathfrak{g},\mathrm{ind}}\left(\mathscr{S}_{\mathfrak{g}} \cup \{\xi\}\right) = \mathfrak{g}\text{-Cod}_{\mathfrak{g},\mathrm{ind}}\left(\mathscr{S}_{\mathfrak{g}}\right)$. But, $\zeta \notin \mathfrak{g}\text{-Cod}_{\mathfrak{g},\mathrm{ind}}\left(\mathscr{S}_{\mathfrak{g}}\right)$ by hypothesis. Thus, $\left(\zeta \notin \mathscr{O}_{\mathfrak{g}}\right) \wedge \left(\zeta \notin \mathscr{O}_{\mathfrak{g}}\right)$, a contradiction. The hypothesis is therefore a contradiction. Hence, $\mathfrak{g}\text{-Cod}_{\mathfrak{g}}\left(\mathscr{S}_{\mathfrak{g}}\right) \supseteq \mathfrak{g}\text{-Cod}_{\mathfrak{g},\mathrm{ind}}\left(\mathscr{S}_{\mathfrak{g}}\right)$.

The relation $\mathfrak{g}\text{-Cod}_{\mathfrak{g}}\left(\mathscr{S}_{\mathfrak{g}}\right) \subseteq \mathfrak{g}\text{-Cod}_{\mathfrak{g},\mathrm{ind}}\left(\mathscr{S}_{\mathfrak{g}}\right)$ is now proved. By hypothesis, let $\left(\xi \notin \mathfrak{g}\text{-Cod}_{\mathfrak{g}}\left(\mathscr{S}_{\mathfrak{g}}\right)\right) \wedge \left(\xi \in \mathfrak{g}\text{-Cod}_{\mathfrak{g},\mathrm{ind}}\left(\mathscr{S}_{\mathfrak{g}}\right)\right)$ hold true. Then, uniqueness is again shown by proving that such hypothesis is a contradiction. Thus, the following cases present themselves:

Case (i.) Suppose $\zeta \notin \mathscr{S}_{\mathfrak{g}}$. Clearly, a $\mathscr{T}_{\mathfrak{g}}$-open set $\mathscr{O}_{\mathfrak{g}} \in \mathscr{T}_{\mathfrak{g}}$ can be found such that $\mathscr{O}_{\mathfrak{g}} = \mathscr{S}_{\mathfrak{g}} \cap \mathfrak{g}\text{-Cod}_{\mathfrak{g}}\left(\mathscr{S}_{\mathfrak{g}}\right)$, and consequently, $\mathscr{O}_{\mathfrak{g}} \subseteq \mathfrak{g}\text{-Cod}_{\mathfrak{g},\mathrm{ind}}\left(\mathscr{S}_{\mathfrak{g}}\right)$. But, by virtue of the supposition and the hypothesis, $\left(\zeta \notin \mathscr{S}_{\mathfrak{g}}\right) \wedge \left(\zeta \notin \mathfrak{g}\text{-Cod}_{\mathfrak{g}}\left(\mathscr{S}_{\mathfrak{g}}\right)\right)$, implying $\zeta \notin \mathfrak{g}\text{-Cod}_{\mathfrak{g},\mathrm{ind}}\left(\mathscr{S}_{\mathfrak{g}}\right)$, a contradiction. The hypothesis is therefore a contradiction.

Case (ii.) Suppose $\zeta \in \mathscr{S}_{\mathfrak{g}}$. Then, $\left(\zeta \in \mathscr{S}_{\mathfrak{g}}\right) \wedge \left(\zeta \in \mathfrak{g}\text{-Cod}_{\mathfrak{g},\mathrm{ind}}\left(\mathscr{S}_{\mathfrak{g}}\right)\right)$ by virtue of the supposition and the hypothesis. Consequently, $\zeta \in \mathfrak{g}\text{-Int}_{\mathfrak{g}}\left(\mathscr{S}_{\mathfrak{g}} \cup \{\zeta\}\right)$. Since $\zeta \in \mathfrak{g}\text{-Cod}_{\mathfrak{g},\mathrm{ind}}\left(\mathscr{S}_{\mathfrak{g}}\right)$ is equivalent to $\zeta \in \mathfrak{g}\text{-Cod}_{\mathfrak{g},\mathrm{ind}}\left(\mathscr{S}_{\mathfrak{g}} \cup \{\zeta\}\right)$ and, on the other hand, $\mathfrak{g}\text{-Int}_{\mathfrak{g}}\left(\mathscr{S}_{\mathfrak{g}} \cup \{\zeta\}\right)$ is equivalent to $\left(\mathscr{S}_{\mathfrak{g}} \cup \{\zeta\}\right) \cap \mathfrak{g}\text{-Cod}_{\mathfrak{g},\mathrm{ind}}\left(\mathscr{S}_{\mathfrak{g}} \cup \{\zeta\}\right) = \mathscr{O}_{\mathfrak{g}}$ for some $\mathscr{T}_{\mathfrak{g}}$-open set $\mathscr{O}_{\mathfrak{g}} \in \mathscr{T}_{\mathfrak{g}}$, it follows that $\zeta \in \mathfrak{g}\text{-Cod}_{\mathfrak{g},\mathrm{ind}}\left(\mathscr{S}_{\mathfrak{g}} \cup \{\xi\}\right) = \mathfrak{g}\text{-Cod}_{\mathfrak{g},\mathrm{ind}}\left(\mathscr{S}_{\mathfrak{g}}\right)$, implying $\zeta \in \mathfrak{g}\text{-Cod}_{\mathfrak{g},\mathrm{ind}}\left(\mathscr{S}_{\mathfrak{g}}\right)$. But, by virtue of the supposition and the hypothesis, $\left(\zeta \in \mathscr{S}_{\mathfrak{g}}\right) \wedge \left(\zeta \in \mathfrak{g}\text{-Cod}_{\mathfrak{g},\mathrm{ind}}\left(\mathscr{S}_{\mathfrak{g}}\right)\right)$, implying $\zeta \in \mathfrak{g}\text{-Cod}_{\mathfrak{g}}\left(\mathscr{S}_{\mathfrak{g}}\right)$, a contradiction. The hypothesis is therefore a contradiction. Hence, $\mathfrak{g}\text{-Cod}_{\mathfrak{g}}\left(\mathscr{S}_{\mathfrak{g}}\right) \subseteq \mathfrak{g}\text{-Cod}_{\mathfrak{g},\mathrm{ind}}\left(\mathscr{S}_{\mathfrak{g}}\right)$. The proof of the lemma is complete. Q.E.D.

On the essential properties of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-derived and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-coderived operators in $\mathscr{T}_{\mathfrak{g}}$-spaces, the discussion of the present section terminates here.

3.2. Iterations. In the preceding section, considerations were given to the study of the essential properties of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-derived and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-coderived operators in $\mathscr{T}_{\mathfrak{g}}$-spaces; now considerations are given to the study of the essential properties of their iterations.

In a $\mathscr{T}_{\mathfrak{g}}$-space, every $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-derived set is contained in all the preceding $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-derived sets and, every $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-coderived set contains all the preceding $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-coderived sets. The theorem follows.

Theorem 3.37. *Let* $\mathfrak{g}\text{-Der}_{\mathfrak{g}}, \mathfrak{g}\text{-Cod}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ *be a* $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$*-derived and a* $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$*-coderived operators, respectively, and let* $\mathscr{S}_{\mathfrak{g}} \in \mathscr{P}(\Omega)$ *be arbitrary in a* $\mathscr{T}_{\mathfrak{g}}$*-space* $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$*. Then:*

- I. $\mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta+1)}\left(\mathscr{S}_{\mathfrak{g}}\right) \subseteq \mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta)}\left(\mathscr{S}_{\mathfrak{g}}\right) \quad (\forall \delta : 1 \preccurlyeq \delta \prec \lambda)$,
- II. $\mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\delta+1)}\left(\mathscr{S}_{\mathfrak{g}}\right) \supseteq \mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\delta)}\left(\mathscr{S}_{\mathfrak{g}}\right) \quad (\forall \delta : 1 \preccurlyeq \delta \prec \lambda)$.

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PROOF. Let $\mathfrak{g}\text{-Der}_{\mathfrak{g}}$, $\mathfrak{g}\text{-Cod}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ be a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-derived and a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-coderived operators, respectively, and let $\mathscr{S}_{\mathfrak{g}} \in \mathscr{P}(\Omega)$ be arbitrary in a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$. Then:

I. Introduce $\mathbb{B} = \{0,1\}$ as Boolean domain and introduce the Boolean-valued propositional formula

$$\mathbb{B} \ni \mathrm{P}(\delta) \overset{\text{def}}{\longleftrightarrow} \mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta+1)}(\mathscr{S}_{\mathfrak{g}}) \subseteq \mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta)}(\mathscr{S}_{\mathfrak{g}}) \quad (\forall \delta :\ 1 \preccurlyeq \delta \prec \lambda).$$

Then, to prove ITEM I., it only suffices to prove that,

$$\big(\forall \delta :\ 1 \preccurlyeq \delta \prec \lambda\big)\big[\big(\mathrm{P}(1) = 1\big) \wedge \big(\mathrm{P}(\delta) = 1 \longrightarrow \mathrm{P}(\delta+1) = 1\big)\big].$$

Case (i.) Let $1 = \delta$. Since $\mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \supseteq \mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$, it follows that

$$\begin{aligned}
\mathfrak{g}\text{-Der}_{\mathfrak{g}} :\ \mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \quad &\longmapsto \quad \big\{\xi \in \mathfrak{T}_{\mathfrak{g}} :\ \xi \in \mathfrak{g}\text{-Cl}_{\mathfrak{g}}\big(\mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \cap \mathfrak{g}\text{-Op}_{\mathfrak{g}}(\{\xi\})\big)\big\} \\
&\subseteq \quad \big\{\xi \in \mathfrak{T}_{\mathfrak{g}} :\ \xi \in \mathfrak{g}\text{-Cl}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})\big\} \\
&\longleftrightarrow \quad \big\{\xi \in \mathfrak{T}_{\mathfrak{g}} :\ \xi \in \mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})\big\} \\
&\longleftrightarrow \quad \big\{\xi \in \mathfrak{T}_{\mathfrak{g}} :\ \xi \in \mathfrak{g}\text{-Cl}_{\mathfrak{g}}\big(\mathscr{S}_{\mathfrak{g}} \cap \mathfrak{g}\text{-Op}_{\mathfrak{g}}(\{\xi\})\big)\big\} \\
&\longleftrightarrow \quad \mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}).
\end{aligned}$$

Thus, $\mathfrak{g}\text{-Der}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \subseteq \mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$. But, $\mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \longleftrightarrow \mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(1)}(\mathscr{S}_{\mathfrak{g}})$ and $\mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(2)}(\mathscr{S}_{\mathfrak{g}}) \longleftrightarrow \mathfrak{g}\text{-Der}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$. Thus, $\mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(2)}(\mathscr{S}_{\mathfrak{g}}) \subseteq \mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(1)}(\mathscr{S}_{\mathfrak{g}})$, implying $\mathrm{P}(1) = 1$. The base case therefore holds.

Case (ii.) Let $1 \prec \delta \prec \lambda$ and assume that the inductive hypothesis $\mathrm{P}(\delta) = 1$ holds true. Then, $\mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta+1)}(\mathscr{S}_{\mathfrak{g}}) \subseteq \mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta)}(\mathscr{S}_{\mathfrak{g}})$ and consequently, it results that $\mathfrak{g}\text{-Der}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta+1)}(\mathscr{S}_{\mathfrak{g}}) \subseteq \mathfrak{g}\text{-Der}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta)}(\mathscr{S}_{\mathfrak{g}})$. But, for each $\eta \in \{\delta, \delta+1\}$,

$$\mathfrak{g}\text{-Der}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\eta)}(\mathscr{S}_{\mathfrak{g}}) \longleftrightarrow \mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(1)} \circ \mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\eta)}(\mathscr{S}_{\mathfrak{g}}) \longleftrightarrow \mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\eta+1)}(\mathscr{S}_{\mathfrak{g}}).$$

Hence, $\mathfrak{g}\text{-Der}_{\mathfrak{g}}^{((\delta+1)+1)}(\mathscr{S}_{\mathfrak{g}}) \subseteq \mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta+1)}(\mathscr{S}_{\mathfrak{g}})$, implying $\mathrm{P}(\delta+1) = 1$. The inductive case therefore holds.

Since $\mathrm{P}(\delta) = 1$ for all δ such that $1 \prec \delta \prec \lambda$, it follows that

$$\begin{aligned}
\mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\lambda+1)}(\mathscr{S}_{\mathfrak{g}}) \quad &\longleftrightarrow \quad \mathfrak{g}\text{-Der}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\lambda)}(\mathscr{S}_{\mathfrak{g}}) \\
&\longleftrightarrow \quad \mathfrak{g}\text{-Der}_{\mathfrak{g}}\Bigg(\bigcap_{\delta \prec \lambda} \mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta)}(\mathscr{S}_{\mathfrak{g}})\Bigg) \\
&\subseteq \quad \mathfrak{g}\text{-Der}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta)}(\mathscr{S}_{\mathfrak{g}}) \longleftrightarrow \mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta+1)}(\mathscr{S}_{\mathfrak{g}}) \\
&\qquad\qquad \subseteq \mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta)}(\mathscr{S}_{\mathfrak{g}})
\end{aligned}$$

for all δ such that $1 \prec \delta \prec \lambda$, from which $\mathrm{P}(\lambda) = 1$ follows.

II. Introduce the Boolean-valued propositional formula

$$\mathbb{B} \ni \mathrm{Q}(\delta) \overset{\text{def}}{\longleftrightarrow} \mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\delta+1)}(\mathscr{S}_{\mathfrak{g}}) \supseteq \mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\delta)}(\mathscr{S}_{\mathfrak{g}}) \quad (\forall \delta :\ 1 \preccurlyeq \delta \prec \lambda).$$

Then, to prove ITEM II., it only suffices to prove that,

$$\big(\forall \delta :\ 1 \preccurlyeq \delta \prec \lambda\big)\big[\big(\mathrm{Q}(0) = 1\big) \wedge \big(\mathrm{Q}(\delta) = 1 \longrightarrow \mathrm{Q}(\delta+1) = 1\big)\big].$$

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Case (i.) Let $1=\delta$. Since $\mathfrak{g}\text{-Int}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})\subseteq\mathfrak{g}\text{-Cod}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$, it results that

$$\begin{aligned}\mathfrak{g}\text{-Cod}_{\mathfrak{g}}:\ \mathfrak{g}\text{-Cod}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) &\longmapsto \big\{\zeta\in\mathfrak{T}_{\mathfrak{g}}:\ \zeta\in\mathfrak{g}\text{-Int}_{\mathfrak{g}}\big(\mathfrak{g}\text{-Cod}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})\cup\{\zeta\}\big)\big\}\\ &\supseteq \big\{\zeta\in\mathfrak{T}_{\mathfrak{g}}:\ \zeta\in\mathfrak{g}\text{-Int}_{\mathfrak{g}}\circ\mathfrak{g}\text{-Int}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})\big\}\\ &\longleftrightarrow \big\{\zeta\in\mathfrak{T}_{\mathfrak{g}}:\ \zeta\in\mathfrak{g}\text{-Int}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})\big\}\\ &\longleftrightarrow \big\{\zeta\in\mathfrak{T}_{\mathfrak{g}}:\ \xi\in\mathfrak{g}\text{-Int}_{\mathfrak{g}}\big(\mathscr{S}_{\mathfrak{g}}\cup\{\zeta\}\big)\big\}\\ &\longleftrightarrow \mathfrak{g}\text{-Cod}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}).\end{aligned}$$

Thus, $\mathfrak{g}\text{-Cod}_{\mathfrak{g}}\circ\mathfrak{g}\text{-Cod}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})\supseteq\mathfrak{g}\text{-Cod}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$. But, $\mathfrak{g}\text{-Cod}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})\longleftrightarrow\mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(1)}(\mathscr{S}_{\mathfrak{g}})$ and $\mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(2)}(\mathscr{S}_{\mathfrak{g}})\longleftrightarrow\mathfrak{g}\text{-Cod}_{\mathfrak{g}}\circ\mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$. Thus, the relation $\mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(2)}(\mathscr{S}_{\mathfrak{g}})\supseteq\mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(1)}(\mathscr{S}_{\mathfrak{g}})$ holds true, implying $\mathrm{Q}(1)=1$. The base case therefore holds.

Case (ii.) Let $1\prec\delta\prec\lambda$ and assume that the inductive hypothesis $\mathrm{Q}(\delta)=1$ holds true. Then, $\mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\delta+1)}(\mathscr{S}_{\mathfrak{g}})\supseteq\mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\delta)}(\mathscr{S}_{\mathfrak{g}})$ and consequently, it results that $\mathfrak{g}\text{-Cod}_{\mathfrak{g}}\circ\mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\delta+1)}(\mathscr{S}_{\mathfrak{g}})\supseteq\mathfrak{g}\text{-Cod}_{\mathfrak{g}}\circ\mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\delta)}(\mathscr{S}_{\mathfrak{g}})$. But, for each $\eta\in\{\delta,\delta+1\}$,

$$\mathfrak{g}\text{-Cod}_{\mathfrak{g}}\circ\mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\eta)}(\mathscr{S}_{\mathfrak{g}})\longleftrightarrow\mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(1)}\circ\mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\eta)}(\mathscr{S}_{\mathfrak{g}})\longleftrightarrow\mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\eta+1)}(\mathscr{S}_{\mathfrak{g}}).$$

Hence, $\mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{((\delta+1)+1)}(\mathscr{S}_{\mathfrak{g}})\supseteq\mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\delta+1)}(\mathscr{S}_{\mathfrak{g}})$, implying $\mathrm{Q}(\delta+1)=1$. The inductive case therefore holds.

Since $\mathrm{Q}(\delta)=1$ for all δ such that $1\prec\delta\prec\lambda$, it follows that

$$\begin{aligned}\mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\lambda+1)}(\mathscr{S}_{\mathfrak{g}}) &\longleftrightarrow \mathfrak{g}\text{-Cod}_{\mathfrak{g}}\circ\mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\lambda)}(\mathscr{S}_{\mathfrak{g}})\\ &\longleftrightarrow \mathfrak{g}\text{-Cod}_{\mathfrak{g}}\Big(\bigcap_{\delta\prec\lambda}\mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\delta)}(\mathscr{S}_{\mathfrak{g}})\Big)\\ &\supseteq \mathfrak{g}\text{-Cod}_{\mathfrak{g}}\circ\mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\delta)}(\mathscr{S}_{\mathfrak{g}})\longleftrightarrow\mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\delta+1)}(\mathscr{S}_{\mathfrak{g}})\\ &\qquad\supseteq\mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\delta)}(\mathscr{S}_{\mathfrak{g}})\end{aligned}$$

for all δ such that $1\prec\delta\prec\lambda$, from which $\mathrm{Q}(\lambda)=1$ follows. The proof of the theorem is complete. Q.E.D.

The corollary stated below is an immediate consequence of the above theorem.

COROLLARY 3.38. *Let* $\mathfrak{g}\text{-Der}_{\mathfrak{g}}$, $\mathfrak{g}\text{-Cod}_{\mathfrak{g}}:\mathscr{P}(\Omega)\longrightarrow\mathscr{P}(\Omega)$ *be a* $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$*-derived and a* $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$*-coderived operators, respectively, and let* $\mathscr{S}_{\mathfrak{g}}\in\mathscr{P}(\Omega)$ *be arbitrary in a* $\mathscr{T}_{\mathfrak{g}}$*-space* $\mathfrak{T}_{\mathfrak{g}}=(\Omega,\mathscr{T}_{\mathfrak{g}})$. *Then:*

- I. $\mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta)}(\mathscr{S}_{\mathfrak{g}})\subseteq\mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})\quad(\forall\delta:\ 1\preccurlyeq\delta\prec\lambda)$,
- II. $\mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\delta)}(\mathscr{S}_{\mathfrak{g}})\supseteq\mathfrak{g}\text{-Cod}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})\quad(\forall\delta:\ 1\preccurlyeq\delta\prec\lambda)$.

In a $\mathscr{T}_{\mathfrak{g}}$-space, just as $\mathfrak{g}\text{-Der}_{\mathfrak{g}}:\mathscr{P}(\Omega)\longrightarrow\mathscr{P}(\Omega)$ is *coarser* (or, *smaller*, *weaker*) than $\text{der}_{\mathfrak{g}}:\mathscr{P}(\Omega)\longrightarrow\mathscr{P}(\Omega)$ (or, $\text{der}_{\mathfrak{g}}:\mathscr{P}(\Omega)\longrightarrow\mathscr{P}(\Omega)$ is *finer* (or, *larger*, *stronger*) than $\mathfrak{g}\text{-Der}_{\mathfrak{g}}:\mathscr{P}(\Omega)\longrightarrow\mathscr{P}(\Omega)$), so is $\mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta)}:\mathscr{P}(\Omega)\longrightarrow\mathscr{P}(\Omega)$ *coarser* (or, *smaller*, *weaker*) than $\text{der}_{\mathfrak{g}}^{(\delta)}:\mathscr{P}(\Omega)\longrightarrow\mathscr{P}(\Omega)$ (or, $\text{der}_{\mathfrak{g}}^{(\delta)}:\mathscr{P}(\Omega)\longrightarrow\mathscr{P}(\Omega)$ *finer* (or, *larger*, *stronger*) than $\mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta)}:\mathscr{P}(\Omega)\longrightarrow\mathscr{P}(\Omega)$); likewise, just as $\mathfrak{g}\text{-Cod}_{\mathfrak{g}}:\mathscr{P}(\Omega)\longrightarrow\mathscr{P}(\Omega)$ is *finer* (or, *larger*, *stronger*) than $\text{cod}_{\mathfrak{g}}:\mathscr{P}(\Omega)\longrightarrow\mathscr{P}(\Omega)$ or, $\text{cod}_{\mathfrak{g}}:\mathscr{P}(\Omega)\longrightarrow\mathscr{P}(\Omega)$ is *coarser* (or, *smaller*, *weaker*) than $\mathfrak{g}\text{-Cod}_{\mathfrak{g}}:\mathscr{P}(\Omega)\longrightarrow\mathscr{P}(\Omega)$, so is $\mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\delta)}:\mathscr{P}(\Omega)\longrightarrow\mathscr{P}(\Omega)$ *finer* (or, *larger*, *stronger*)

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than $\mathrm{cod}_{\mathfrak{g}}^{(\delta)} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ (or, $\mathrm{cod}_{\mathfrak{g}}^{(\delta)} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ *coarser* (or, *smaller*, *weaker*) than $\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}^{(\delta)} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$). Accordingly, the proposition follows.

Proposition 3.39. *If* $\mathfrak{g}\text{-}\mathbf{Dc}_{\mathfrak{g}} \in \mathfrak{g}\text{-}\mathrm{DC}[\mathfrak{T}_{\mathfrak{g}}]$ *be a given pair of* $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$*-operators* $\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}$, $\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ *and* $\mathbf{dc}_{\mathfrak{g}} \in \mathrm{DC}[\mathfrak{T}_{\mathfrak{g}}]$ *be a given pair of* $\mathfrak{T}_{\mathfrak{g}}$*-operators* $\mathrm{der}_{\mathfrak{g}}$, $\mathrm{cod}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$, *and let* $\mathscr{S}_{\mathfrak{g}} \in \mathscr{P}(\Omega)$ *be arbitrary in a* $\mathscr{T}_{\mathfrak{g}}$*-space* $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$, *then:*

- I. $\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}^{(\delta)}(\mathscr{S}_{\mathfrak{g}}) \subseteq \mathrm{der}_{\mathfrak{g}}^{(\delta)}(\mathscr{S}_{\mathfrak{g}}) \quad (\forall \delta : 1 \preccurlyeq \delta \prec \lambda)$,
- II. $\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}^{(\delta)}(\mathscr{S}_{\mathfrak{g}}) \supseteq \mathrm{cod}_{\mathfrak{g}}^{(\delta)}(\mathscr{S}_{\mathfrak{g}}) \quad (\forall \delta : 1 \preccurlyeq \delta \prec \lambda)$.

Proof. Let $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$ be a $\mathscr{T}_{\mathfrak{g}}$-space. Suppose $\mathfrak{g}\text{-}\mathbf{Dc}_{\mathfrak{g}} \in \mathfrak{g}\text{-}\mathrm{DC}[\mathfrak{T}_{\mathfrak{g}}]$ and $\mathbf{dc}_{\mathfrak{g}} \in \mathrm{DC}[\mathfrak{T}_{\mathfrak{g}}]$ be given and $\mathscr{S}_{\mathfrak{g}} \in \mathscr{P}(\Omega)$ be arbitrary. Then:

I. Introduce $\mathbb{B} = \{0, 1\}$ as Boolean domain and introduce the Boolean-valued propositional formula

$$\mathbb{B} \ni \mathrm{P}(\delta) \quad \overset{\mathrm{def}}{\longleftrightarrow} \quad \mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}^{(\delta)}(\mathscr{S}_{\mathfrak{g}}) \subseteq \mathrm{der}_{\mathfrak{g}}^{(\delta)}(\mathscr{S}_{\mathfrak{g}}) \quad (\forall \delta : 1 \preccurlyeq \delta \prec \lambda).$$

Then, to prove Item I., it only suffices to prove that,

$$\big(\forall \delta : 1 \preccurlyeq \delta \prec \lambda\big)\big[\big(\mathrm{P}(1) = 1\big) \wedge \big(\mathrm{P}(\delta) = 1 \longrightarrow \mathrm{P}(\delta + 1) = 1\big)\big].$$

Case (*i.*) Let $1 = \delta$. Then, $\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \subseteq \mathrm{der}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$ holds true, implying $\mathrm{P}(1) = 1$. The base case therefore holds.

Case (*ii.*) Let $1 \prec \delta \prec \lambda$ and assume that the inductive hypothesis $\mathrm{P}(\delta) = 1$ holds true. Then, $\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}^{(\delta)}(\mathscr{S}_{\mathfrak{g}}) \subseteq \mathrm{der}_{\mathfrak{g}}^{(\delta)}(\mathscr{S}_{\mathfrak{g}})$ and consequently, it follows that

$$\begin{aligned}
\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}^{(\delta+1)}(\mathscr{S}_{\mathfrak{g}}) \quad &\longleftrightarrow \quad \mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}} \circ \mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}^{(\delta)}(\mathscr{S}_{\mathfrak{g}}) \\
&\subseteq \quad \mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}} \circ \mathrm{der}_{\mathfrak{g}}^{(\delta)}(\mathscr{S}_{\mathfrak{g}}) \\
&\subseteq \quad \mathrm{der}_{\mathfrak{g}} \circ \mathrm{der}_{\mathfrak{g}}^{(\delta)}(\mathscr{S}_{\mathfrak{g}}) \longleftrightarrow \mathrm{der}_{\mathfrak{g}}^{(\delta+1)}(\mathscr{S}_{\mathfrak{g}}).
\end{aligned}$$

Hence, $\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}^{(\delta+1)}(\mathscr{S}_{\mathfrak{g}}) \subseteq \mathrm{der}_{\mathfrak{g}}^{(\delta+1)}(\mathscr{S}_{\mathfrak{g}})$, implying $\mathrm{P}(\delta + 1) = 1$. The inductive case therefore holds.

Since $\mathrm{P}(\delta) = 1$ for all δ such that $1 \prec \delta \prec \lambda$, it follows that

$$\begin{aligned}
\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}^{(\lambda+1)}(\mathscr{S}_{\mathfrak{g}}) \quad &\longleftrightarrow \quad \mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}} \circ \mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}^{(\lambda)}(\mathscr{S}_{\mathfrak{g}}) \\
&\subseteq \quad \mathrm{der}_{\mathfrak{g}} \circ \mathrm{der}_{\mathfrak{g}}^{(\lambda)}(\mathscr{S}_{\mathfrak{g}}) \\
&\longleftrightarrow \quad \mathrm{der}_{\mathfrak{g}}\Big(\bigcap_{\delta \prec \lambda} \mathrm{der}_{\mathfrak{g}}^{(\delta)}(\mathscr{S}_{\mathfrak{g}})\Big) \\
&\subseteq \quad \mathrm{der}_{\mathfrak{g}} \circ \mathrm{der}_{\mathfrak{g}}^{(\delta)}(\mathscr{S}_{\mathfrak{g}}) \longleftrightarrow \mathrm{der}_{\mathfrak{g}}^{(\delta+1)}(\mathscr{S}_{\mathfrak{g}}) \\
&\qquad\qquad\qquad\qquad\qquad \subseteq \mathrm{der}_{\mathfrak{g}}^{(\delta)}(\mathscr{S}_{\mathfrak{g}})
\end{aligned}$$

for all δ such that $1 \prec \delta \prec \lambda$, from which $\mathrm{P}(\lambda) = 1$ follows.

II. Introduce $\mathbb{B} = \{0, 1\}$ as Boolean domain and introduce the Boolean-valued propositional formula

$$\mathbb{B} \ni \mathrm{Q}(\delta) \quad \overset{\mathrm{def}}{\longleftrightarrow} \quad \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}^{(\delta)}(\mathscr{S}_{\mathfrak{g}}) \supseteq \mathrm{cod}_{\mathfrak{g}}^{(\delta)}(\mathscr{S}_{\mathfrak{g}}) \quad (\forall \delta : 1 \preccurlyeq \delta \prec \lambda).$$

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Then, to prove ITEM II., it only suffices to prove that,

$$\big(\forall\delta:\ 1\preccurlyeq\delta\prec\lambda\big)\big[\big(\mathrm{Q}(1)=1\big)\wedge\big(\mathrm{Q}(\delta)=1\ \longrightarrow\ \mathrm{P}(\delta+1)=1\big)\big].$$

Case (i.) Let $1=\delta$. Then, $\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})\supseteq\mathrm{cod}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$ holds true, implying $\mathrm{Q}(1)=1$. The base case therefore holds.

Case (ii.) Let $1\prec\delta\prec\lambda$ and assume that the inductive hypothesis $\mathrm{Q}(\delta)=1$ holds true. Then, $\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}^{(\delta)}(\mathscr{S}_{\mathfrak{g}})\supseteq\mathrm{cod}_{\mathfrak{g}}^{(\delta)}(\mathscr{S}_{\mathfrak{g}})$ and consequently, it follows that

$$\begin{aligned}\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}^{(\delta+1)}(\mathscr{S}_{\mathfrak{g}})\ &\longleftrightarrow\ \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}\circ\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}^{(\delta)}(\mathscr{S}_{\mathfrak{g}})\\ &\supseteq\ \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}\circ\mathrm{cod}_{\mathfrak{g}}^{(\delta)}(\mathscr{S}_{\mathfrak{g}})\\ &\supseteq\ \mathrm{cod}_{\mathfrak{g}}\circ\mathrm{cod}_{\mathfrak{g}}^{(\delta)}(\mathscr{S}_{\mathfrak{g}})\ \longleftrightarrow\ \mathrm{cod}_{\mathfrak{g}}^{(\delta+1)}(\mathscr{S}_{\mathfrak{g}}).\end{aligned}$$

Hence, $\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}^{(\delta+1)}(\mathscr{S}_{\mathfrak{g}})\supseteq\mathrm{cod}_{\mathfrak{g}}^{(\delta+1)}(\mathscr{S}_{\mathfrak{g}})$, implying $\mathrm{Q}(\delta+1)=1$. The inductive case therefore holds.

Since $\mathrm{Q}(\delta)=1$ for all δ such that $1\prec\delta\prec\lambda$, it follows that

$$\begin{aligned}\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}^{(\lambda+1)}(\mathscr{S}_{\mathfrak{g}})\ &\longleftrightarrow\ \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}\circ\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}^{(\lambda)}(\mathscr{S}_{\mathfrak{g}})\\ &\supseteq\ \mathrm{cod}_{\mathfrak{g}}\circ\mathrm{cod}_{\mathfrak{g}}^{(\lambda)}(\mathscr{S}_{\mathfrak{g}})\\ &\longleftrightarrow\ \mathrm{cod}_{\mathfrak{g}}\Big(\bigcap_{\delta\prec\lambda}\mathrm{cod}_{\mathfrak{g}}^{(\delta)}(\mathscr{S}_{\mathfrak{g}})\Big)\\ &\supseteq\ \mathrm{cod}_{\mathfrak{g}}\circ\mathrm{cod}_{\mathfrak{g}}^{(\delta)}(\mathscr{S}_{\mathfrak{g}})\longleftrightarrow\mathrm{cod}_{\mathfrak{g}}^{(\delta+1)}(\mathscr{S}_{\mathfrak{g}})\\ &\qquad\qquad\qquad\supseteq\mathrm{cod}_{\mathfrak{g}}^{(\delta)}(\mathscr{S}_{\mathfrak{g}})\end{aligned}$$

for all δ such that $1\prec\delta\prec\lambda$, from which $\mathrm{Q}(\lambda)=1$ follows. The proof of the proposition is complete. Q.E.D.

For any δ such that $1\preccurlyeq\delta\prec\lambda$, $\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}^{(\delta)}:\mathscr{P}(\Omega)\longrightarrow\mathscr{P}(\Omega)$ is *coarser* (or, *smaller*, *weaker*) than $\mathrm{der}_{\mathfrak{g}}:\mathscr{P}(\Omega)\longrightarrow\mathscr{P}(\Omega)$ or, $\mathrm{der}_{\mathfrak{g}}:\mathscr{P}(\Omega)\longrightarrow\mathscr{P}(\Omega)$ is *finer* (or, *larger*, *stronger*) than $\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}^{(\delta)}:\mathscr{P}(\Omega)\longrightarrow\mathscr{P}(\Omega)$; $\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}^{(\delta)}:\mathscr{P}(\Omega)\longrightarrow\mathscr{P}(\Omega)$ is *finer* (or, *larger*, *stronger*) than $\mathrm{cod}_{\mathfrak{g}}:\mathscr{P}(\Omega)\longrightarrow\mathscr{P}(\Omega)$ or, $\mathrm{cod}_{\mathfrak{g}}:\mathscr{P}(\Omega)\longrightarrow\mathscr{P}(\Omega)$ is *coarser* (or, *smaller*, *weaker*) than $\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}^{(\delta)}:\mathscr{P}(\Omega)\longrightarrow\mathscr{P}(\Omega)$. Accordingly, the following corollary is an immediate consequence of the above proposition.

COROLLARY 3.40. *If* $\mathfrak{g}\text{-}\mathbf{Dc}_{\mathfrak{g}}\in\mathfrak{g}\text{-}\mathrm{DC}[\mathfrak{T}_{\mathfrak{g}}]$ *be a given pair of* $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}$*-operators* $\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}$, $\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}:\mathscr{P}(\Omega)\longrightarrow\mathscr{P}(\Omega)$ *and* $\mathbf{dc}_{\mathfrak{g}}\in\mathrm{DC}[\mathfrak{T}_{\mathfrak{g}}]$ *be a given pair of* $\mathfrak{T}_{\mathfrak{g}}$*-operators* $\mathrm{der}_{\mathfrak{g}}$, $\mathrm{cod}_{\mathfrak{g}}:\mathscr{P}(\Omega)\longrightarrow\mathscr{P}(\Omega)$, *and let* $\mathscr{S}_{\mathfrak{g}}\in\mathscr{P}(\Omega)$ *be arbitrary in a* $\mathscr{T}_{\mathfrak{g}}$*-space* $\mathfrak{T}_{\mathfrak{g}}=(\Omega,\mathscr{T}_{\mathfrak{g}})$, *then:*

- I. $\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}^{(\delta)}(\mathscr{S}_{\mathfrak{g}})\subseteq\mathrm{der}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})\quad(\forall\delta:\ 1\preccurlyeq\delta\prec\lambda)$,
- II. $\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}^{(\delta)}(\mathscr{S}_{\mathfrak{g}})\supseteq\mathrm{cod}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})\quad(\forall\delta:\ 1\preccurlyeq\delta\prec\lambda)$.

For any δ such that $1\preccurlyeq\delta\prec\lambda$, the notions of δ^{th}-order $\mathfrak{T}_{\mathfrak{g}}$, $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}$-derived set operators can be interrelated among themselves and presented δ^{th}*-order* $\mathfrak{T}_{\mathfrak{g}}$*,* $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}$*-derived set operators finness-coarseness diagrams*; similarly, the notions of δ^{th}-order $\mathfrak{T}_{\mathfrak{g}}$, $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}$-coderived set operators can be interrelated among themselves and

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presented δ^{th}*-order* $\mathfrak{T}_{\mathfrak{g}}$, $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$*-coderived set operators finness-coarseness diagrams.* A further corollary follows.

Corollary 3.41. *If* $\mathfrak{g}\text{-}\mathbf{Dc}_{\mathfrak{g}} \in \mathfrak{g}\text{-}\mathrm{DC}[\mathfrak{T}_{\mathfrak{g}}]$ *be a given pair of* $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$*-operators* $\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}$, $\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ *and* $\mathbf{dc}_{\mathfrak{g}} \in \mathrm{DC}[\mathfrak{T}_{\mathfrak{g}}]$ *be a given pair of* $\mathfrak{T}_{\mathfrak{g}}$*-operators* $\mathrm{der}_{\mathfrak{g}}$, $\mathrm{cod}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ *in a* $\mathscr{T}_{\mathfrak{g}}$*-space* $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$*, then:*

- I. *For any* $\mathscr{R}_{\mathfrak{g}} \in \mathscr{P}(\Omega)$,

$$\begin{array}{ccc} \mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}^{(\delta)}(\mathscr{R}_{\mathfrak{g}}) \subseteq \mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}) & \longrightarrow & \mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}^{(\delta)}(\mathscr{R}_{\mathfrak{g}}) \subseteq \mathrm{der}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}) \\ & \nwarrow & \uparrow \\ & & \mathrm{der}_{\mathfrak{g}}^{(\delta)}(\mathscr{R}_{\mathfrak{g}}) \subseteq \mathrm{der}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}) \quad (\forall\delta:\ 1 \preccurlyeq \delta \prec \lambda). \end{array} \tag{3.19}$$

- II. *For any* $\mathscr{S}_{\mathfrak{g}} \in \mathscr{P}(\Omega)$,

$$\begin{array}{ccc} \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}^{(\delta)}(\mathscr{S}_{\mathfrak{g}}) \supseteq \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) & \longleftarrow & \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}^{(\delta)}(\mathscr{S}_{\mathfrak{g}}) \supseteq \mathrm{cod}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \\ & \searrow & \downarrow \\ & & \mathrm{cod}_{\mathfrak{g}}^{(\delta)}(\mathscr{S}_{\mathfrak{g}}) \supseteq \mathrm{cod}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \quad (\forall\delta:\ 1 \preccurlyeq \delta \prec \lambda). \end{array} \tag{3.20}$$

For any δ such that $1 \preccurlyeq \delta \prec \lambda$, the δ^{th}-order $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-derived set operator is $\emptyset$-*grounded* (alternatively, $\emptyset$-*preserving*); the δ^{th}-order $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-coderived set operator is Ω-*grounded* (alternatively, Ω-*preserving*). These are embodied in the following theorem.

Theorem 3.42. *Let* $\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}$, $\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ *be a* $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$*-derived and a* $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$*-coderived operators, respectively, in a strong* $\mathscr{T}_{\mathfrak{g}}$*-space* $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$*. Then:*

- I. $\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}^{(\delta)}(\emptyset) = \emptyset \quad (\forall\delta:\ 1 \preccurlyeq \delta \prec \lambda)$,
- II. $\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}^{(\delta)}(\Omega) = \Omega \quad (\forall\delta:\ 1 \preccurlyeq \delta \prec \lambda)$.

Proof. Let $\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}$, $\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ be a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-derived and a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-coderived operators, respectively, in a strong $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$. Then:

I. Introduce $\mathbb{B} = \{0, 1\}$ as Boolean domain and introduce the Boolean-valued propositional formula

$$\mathbb{B} \ni \mathrm{P}(\delta) \quad \stackrel{\text{def}}{\longleftrightarrow} \quad \mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}^{(\delta)}(\emptyset) = \emptyset \quad (\forall\delta:\ 1 \preccurlyeq \delta \prec \lambda).$$

Then, to prove Item I., it only suffices to prove that,

$$\big(\forall\delta:\ 1 \preccurlyeq \delta \prec \lambda\big)\Big[\big(\mathrm{P}(1) = 1\big) \wedge \big(\mathrm{P}(\delta) = 1 \ \longrightarrow\ \mathrm{P}(\delta+1) = 1\big)\Big].$$

Case (i.) Let $1 = \delta$. Then, $\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}^{(1)}(\emptyset) \longleftrightarrow \mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}(\emptyset) = \emptyset$. Thus, $\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}^{(1)}(\emptyset) = \emptyset$, implying $\mathrm{P}(1) = 1$. The base case therefore holds.

Case (ii.) Let $1 \prec \delta \prec \lambda$ and assume that the inductive hypothesis $\mathrm{P}(\delta) = 1$ holds true. Then, $\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}^{(\delta)}(\emptyset) = \emptyset$ and consequently, it follows that

$$\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}^{(\delta+1)}(\emptyset) \longleftrightarrow \mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}} \circ \mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}^{(\delta)}(\emptyset) \longleftrightarrow \mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}(\emptyset) = \emptyset.$$

Hence, $\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}^{(\delta+1)}(\emptyset) = \emptyset$, implying $\mathrm{P}(\delta+1) = 1$. The inductive case therefore holds.

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Since $\mathrm{P}(\delta) = 1$ for all δ such that $1 \prec \delta \prec \lambda$, it follows that

$$\begin{aligned} \mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}^{(\lambda+1)}(\emptyset) &\longleftrightarrow \mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}} \circ \mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}^{(\lambda)}(\emptyset) \\ &\longleftrightarrow \mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}\Big(\bigcap_{\delta\prec\lambda} \mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}^{(\delta)}(\emptyset)\Big) \longleftrightarrow \mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}(\emptyset) = \emptyset \end{aligned}$$

for all δ such that $1 \prec \delta \prec \lambda$, from which $\mathrm{P}(\lambda) = 1$ follows.

II. Introduce $\mathbb{B} = \{0, 1\}$ as Boolean domain and introduce the Boolean-valued propositional formula

$$\mathbb{B} \ni \mathrm{Q}(\delta) \stackrel{\text{def}}{\longleftrightarrow} \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}^{(\delta)}(\Omega) = \Omega \quad (\forall \delta :\ 1 \preccurlyeq \delta \prec \lambda).$$

Then, to prove ITEM II., it only suffices to prove that,

$$(\forall \delta :\ 1 \preccurlyeq \delta \prec \lambda)\big[\big(\mathrm{Q}(1) = 1\big) \wedge \big(\mathrm{Q}(\delta) = 1 \longrightarrow \mathrm{Q}(\delta+1) = 1\big)\big].$$

Case (i.) Let $1 = \delta$. Then, $\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}^{(1)}(\Omega) \longleftrightarrow \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}(\Omega) = \Omega$. Thus, $\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}^{(1)}(\Omega) = \Omega$, implying $\mathrm{Q}(1) = 1$. The base case therefore holds.

Case (ii.) Let $1 \prec \delta \prec \lambda$ and assume that the inductive hypothesis $\mathrm{Q}(\delta) = 1$ holds true. Then, $\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}^{(\delta)}(\Omega) = \Omega$ and consequently, it follows that

$$\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}^{(\delta+1)}(\Omega) \longleftrightarrow \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}} \circ \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}^{(\delta)}(\Omega) \longleftrightarrow \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}(\Omega) = \Omega.$$

Hence, $\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}^{(\delta+1)}(\Omega) = \Omega$, implying $\mathrm{Q}(\delta+1) = 1$. The inductive case therefore holds.

Since $\mathrm{Q}(\delta) = 1$ for all δ such that $1 \prec \delta \prec \lambda$, it follows that

$$\begin{aligned} \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}^{(\lambda+1)}(\Omega) &\longleftrightarrow \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}} \circ \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}^{(\lambda)}(\Omega) \\ &\longleftrightarrow \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}\Big(\bigcap_{\delta\prec\lambda} \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}^{(\delta)}(\Omega)\Big) \longleftrightarrow \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}(\Omega) = \Omega \end{aligned}$$

for all δ such that $1 \prec \delta \prec \lambda$, from which $\mathrm{Q}(\lambda) = 1$ follows. The proof of the theorem is complete. Q.E.D.

For any δ such that $1 \preccurlyeq \delta \prec \lambda$, the δ^{th}-order $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}$-derived set operator is $\cup$-*additive* (alternatively, $\cup$-*distributive*); the δ^{th}-order $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}$-coderived set operator is $\cap$-*additive* (alternatively, $\cap$-*distributive*). The theorem follows.

THEOREM 3.43. *Let $\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}, \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ be a $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}$-derived and a $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}$-coderived operators, respectively, and let $(\mathscr{R}_{\mathfrak{g}}, \mathscr{S}_{\mathfrak{g}}) \in \times_{\alpha\in I_2^*} \mathscr{P}(\Omega)$ be arbitrary in a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$. Then:*

- I. $\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}^{(\delta)}(\mathscr{R}_{\mathfrak{g}} \cup \mathscr{S}_{\mathfrak{g}}) = \bigcup_{\mathscr{W}_{\mathfrak{g}}=\mathscr{R}_{\mathfrak{g}},\mathscr{S}_{\mathfrak{g}}} \mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}^{(\delta)}(\mathscr{W}_{\mathfrak{g}}) \quad (\forall \delta :\ 1 \preccurlyeq \delta \prec \lambda)$,
- II. $\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}^{(\delta)}(\mathscr{R}_{\mathfrak{g}} \cap \mathscr{S}_{\mathfrak{g}}) = \bigcap_{\mathscr{W}_{\mathfrak{g}}=\mathscr{R}_{\mathfrak{g}},\mathscr{S}_{\mathfrak{g}}} \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}^{(\delta)}(\mathscr{W}_{\mathfrak{g}}) \quad (\forall \delta :\ 1 \preccurlyeq \delta \prec \lambda)$.

PROOF. Let $\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}, \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ be a $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}$-derived and a $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}$-coderived operators, respectively, and let $(\mathscr{R}_{\mathfrak{g}}, \mathscr{S}_{\mathfrak{g}}) \in \times_{\alpha\in I_2^*} \mathscr{P}(\Omega)$ be arbitrary in a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$. Then:

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I. Introduce $\mathbb{B}=\{0,1\}$ as Boolean domain and introduce the Boolean-valued propositional formula

$$\mathbb{B}\ni \mathrm{P}(\delta)\quad\stackrel{\text{def}}{\longleftrightarrow}\quad \mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}^{(\delta)}(\mathscr{R}_{\mathfrak{g}}\cup\mathscr{S}_{\mathfrak{g}})=\bigcup_{\mathscr{W}_{\mathfrak{g}}=\mathscr{R}_{\mathfrak{g}},\mathscr{S}_{\mathfrak{g}}}\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}^{(\delta)}(\mathscr{W}_{\mathfrak{g}})\quad(\forall\delta:1\preccurlyeq\delta\prec\lambda).$$

Then, to prove ITEM I., it only suffices to prove that,

$$\big(\forall\delta:\ 1\preccurlyeq\delta\prec\lambda\big)\big[\big(\mathrm{P}(1)=1\big)\wedge\big(\mathrm{P}(\delta)=1\ \longrightarrow\ \mathrm{P}(\delta+1)=1\big)\big].$$

Case (i.) Let $1=\delta$. Then, $\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}\cup\mathscr{S}_{\mathfrak{g}})=\bigcup_{\mathscr{W}_{\mathfrak{g}}=\mathscr{R}_{\mathfrak{g}},\mathscr{S}_{\mathfrak{g}}}\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}(\mathscr{W}_{\mathfrak{g}})$ holds true, implying $\mathrm{P}(1)=1$. The base case therefore holds.

Case (ii.) Let $1\prec\delta\prec\lambda$ and assume that the inductive hypothesis $\mathrm{P}(\delta)=1$ holds true. Then, $\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}^{(\delta)}(\mathscr{R}_{\mathfrak{g}}\cup\mathscr{S}_{\mathfrak{g}})=\bigcup_{\mathscr{W}_{\mathfrak{g}}=\mathscr{R}_{\mathfrak{g}},\mathscr{S}_{\mathfrak{g}}}\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}^{(\delta)}(\mathscr{W}_{\mathfrak{g}})$ and consequently, it follows that

$$\begin{aligned}\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}^{(\delta+1)}(\mathscr{R}_{\mathfrak{g}}\cup\mathscr{S}_{\mathfrak{g}})\quad&\longleftrightarrow\quad \mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}\circ\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}^{(\delta)}(\mathscr{R}_{\mathfrak{g}}\cup\mathscr{S}_{\mathfrak{g}})\\&=\quad \mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}\Bigg(\bigcup_{\mathscr{W}_{\mathfrak{g}}=\mathscr{R}_{\mathfrak{g}},\mathscr{S}_{\mathfrak{g}}}\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}^{(\delta)}(\mathscr{W}_{\mathfrak{g}})\Bigg)\\&=\quad\bigcup_{\mathscr{W}_{\mathfrak{g}}=\mathscr{R}_{\mathfrak{g}},\mathscr{S}_{\mathfrak{g}}}\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}\circ\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}^{(\delta)}(\mathscr{W}_{\mathfrak{g}})\\&\longleftrightarrow\quad\bigcup_{\mathscr{W}_{\mathfrak{g}}=\mathscr{R}_{\mathfrak{g}},\mathscr{S}_{\mathfrak{g}}}\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}^{(\delta+1)}(\mathscr{W}_{\mathfrak{g}}).\end{aligned}$$

Hence, $\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}^{(\delta+1)}(\mathscr{R}_{\mathfrak{g}}\cup\mathscr{S}_{\mathfrak{g}})=\bigcup_{\mathscr{W}_{\mathfrak{g}}=\mathscr{R}_{\mathfrak{g}},\mathscr{S}_{\mathfrak{g}}}\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}^{(\delta+1)}(\mathscr{W}_{\mathfrak{g}})$, implying $\mathrm{P}(\delta+1)=1$. The inductive case therefore holds.

Since $\mathrm{P}(\delta)=1$ for all δ such that $1\prec\delta\prec\lambda$, it follows that $\mathrm{P}(\lambda)=1$ states that

$$\bigcap_{\delta\prec\lambda}\Bigg(\bigcap_{\mathscr{W}_{\mathfrak{g}}=\mathscr{R}_{\mathfrak{g}},\mathscr{S}_{\mathfrak{g}}}\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}^{(\delta)}(\mathscr{W}_{\mathfrak{g}})\Bigg)\quad\longleftrightarrow\quad\bigcap_{\mathscr{W}_{\mathfrak{g}}=\mathscr{R}_{\mathfrak{g}},\mathscr{S}_{\mathfrak{g}}}\Bigg(\bigcap_{\delta\prec\lambda}\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}^{(\delta)}(\mathscr{W}_{\mathfrak{g}})\Bigg),$$

and it is evident that any element in $\bigcap_{\delta\prec\lambda}\Big(\bigcap_{\mathscr{W}_{\mathfrak{g}}=\mathscr{R}_{\mathfrak{g}},\mathscr{S}_{\mathfrak{g}}}\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}^{(\delta)}(\mathscr{W}_{\mathfrak{g}})\Big)$ is contained in $\bigcap_{\mathscr{W}_{\mathfrak{g}}=\mathscr{R}_{\mathfrak{g}},\mathscr{S}_{\mathfrak{g}}}\Big(\bigcap_{\delta\prec\lambda}\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}^{(\delta)}(\mathscr{W}_{\mathfrak{g}})\Big)$. Thus, to prove that any element in $\bigcap_{\mathscr{W}_{\mathfrak{g}}=\mathscr{R}_{\mathfrak{g}},\mathscr{S}_{\mathfrak{g}}}\Big(\bigcap_{\delta\prec\lambda}\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}^{(\delta)}(\mathscr{W}_{\mathfrak{g}})\Big)$ is also in $\bigcap_{\delta\prec\lambda}\Big(\bigcap_{\mathscr{W}_{\mathfrak{g}}=\mathscr{R}_{\mathfrak{g}},\mathscr{S}_{\mathfrak{g}}}\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}^{(\delta)}(\mathscr{W}_{\mathfrak{g}})\Big)$, let $\xi\in\bigcap_{\mathscr{W}_{\mathfrak{g}}=\mathscr{R}_{\mathfrak{g}},\mathscr{S}_{\mathfrak{g}}}\Big(\bigcap_{\delta\prec\lambda}\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}^{(\delta)}(\mathscr{W}_{\mathfrak{g}})\Big)$ such that, for some $(\alpha,\beta)\prec(\lambda,\lambda)$ where $\alpha\preccurlyeq\beta$, say, the statement $\xi\in\bigcap_{\mathscr{W}_{\mathfrak{g}}=\mathscr{R}_{\mathfrak{g}},\mathscr{S}_{\mathfrak{g}}}\Big(\bigcap_{\delta=\alpha,\beta}\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}^{(\delta)}(\mathscr{W}_{\mathfrak{g}})\Big)$ holds true. Then, $\xi\in\bigcap_{\mathscr{W}_{\mathfrak{g}}=\mathscr{R}_{\mathfrak{g}},\mathscr{S}_{\mathfrak{g}}}\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}^{(\alpha)}(\mathscr{W}_{\mathfrak{g}})$ and therefore $\xi\in\bigcap_{\delta\prec\lambda}\Big(\bigcap_{\mathscr{W}_{\mathfrak{g}}=\mathscr{R}_{\mathfrak{g}},\mathscr{S}_{\mathfrak{g}}}\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}^{(\delta)}(\mathscr{W}_{\mathfrak{g}})\Big)$, implying $\mathrm{P}(\lambda)=1$ holds.

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II. Introduce $\mathbb{B} = \{0,1\}$ as Boolean domain and introduce the Boolean-valued propositional formula

$$\mathbb{B} \ni \mathrm{Q}(\delta) \quad \overset{\mathrm{def}}{\longleftrightarrow} \quad \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}^{(\delta)}\left(\mathscr{R}_{\mathfrak{g}} \cap \mathscr{S}_{\mathfrak{g}}\right) = \bigcap_{\mathscr{W}_{\mathfrak{g}}=\mathscr{R}_{\mathfrak{g}},\mathscr{S}_{\mathfrak{g}}} \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}^{(\delta)}\left(\mathscr{W}_{\mathfrak{g}}\right) \ (\forall\delta : 1 \preccurlyeq \delta \prec \lambda).$$

Then, to prove ITEM II., it only suffices to prove that,

$$\left(\forall\delta :\ 1 \preccurlyeq \delta \prec \lambda\right)\left[\left(\mathrm{Q}(1) = 1\right) \wedge \left(\mathrm{Q}(\delta) = 1 \ \longrightarrow\ \mathrm{Q}(\delta+1) = 1\right)\right].$$

Case (*i.*) Let $1 = \delta$. Then, $\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}\left(\mathscr{R}_{\mathfrak{g}} \cap \mathscr{S}_{\mathfrak{g}}\right) = \bigcap_{\mathscr{W}_{\mathfrak{g}}=\mathscr{R}_{\mathfrak{g}},\mathscr{S}_{\mathfrak{g}}} \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}\left(\mathscr{W}_{\mathfrak{g}}\right)$ holds true, implying $\mathrm{Q}(1) = 1$. The base case therefore holds.

Case (*ii.*) Let $1 \prec \delta \prec \lambda$ and assume that the inductive hypothesis $\mathrm{Q}(\delta) = 1$ holds true. Then, $\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}^{(\delta)}\left(\mathscr{R}_{\mathfrak{g}} \cap \mathscr{S}_{\mathfrak{g}}\right) = \bigcap_{\mathscr{W}_{\mathfrak{g}}=\mathscr{R}_{\mathfrak{g}},\mathscr{S}_{\mathfrak{g}}} \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}^{(\delta)}\left(\mathscr{W}_{\mathfrak{g}}\right)$ and consequently, it follows that

$$\begin{aligned}
\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}^{(\delta+1)}\left(\mathscr{R}_{\mathfrak{g}} \cap \mathscr{S}_{\mathfrak{g}}\right) \quad &\longleftrightarrow \quad \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}} \circ \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}^{(\delta)}\left(\mathscr{R}_{\mathfrak{g}} \cap \mathscr{S}_{\mathfrak{g}}\right) \\
&= \quad \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}\left(\bigcap_{\mathscr{W}_{\mathfrak{g}}=\mathscr{R}_{\mathfrak{g}},\mathscr{S}_{\mathfrak{g}}} \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}^{(\delta)}\left(\mathscr{W}_{\mathfrak{g}}\right)\right) \\
&= \quad \bigcap_{\mathscr{W}_{\mathfrak{g}}=\mathscr{R}_{\mathfrak{g}},\mathscr{S}_{\mathfrak{g}}} \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}} \circ \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}^{(\delta)}\left(\mathscr{W}_{\mathfrak{g}}\right) \\
&\longleftrightarrow \quad \bigcap_{\mathscr{W}_{\mathfrak{g}}=\mathscr{R}_{\mathfrak{g}},\mathscr{S}_{\mathfrak{g}}} \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}^{(\delta+1)}\left(\mathscr{W}_{\mathfrak{g}}\right).
\end{aligned}$$

Hence, $\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}^{(\delta+1)}\left(\mathscr{R}_{\mathfrak{g}} \cap \mathscr{S}_{\mathfrak{g}}\right) = \bigcap_{\mathscr{W}_{\mathfrak{g}}=\mathscr{R}_{\mathfrak{g}},\mathscr{S}_{\mathfrak{g}}} \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}^{(\delta+1)}\left(\mathscr{W}_{\mathfrak{g}}\right)$, implying $\mathrm{Q}(\delta+1) = 1$. The inductive case therefore holds.

Since $\mathrm{Q}(\delta) = 1$ for all δ such that $1 \prec \delta \prec \lambda$, it follows that $\mathrm{Q}(\lambda) = 1$ states that

$$\bigcap_{\delta\prec\lambda}\left(\bigcap_{\mathscr{W}_{\mathfrak{g}}=\mathscr{R}_{\mathfrak{g}},\mathscr{S}_{\mathfrak{g}}} \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}^{(\delta)}\left(\mathscr{W}_{\mathfrak{g}}\right)\right) \quad \longleftrightarrow \quad \bigcap_{\mathscr{W}_{\mathfrak{g}}=\mathscr{R}_{\mathfrak{g}},\mathscr{S}_{\mathfrak{g}}}\left(\bigcap_{\delta\prec\lambda} \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}^{(\delta)}\left(\mathscr{W}_{\mathfrak{g}}\right)\right),$$

and it is evident that any element in $\bigcap_{\delta\prec\lambda}\left(\bigcap_{\mathscr{W}_{\mathfrak{g}}=\mathscr{R}_{\mathfrak{g}},\mathscr{S}_{\mathfrak{g}}} \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}^{(\delta)}\left(\mathscr{W}_{\mathfrak{g}}\right)\right)$ is contained in $\bigcap_{\mathscr{W}_{\mathfrak{g}}=\mathscr{R}_{\mathfrak{g}},\mathscr{S}_{\mathfrak{g}}}\left(\bigcap_{\delta\prec\lambda} \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}^{(\delta)}\left(\mathscr{W}_{\mathfrak{g}}\right)\right)$. Thus, to prove that any element in $\bigcap_{\mathscr{W}_{\mathfrak{g}}=\mathscr{R}_{\mathfrak{g}},\mathscr{S}_{\mathfrak{g}}}\left(\bigcap_{\delta\prec\lambda} \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}^{(\delta)}\left(\mathscr{W}_{\mathfrak{g}}\right)\right)$ is also in $\bigcap_{\delta\prec\lambda}\left(\bigcap_{\mathscr{W}_{\mathfrak{g}}=\mathscr{R}_{\mathfrak{g}},\mathscr{S}_{\mathfrak{g}}} \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}^{(\delta)}\left(\mathscr{W}_{\mathfrak{g}}\right)\right)$, let $\zeta \in \bigcap_{\mathscr{W}_{\mathfrak{g}}=\mathscr{R}_{\mathfrak{g}},\mathscr{S}_{\mathfrak{g}}}\left(\bigcap_{\delta\prec\lambda} \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}^{(\delta)}\left(\mathscr{W}_{\mathfrak{g}}\right)\right)$ such that, for some $(\alpha,\beta) \prec (\lambda,\lambda)$ where $\alpha \preccurlyeq \beta$, say, the statement $\zeta \in \bigcap_{\mathscr{W}_{\mathfrak{g}}=\mathscr{R}_{\mathfrak{g}},\mathscr{S}_{\mathfrak{g}}}\left(\bigcap_{\delta=\alpha,\beta} \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}^{(\delta)}\left(\mathscr{W}_{\mathfrak{g}}\right)\right)$ holds true. Then, $\zeta \in \bigcap_{\mathscr{W}_{\mathfrak{g}}=\mathscr{R}_{\mathfrak{g}},\mathscr{S}_{\mathfrak{g}}} \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}^{(\alpha)}\left(\mathscr{W}_{\mathfrak{g}}\right)$ and therefore, it follows that $\zeta \in \bigcap_{\delta\prec\lambda}\left(\bigcap_{\mathscr{W}_{\mathfrak{g}}=\mathscr{R}_{\mathfrak{g}},\mathscr{S}_{\mathfrak{g}}} \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}^{(\delta)}\left(\mathscr{W}_{\mathfrak{g}}\right)\right)$, implying $\mathrm{Q}(\lambda) = 1$ holds. The proof of the theorem is complete. Q.E.D.

The corollary stated below is an immediate consequence of the above theorem.

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COROLLARY 3.44. *If* $\mathfrak{g}\text{-}\mathbf{Dc}_{\mathfrak{g}} \in \mathfrak{g}\text{-DC}[\mathfrak{T}_{\mathfrak{g}}]$ *be a given pair of* $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$*-operators* $\mathfrak{g}\text{-Der}_{\mathfrak{g}}$, $\mathfrak{g}\text{-Cod}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$, $\mathbf{dc}_{\mathfrak{g}} \in \mathrm{DC}[\mathfrak{T}_{\mathfrak{g}}]$ *be a given pair of* $\mathfrak{T}_{\mathfrak{g}}$*-operators* $\mathrm{der}_{\mathfrak{g}}$, $\mathrm{cod}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$, *and* $(\mathscr{R}_{\mathfrak{g}}, \mathscr{S}_{\mathfrak{g}}) \in \times_{\alpha \in I_2^*} \mathscr{P}(\Omega)$ *be arbitrary in a* $\mathscr{T}_{\mathfrak{g}}$*-space* $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$, *then:*

- I. $\mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta)}(\mathscr{R}_{\mathfrak{g}} \cup \mathscr{S}_{\mathfrak{g}}) \subseteq \bigcup_{\mathscr{W}_{\mathfrak{g}} = \mathscr{R}_{\mathfrak{g}}, \mathscr{S}_{\mathfrak{g}}} \mathrm{der}_{\mathfrak{g}}^{(\delta)}(\mathscr{W}_{\mathfrak{g}}) \quad (\forall \delta : 1 \preccurlyeq \delta \prec \lambda)$,
- II. $\mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\delta)}(\mathscr{R}_{\mathfrak{g}} \cap \mathscr{S}_{\mathfrak{g}}) \supseteq \bigcap_{\mathscr{W}_{\mathfrak{g}} = \mathscr{R}_{\mathfrak{g}}, \mathscr{S}_{\mathfrak{g}}} \mathrm{cod}_{\mathfrak{g}}^{(\delta)}(\mathscr{W}_{\mathfrak{g}}) \quad (\forall \delta : 1 \preccurlyeq \delta \prec \lambda)$.

For any (δ, η) such that $1 \preccurlyeq \delta \prec \eta \prec \lambda$, $\mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\eta)} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ is *coarser* (or, *smaller*, *weaker*) than $\mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta)} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ or, $\mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta)} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ is *finer* (or, *larger*, *stronger*) than $\mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\eta)} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$; $\mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\eta)} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ is *finer* (or, *larger*, *stronger*) than $\mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\delta)} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ or, $\mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\delta)} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ is *coarser* (or, *smaller*, *weaker*) than $\mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\eta)} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$. Accordingly, the proposition follows.

PROPOSITION 3.45. *Let* $\mathfrak{g}\text{-Der}_{\mathfrak{g}}$, $\mathfrak{g}\text{-Cod}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ *be a* $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$*-derived and a* $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$*-coderived operators, respectively, and let* $\mathscr{S}_{\mathfrak{g}} \in \mathscr{P}(\Omega)$ *be arbitrary in a* $\mathscr{T}_{\mathfrak{g}}$*-space* $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$. *Then:*

- I. $\mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\eta)}(\mathscr{S}_{\mathfrak{g}}) \subseteq \mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta)}(\mathscr{S}_{\mathfrak{g}}) \quad \big(\forall (\delta, \eta) : 1 \preccurlyeq \delta \prec \eta \prec \lambda\big)$,
- II. $\mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\eta)}(\mathscr{S}_{\mathfrak{g}}) \supseteq \mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\delta)}(\mathscr{S}_{\mathfrak{g}}) \quad \big(\forall (\delta, \eta) : 1 \preccurlyeq \delta \prec \eta \prec \lambda\big)$.

PROOF. Let $\mathfrak{g}\text{-Der}_{\mathfrak{g}}$, $\mathfrak{g}\text{-Cod}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ be a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-derived and a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-coderived operators, respectively, and let $\mathscr{S}_{\mathfrak{g}} \in \mathscr{P}(\Omega)$ be arbitrary in a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$. Then:

I. Set $\eta = \delta + \varepsilon$, where $1 \preccurlyeq \varepsilon$, introduce $\mathbb{B} = \{0, 1\}$ as Boolean domain and introduce the Boolean-valued propositional formula

$$\mathbb{B} \ni \mathrm{P}(\varepsilon) \quad \stackrel{\mathrm{def}}{\longleftrightarrow} \quad \mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta+\varepsilon)}(\mathscr{S}_{\mathfrak{g}}) \subseteq \mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta)}(\mathscr{S}_{\mathfrak{g}}) \quad (\forall \varepsilon : 1 \preccurlyeq \varepsilon).$$

Then, to prove ITEM I., it only suffices to prove that,

$$\big(\forall \varepsilon : 1 \preccurlyeq \varepsilon\big)\big[\big(\mathrm{P}(1) = 1\big) \wedge \big(\mathrm{P}(\varepsilon) = 1 \longrightarrow \mathrm{P}(\varepsilon + 1) = 1\big)\big].$$

Case (i.) Let $1 = \varepsilon$. Then, $\mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta+1)}(\mathscr{S}_{\mathfrak{g}}) \subseteq \mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta)}(\mathscr{S}_{\mathfrak{g}})$, implying $\mathrm{P}(1) = 1$. The base case therefore holds.

Case (ii.) Let $1 \prec \varepsilon$ and assume that the inductive hypothesis $\mathrm{P}(\varepsilon) = 1$ holds true. Then, $\mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta+\varepsilon)}(\mathscr{S}_{\mathfrak{g}}) \subseteq \mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta)}(\mathscr{S}_{\mathfrak{g}})$ and consequently, it results that $\mathfrak{g}\text{-Der}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta+\varepsilon)}(\mathscr{S}_{\mathfrak{g}}) \subseteq \mathfrak{g}\text{-Der}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta)}(\mathscr{S}_{\mathfrak{g}})$. But,

$$\begin{aligned} \mathfrak{g}\text{-Der}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta+\varepsilon)}(\mathscr{S}_{\mathfrak{g}}) \quad &\longleftrightarrow \quad \mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta+(\varepsilon+1))}(\mathscr{S}_{\mathfrak{g}}) \\ &\subseteq \quad \mathfrak{g}\text{-Der}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta)}(\mathscr{S}_{\mathfrak{g}}) \\ &\longleftrightarrow \quad \mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta+1)}(\mathscr{S}_{\mathfrak{g}}) \subseteq \mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta)}(\mathscr{S}_{\mathfrak{g}}). \end{aligned}$$

Hence, $\mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta+(\varepsilon+1))}(\mathscr{S}_{\mathfrak{g}}) \subseteq \mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta)}(\mathscr{S}_{\mathfrak{g}})$, implying $\mathrm{P}(\varepsilon + 1) = 1$. The inductive case therefore holds.

Since $\mathrm{P}(\delta) = 1$ for all δ such that $1 \prec \delta \prec \eta \prec \lambda$, it follows that

$$\begin{aligned}\mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\lambda+1)}(\mathscr{S}_{\mathfrak{g}}) &\longleftrightarrow \mathfrak{g}\text{-Der}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\lambda)}(\mathscr{S}_{\mathfrak{g}})\\ &\longleftrightarrow \mathfrak{g}\text{-Der}_{\mathfrak{g}}\Big(\bigcap_{\eta \prec \lambda} \mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\eta)}(\mathscr{S}_{\mathfrak{g}})\Big)\\ &\subseteq \mathfrak{g}\text{-Der}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\eta)}(\mathscr{S}_{\mathfrak{g}})\\ &\longleftrightarrow \mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\eta+1)}(\mathscr{S}_{\mathfrak{g}}) \subseteq \mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta+1)}(\mathscr{S}_{\mathfrak{g}})\\ &\qquad\qquad \subseteq \mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\eta)}(\mathscr{S}_{\mathfrak{g}}) \subseteq \mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta)}(\mathscr{S}_{\mathfrak{g}})\end{aligned}$$

for all δ such that $1 \prec \delta \prec \eta \prec \lambda$, from which $\mathrm{P}(\lambda) = 1$ follows.

II. Set $\eta = \delta + \varepsilon$, where $1 \preccurlyeq \varepsilon$, introduce $\mathbb{B} = \{0, 1\}$ as Boolean domain and introduce the Boolean-valued propositional formula

$$\mathbb{B} \ni \mathrm{Q}(\varepsilon) \quad \overset{\text{def}}{\longleftrightarrow} \quad \mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\delta+\varepsilon)}(\mathscr{S}_{\mathfrak{g}}) \supseteq \mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\delta)}(\mathscr{S}_{\mathfrak{g}}) \quad (\forall \varepsilon : \ 1 \preccurlyeq \varepsilon).$$

Then, to prove Item II., it only suffices to prove that,

$$\big(\forall \varepsilon : \ 1 \preccurlyeq \varepsilon\big)\big[\big(\mathrm{Q}(1) = 1\big) \wedge \big(\mathrm{Q}(\varepsilon) = 1 \ \longrightarrow \ \mathrm{Q}(\varepsilon + 1) = 1\big)\big].$$

Case (i.) Let $1 = \varepsilon$. Then, $\mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\delta+1)}(\mathscr{S}_{\mathfrak{g}}) \supseteq \mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\delta)}(\mathscr{S}_{\mathfrak{g}})$, implying $\mathrm{Q}(1) = 1$. The base case therefore holds.

Case (ii.) Let $1 \prec \varepsilon$ and assume that the inductive hypothesis $\mathrm{Q}(\varepsilon) = 1$ holds true. Then, $\mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\delta+\varepsilon)}(\mathscr{S}_{\mathfrak{g}}) \supseteq \mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\delta)}(\mathscr{S}_{\mathfrak{g}})$ and consequently, it results that $\mathfrak{g}\text{-Cod}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\delta+\varepsilon)}(\mathscr{S}_{\mathfrak{g}}) \supseteq \mathfrak{g}\text{-Cod}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\delta)}(\mathscr{S}_{\mathfrak{g}})$. But,

$$\begin{aligned}\mathfrak{g}\text{-Cod}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\delta+\varepsilon)}(\mathscr{S}_{\mathfrak{g}}) &\longleftrightarrow \mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\delta+(\varepsilon+1))}(\mathscr{S}_{\mathfrak{g}})\\ &\supseteq \mathfrak{g}\text{-Cod}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\delta)}(\mathscr{S}_{\mathfrak{g}})\\ &\longleftrightarrow \mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\delta+1)}(\mathscr{S}_{\mathfrak{g}}) \supseteq \mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\delta)}(\mathscr{S}_{\mathfrak{g}}).\end{aligned}$$

Hence, $\mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\delta+(\varepsilon+1))}(\mathscr{S}_{\mathfrak{g}}) \supseteq \mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\delta)}(\mathscr{S}_{\mathfrak{g}})$, implying $\mathrm{Q}(\varepsilon + 1) = 1$. The inductive case therefore holds.

Since $\mathrm{Q}(\delta) = 1$ for all δ such that $1 \prec \delta \prec \eta \prec \lambda$, it follows that

$$\begin{aligned}\mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\lambda+1)}(\mathscr{S}_{\mathfrak{g}}) &\longleftrightarrow \mathfrak{g}\text{-Cod}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\lambda)}(\mathscr{S}_{\mathfrak{g}})\\ &\longleftrightarrow \mathfrak{g}\text{-Cod}_{\mathfrak{g}}\Big(\bigcap_{\eta \prec \lambda} \mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\eta)}(\mathscr{S}_{\mathfrak{g}})\Big)\\ &\supseteq \mathfrak{g}\text{-Cod}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\eta)}(\mathscr{S}_{\mathfrak{g}})\\ &\longleftrightarrow \mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\eta+1)}(\mathscr{S}_{\mathfrak{g}}) \supseteq \mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\delta+1)}(\mathscr{S}_{\mathfrak{g}})\\ &\qquad\qquad \supseteq \mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\eta)}(\mathscr{S}_{\mathfrak{g}}) \supseteq \mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\delta)}(\mathscr{S}_{\mathfrak{g}})\end{aligned}$$

for all δ such that $1 \prec \delta \prec \eta \prec \lambda$, from which $\mathrm{Q}(\lambda) = 1$ follows. The proof of the proposition is complete. Q.E.D.

The corollary stated below is an immediate consequence of the above proposition.

COROLLARY 3.46. *If* $\mathfrak{g}\text{-}\mathbf{Dc}_{\mathfrak{g}} \in \mathfrak{g}\text{-}\mathrm{DC}[\mathfrak{T}_{\mathfrak{g}}]$ *be a given pair of* $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$*-operators* $\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}$, $\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ *and* $\mathbf{dc}_{\mathfrak{g}} \in \mathrm{DC}[\mathfrak{T}_{\mathfrak{g}}]$ *be a given pair of* $\mathfrak{T}_{\mathfrak{g}}$*-operators* $\mathrm{der}_{\mathfrak{g}}$, $\mathrm{cod}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$, *and let* $\mathscr{S}_{\mathfrak{g}} \in \mathscr{P}(\Omega)$ *be arbitrary in a* $\mathscr{T}_{\mathfrak{g}}$*-space* $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$, *then:*

- I. $\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}^{(\eta)}(\mathscr{S}_{\mathfrak{g}}) \subseteq \mathrm{der}_{\mathfrak{g}}^{(\delta)}(\mathscr{S}_{\mathfrak{g}}) \quad \big(\forall(\delta,\eta) : \ 1 \preccurlyeq \delta \prec \eta \prec \lambda\big)$,
- II. $\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}^{(\eta)}(\mathscr{S}_{\mathfrak{g}}) \supseteq \mathrm{cod}_{\mathfrak{g}}^{(\delta)}(\mathscr{S}_{\mathfrak{g}}) \quad \big(\forall(\delta,\eta) : \ 1 \preccurlyeq \delta \prec \eta \prec \lambda\big)$.

For any (δ,η) such that $1 \preccurlyeq \delta \prec \eta \prec \lambda$, the $(\delta+\eta)^{\mathrm{th}}$-order $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-derived set operator is equivalent to the composition of the δ^{th}-order and the η^{th}-order of the $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-derived set operator; likewise, the $(\delta+\eta)^{\mathrm{th}}$-order $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-coderived set operator is equivalent to the composition of the δ^{th}-order and the η^{th}-order of the $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-coderived set operator. The proposition follows.

PROPOSITION 3.47. *Let* $\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}$, $\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ *be a* $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$*-derived and a* $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$*-coderived operators, respectively, and let* $\mathscr{S}_{\mathfrak{g}} \in \mathscr{P}(\Omega)$ *be arbitrary in a* $\mathscr{T}_{\mathfrak{g}}$*-space* $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$. *Then:*

- I. $\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}^{(\delta+\eta)}(\mathscr{S}_{\mathfrak{g}}) = \mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}^{(\delta)} \circ \mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}^{(\eta)}(\mathscr{S}_{\mathfrak{g}}) \quad \big(\forall(\delta,\eta) : \ (1,1) \preccurlyeq (\delta,\eta)\big)$,
- II. $\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}^{(\delta+\eta)}(\mathscr{S}_{\mathfrak{g}}) = \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}^{(\delta)} \circ \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}^{(\eta)}(\mathscr{S}_{\mathfrak{g}}) \quad \big(\forall(\delta,\eta) : \ (1,1) \preccurlyeq (\delta,\eta)\big)$,

where $(\delta,\eta) \prec (\lambda,\lambda)$.

PROOF. Let $\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}$, $\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ be a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-derived and a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-coderived operators, respectively, and let $\mathscr{S}_{\mathfrak{g}} \in \mathscr{P}(\Omega)$ be arbitrary in a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$. Then:

I. Introduce $\mathbb{B} = \{0,1\}$ as Boolean domain and introduce the Boolean-valued propositional formula

$$\mathbb{B} \ni \mathrm{P}(\delta,\eta) \stackrel{\mathrm{def}}{\longleftrightarrow} \mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}^{(\delta+\eta)}(\mathscr{S}_{\mathfrak{g}}) = \mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}^{(\delta)} \circ \mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}^{(\eta)}(\mathscr{S}_{\mathfrak{g}}) \quad \big(\forall(\delta,\eta) : \ (1,1) \preccurlyeq (\delta,\eta) \prec (\lambda,\lambda)\big).$$

Then, to prove ITEM I., it only suffices to prove that,

$$\big(\forall(\delta,\eta) : \ (1,1) \preccurlyeq (\delta,\eta) \prec (\lambda,\lambda)\big) \\ \big[\big(\mathrm{P}(1,1) = 1\big) \wedge \big(\mathrm{P}(\delta,\eta) = 1 \ \longrightarrow \ \mathrm{P}(\delta+1,\eta+1) = 1\big)\big].$$

Case (i.) Let $(1,1) = (\delta,\eta)$. Then,

$$\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}^{(2)}(\mathscr{S}_{\mathfrak{g}}) \longleftrightarrow \mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}^{(1+1)}(\mathscr{S}_{\mathfrak{g}}) = \mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}^{(1)} \circ \mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}^{(1)}(\mathscr{S}_{\mathfrak{g}}) \longleftrightarrow \mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}^{(2)}(\mathscr{S}_{\mathfrak{g}}),$$

implying $\mathrm{P}(1,1) = 1$. The base case therefore holds.

Case (ii.) Let $(1,1) \prec (\delta,\eta) \prec (\lambda,\lambda)$ and assume that the inductive hypothesis $\mathrm{P}(\delta,\eta) = 1$ holds true. Then, $\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}^{(\delta+\eta)}(\mathscr{S}_{\mathfrak{g}}) = \mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}^{(\delta)} \circ \mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}^{(\eta)}(\mathscr{S}_{\mathfrak{g}})$ and consequently, $\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}^{(2)} \circ \mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}^{(\delta+\eta)}(\mathscr{S}_{\mathfrak{g}}) = \mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}^{(2)} \circ \mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}^{(\delta)} \circ \mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}^{(\eta)}(\mathscr{S}_{\mathfrak{g}})$. But,

$\mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(2)} \circ \mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta+\eta)}(\mathscr{S}_{\mathfrak{g}}) \longleftrightarrow \mathfrak{g}\text{-Der}_{\mathfrak{g}}^{((\delta+1)+(\eta+1))}(\mathscr{S}_{\mathfrak{g}})$ and,

$$\begin{gathered}\mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(2)} \circ \mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta)} \circ \mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\eta)}(\mathscr{S}_{\mathfrak{g}}) \\ \updownarrow \\ \mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(2)} \circ \mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta-1)} \circ \mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(1)} \circ \mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\eta)}(\mathscr{S}_{\mathfrak{g}}) \\ \updownarrow \\ \mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta+1)} \circ \mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\eta+1)}(\mathscr{S}_{\mathfrak{g}}).\end{gathered}$$

Hence, it follows that $\mathfrak{g}\text{-Der}_{\mathfrak{g}}^{((\delta+1)+(\eta+1))}(\mathscr{S}_{\mathfrak{g}}) = \mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta+1)} \circ \mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\eta+1)}(\mathscr{S}_{\mathfrak{g}})$, implying $\mathrm{P}(\delta+1, \eta+1) = 1$. The inductive case therefore holds.

Suppose $\mathrm{P}(\delta, \eta) = 1$ holds for all (δ, η) such that $(1,1) \prec (\delta, \eta) \prec (\lambda, \lambda)$. Then,

$$\begin{array}{ccc}\displaystyle\bigcap_{\delta+\eta\prec\lambda+\lambda} \mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta+\eta)}(\mathscr{S}_{\mathfrak{g}}) & = & \displaystyle\bigcap_{\delta+\eta\prec\lambda+\lambda} \mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta)} \circ \mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\eta)}(\mathscr{S}_{\mathfrak{g}}) \\ & \updownarrow & \\ \mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\lambda+\lambda)}(\mathscr{S}_{\mathfrak{g}}) & = & \mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\lambda)} \circ \mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\lambda)}(\mathscr{S}_{\mathfrak{g}}),\end{array}$$

from which $\mathrm{P}(\lambda, \lambda) = 1$ follows.

II. Introduce $\mathbb{B} = \{0,1\}$ as Boolean domain and introduce the Boolean-valued propositional formula

$$\mathbb{B} \ni \mathrm{Q}(\delta, \eta) \stackrel{\text{def}}{\longleftrightarrow} \mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\delta+\eta)}(\mathscr{S}_{\mathfrak{g}}) = \mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\delta)} \circ \mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\eta)}(\mathscr{S}_{\mathfrak{g}}) \\ \big(\forall(\delta, \eta) : \ (1,1) \preccurlyeq (\delta, \eta) \prec (\lambda, \lambda)\big).$$

Then, to prove ITEM II., it only suffices to prove that,

$$\big(\forall(\delta, \eta) : \ (1,1) \preccurlyeq (\delta, \eta) \prec (\lambda, \lambda)\big) \\ \big[\big(\mathrm{Q}(1,1) = 1\big) \wedge \big(\mathrm{Q}(\delta, \eta) = 1 \ \longrightarrow \ \mathrm{Q}(\delta+1, \eta+1) = 1\big)\big].$$

Case (i.) Let $(1,1) = (\delta, \eta)$. Then,

$$\mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(2)}(\mathscr{S}_{\mathfrak{g}}) \longleftrightarrow \mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(1+1)}(\mathscr{S}_{\mathfrak{g}}) = \mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(1)} \circ \mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(1)}(\mathscr{S}_{\mathfrak{g}}) \longleftrightarrow \mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(2)}(\mathscr{S}_{\mathfrak{g}}),$$

implying $\mathrm{Q}(1,1) = 1$. The base case therefore holds.

Case (ii.) Let $(1,1) \prec (\delta, \eta) \prec (\lambda, \lambda)$ and assume that the inductive hypothesis $\mathrm{Q}(\delta, \eta) = 1$ holds true. Then, $\mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\delta+\eta)}(\mathscr{S}_{\mathfrak{g}}) = \mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\delta)} \circ \mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\eta)}(\mathscr{S}_{\mathfrak{g}})$ and consequently, $\mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(2)} \circ \mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\delta+\eta)}(\mathscr{S}_{\mathfrak{g}}) = \mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(2)} \circ \mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\delta)} \circ \mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\eta)}(\mathscr{S}_{\mathfrak{g}})$. But,

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$\mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(2)} \circ \mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\delta+\eta)}(\mathscr{S}_{\mathfrak{g}}) \longleftrightarrow \mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{((\delta+1)+(\eta+1))}(\mathscr{S}_{\mathfrak{g}})$ and,

$$\begin{gathered}\mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(2)} \circ \mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\delta)} \circ \mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\eta)}(\mathscr{S}_{\mathfrak{g}}) \\ \updownarrow \\ \mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(2)} \circ \mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\delta-1)} \circ \mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(1)} \circ \mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\eta)}(\mathscr{S}_{\mathfrak{g}}) \\ \updownarrow \\ \mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\delta+1)} \circ \mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\eta+1)}(\mathscr{S}_{\mathfrak{g}}).\end{gathered}$$

Hence, it follows that $\mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{((\delta+1)+(\eta+1))}(\mathscr{S}_{\mathfrak{g}}) = \mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\delta+1)} \circ \mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\eta+1)}(\mathscr{S}_{\mathfrak{g}})$, implying $\mathrm{Q}(\delta+1,\eta+1)=1$. The inductive case therefore holds.

Suppose $\mathrm{Q}(\delta,\eta)=1$ holds for all (δ,η) such that $(1,1)\prec(\delta,\eta)\prec(\lambda,\lambda)$. Then,

$$\begin{gathered}\bigcap_{\delta+\eta\prec\lambda+\lambda} \mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\delta+\eta)}(\mathscr{S}_{\mathfrak{g}}) = \bigcap_{\delta+\eta\prec\lambda+\lambda} \mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\delta)} \circ \mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\eta)}(\mathscr{S}_{\mathfrak{g}}) \\ \updownarrow \\ \mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\lambda+\lambda)}(\mathscr{S}_{\mathfrak{g}}) = \mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\lambda)} \circ \mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\lambda)}(\mathscr{S}_{\mathfrak{g}}),\end{gathered}$$

from which $\mathrm{Q}(\lambda,\lambda)=1$ follows. The proof of the proposition is complete. Q.E.D.

The corollary stated below is an immediate consequence of the above proposition.

Corollary 3.48. *If* $\mathfrak{g}\text{-}\mathbf{Dc}_{\mathfrak{g}} \in \mathfrak{g}\text{-DC}[\mathfrak{T}_{\mathfrak{g}}]$ *be a given pair of* $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$*-operators* $\mathfrak{g}\text{-Der}_{\mathfrak{g}}$, $\mathfrak{g}\text{-Cod}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ *and* $\mathbf{dc}_{\mathfrak{g}} \in \mathrm{DC}[\mathfrak{T}_{\mathfrak{g}}]$ *be a given pair of* $\mathfrak{T}_{\mathfrak{g}}$*-operators* $\mathrm{der}_{\mathfrak{g}}$, $\mathrm{cod}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$, *and let* $\mathscr{S}_{\mathfrak{g}} \in \mathscr{P}(\Omega)$ *be arbitrary in a* $\mathscr{T}_{\mathfrak{g}}$*-space* $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$*, then:*

- I. $\mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta+\eta)}(\mathscr{S}_{\mathfrak{g}}) \subseteq \mathrm{der}_{\mathfrak{g}}^{(\delta)} \circ \mathrm{der}_{\mathfrak{g}}^{(\eta)}(\mathscr{S}_{\mathfrak{g}}) \quad \big(\forall(\delta,\eta):\ (1,1)\preccurlyeq(\delta,\eta)\big)$,
- II. $\mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\delta+\eta)}(\mathscr{S}_{\mathfrak{g}}) \supseteq \mathrm{cod}_{\mathfrak{g}}^{(\delta)} \circ \mathrm{cod}_{\mathfrak{g}}^{(\eta)}(\mathscr{S}_{\mathfrak{g}}) \quad \big(\forall(\delta,\eta):\ (1,1)\preccurlyeq(\delta,\eta)\big)$,

where $(\delta,\eta)\prec(\lambda,\lambda)$.

For any (δ,η) such that $(1,1)\preccurlyeq(\delta,\eta)\prec(\lambda,\lambda)$, the $\delta\eta^{\text{th}}$-order $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-derived set operator is equivalent to the η^{th}-order of the δ^{th}-order of the $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-derived set operator; likewise, the $\delta\eta^{\text{th}}$-order $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-coderived set operator is equivalent to the η^{th}-order of the δ^{th}-order of the $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-coderived set operator. Accordingly, the following proposition presents itself.

Proposition 3.49. *Let* $\mathfrak{g}\text{-Der}_{\mathfrak{g}}$, $\mathfrak{g}\text{-Cod}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ *be a* $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$*-derived and a* $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$*-coderived operators, respectively, and let* $\mathscr{S}_{\mathfrak{g}} \in \mathscr{P}(\Omega)$ *be arbitrary in a* $\mathscr{T}_{\mathfrak{g}}$*-space* $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$*. Then:*

- I. $\mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta\eta)}(\mathscr{S}_{\mathfrak{g}}) = \big(\mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta)}\big)^{(\eta)}(\mathscr{S}_{\mathfrak{g}}) \ \big(\forall(\delta,\eta):\ (1,1)\preccurlyeq(\delta,\eta)\prec(\lambda,\lambda)\big)$,
- II. $\mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\delta\eta)}(\mathscr{S}_{\mathfrak{g}}) = \big(\mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\delta)}\big)^{(\eta)}(\mathscr{S}_{\mathfrak{g}}) \ \big(\forall(\delta,\eta):\ (1,1)\preccurlyeq(\delta,\eta)\prec(\lambda,\lambda)\big)$.

Proof. Let $\mathfrak{g}\text{-Der}_{\mathfrak{g}}$, $\mathfrak{g}\text{-Cod}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ be a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-derived and a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-coderived operators, respectively, and let $\mathscr{S}_{\mathfrak{g}} \in \mathscr{P}(\Omega)$ be arbitrary in a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$. Then:

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I. Introduce $\mathbb{B} = \{0,1\}$ as Boolean domain and introduce the Boolean-valued propositional formula

$$\mathbb{B} \ni \mathrm{P}(\eta) \overset{\text{def}}{\longleftrightarrow} \mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}^{(\delta\eta)}(\mathscr{S}_{\mathfrak{g}}) = \left(\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}^{(\delta)}\right)^{(\eta)}(\mathscr{S}_{\mathfrak{g}}) \quad (\forall \eta:\ 1 \preccurlyeq \eta \prec \lambda).$$

Then, to prove ITEM I., it only suffices to prove that,

$$(\forall \eta:\ 1 \preccurlyeq \eta \prec \lambda)\left[(\mathrm{P}(1) = 1) \wedge (\mathrm{P}(\eta) = 1 \longrightarrow \mathrm{P}(\eta+1) = 1)\right].$$

Case (*i.*) Let $1 = \eta$. Then,

$$\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}^{(\delta)}(\mathscr{S}_{\mathfrak{g}}) \longleftrightarrow \mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}^{(\delta\times 1)}(\mathscr{S}_{\mathfrak{g}}) = \left(\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}^{(\delta)}\right)^{(1)}(\mathscr{S}_{\mathfrak{g}}) \longleftrightarrow \mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}^{(\delta)}(\mathscr{S}_{\mathfrak{g}}),$$

implying $\mathrm{P}(1) = 1$. The base case therefore holds.

Case (*ii.*) Let $1 \prec \eta \prec \lambda$ and assume that the inductive hypothesis $\mathrm{P}(\eta) = 1$ holds true. Then, $\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}^{(\delta\eta)}(\mathscr{S}_{\mathfrak{g}}) = \left(\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}^{(\delta)}\right)^{(\eta)}(\mathscr{S}_{\mathfrak{g}})$ and consequently, it results that $\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}^{(\delta)} \circ \mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}^{(\delta\eta)}(\mathscr{S}_{\mathfrak{g}}) = \mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}^{(\delta)} \circ \left(\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}^{(\delta)}\right)^{(\eta)}(\mathscr{S}_{\mathfrak{g}})$. But, the relation $\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}^{(\eta)} \circ \mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}^{(\delta\eta)}(\mathscr{S}_{\mathfrak{g}}) \longleftrightarrow \mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}^{(\delta(\eta+1))}(\mathscr{S}_{\mathfrak{g}})$ holds true and on the other hand, the relation $\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}^{(\delta)} \circ \left(\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}^{(\delta)}\right)^{(\eta)}(\mathscr{S}_{\mathfrak{g}}) \longleftrightarrow \left(\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}^{(\delta)}\right)^{(\eta+1)}(\mathscr{S}_{\mathfrak{g}})$ also holds true. Hence, it follows that $\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}^{(\delta(\eta+1))}(\mathscr{S}_{\mathfrak{g}}) = \left(\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}^{(\delta)}\right)^{(\eta+1)}(\mathscr{S}_{\mathfrak{g}})$, implying $\mathrm{P}(\eta+1) = 1$. The inductive case therefore holds.

Suppose $\mathrm{P}(\delta,\eta) = 1$ holds for all (δ,η) such that $(1,1) \prec (\delta,\eta) \prec (\lambda,\lambda)$. Then,

$$\begin{array}{ccc}
\bigcap_{\delta\prec\lambda}\left(\bigcap_{\eta\prec\lambda} \mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}^{(\delta\eta)}(\mathscr{S}_{\mathfrak{g}})\right) & = & \bigcap_{\delta\prec\lambda}\left(\bigcap_{\eta\prec\lambda} \left(\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}^{(\delta)}\right)^{(\eta)}(\mathscr{S}_{\mathfrak{g}})\right) \\
& \updownarrow & \\
\bigcap_{\delta\prec\lambda} \mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}^{(\delta\lambda)}(\mathscr{S}_{\mathfrak{g}}) & = & \bigcap_{\delta\prec\lambda} \left(\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}^{(\delta)}\right)^{(\lambda)}(\mathscr{S}_{\mathfrak{g}}) \\
& \updownarrow & \\
\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}^{(\lambda\times\lambda)}(\mathscr{S}_{\mathfrak{g}}) & = & \left(\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}^{(\lambda)}\right)^{(\lambda)}(\mathscr{S}_{\mathfrak{g}})
\end{array}$$

from which $\mathrm{P}(\lambda,\lambda) = 1$ follows.

II. Introduce $\mathbb{B} = \{0,1\}$ as Boolean domain and introduce the Boolean-valued propositional formula

$$\mathbb{B} \ni \mathrm{Q}(\eta) \overset{\text{def}}{\longleftrightarrow} \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}^{(\delta\eta)}(\mathscr{S}_{\mathfrak{g}}) = \left(\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}^{(\delta)}\right)^{(\eta)}(\mathscr{S}_{\mathfrak{g}}) \quad (\forall \eta:\ 1 \preccurlyeq \eta \prec \lambda).$$

Then, to prove ITEM II., it only suffices to prove that,

$$(\forall \eta:\ 1 \preccurlyeq \eta \prec \lambda)\left[(\mathrm{Q}(1) = 1) \wedge (\mathrm{Q}(\eta) = 1 \longrightarrow \mathrm{Q}(\eta+1) = 1)\right].$$

Case (*i.*) Let $1 = \eta$. Then,

$$\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}^{(\delta)}(\mathscr{S}_{\mathfrak{g}}) \longleftrightarrow \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}^{(\delta\times 1)}(\mathscr{S}_{\mathfrak{g}}) = \left(\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}^{(\delta)}\right)^{(1)}(\mathscr{S}_{\mathfrak{g}}) \longleftrightarrow \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}^{(\delta)}(\mathscr{S}_{\mathfrak{g}}),$$

implying $\mathrm{Q}(1) = 1$. The base case therefore holds.

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Case (ii.) Let $1 \prec \eta \prec \lambda$ and assume that the inductive hypothesis $\mathrm{Q}(\eta) = 1$ holds true. Then, $\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}^{(\delta\eta)}(\mathscr{S}_{\mathfrak{g}}) = \left(\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}^{(\delta)}\right)^{(\eta)}(\mathscr{S}_{\mathfrak{g}})$ and consequently, it results that $\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}^{(\delta)} \circ \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}^{(\delta\eta)}(\mathscr{S}_{\mathfrak{g}}) = \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}^{(\delta)} \circ \left(\left(\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}^{(\delta)}\right)^{(\eta)}(\mathscr{S}_{\mathfrak{g}})\right)$. But, the relation $\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}^{(\eta)} \circ \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}^{(\delta\eta)}(\mathscr{S}_{\mathfrak{g}}) \longleftrightarrow \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}^{(\delta(\eta+1))}(\mathscr{S}_{\mathfrak{g}})$ holds true and on the other hand, the relation $\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}^{(\delta)} \circ \left(\left(\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}^{(\delta)}\right)^{(\eta)}(\mathscr{S}_{\mathfrak{g}})\right) \longleftrightarrow \left(\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}^{(\delta)}\right)^{(\eta+1)}(\mathscr{S}_{\mathfrak{g}})$ also holds true. Hence, it follows that $\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}^{(\delta(\eta+1))}(\mathscr{S}_{\mathfrak{g}}) = \left(\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}^{(\delta)}\right)^{(\eta+1)}(\mathscr{S}_{\mathfrak{g}})$, implying $\mathrm{Q}(\eta+1) = 1$. The inductive case therefore holds.

Suppose $\mathrm{Q}(\delta,\eta) = 1$ holds for all (δ,η) such that $(1,1) \prec (\delta,\eta) \prec (\lambda,\lambda)$. Then,

$$\begin{array}{ccc}
\bigcap_{\delta\prec\lambda}\left(\bigcap_{\eta\prec\lambda}\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}^{(\delta\eta)}(\mathscr{S}_{\mathfrak{g}})\right) & = & \bigcap_{\delta\prec\lambda}\left(\bigcap_{\eta\prec\lambda}\left(\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}^{(\delta)}\right)^{(\eta)}(\mathscr{S}_{\mathfrak{g}})\right) \\
& \updownarrow & \\
\bigcap_{\delta\prec\lambda}\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}^{(\delta\lambda)}(\mathscr{S}_{\mathfrak{g}}) & = & \bigcap_{\delta\prec\lambda}\left(\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}^{(\delta)}\right)^{(\lambda)}(\mathscr{S}_{\mathfrak{g}}) \\
& \updownarrow & \\
\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}^{(\lambda\times\lambda)}(\mathscr{S}_{\mathfrak{g}}) & = & \left(\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}^{(\lambda)}\right)^{(\lambda)}(\mathscr{S}_{\mathfrak{g}})
\end{array}$$

from which $\mathrm{Q}(\lambda,\lambda) = 1$ follows. The proof of the proposition is complete. Q.E.D.

An immediate consequence of the above proposition is the following corollary.

COROLLARY 3.50. *If* $\mathfrak{g}\text{-}\mathbf{Dc}_{\mathfrak{g}} \in \mathfrak{g}\text{-}\mathrm{DC}[\mathfrak{T}_{\mathfrak{g}}]$ *be a given pair of* $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$*-operators* $\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}$, $\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ *and* $\mathbf{dc}_{\mathfrak{g}} \in \mathrm{DC}[\mathfrak{T}_{\mathfrak{g}}]$ *be a given pair of* $\mathfrak{T}_{\mathfrak{g}}$*-operators* $\mathrm{der}_{\mathfrak{g}}$, $\mathrm{cod}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$, *and let* $\mathscr{S}_{\mathfrak{g}} \in \mathscr{P}(\Omega)$ *be arbitrary in a* $\mathscr{T}_{\mathfrak{g}}$*-space* $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$, *then:*

- I. $\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}^{(\delta\eta)}(\mathscr{S}_{\mathfrak{g}}) \subseteq \left(\mathrm{der}_{\mathfrak{g}}^{(\delta)}\right)^{(\eta)}(\mathscr{S}_{\mathfrak{g}}) \quad \left(\forall(\delta,\eta) : \ (1,1) \preccurlyeq (\delta,\eta) \prec (\lambda,\lambda)\right)$,
- II. $\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}^{(\delta\eta)}(\mathscr{S}_{\mathfrak{g}}) \supseteq \left(\mathrm{cod}_{\mathfrak{g}}^{(\delta)}\right)^{(\eta)}(\mathscr{S}_{\mathfrak{g}}) \quad \left(\forall(\delta,\eta) : \ (1,1) \preccurlyeq (\delta,\eta) \prec (\lambda,\lambda)\right)$.

For any δ such that $1 \preccurlyeq \delta \prec \lambda$, the union of a $\mathfrak{T}_{\mathfrak{g}}$-set and its $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-derived set includes the image of the $\mathfrak{T}_{\mathfrak{g}}$-set under the δ^{th}-order $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-derived set operator composition with itself; the intersection of a $\mathfrak{T}_{\mathfrak{g}}$-set and its $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-coderived set is included in the image of the $\mathfrak{T}_{\mathfrak{g}}$-set under the δ^{th}-order $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-coderived set operator composition with itself. These are embodied in the following theorem.

THEOREM 3.51. *Let* $\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}$, $\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ *be a* $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$*-derived and a* $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$*-coderived operators, respectively, and let* $\mathscr{S}_{\mathfrak{g}} \in \mathscr{P}(\Omega)$ *be arbitrary in a* $\mathscr{T}_{\mathfrak{g}}$*-space* $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$. *Then:*

- I. $\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}^{(\delta)} \circ \mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}^{(\delta)}(\mathscr{S}_{\mathfrak{g}}) \subseteq \mathscr{S}_{\mathfrak{g}} \cup \mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \quad (\forall\delta : \ 1 \preccurlyeq \delta \prec \lambda)$,
- II. $\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}^{(\delta)} \circ \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}^{(\delta)}(\mathscr{S}_{\mathfrak{g}}) \supseteq \mathscr{S}_{\mathfrak{g}} \cap \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \quad (\forall\delta : \ 1 \preccurlyeq \delta \prec \lambda)$.

PROOF. Let $\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}$, $\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ be a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-derived and a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-coderived operators, respectively, and let $\mathscr{S}_{\mathfrak{g}} \in \mathscr{P}(\Omega)$ be arbitrary in a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$. Then:

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I. Introduce $\mathbb{B}=\{0,1\}$ as Boolean domain and introduce the Boolean-valued propositional formula

$$\mathbb{B}\ni \mathrm{P}(\delta) \overset{\text{def}}{\longleftrightarrow} \mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta)}\circ\mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta)}(\mathscr{S}_{\mathfrak{g}})\subseteq\mathscr{S}_{\mathfrak{g}}\cup\mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \quad (\forall\delta:\ 1\preccurlyeq\delta\prec\lambda).$$

Then, to prove Item I., it only suffices to prove that,

$$(\forall\delta:\ 1\preccurlyeq\delta\prec\lambda)\big[\big(\mathrm{P}(1)=1\big)\wedge\big(\mathrm{P}(\delta)=1\ \longrightarrow\ \mathrm{P}(\delta+1)=1\big)\big].$$

Case (i.) Let $1=\delta$. Then, $\mathfrak{g}\text{-Der}_{\mathfrak{g}}\circ\mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})\subseteq\mathscr{S}_{\mathfrak{g}}\cup\mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$ holds true, implying $\mathrm{P}(1)=1$. The base case therefore holds.

Case (ii.) Let $1\prec\delta\prec\lambda$ and assume that the inductive hypothesis $\mathrm{P}(\delta)=1$ holds true. Then, $\mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta)}\circ\mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta)}(\mathscr{S}_{\mathfrak{g}})\subseteq\mathscr{S}_{\mathfrak{g}}\cup\mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$ and consequently, it follows that $\mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(2)}\circ\mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta)}\circ\mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta)}(\mathscr{S}_{\mathfrak{g}})\subseteq\mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(2)}\big(\mathscr{S}_{\mathfrak{g}}\cup\mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})\big)$. But, $\mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(2)}\circ\mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta)}\circ\mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta)}(\mathscr{S}_{\mathfrak{g}})\longleftrightarrow\mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta+1)}\circ\mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta+1)}(\mathscr{S}_{\mathfrak{g}})$ and, on the other hand, $\mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(2)}\big(\mathscr{S}_{\mathfrak{g}}\cup\mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})\big)\subseteq\mathfrak{g}\text{-Der}_{\mathfrak{g}}\big(\mathscr{S}_{\mathfrak{g}}\cup\mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})\big)\subseteq\mathscr{S}_{\mathfrak{g}}\cup\mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$. Hence, $\mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta+1)}\circ\mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta+1)}(\mathscr{S}_{\mathfrak{g}})\subseteq\mathscr{S}_{\mathfrak{g}}\cup\mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$, implying $\mathrm{P}(\delta+1)=1$. The inductive case therefore holds.

Suppose $\mathrm{P}(\delta)=1$ holds for all δ such that $1\prec\delta\prec\lambda$. Then,

$$\begin{aligned}
\mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\lambda+1)}\circ\mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\lambda+1)}(\mathscr{S}_{\mathfrak{g}}) &\longleftrightarrow \mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(2)}\circ\mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\lambda)}\circ\mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\lambda)}(\mathscr{S}_{\mathfrak{g}})\\
&\longleftrightarrow \mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(2)}\circ\mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\lambda+\lambda)}(\mathscr{S}_{\mathfrak{g}})\\
&\longleftrightarrow \mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(2)}\Bigg(\bigcap_{\delta+\delta\prec\lambda+\lambda}\mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta+\delta)}(\mathscr{S}_{\mathfrak{g}})\Bigg)\\
&\subseteq \mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(2)}\circ\mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta+\delta)}(\mathscr{S}_{\mathfrak{g}})\\
&\longleftrightarrow \mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta+1)}\circ\mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta+1)}(\mathscr{S}_{\mathfrak{g}})\\
&\subseteq \mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta)}\circ\mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta)}(\mathscr{S}_{\mathfrak{g}})\subseteq\mathscr{S}_{\mathfrak{g}}\cup\mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})
\end{aligned}$$

from which $\mathrm{P}(\lambda)=1$ follows.

II. Introduce $\mathbb{B}=\{0,1\}$ as Boolean domain and introduce the Boolean-valued propositional formula

$$\mathbb{B}\ni \mathrm{Q}(\delta) \overset{\text{def}}{\longleftrightarrow} \mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\delta)}\circ\mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\delta)}(\mathscr{S}_{\mathfrak{g}})\supseteq\mathscr{S}_{\mathfrak{g}}\cap\mathfrak{g}\text{-Cod}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \quad (\forall\delta:\ 1\preccurlyeq\delta\prec\lambda).$$

Then, to prove Item II., it only suffices to prove that,

$$(\forall\delta:\ 1\preccurlyeq\delta\prec\lambda)\big[\big(\mathrm{Q}(1)=1\big)\wedge\big(\mathrm{Q}(\delta)=1\ \longrightarrow\ \mathrm{Q}(\delta+1)=1\big)\big].$$

Case (i.) Let $1=\delta$. Then, $\mathfrak{g}\text{-Cod}_{\mathfrak{g}}\circ\mathfrak{g}\text{-Cod}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})\supseteq\mathscr{S}_{\mathfrak{g}}\cap\mathfrak{g}\text{-Cod}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$ holds true, implying $\mathrm{Q}(1)=1$. The base case therefore holds.

Case (ii.) Let $1\prec\delta\prec\lambda$ and assume that the inductive hypothesis $\mathrm{Q}(\delta)=1$ holds true. Then, $\mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\delta)}\circ\mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\delta)}(\mathscr{S}_{\mathfrak{g}})\supseteq\mathscr{S}_{\mathfrak{g}}\cap\mathfrak{g}\text{-Cod}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$ and consequently, it follows that $\mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(2)}\circ\mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\delta)}\circ\mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\delta)}(\mathscr{S}_{\mathfrak{g}})\supseteq\mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(2)}\big(\mathscr{S}_{\mathfrak{g}}\cap\mathfrak{g}\text{-Cod}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})\big)$. But, $\mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(2)}\circ\mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\delta)}\circ\mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\delta)}(\mathscr{S}_{\mathfrak{g}})\longleftrightarrow\mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\delta+1)}\circ\mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\delta+1)}(\mathscr{S}_{\mathfrak{g}})$ and, on the other hand, $\mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(2)}\big(\mathscr{S}_{\mathfrak{g}}\cap\mathfrak{g}\text{-Cod}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})\big)\supseteq\mathfrak{g}\text{-Cod}_{\mathfrak{g}}\big(\mathscr{S}_{\mathfrak{g}}\cap\mathfrak{g}\text{-Cod}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})\big)\supseteq\mathscr{S}_{\mathfrak{g}}\cap$

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$\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$. Hence, $\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}^{(\delta+1)}\circ\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}^{(\delta+1)}(\mathscr{S}_{\mathfrak{g}})\supseteq\mathscr{S}_{\mathfrak{g}}\cap\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$, implying $\mathrm{Q}(\delta+1)=1$. The inductive case therefore holds.

Suppose $\mathrm{Q}(\delta)=1$ holds for all δ such that $1\prec\delta\prec\lambda$. Then,

$$\begin{aligned}
\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}^{(\lambda+1)}\circ\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}^{(\lambda+1)}(\mathscr{S}_{\mathfrak{g}}) &\longleftrightarrow \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}^{(2)}\circ\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}^{(\lambda)}\circ\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}^{(\lambda)}(\mathscr{S}_{\mathfrak{g}})\\
&\longleftrightarrow \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}^{(2)}\circ\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}^{(\lambda+\lambda)}(\mathscr{S}_{\mathfrak{g}})\\
&\longleftrightarrow \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}^{(2)}\Bigg(\bigcap_{\delta+\delta\prec\lambda+\lambda}\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}^{(\delta+\delta)}(\mathscr{S}_{\mathfrak{g}})\Bigg)\\
&\subseteq \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}^{(2)}\circ\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}^{(\delta+\delta)}(\mathscr{S}_{\mathfrak{g}})\\
&\longleftrightarrow \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}^{(\delta+1)}\circ\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}^{(\delta+1)}(\mathscr{S}_{\mathfrak{g}})\\
&\supseteq \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}^{(\delta)}\circ\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}^{(\delta)}(\mathscr{S}_{\mathfrak{g}})\\
&\supseteq \mathscr{S}_{\mathfrak{g}}\cap\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})
\end{aligned}$$

from which $\mathrm{Q}(\lambda)=1$ follows. The proof of the theorem is complete. Q.E.D.

The following corollary is an immediate consequence of the above theorem.

COROLLARY 3.52. *If* $\mathfrak{g}\text{-}\mathbf{Dc}_{\mathfrak{g}}\in\mathfrak{g}\text{-}\mathrm{DC}[\mathfrak{T}_{\mathfrak{g}}]$ *be a given pair of* $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$*-operators* $\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}$, $\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}:\mathscr{P}(\Omega)\longrightarrow\mathscr{P}(\Omega)$ *and* $\mathbf{dc}_{\mathfrak{g}}\in\mathrm{DC}[\mathfrak{T}_{\mathfrak{g}}]$ *be a given pair of* $\mathfrak{T}_{\mathfrak{g}}$*-operators* $\mathrm{der}_{\mathfrak{g}}$, $\mathrm{cod}_{\mathfrak{g}}:\mathscr{P}(\Omega)\longrightarrow\mathscr{P}(\Omega)$, *and let* $\mathscr{S}_{\mathfrak{g}}\in\mathscr{P}(\Omega)$ *be arbitrary in a* $\mathscr{T}_{\mathfrak{g}}$*-space* $\mathfrak{T}_{\mathfrak{g}}=(\Omega,\mathscr{T}_{\mathfrak{g}})$, *then:*

- I. $\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}^{(\delta)}\circ\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}^{(\delta)}(\mathscr{S}_{\mathfrak{g}})\subseteq\mathscr{S}_{\mathfrak{g}}\cup\mathrm{der}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})\quad(\forall\delta:\ 1\preccurlyeq\delta\prec\lambda)$,
- II. $\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}^{(\delta)}\circ\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}^{(\delta)}(\mathscr{S}_{\mathfrak{g}})\supseteq\mathscr{S}_{\mathfrak{g}}\cap\mathrm{cod}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})\quad(\forall\delta:\ 1\preccurlyeq\delta\prec\lambda)$.

For any δ such $1\preccurlyeq\delta\prec\lambda$, the image of a $\mathfrak{T}_{\mathfrak{g}}$-set under the δ^{th}-order $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-derived operator is equivalent to the image of the relative complement of any $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-derived unit set in the $\mathfrak{T}_{\mathfrak{g}}$-set under the δ^{th}-order $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-derived operator; the image of the $\mathfrak{T}_{\mathfrak{g}}$-set under the δ^{th}-order $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-coderived operator is equivalent to the image of the union of the $\mathfrak{T}_{\mathfrak{g}}$-set and any $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-coderived unit set under the δ^{th}-order $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-coderived operator. The theorem follows.

THEOREM 3.53. *Let* $\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}$, $\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}:\mathscr{P}(\Omega)\longrightarrow\mathscr{P}(\Omega)$ *be a* $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$*-derived and a* $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$*-coderived operators, respectively, and let* $(\{\xi\},\mathscr{S}_{\mathfrak{g}})\in\times_{\alpha\in I_2^*}\mathscr{P}(\Omega)$ *be arbitrary in a* $\mathscr{T}_{\mathfrak{g}}$*-space* $\mathfrak{T}_{\mathfrak{g}}=(\Omega,\mathscr{T}_{\mathfrak{g}})$. *Then:*

- I. $\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}^{(\delta)}(\mathscr{S}_{\mathfrak{g}})=\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}^{(\delta)}(\mathscr{S}_{\mathfrak{g}}\cap\mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}(\{\xi\}))\quad(\forall\delta:\ 1\preccurlyeq\delta\prec\lambda)$,
- II. $\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}^{(\delta)}(\mathscr{S}_{\mathfrak{g}})=\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}^{(\delta)}(\mathscr{S}_{\mathfrak{g}}\cup\{\xi\})\quad(\forall\delta:\ 1\preccurlyeq\delta\prec\lambda)$.

PROOF. Let $\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}$, $\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}:\mathscr{P}(\Omega)\longrightarrow\mathscr{P}(\Omega)$ be a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-derived and a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-coderived operators, respectively, and let $(\{\xi\},\mathscr{S}_{\mathfrak{g}})\in\times_{\alpha\in I_2^*}\mathscr{P}(\Omega)$ be arbitrary in a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}}=(\Omega,\mathscr{T}_{\mathfrak{g}})$. Then:

I. Introduce $\mathbb{B}=\{0,1\}$ as Boolean domain and introduce the Boolean-valued propositional formula

$$\mathbb{B}\ni\mathrm{P}(\delta)\ \stackrel{\mathrm{def}}{\longleftrightarrow}\ \mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}^{(\delta)}(\mathscr{S}_{\mathfrak{g}})=\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}^{(\delta)}(\mathscr{S}_{\mathfrak{g}}\cap\mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}(\{\xi\}))\quad(\forall\delta:\ 1\preccurlyeq\delta\prec\lambda).$$

Then, to prove ITEM I., it only suffices to prove that,

$$(\forall\delta:\ 1\preccurlyeq\delta\prec\lambda)\big[(\mathrm{P}(1)=1)\wedge(\mathrm{P}(\delta)=1\ \longrightarrow\ \mathrm{P}(\delta+1)=1)\big].$$

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Case (*i.*) Let $1=\delta$. Then, $\mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) = \mathfrak{g}\text{-Der}_{\mathfrak{g}}\big(\mathscr{S}_{\mathfrak{g}} \cap \mathfrak{g}\text{-Op}_{\mathfrak{g}}(\{\xi\})\big)$ holds true, implying $\mathrm{P}(1)=1$. The base case therefore holds.

Case (*ii.*) Let $1 \prec \delta \prec \lambda$ and assume that the inductive hypothesis $\mathrm{P}(\delta)=1$ holds true. Then, $\mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta)}(\mathscr{S}_{\mathfrak{g}}) = \mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta)}\big(\mathscr{S}_{\mathfrak{g}} \cap \mathfrak{g}\text{-Op}_{\mathfrak{g}}(\{\xi\})\big)$ and consequently, it follows that $\mathfrak{g}\text{-Der}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta)}(\mathscr{S}_{\mathfrak{g}}) = \mathfrak{g}\text{-Der}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta)}\big(\mathscr{S}_{\mathfrak{g}} \cap \mathfrak{g}\text{-Op}_{\mathfrak{g}}(\{\xi\})\big)$. But, $\mathfrak{g}\text{-Der}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta)}(\mathscr{S}_{\mathfrak{g}}) \longleftrightarrow \mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta+1)}(\mathscr{S}_{\mathfrak{g}})$ and, on the other hand, the relation $\mathfrak{g}\text{-Der}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta)}\big(\mathscr{S}_{\mathfrak{g}} \cap \mathfrak{g}\text{-Op}_{\mathfrak{g}}(\{\xi\})\big) \longleftrightarrow \mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta+1)}\big(\mathscr{S}_{\mathfrak{g}} \cap \mathfrak{g}\text{-Op}_{\mathfrak{g}}(\{\xi\})\big)$ also holds true. Hence, $\mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta+1)}(\mathscr{S}_{\mathfrak{g}}) = \mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta+1)}\big(\mathscr{S}_{\mathfrak{g}} \cap \mathfrak{g}\text{-Op}_{\mathfrak{g}}(\{\xi\})\big)$, implying $\mathrm{P}(\delta+1)=1$. The inductive case therefore holds.

Suppose $\mathrm{P}(\delta)=1$ holds for all δ such that $1 \prec \delta \prec \lambda$. Then,

$$\begin{array}{ccc} \bigcap_{\delta\prec\lambda} \mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta)}(\mathscr{S}_{\mathfrak{g}}) & = & \bigcap_{\delta\prec\lambda} \mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta)}\big(\mathscr{S}_{\mathfrak{g}} \cap \mathfrak{g}\text{-Op}_{\mathfrak{g}}(\{\xi\})\big) \\ & \updownarrow & \\ \mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\lambda)}(\mathscr{S}_{\mathfrak{g}}) & = & \mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\lambda)}\big(\mathscr{S}_{\mathfrak{g}} \cap \mathfrak{g}\text{-Op}_{\mathfrak{g}}(\{\xi\})\big) \end{array}$$

from which $\mathrm{P}(\lambda)=1$ follows.

II. Introduce $\mathbb{B}=\{0,1\}$ as Boolean domain and introduce the Boolean-valued propositional formula

$$\mathbb{B} \ni \mathrm{Q}(\delta) \quad \stackrel{\text{def}}{\longleftrightarrow} \quad \mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\delta)}\big(\mathscr{S}_{\mathfrak{g}}\big) = \mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\delta)}\big(\mathscr{S}_{\mathfrak{g}} \cup \{\xi\}\big) \quad (\forall \delta:\ 1 \preccurlyeq \delta \prec \lambda).$$

Then, to prove ITEM II., it only suffices to prove that,

$$\big(\forall \delta:\ 1 \preccurlyeq \delta \prec \lambda\big)\big[\big(\mathrm{Q}(1)=1\big) \wedge \big(\mathrm{Q}(\delta)=1 \ \longrightarrow\ \mathrm{Q}(\delta+1)=1\big)\big].$$

Case (*i.*) Let $1=\delta$. Then, $\mathfrak{g}\text{-Cod}_{\mathfrak{g}}\big(\mathscr{S}_{\mathfrak{g}}\big) = \mathfrak{g}\text{-Cod}_{\mathfrak{g}}\big(\mathscr{S}_{\mathfrak{g}} \cup \{\xi\}\big)$ holds true, implying $\mathrm{Q}(1)=1$. The base case therefore holds.

Case (*ii.*) Let $1 \prec \delta \prec \lambda$ and assume that the inductive hypothesis $\mathrm{Q}(\delta)=1$ holds true. Then, $\mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\delta)}\big(\mathscr{S}_{\mathfrak{g}}\big) = \mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\delta)}\big(\mathscr{S}_{\mathfrak{g}} \cup \{\xi\}\big)$ and consequently, it follows that $\mathfrak{g}\text{-Cod}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\delta)}\big(\mathscr{S}_{\mathfrak{g}}\big) = \mathfrak{g}\text{-Cod}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\delta)}\big(\mathscr{S}_{\mathfrak{g}} \cup \{\xi\}\big)$. But on the one hand, $\mathfrak{g}\text{-Cod}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\delta)}\big(\mathscr{S}_{\mathfrak{g}}\big) \longleftrightarrow \mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\delta+1)}\big(\mathscr{S}_{\mathfrak{g}}\big)$ and, on the other hand, the relation $\mathfrak{g}\text{-Cod}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\delta)}\big(\mathscr{S}_{\mathfrak{g}} \cup \{\xi\}\big) \longleftrightarrow \mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\delta+1)}\big(\mathscr{S}_{\mathfrak{g}} \cup \{\xi\}\big)$ also holds true. Hence, $\mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\delta+1)}\big(\mathscr{S}_{\mathfrak{g}}\big) = \mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\delta+1)}\big(\mathscr{S}_{\mathfrak{g}} \cup \{\xi\}\big)$, implying $\mathrm{Q}(\delta+1)=1$. The inductive case therefore holds.

Suppose $\mathrm{Q}(\delta)=1$ holds for all δ such that $1 \prec \delta \prec \lambda$. Then,

$$\begin{array}{ccc} \bigcap_{\delta\prec\lambda} \mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\delta)}(\mathscr{S}_{\mathfrak{g}}) & = & \bigcap_{\delta\prec\lambda} \mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\delta)}\big(\mathscr{S}_{\mathfrak{g}} \cup \{\xi\}\big) \\ & \updownarrow & \\ \mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\lambda)}(\mathscr{S}_{\mathfrak{g}}) & = & \mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\lambda)}\big(\mathscr{S}_{\mathfrak{g}} \cup \{\xi\}\big) \end{array}$$

from which $\mathrm{Q}(\lambda)=1$ follows. The proof of the theorem is complete. Q.E.D.

The corollary stated below is an immediate consequence of the above theorem.

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COROLLARY 3.54. *If* $\mathfrak{g}\text{-}\mathbf{Dc}_{\mathfrak{g}} \in \mathfrak{g}\text{-DC}[\mathfrak{T}_{\mathfrak{g}}]$ *be a given pair of* $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$*-operators* $\mathfrak{g}\text{-Der}_{\mathfrak{g}}$, $\mathfrak{g}\text{-Cod}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ *and* $\mathbf{dc}_{\mathfrak{g}} \in \mathrm{DC}[\mathfrak{T}_{\mathfrak{g}}]$ *be a given pair of* $\mathfrak{T}_{\mathfrak{g}}$*-operators* $\mathrm{der}_{\mathfrak{g}}$, $\mathrm{cod}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$, *and let* $\left(\{\xi\}, \mathscr{S}_{\mathfrak{g}}\right) \in \times_{\alpha \in I_2^*} \mathscr{P}(\Omega)$ *be arbitrary in a* $\mathscr{T}_{\mathfrak{g}}$*-space* $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$*, then:*

- I. $\mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta)}(\mathscr{S}_{\mathfrak{g}}) \subseteq \mathrm{der}_{\mathfrak{g}}^{(\delta)}\left(\mathscr{S}_{\mathfrak{g}} \cap \mathfrak{g}\text{-Op}_{\mathfrak{g}}(\{\xi\})\right) \quad (\forall \delta : 1 \preccurlyeq \delta \prec \lambda)$,
- II. $\mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\delta)}\left(\mathscr{S}_{\mathfrak{g}}\right) \supseteq \mathrm{cod}_{\mathfrak{g}}^{(\delta)}\left(\mathscr{S}_{\mathfrak{g}} \cup \{\xi\}\right) \quad (\forall \delta : 1 \preccurlyeq \delta \prec \lambda)$.

Our second research objective concerning the definitions and the essential properties of the concepts of δ^{th}-order derivative $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-derived and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-coderived operators defined by transfinite recursion on the class of successor ordinals in $\mathscr{T}_{\mathfrak{g}}$-spaces is now complete. Of the notions of the δ^{th}-iterates of the $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-derived and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-coderived operators in $\mathscr{T}_{\mathfrak{g}}$-spaces, we conclude the present section with two corollaries and two axiomatic definitions derived from these two corollaries.

The first corollary stated below gives the necessary and sufficient condition for a δ^{th}-order $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-derived operator to be a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-derived operator.

COROLLARY 3.55. *A necessary and sufficient condition for the* δ^{th}*-iterate* $\mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta)} : \mathscr{S}_{\mathfrak{g}} \in \mathscr{P}(\Omega) \longmapsto \mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta)}(\mathscr{S}_{\mathfrak{g}})$ *of* $\mathfrak{g}\text{-Der}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ *to be a* $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$*-derived operator in a strong* $\mathscr{T}_{\mathfrak{g}}$*-space* $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$ *is that, for every* $\left(\{\xi\}, \mathscr{R}_{\mathfrak{g}}, \mathscr{S}_{\mathfrak{g}}\right) \in \times_{\alpha \in I_3^*} \mathscr{P}(\Omega)$*, it satisfies:*

- I. $\mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta)}(\emptyset) = \emptyset \quad (\forall \delta : 1 \preccurlyeq \delta \prec \lambda)$,
- II. $\mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta)}(\mathscr{R}_{\mathfrak{g}}) = \mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta)}\left(\mathscr{R}_{\mathfrak{g}} \cap \mathfrak{g}\text{-Op}_{\mathfrak{g}}(\{\xi\})\right) \quad (\forall \delta : 1 \preccurlyeq \delta \prec \lambda)$,
- III. $\mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta)} \circ \mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta)}(\mathscr{R}_{\mathfrak{g}}) \subseteq \mathscr{R}_{\mathfrak{g}} \cup \mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}) \quad (\forall \delta : 1 \preccurlyeq \delta \prec \lambda)$,
- IV. $\mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta)}(\mathscr{R}_{\mathfrak{g}} \cup \mathscr{S}_{\mathfrak{g}}) = \bigcup_{\mathscr{U}_{\mathfrak{g}} = \mathscr{R}_{\mathfrak{g}}, \mathscr{S}_{\mathfrak{g}}} \mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta)}(\mathscr{U}_{\mathfrak{g}}) \quad (\forall \delta : 1 \preccurlyeq \delta \prec \lambda)$.

The second corollary stated below gives the necessary and sufficient condition for a δ^{th}-order $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-coderived operator to be a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-coderived operator.

COROLLARY 3.56. *A necessary and sufficient condition for the* δ^{th}*-iterate* $\mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\delta)} : \mathscr{S}_{\mathfrak{g}} \in \mathscr{P}(\Omega) \longmapsto \mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\delta)}(\mathscr{S}_{\mathfrak{g}})$ *of* $\mathfrak{g}\text{-Cod}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ *to be a* $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$*-coderived operator in a* $\mathscr{T}_{\mathfrak{g}}$*-space* $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$ *is that, for every* $\left(\{\xi\}, \mathscr{R}_{\mathfrak{g}}, \mathscr{S}_{\mathfrak{g}}\right) \in \times_{\alpha \in I_3^*} \mathscr{P}(\Omega)$*, it satisfies:*

- I. $\mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\delta)}(\Omega) = \Omega \quad (\forall \delta : 1 \preccurlyeq \delta \prec \lambda)$,
- II. $\mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\delta)}\left(\mathscr{R}_{\mathfrak{g}}\right) = \mathrm{cod}_{\mathfrak{g}}^{(\delta)}\left(\mathscr{R}_{\mathfrak{g}} \cup \{\xi\}\right) \quad (\forall \delta : 1 \preccurlyeq \delta \prec \lambda)$,
- III. $\mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\delta)} \circ \mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\delta)}(\mathscr{R}_{\mathfrak{g}}) \supseteq \mathscr{R}_{\mathfrak{g}} \cap \mathfrak{g}\text{-Cod}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}) \quad (\forall \delta : 1 \preccurlyeq \delta \prec \lambda)$,
- IV. $\mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\delta)}(\mathscr{R}_{\mathfrak{g}} \cap \mathscr{S}_{\mathfrak{g}}) = \bigcap_{\mathscr{U}_{\mathfrak{g}} = \mathscr{R}_{\mathfrak{g}}, \mathscr{S}_{\mathfrak{g}}} \mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\delta)}(\mathscr{U}_{\mathfrak{g}}) \quad (\forall \delta : 1 \preccurlyeq \delta \prec \lambda)$.

Hence, in a strong $\mathscr{T}_{\mathfrak{g}}$-space, for the δ^{th}-iterate of a set-valued map $\mathfrak{g}\text{-Der}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ on $\mathscr{P}(\Omega)$ ranging in $\mathscr{P}(\Omega)$ to be characterized as a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-derived operator it must necessarily and sufficiently satisfy a list of *derived set* $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$*-derived operator conditions* (ITEMS I.–IV. of COR. 3.55), and similarly, for the δ^{th}-iterate of a set-valued map $\mathfrak{g}\text{-Der}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ on $\mathscr{P}(\Omega)$ ranging in $\mathscr{P}(\Omega)$ to be characterized as a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-coderived operator it must necessarily and sufficiently satisfy a list of *derived set* $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$*-coderived operator conditions* (ITEMS V.–VIII. of COR. 3.56).

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Evidently, Items I., II., III. and IV. of Cor. 3.55 state that the δ^{th}-iterate of the $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-derived operator $\mathfrak{g}\text{-Der}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ is $\emptyset$-*grounded* (alternatively, $\emptyset$-*preserving*), ξ-*invariant* (alternatively, ξ-*unaffected*), $\mathfrak{g}\text{-Cl}_{\mathfrak{g}}$-*intensive* and $\cup$-*additive* (alternatively, $\cup$-*distributive*), respectively. On the other hand, Items I., II., III. and IV. of Cor. 3.56 state that the δ^{th}-iterate of the $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-coderived operator $\mathfrak{g}\text{-Cod}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ is Ω-*grounded* (alternatively, Ω-*preserving*), ζ-*invariant* (alternatively, ζ-*unaffected*), $\mathfrak{g}\text{-Int}_{\mathfrak{g}}$-*extensive* and $\cap$-*additive* (alternatively, $\cap$-*distributive*), respectively.

Viewing the δ^{th}-order derived set $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-derived operator conditions (Items I.–IV. of Cor. 3.55 above) as δ^{th}-*order* $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-*derived operator axioms*, the axiomatic definition of the concept of a δ^{th}-order $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-derived operator, then, can be defined as a δ^{th}-order set-valued map $\mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta)} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ on $\mathscr{P}(\Omega)$ ranging in $\mathscr{P}(\Omega)$ satisfying a list of δ^{th}-*order* $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-*derived operator axioms*. The axiomatic definition of the concept of a δ^{th}-order $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-derived operator in strong $\mathscr{T}_{\mathfrak{g}}$-spaces follows.

Definition 3.57 (Axiomatic Definition: $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-Derived Operator). The δ^{th}-iterate $\mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta)} : \mathscr{S}_{\mathfrak{g}} \in \mathscr{P}(\Omega) \longmapsto \mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta)}(\mathscr{S}_{\mathfrak{g}})$ of $\mathfrak{g}\text{-Der}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ is called a "$\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-derived operator of δ^{th} order" on $\mathscr{P}(\Omega)$ ranging in $\mathscr{P}(\Omega)$ for some ordinal δ such that $1 \preccurlyeq \delta \prec \lambda$ if and only if, for any $\left(\{\xi\}, \mathscr{R}_{\mathfrak{g}}, \mathscr{S}_{\mathfrak{g}}\right) \in \times_{\alpha\in I_3^*} \mathscr{P}(\Omega)$ such that $\{\xi\} \subset \mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}})$, it satisfies each "$\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-derived operator axiom" in $\mathrm{AX}\big[\mathfrak{g}\text{-DE}^{(\delta)}[\mathfrak{T}_{\mathfrak{g}}];\mathbb{B}\big] \stackrel{\text{def}}{=} \big\{\mathrm{Ax}_{\mathrm{DE},\nu}\big(\mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta)}\big) : \nu \in I_4^*\big\}$, where the mapping $\mathrm{Ax}_{\mathrm{DE},\nu} : \mathfrak{g}\text{-DE}^{(\delta)}[\mathfrak{T}_{\mathfrak{g}}] \longrightarrow \mathbb{B} \stackrel{\text{def}}{=} \{0,1\}$, $\nu \in I_4^*$, is defined as thus:

- $\mathrm{Ax}_{\mathrm{DE},1}\big(\mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta)}\big) \stackrel{\text{def}}{\Longleftrightarrow} \mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta)}(\emptyset) = \emptyset$,
- $\mathrm{Ax}_{\mathrm{DE},2}\big(\mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta)}\big) \stackrel{\text{def}}{\Longleftrightarrow} \mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta)}(\mathscr{R}_{\mathfrak{g}}) = \mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta)}\big(\mathscr{R}_{\mathfrak{g}} \cap \mathfrak{g}\text{-Op}_{\mathfrak{g}}(\{\xi\})\big)$,
- $\mathrm{Ax}_{\mathrm{DE},3}\big(\mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta)}\big) \stackrel{\text{def}}{\Longleftrightarrow} \mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta)} \circ \mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta)}(\mathscr{R}_{\mathfrak{g}}) \subseteq \mathscr{R}_{\mathfrak{g}} \cup \mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}})$,
- $\mathrm{Ax}_{\mathrm{DE},4}\big(\mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta)}\big) \stackrel{\text{def}}{\Longleftrightarrow} \mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta)}(\mathscr{R}_{\mathfrak{g}} \cup \mathscr{S}_{\mathfrak{g}}) = \bigcup_{\mathscr{U}_{\mathfrak{g}}=\mathscr{R}_{\mathfrak{g}},\mathscr{S}_{\mathfrak{g}}} \mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta)}(\mathscr{U}_{\mathfrak{g}})$.

Similarly, viewing the δ^{th}-order derived set $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-coderived operator conditions (Items I.–IV. of Cor. 3.56 above) as δ^{th}-*order* $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-*coderived operator axioms*, the axiomatic definition of the concept of a δ^{th}-order $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-coderived operator, then, can be defined as a δ^{th}-order set-valued map $\mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\delta)} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ on $\mathscr{P}(\Omega)$ ranging in $\mathscr{P}(\Omega)$ satisfying a list of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-*coderived operator axioms*. The axiomatic definition of the concept of a δ^{th}-order $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-coderived operator in $\mathscr{T}_{\mathfrak{g}}$-spaces follows.

Definition 3.58 (Axiomatic Definition: $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-Coderived Operator). The δ^{th}-iterate $\mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\delta)} : \mathscr{S}_{\mathfrak{g}} \in \mathscr{P}(\Omega) \longmapsto \mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\delta)}(\mathscr{S}_{\mathfrak{g}})$ of $\mathfrak{g}\text{-Cod}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ is called a "$\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-coderived operator of δ^{th} order" on $\mathscr{P}(\Omega)$ ranging in $\mathscr{P}(\Omega)$ for some ordinal δ such that $1 \preccurlyeq \delta \prec \lambda$ if and only if, for any $\left(\{\xi\}, \mathscr{R}_{\mathfrak{g}}, \mathscr{S}_{\mathfrak{g}}\right) \in \times_{\alpha\in I_3^*} \mathscr{P}(\Omega)$ such that $\{\xi\} \subset \mathfrak{g}\text{-Cod}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}})$, it satisfies each "$\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-derived operator axiom" in $\mathrm{AX}\big[\mathfrak{g}\text{-CD}^{(\delta)}[\mathfrak{T}_{\mathfrak{g}}];\mathbb{B}\big] \stackrel{\text{def}}{=} \big\{\mathrm{Ax}_{\mathrm{CD},\nu}\big(\mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\delta)}\big) : \nu \in I_4^*\big\}$, where the mapping $\mathrm{Ax}_{\mathrm{CD},\nu} : \mathfrak{g}\text{-CD}^{(\delta)}[\mathfrak{T}_{\mathfrak{g}}] \longrightarrow \mathbb{B} \stackrel{\text{def}}{=} \{0,1\}$, $\nu \in I_4^*$, is defined as thus:

- $\mathrm{Ax}_{\mathrm{CD},1}\big(\mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\delta)}\big) \stackrel{\text{def}}{\Longleftrightarrow} \mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\delta)}(\Omega) = \Omega$,
- $\mathrm{Ax}_{\mathrm{CD},2}\big(\mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\delta)}\big) \stackrel{\text{def}}{\Longleftrightarrow} \mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\delta)}(\mathscr{R}_{\mathfrak{g}}) = \mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\delta)}\big(\mathscr{R}_{\mathfrak{g}} \cup \{\xi\}\big)$,

 doi:10.20944/preprints202011.0747.v1

- $\mathrm{Ax}_{\mathrm{CD},3}\big(\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}^{(\delta)}\big) \overset{\text{def}}{\longleftrightarrow} \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}^{(\delta)} \circ \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}^{(\delta)}(\mathscr{R}_{\mathfrak{g}}) \supseteq \mathscr{R}_{\mathfrak{g}} \cap \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}})$,
- $\mathrm{Ax}_{\mathrm{CD},4}\big(\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}^{(\delta)}\big) \overset{\text{def}}{\longleftrightarrow} \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}^{(\delta)}(\mathscr{R}_{\mathfrak{g}} \cap \mathscr{S}_{\mathfrak{g}}) = \bigcap_{\mathscr{U}_{\mathfrak{g}}=\mathscr{R}_{\mathfrak{g}},\mathscr{S}_{\mathfrak{g}}} \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}^{(\delta)}(\mathscr{U}_{\mathfrak{g}})$.

On the essential properties of the δ^{th}-order derivative $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-derived and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-coderived operators defined by transfinite recursion on the class of successor ordinals in $\mathscr{T}_{\mathfrak{g}}$-spaces, the discussion of the present section terminates here.

3.3. Ranks: Openness and Closedness. It is the intention of this section to investigate some of the basic properties of the notions of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-open and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-closed sets of rank δ in $\mathscr{T}_{\mathfrak{g}}$-spaces.

The following lemma in which are proved that $\mathfrak{T}_{\mathfrak{g}}$-openness and $\mathfrak{T}_{\mathfrak{g}}$-closedness imply $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-openness and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-closedness, respectively, will be helpful in the sequel.

Lemma 3.59. *If $\big(\mathscr{O}_{\mathfrak{g}}^{(\delta)}, \mathscr{K}_{\mathfrak{g}}^{(\delta)}\big) \subset \mathfrak{T}_{\mathfrak{g}} \times \mathfrak{T}_{\mathfrak{g}}$ be a pair of $\mathfrak{T}_{\mathfrak{g}}$-open and $\mathfrak{T}_{\mathfrak{g}}$-closed sets of rank δ in a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$, then $\big(\mathscr{O}_{\mathfrak{g}}^{(\delta)}, \mathscr{K}_{\mathfrak{g}}^{(\delta)}\big)$ is also a pair of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-open and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-closed sets of rank δ in $\mathfrak{T}_{\mathfrak{g}}$.*

Proof. Let $\big(\mathscr{O}_{\mathfrak{g}}^{(\delta)}, \mathscr{K}_{\mathfrak{g}}^{(\delta)}\big) \subset \mathfrak{T}_{\mathfrak{g}} \times \mathfrak{T}_{\mathfrak{g}}$ be a pair of $\mathfrak{T}_{\mathfrak{g}}$-open and $\mathfrak{T}_{\mathfrak{g}}$-closed sets of rank δ in a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$. Then, there exist $\mathscr{O}_{\mathfrak{g},1}^{(\delta)}$, $\mathscr{O}_{\mathfrak{g},2}^{(\delta)}$, … such that

$$\begin{aligned}
\mathscr{O}_{\mathfrak{g}}^{(\delta)} \subseteq \mathrm{int}_{\mathfrak{g}}^{(\delta)}\big(\mathscr{O}_{\mathfrak{g}}^{(\delta)}\big) \quad &= \bigcap_{\mathscr{W}_{\mathfrak{g}}\in \mathrm{CD}^{(\delta)}\left[\mathscr{O}_{\mathfrak{g}}^{(\delta)};\mathfrak{T}_{\mathfrak{g}}\right]} \mathscr{W}_{\mathfrak{g}} \\
&\longleftrightarrow \mathrm{cod}_{\mathfrak{g}}^{(\delta)}\big(\mathscr{O}_{\mathfrak{g},1}^{(\delta)}\big) \cap \mathrm{cod}_{\mathfrak{g}}^{(\delta)}\big(\mathscr{O}_{\mathfrak{g},2}^{(\delta)}\big) \cap \cdots \\
&\subseteq \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}^{(\delta)}\big(\mathscr{O}_{\mathfrak{g},1}^{(\delta)}\big) \cap \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}^{(\delta)}\big(\mathscr{O}_{\mathfrak{g},2}^{(\delta)}\big) \cap \cdots \\
&\longleftrightarrow \bigcap_{\mathscr{W}_{\mathfrak{g}}\in \mathfrak{g}\text{-}\mathrm{CD}^{(\delta)}\left[\mathscr{O}_{\mathfrak{g}}^{(\delta)};\mathfrak{T}_{\mathfrak{g}}\right]} \mathscr{W}_{\mathfrak{g}} = \mathfrak{g}\text{-}\mathrm{Int}_{\mathfrak{g}}^{(\delta)}\big(\mathscr{O}_{\mathfrak{g}}^{(\delta)}\big),
\end{aligned}$$

implying that $\mathscr{O}_{\mathfrak{g}}^{(\delta)}$ is also a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-open set of rank δ in $\mathfrak{T}_{\mathfrak{g}}$. Similarly, there exist $\mathscr{K}_{\mathfrak{g},1}^{(\delta)}$, $\mathscr{K}_{\mathfrak{g},2}^{(\delta)}$, … such that

$$\begin{aligned}
\mathscr{K}_{\mathfrak{g}}^{(\delta)} \supseteq \mathrm{cl}_{\mathfrak{g}}^{(\delta)}\big(\mathscr{K}_{\mathfrak{g}}^{(\delta)}\big) \quad &= \bigcup_{\mathscr{W}_{\mathfrak{g}}\in \mathrm{DE}^{(\delta)}\left[\mathscr{K}_{\mathfrak{g}}^{(\delta)};\mathfrak{T}_{\mathfrak{g}}\right]} \mathscr{W}_{\mathfrak{g}} \\
&\longleftrightarrow \mathrm{der}_{\mathfrak{g}}^{(\delta)}\big(\mathscr{K}_{\mathfrak{g},1}^{(\delta)}\big) \cap \mathrm{der}_{\mathfrak{g}}^{(\delta)}\big(\mathscr{K}_{\mathfrak{g},2}^{(\delta)}\big) \cap \cdots \\
&\supseteq \mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}^{(\delta)}\big(\mathscr{K}_{\mathfrak{g},1}^{(\delta)}\big) \cap \mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}^{(\delta)}\big(\mathscr{K}_{\mathfrak{g},2}^{(\delta)}\big) \cap \cdots \\
&\longleftrightarrow \bigcup_{\mathscr{W}_{\mathfrak{g}}\in \mathfrak{g}\text{-}\mathrm{DE}^{(\delta)}\left[\mathscr{K}_{\mathfrak{g}}^{(\delta)};\mathfrak{T}_{\mathfrak{g}}\right]} \mathscr{W}_{\mathfrak{g}} = \mathfrak{g}\text{-}\mathrm{Cl}_{\mathfrak{g}}^{(\delta)}\big(\mathscr{K}_{\mathfrak{g}}^{(\delta)}\big),
\end{aligned}$$

implying that $\mathscr{K}_{\mathfrak{g}}^{(\delta)}$ is also a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-closed set of rank δ in $\mathfrak{T}_{\mathfrak{g}}$. The proof of the lemma is complete. Q.E.D.

In a $\mathscr{T}_{\mathfrak{g}}$-space, $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-open, closed sets of rank δ are also $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-open, closed sets of rank $\delta\eta$, respectively, η being such that $0 \prec \eta$, as proved in the following theorem.

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THEOREM 3.60. *If* $\left(\mathscr{O}_{\mathfrak{g}}^{(\delta)},\mathscr{K}_{\mathfrak{g}}^{(\delta)}\right)\subset\mathfrak{T}_{\mathfrak{g}}\times\mathfrak{T}_{\mathfrak{g}}$ *be a pair of* $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$*-open and* $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$*-closed sets of rank* δ *in a* $\mathscr{T}_{\mathfrak{g}}$*-space* $\mathfrak{T}_{\mathfrak{g}}=(\Omega,\mathscr{T}_{\mathfrak{g}})$*, then* $\left(\mathscr{O}_{\mathfrak{g}}^{(\delta)},\mathscr{K}_{\mathfrak{g}}^{(\delta)}\right)$ *is also a pair of* $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$*-open and* $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$*-closed sets of rank* $\delta\eta$*,* $0\prec\eta$*, in* $\mathfrak{T}_{\mathfrak{g}}$*.*

PROOF. Let $\left(\mathscr{O}_{\mathfrak{g}}^{(\delta)},\mathscr{K}_{\mathfrak{g}}^{(\delta)}\right)\subset\mathfrak{T}_{\mathfrak{g}}\times\mathfrak{T}_{\mathfrak{g}}$ be a pair of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-open and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-closed sets of rank δ in a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}}=(\Omega,\mathscr{T}_{\mathfrak{g}})$. Then,

$$\begin{aligned}\mathscr{O}_{\mathfrak{g}}^{(\delta)}\subseteq\mathfrak{g}\text{-Int}_{\mathfrak{g}}^{(\delta)}\big(\mathscr{O}_{\mathfrak{g}}^{(\delta)}\big)&=\bigcap_{\mathscr{W}_{\mathfrak{g}}\in\mathfrak{g}\text{-CD}^{(\delta)}\left[\mathscr{O}_{\mathfrak{g}}^{(\delta)};\mathfrak{T}_{\mathfrak{g}}\right]}\mathscr{W}_{\mathfrak{g}},\\ \mathscr{K}_{\mathfrak{g}}^{(\delta)}\supseteq\mathfrak{g}\text{-Cl}_{\mathfrak{g}}^{(\delta)}\big(\mathscr{K}_{\mathfrak{g}}^{(\delta)}\big)&=\bigcup_{\mathscr{W}_{\mathfrak{g}}\in\mathfrak{g}\text{-DE}^{(\delta)}\left[\mathscr{K}_{\mathfrak{g}}^{(\delta)};\mathfrak{T}_{\mathfrak{g}}\right]}\mathscr{W}_{\mathfrak{g}}.\end{aligned}$$

But $\big(\mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\delta\eta)}\big(\mathscr{O}_{\mathfrak{g}}^{(\delta)}\big),\mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta)}\big(\mathscr{K}_{\mathfrak{g}}^{(\delta)}\big)\big)\supseteq\big(\mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\delta)}\big(\mathscr{O}_{\mathfrak{g}}^{(\delta)}\big),\mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta\eta)}\big(\mathscr{K}_{\mathfrak{g}}^{(\delta)}\big)\big)$, where $0\prec\eta$. Consequently,

$$\begin{aligned}\bigcap_{\mathscr{W}_{\mathfrak{g}}\in\mathfrak{g}\text{-CD}^{(\delta)}\left[\mathscr{O}_{\mathfrak{g}}^{(\delta)};\mathfrak{T}_{\mathfrak{g}}\right]}\mathscr{W}_{\mathfrak{g}}&\subseteq\bigcap_{\mathscr{W}_{\mathfrak{g}}\in\mathfrak{g}\text{-CD}^{(\delta\eta)}\left[\mathscr{O}_{\mathfrak{g}}^{(\delta)};\mathfrak{T}_{\mathfrak{g}}\right]}\mathscr{W}_{\mathfrak{g}}\longleftrightarrow\mathfrak{g}\text{-Int}_{\mathfrak{g}}^{(\delta\eta)}\big(\mathscr{O}_{\mathfrak{g}}^{(\delta)}\big),\\ \bigcup_{\mathscr{W}_{\mathfrak{g}}\in\mathfrak{g}\text{-DE}^{(\delta)}\left[\mathscr{K}_{\mathfrak{g}}^{(\delta)};\mathfrak{T}_{\mathfrak{g}}\right]}\mathscr{W}_{\mathfrak{g}}&\supseteq\bigcup_{\mathscr{W}_{\mathfrak{g}}\in\mathfrak{g}\text{-DE}^{(\delta\eta)}\left[\mathscr{K}_{\mathfrak{g}}^{(\delta)};\mathfrak{T}_{\mathfrak{g}}\right]}\mathscr{W}_{\mathfrak{g}}\longleftrightarrow\mathfrak{g}\text{-Cl}_{\mathfrak{g}}^{(\delta\eta)}\big(\mathscr{O}_{\mathfrak{g}}^{(\delta)}\big).\end{aligned}$$

Hence,

$$\begin{aligned}\mathscr{O}_{\mathfrak{g}}^{(\delta)}\subseteq\mathfrak{g}\text{-Int}_{\mathfrak{g}}^{(\delta\eta)}\big(\mathscr{O}_{\mathfrak{g}}^{(\delta)}\big)&=\bigcap_{\mathscr{W}_{\mathfrak{g}}\in\mathfrak{g}\text{-CD}^{(\delta\eta)}\left[\mathscr{O}_{\mathfrak{g}}^{(\delta)};\mathfrak{T}_{\mathfrak{g}}\right]}\mathscr{W}_{\mathfrak{g}},\\ \mathscr{K}_{\mathfrak{g}}^{(\delta)}\supseteq\mathfrak{g}\text{-Cl}_{\mathfrak{g}}^{(\delta\eta)}\big(\mathscr{K}_{\mathfrak{g}}^{(\delta)}\big)&=\bigcup_{\mathscr{W}_{\mathfrak{g}}\in\mathfrak{g}\text{-DE}^{(\delta\eta)}\left[\mathscr{K}_{\mathfrak{g}}^{(\delta)};\mathfrak{T}_{\mathfrak{g}}\right]}\mathscr{W}_{\mathfrak{g}},\end{aligned}$$

implying that $\left(\mathscr{O}_{\mathfrak{g}}^{(\delta)},\mathscr{K}_{\mathfrak{g}}^{(\delta)}\right)$ is also a pair of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-open and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-closed sets of rank $\delta\eta$, $0\prec\eta$, in $\mathfrak{T}_{\mathfrak{g}}$. The proof of the theorem is complete. Q.E.D.

In a $\mathscr{T}_{\mathfrak{g}}$-space, $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-open sets of rank δ under $\cup$-operation and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-closed sets of rank δ under $\cap$-operation are both *rank-preserving*, as proved in the following theorem.

THEOREM 3.61. *If* $\mathscr{O}_{\mathfrak{g},1}^{(\delta)}$*,* $\mathscr{O}_{\mathfrak{g},2}^{(\delta)}$*, …,* $\mathscr{O}_{\mathfrak{g},\alpha}^{(\delta)}\subset\mathfrak{T}_{\mathfrak{g}}$ *be* $\alpha<\infty$ $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$*-open sets of rank* δ *and* $\mathscr{K}_{\mathfrak{g},1}^{(\delta)}$*,* $\mathscr{K}_{\mathfrak{g},2}^{(\delta)}$*, …,* $\mathscr{K}_{\mathfrak{g},\beta}^{(\delta)}\subset\mathfrak{T}_{\mathfrak{g}}$ *be* $\beta<\infty$ $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$*-closed sets of rank* δ *in a* $\mathscr{T}_{\mathfrak{g}}$*-space* $\mathfrak{T}_{\mathfrak{g}}=(\Omega,\mathscr{T}_{\mathfrak{g}})$*, then:*

- I. $\mathscr{O}_{\mathfrak{g}}^{(\delta)}=\bigcup_{\nu\in I_{\alpha}^{*}}\mathscr{O}_{\mathfrak{g},\nu}^{(\delta)}$ *is likewise* $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$*-open of rank* δ *in* $\mathfrak{T}_{\mathfrak{g}}$*,*
- II. $\mathscr{K}_{\mathfrak{g}}^{(\delta)}=\bigcap_{\nu\in I_{\beta}^{*}}\mathscr{K}_{\mathfrak{g},\nu}^{(\delta)}$ *is likewise* $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$*-closed of rank* δ *in* $\mathfrak{T}_{\mathfrak{g}}$*.*

PROOF. Let $\mathscr{O}_{\mathfrak{g},1}^{(\delta)}$, $\mathscr{O}_{\mathfrak{g},2}^{(\delta)}$, …, $\mathscr{O}_{\mathfrak{g},\alpha}^{(\delta)}\subset\mathfrak{T}_{\mathfrak{g}}$ be $\alpha<\infty$ $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-open sets of rank δ and let $\mathscr{K}_{\mathfrak{g},1}^{(\delta)}$, $\mathscr{K}_{\mathfrak{g},2}^{(\delta)}$, …, $\mathscr{K}_{\mathfrak{g},\beta}^{(\delta)}\subset\mathfrak{T}_{\mathfrak{g}}$ be $\beta<\infty$ $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-closed sets of rank δ in a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}}=(\Omega,\mathscr{T}_{\mathfrak{g}})$. Then:

 doi:10.20944/preprints202011.0747.v1

I. For any $\mu \in I_{\alpha}^{*}$,

$$\mathscr{O}_{\mathfrak{g},\mu}^{(\delta)} \subseteq \mathfrak{g}\text{-}\mathrm{Int}_{\mathfrak{g}}^{(\delta)}\big(\mathscr{O}_{\mathfrak{g},\mu}^{(\delta)}\big) = \bigcap_{\mathscr{W}_{\mathfrak{g}} \in \mathfrak{g}\text{-}\mathrm{CD}^{(\delta)}\left[\mathscr{O}_{\mathfrak{g},\mu}^{(\delta)};\mathfrak{T}_{\mathfrak{g}}\right]} \mathscr{W}_{\mathfrak{g}}.$$

Consequently,

$$\begin{aligned}
\mathscr{O}_{\mathfrak{g}}^{(\delta)} = \textstyle\bigcup_{\nu\in I_{\alpha}^{*}} \mathscr{O}_{\mathfrak{g},\mu}^{(\delta)} &\subseteq \bigcup_{\nu\in I_{\alpha}^{*}} \mathfrak{g}\text{-}\mathrm{Int}_{\mathfrak{g}}^{(\delta)}\big(\mathscr{O}_{\mathfrak{g},\mu}^{(\delta)}\big)\\
&\subseteq \mathfrak{g}\text{-}\mathrm{Int}_{\mathfrak{g}}^{(\delta)}\Bigg(\bigcup_{\nu\in I_{\alpha}^{*}} \mathscr{O}_{\mathfrak{g},\mu}^{(\delta)}\Bigg) \longleftrightarrow \mathfrak{g}\text{-}\mathrm{Int}_{\mathfrak{g}}^{(\delta)}\big(\mathscr{O}_{\mathfrak{g}}^{(\delta)}\big)\\
&\qquad\longleftrightarrow \bigcap_{\mathscr{W}_{\mathfrak{g}} \in \mathfrak{g}\text{-}\mathrm{CD}^{(\delta)}\left[\mathscr{O}_{\mathfrak{g}}^{(\delta)};\mathfrak{T}_{\mathfrak{g}}\right]} \mathscr{W}_{\mathfrak{g}}.
\end{aligned}$$

Thus, $\mathscr{O}_{\mathfrak{g}}^{(\delta)} = \bigcup_{\nu\in I_{\alpha}^{*}} \mathscr{O}_{\mathfrak{g},\nu}^{(\delta)}$ is likewise $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-open of rank δ in $\mathfrak{T}_{\mathfrak{g}}$.

II. For any $\mu \in I_{\beta}^{*}$,

$$\mathscr{K}_{\mathfrak{g},\mu}^{(\delta)} \supseteq \mathfrak{g}\text{-}\mathrm{Cl}_{\mathfrak{g}}^{(\delta)}\big(\mathscr{K}_{\mathfrak{g},\mu}^{(\delta)}\big) = \bigcup_{\mathscr{W}_{\mathfrak{g}} \in \mathfrak{g}\text{-}\mathrm{DE}^{(\delta)}\left[\mathscr{K}_{\mathfrak{g},\mu}^{(\delta)};\mathfrak{T}_{\mathfrak{g}}\right]} \mathscr{W}_{\mathfrak{g}}.$$

Consequently,

$$\begin{aligned}
\mathscr{K}_{\mathfrak{g}}^{(\delta)} = \textstyle\bigcap_{\nu\in I_{\beta}^{*}} \mathscr{K}_{\mathfrak{g},\mu}^{(\delta)} &\supseteq \bigcap_{\nu\in I_{\beta}^{*}} \mathfrak{g}\text{-}\mathrm{Cl}_{\mathfrak{g}}^{(\delta)}\big(\mathscr{K}_{\mathfrak{g},\mu}^{(\delta)}\big)\\
&\supseteq \mathfrak{g}\text{-}\mathrm{Cl}_{\mathfrak{g}}^{(\delta)}\Bigg(\bigcap_{\nu\in I_{\beta}^{*}} \mathscr{K}_{\mathfrak{g},\mu}^{(\delta)}\Bigg) \longleftrightarrow \mathfrak{g}\text{-}\mathrm{Cl}_{\mathfrak{g}}^{(\delta)}\big(\mathscr{K}_{\mathfrak{g}}^{(\delta)}\big)\\
&\qquad\longleftrightarrow \bigcap_{\mathscr{W}_{\mathfrak{g}} \in \mathfrak{g}\text{-}\mathrm{DE}^{(\delta)}\left[\mathscr{K}_{\mathfrak{g}}^{(\delta)};\mathfrak{T}_{\mathfrak{g}}\right]} \mathscr{W}_{\mathfrak{g}}.
\end{aligned}$$

Hence, $\mathscr{K}_{\mathfrak{g}}^{(\delta)} = \bigcap_{\nu\in I_{\beta}^{*}} \mathscr{K}_{\mathfrak{g},\nu}^{(\delta)}$ is likewise $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-closed of rank δ in $\mathfrak{T}_{\mathfrak{g}}$. The proof of the theorem is complete. Q.E.D.

An immediate consequence of the above lemma and theorem is the following corollary.

COROLLARY 3.62. *If* $\mathscr{O}_{\mathfrak{g},1}^{(\delta)}, \mathscr{O}_{\mathfrak{g},2}^{(\delta)}, \ldots, \mathscr{O}_{\mathfrak{g},\alpha}^{(\delta)} \subset \mathfrak{T}_{\mathfrak{g}}$ *be* $\alpha < \infty$ $\mathfrak{T}_{\mathfrak{g}}$*-open sets of rank* δ *and* $\mathscr{K}_{\mathfrak{g},1}^{(\delta)}, \mathscr{K}_{\mathfrak{g},2}^{(\delta)}, \ldots, \mathscr{K}_{\mathfrak{g},\beta}^{(\delta)} \subset \mathfrak{T}_{\mathfrak{g}}$ *be* $\beta < \infty$ $\mathfrak{T}_{\mathfrak{g}}$*-closed sets of rank* δ *in a* $\mathscr{T}_{\mathfrak{g}}$*-space* $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$*, then:*

- I. $\mathscr{O}_{\mathfrak{g}}^{(\delta)} = \bigcup_{\nu\in I_{\alpha}^{*}} \mathscr{O}_{\mathfrak{g},\nu}^{(\delta)}$ *is also* $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$*-open of rank* δ *in* $\mathfrak{T}_{\mathfrak{g}}$*,*
- II. $\mathscr{K}_{\mathfrak{g}}^{(\delta)} = \bigcap_{\nu\in I_{\beta}^{*}} \mathscr{K}_{\mathfrak{g},\nu}^{(\delta)}$ *is also* $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$*-closed of rank* δ *in* $\mathfrak{T}_{\mathfrak{g}}$*.*

In a $\mathscr{T}_{\mathfrak{g}}$-space, the $\delta\eta^{\text{th}}$ $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-derived set of each member of any subcollection of the power set of a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-closed set of rank δ is also one element of the power set; likewise, the $\delta\eta^{\text{th}}$ $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-coderived set of each member of any subcollection of the

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power set of a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-open set of rank δ is also one element of the power set, η being such that $0 \prec \eta$, as proved in the following proposition.

Proposition 3.63. *Let* $\big(\mathscr{O}_{\mathfrak{g}}^{(\delta)}, \mathscr{P}\big(\mathscr{O}_{\mathfrak{g}}^{(\delta)}\big)\big)$ *and* $\big(\mathscr{K}_{\mathfrak{g}}^{(\delta)}, \mathscr{P}\big(\mathscr{K}_{\mathfrak{g}}^{(\delta)}\big)\big)$ *be pairs of a* $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-*open set of rank* δ *and its power set and a* $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-*closed set of rank* δ *and its power set, respectively, in a* $\mathscr{T}_{\mathfrak{g}}$-*space* $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$. *Suppose* $\big(\mathscr{C}\big(\mathscr{O}_{\mathfrak{g}}^{(\delta)}\big), \mathscr{C}\big(\mathscr{K}_{\mathfrak{g}}^{(\delta)}\big)\big) \subseteq \big(\mathscr{P}\big(\mathscr{O}_{\mathfrak{g}}^{(\delta)}\big), \mathscr{P}\big(\mathscr{K}_{\mathfrak{g}}^{(\delta)}\big)\big)$ *be a pair of subcollections of* $\big(\mathscr{P}\big(\mathscr{O}_{\mathfrak{g}}^{(\delta)}\big), \mathscr{P}\big(\mathscr{K}_{\mathfrak{g}}^{(\delta)}\big)\big)$, *then the following statements hold for any* η *such that* $0 \prec \eta$:

- I. $\big(\forall \mathscr{O}_{\mathfrak{g}} \in \mathscr{C}\big(\mathscr{O}_{\mathfrak{g}}^{(\delta)}\big)\big)\big(\exists \mathscr{U}_{\mathfrak{g}} \in \mathscr{P}\big(\mathscr{O}_{\mathfrak{g}}^{(\delta)}\big)\big)\big[\mathscr{U}_{\mathfrak{g}} = \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}^{(\delta\eta)}(\mathscr{O}_{\mathfrak{g}})\big]$.
- II. $\big(\forall \mathscr{K}_{\mathfrak{g}} \in \mathscr{C}\big(\mathscr{K}_{\mathfrak{g}}^{(\delta)}\big)\big)\big(\exists \mathscr{V}_{\mathfrak{g}} \in \mathscr{P}\big(\mathscr{K}_{\mathfrak{g}}^{(\delta)}\big)\big)\big[\mathscr{V}_{\mathfrak{g}} = \mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}^{(\delta\eta)}(\mathscr{K}_{\mathfrak{g}})\big]$.

Proof. Let $\big(\mathscr{O}_{\mathfrak{g}}^{(\delta)}, \mathscr{P}\big(\mathscr{O}_{\mathfrak{g}}^{(\delta)}\big)\big)$ and $\big(\mathscr{K}_{\mathfrak{g}}^{(\delta)}, \mathscr{P}\big(\mathscr{K}_{\mathfrak{g}}^{(\delta)}\big)\big)$ be pairs of a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-open set of rank δ and its power set and a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-closed set of rank δ and its power set, respectively, in a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$. Further, suppose $\big(\mathscr{C}\big(\mathscr{O}_{\mathfrak{g}}^{(\delta)}\big), \mathscr{C}\big(\mathscr{K}_{\mathfrak{g}}^{(\delta)}\big)\big) \subseteq \big(\mathscr{P}\big(\mathscr{O}_{\mathfrak{g}}^{(\delta)}\big), \mathscr{P}\big(\mathscr{K}_{\mathfrak{g}}^{(\delta)}\big)\big)$ be a pair of subcollections of $\big(\mathscr{P}\big(\mathscr{O}_{\mathfrak{g}}^{(\delta)}\big), \mathscr{P}\big(\mathscr{K}_{\mathfrak{g}}^{(\delta)}\big)\big)$. Then:

I. For any $(\eta, \mathscr{O}_{\mathfrak{g}})$ such that $\mathscr{O}_{\mathfrak{g}} \in \mathscr{C}\big(\mathscr{O}_{\mathfrak{g}}^{(\delta)}\big) \subseteq \mathscr{P}\big(\mathscr{O}_{\mathfrak{g}}^{(\delta)}\big)$ and $0 \prec \eta$,

$$\begin{aligned}\mathscr{O}_{\mathfrak{g}}^{(\delta)} \subseteq \mathfrak{g}\text{-}\mathrm{Int}_{\mathfrak{g}}^{(\delta)}\big(\mathscr{O}_{\mathfrak{g}}^{(\delta)}\big) &\subseteq \mathfrak{g}\text{-}\mathrm{Int}_{\mathfrak{g}}^{(\delta\eta)}\big(\mathscr{O}_{\mathfrak{g}}^{(\delta)}\big)\\ &= \bigcap_{\mathscr{W}_{\mathfrak{g}} \in \mathfrak{g}\text{-}\mathrm{CD}^{(\delta\eta)}\left[\mathscr{O}_{\mathfrak{g}}^{(\delta)}; \mathfrak{T}_{\mathfrak{g}}\right]} \mathscr{W}_{\mathfrak{g}} \subseteq \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}^{(\delta\eta)}\big(\mathscr{O}_{\mathfrak{g}}^{(\delta)}\big),\end{aligned}$$

implying $\mathscr{O}_{\mathfrak{g}}^{(\delta)} \subseteq \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}^{(\delta\eta)}\big(\mathscr{O}_{\mathfrak{g}}^{(\delta)}\big)$. But, $\mathscr{O}_{\mathfrak{g}} \in \mathscr{C}\big(\mathscr{O}_{\mathfrak{g}}^{(\delta)}\big) \subseteq \mathscr{P}\big(\mathscr{O}_{\mathfrak{g}}^{(\delta)}\big)$ and consequently, there exists $\mathscr{U}_{\mathfrak{g}} \in \mathscr{P}\big(\mathscr{O}_{\mathfrak{g}}^{(\delta)}\big)$ such that $\mathscr{U}_{\mathfrak{g}} = \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}^{(\delta\eta)}(\mathscr{O}_{\mathfrak{g}})$.

II. For any $(\eta, \mathscr{K}_{\mathfrak{g}})$ such that $\mathscr{K}_{\mathfrak{g}} \in \mathscr{C}\big(\mathscr{K}_{\mathfrak{g}}^{(\delta)}\big) \subseteq \mathscr{P}\big(\mathscr{K}_{\mathfrak{g}}^{(\delta)}\big)$ and $0 \prec \eta$,

$$\begin{aligned}\mathscr{K}_{\mathfrak{g}}^{(\delta)} \supseteq \mathfrak{g}\text{-}\mathrm{Cl}_{\mathfrak{g}}^{(\delta)}\big(\mathscr{K}_{\mathfrak{g}}^{(\delta)}\big) &\supseteq \mathfrak{g}\text{-}\mathrm{Cl}_{\mathfrak{g}}^{(\delta\eta)}\big(\mathscr{K}_{\mathfrak{g}}^{(\delta)}\big)\\ &= \bigcup_{\mathscr{W}_{\mathfrak{g}} \in \mathfrak{g}\text{-}\mathrm{DE}^{(\delta\eta)}\left[\mathscr{K}_{\mathfrak{g}}^{(\delta)}; \mathfrak{T}_{\mathfrak{g}}\right]} \mathscr{W}_{\mathfrak{g}} \supseteq \mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}^{(\delta\eta)}\big(\mathscr{K}_{\mathfrak{g}}^{(\delta)}\big),\end{aligned}$$

implying $\mathscr{K}_{\mathfrak{g}}^{(\delta)} \supseteq \mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}^{(\delta\eta)}\big(\mathscr{K}_{\mathfrak{g}}^{(\delta)}\big)$. But, $\mathscr{K}_{\mathfrak{g}} \in \mathscr{C}\big(\mathscr{K}_{\mathfrak{g}}^{(\delta)}\big) \subseteq \mathscr{P}\big(\mathscr{K}_{\mathfrak{g}}^{(\delta)}\big)$ and consequently, there exists $\mathscr{V}_{\mathfrak{g}} \in \mathscr{P}\big(\mathscr{K}_{\mathfrak{g}}^{(\delta)}\big)$ such that $\mathscr{V}_{\mathfrak{g}} = \mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}^{(\delta\eta)}(\mathscr{K}_{\mathfrak{g}})$. The proof of the proposition is complete. Q.E.D.

Any set $\{\delta_\nu : \nu \in I_\alpha^*\}$ of positive ordinals has a least common (positive) right multiple and a greatest common right divisor. In what follows, the expressions $\mathrm{lcm}(\delta_\nu : \nu \in I_\alpha^*)$ and $\mathrm{gcd}(\delta_\nu : \nu \in I_\alpha^*)$, respectively, will stand for the *least common multiple* and *greatest common divisor* of $\delta_1, \delta_2, \ldots, \delta_\alpha$. In a $\mathscr{T}_{\mathfrak{g}}$-space, $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-open sets of ranks $\delta_1, \delta_2, \ldots, \delta_\alpha$ under $\cup$-operation and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-closed sets of ranks $\delta_1, \delta_2, \ldots, \delta_\beta$ under $\cap$-operation are, in general, both *not rank-preserving*; they become $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-open of rank $\delta = \mathrm{lcm}(\delta_\nu : \nu \in I_\alpha^*)$ and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-closed of rank $\delta = \mathrm{lcm}(\delta_\nu : \nu \in I_\beta^*)$, as proved in the following theorem.

Theorem 3.64. *If* $\mathscr{O}_{\mathfrak{g}}^{(\delta_1)}, \mathscr{O}_{\mathfrak{g}}^{(\delta_2)}, \ldots, \mathscr{O}_{\mathfrak{g}}^{(\delta_\alpha)} \subset \mathfrak{T}_{\mathfrak{g}}$ *be* $\alpha < \infty$ $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-*open sets of rank* $\delta_1, \delta_2, \ldots, \delta_\alpha$, *and* $\mathscr{K}_{\mathfrak{g}}^{(\delta_1)}, \mathscr{K}_{\mathfrak{g}}^{(\delta_2)}, \ldots, \mathscr{K}_{\mathfrak{g}}^{(\delta_\beta)} \subset \mathfrak{T}_{\mathfrak{g}}$ *be* $\beta < \infty$ $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-*closed sets of rank* $\delta_1, \delta_2, \ldots, \delta_\alpha$, *respectively, in a* $\mathscr{T}_{\mathfrak{g}}$-*space* $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$, *then:*

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- I. $\mathscr{O}_{\mathfrak{g}}^{(\delta)} = \bigcup_{\nu\in I_{\alpha}^{*}} \mathscr{O}_{\mathfrak{g}}^{(\delta_{\nu})}$ *is* $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$*-open of rank* $\delta = \operatorname{lcm}(\delta_{\nu}:\ \nu\in I_{\alpha}^{*})$ *in* $\mathfrak{T}_{\mathfrak{g}}$,
- II. $\mathscr{K}_{\mathfrak{g}}^{(\delta)} = \bigcap_{\nu\in I_{\beta}^{*}} \mathscr{K}_{\mathfrak{g}}^{(\delta_{\nu})}$ *is* $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$*-closed of rank* $\delta = \operatorname{lcm}\big(\delta_{\nu}:\ \nu\in I_{\beta}^{*}\big)$ *in* $\mathfrak{T}_{\mathfrak{g}}$.

Proof. Let $\mathscr{O}_{\mathfrak{g}}^{(\delta_1)}$, $\mathscr{O}_{\mathfrak{g}}^{(\delta_2)}$, …, $\mathscr{O}_{\mathfrak{g}}^{(\delta_\alpha)} \subset \mathfrak{T}_{\mathfrak{g}}$ be $\alpha<\infty$ $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-open sets of rank δ_1, δ_2, …, δ_α, and let $\mathscr{K}_{\mathfrak{g}}^{(\delta_1)}$, $\mathscr{K}_{\mathfrak{g}}^{(\delta_2)}$, …, $\mathscr{K}_{\mathfrak{g}}^{(\delta_\beta)} \subset \mathfrak{T}_{\mathfrak{g}}$ be $\beta<\infty$ $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-closed sets of rank δ_1, δ_2, …, δ_α, respectively, in a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$. Then:

I. Since $\mathscr{O}_{\mathfrak{g}}^{(\delta_1)}$, $\mathscr{O}_{\mathfrak{g}}^{(\delta_2)}$, …, $\mathscr{O}_{\mathfrak{g}}^{(\delta_\alpha)}$ are $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-open sets of rank δ_1, δ_2, …, δ_α, they are also $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-open sets of rank $\delta_1\eta_1$, $\delta_2\eta_2$, …, $\delta_\alpha\eta_\alpha$ for some η_1, η_2, …, η_α satisfying $(0,0,\ldots,0) \prec (\eta_1,\eta_2,\ldots,\eta_\alpha)$ and $\delta = \operatorname{lcm}(\delta_{\nu}:\ \nu\in I_{\alpha}^{*}) = \delta_1\eta_1 = \delta_2\eta_2 = \cdots = \delta_\alpha\eta_\alpha$. Thus, $\mathscr{O}_{\mathfrak{g}}^{(\delta)} = \bigcup_{\nu\in I_{\alpha}^{*}} \mathscr{O}_{\mathfrak{g}}^{(\delta_{\nu})}$ is $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-open of rank $\delta = \operatorname{lcm}(\delta_{\nu}:\ \nu\in I_{\alpha}^{*})$ in $\mathfrak{T}_{\mathfrak{g}}$.

II. Since $\mathscr{K}_{\mathfrak{g}}^{(\delta_1)}$, $\mathscr{K}_{\mathfrak{g}}^{(\delta_2)}$, …, $\mathscr{K}_{\mathfrak{g}}^{(\delta_\beta)}$ are $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-closed sets of rank δ_1, δ_2, …, δ_β, they are also $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-closed sets of rank $\delta_1\eta_1$, $\delta_2\eta_2$, …, $\delta_\beta\eta_\beta$ for some η_1, η_2, …, η_β satisfying $(0,0,\ldots,0) \prec (\eta_1,\eta_2,\ldots,\eta_\beta)$ and $\delta = \operatorname{lcm}\big(\delta_{\nu}:\ \nu\in I_{\beta}^{*}\big) = \delta_1\eta_1 = \delta_2\eta_2 = \cdots = \delta_\beta\eta_\beta$. Hence, $\mathscr{K}_{\mathfrak{g}}^{(\delta)} = \bigcap_{\nu\in I_{\beta}^{*}} \mathscr{K}_{\mathfrak{g}}^{(\delta_{\nu})}$ is $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-closed of rank $\delta = \operatorname{lcm}\big(\delta_{\nu}:\ \nu\in I_{\beta}^{*}\big)$ in $\mathfrak{T}_{\mathfrak{g}}$. The proof of the theorem is complete. Q.E.D.

An immediate consequence of the above lemma and theorem is the following corollary.

Corollary 3.65. *If* $\mathscr{O}_{\mathfrak{g}}^{(\delta_1)}$, $\mathscr{O}_{\mathfrak{g}}^{(\delta_2)}$, …, $\mathscr{O}_{\mathfrak{g}}^{(\delta_\alpha)} \subset \mathfrak{T}_{\mathfrak{g}}$ *be* $\alpha<\infty$ $\mathfrak{T}_{\mathfrak{g}}$*-open sets of rank* δ_1, δ_2, …, δ_α, *and* $\mathscr{K}_{\mathfrak{g}}^{(\delta_1)}$, $\mathscr{K}_{\mathfrak{g}}^{(\delta_2)}$, …, $\mathscr{K}_{\mathfrak{g}}^{(\delta_\beta)} \subset \mathfrak{T}_{\mathfrak{g}}$ *be* $\beta<\infty$ $\mathfrak{T}_{\mathfrak{g}}$*-closed sets of rank* δ_1, δ_2, …, δ_α, *respectively, in a* $\mathscr{T}_{\mathfrak{g}}$*-space* $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$, *then:*

- I. $\mathscr{O}_{\mathfrak{g}}^{(\delta)} = \bigcup_{\nu\in I_{\alpha}^{*}} \mathscr{O}_{\mathfrak{g}}^{(\delta_{\nu})}$ *is also* $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$*-open of rank* $\delta = \operatorname{lcm}(\delta_{\nu}:\ \nu\in I_{\alpha}^{*})$ *in* $\mathfrak{T}_{\mathfrak{g}}$,
- II. $\mathscr{K}_{\mathfrak{g}}^{(\delta)} = \bigcap_{\nu\in I_{\beta}^{*}} \mathscr{K}_{\mathfrak{g}}^{(\delta_{\nu})}$ *is also* $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$*-closed of rank* $\delta = \operatorname{lcm}\big(\delta_{\nu}:\ \nu\in I_{\beta}^{*}\big)$ *in* $\mathfrak{T}_{\mathfrak{g}}$.

The results presented after the following definition make great use of the elements contained in the definition.

Definition 3.66. Let $\big(\mathscr{O}_{\mathfrak{g}}^{(\delta)}, \mathscr{K}_{\mathfrak{g}}^{(\eta)}\big) \subset \mathfrak{T}_{\mathfrak{g}} \times \mathfrak{T}_{\mathfrak{g}}$ be a pair of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-open set of rank δ and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-closed set of rank η, $(0,0) \prec (\delta,\eta)$, in a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$. Then, for some (γ,σ) such that $(0,0) \prec (\gamma,\sigma)$:

- I. $\mathscr{P}\big(\mathscr{O}_{\mathfrak{g}}^{(\delta)}\big) \stackrel{\text{def}}{=} \big\{\mathscr{O}_{\mathfrak{g}}:\ \mathscr{O}_{\mathfrak{g}} \subseteq \mathscr{O}_{\mathfrak{g}}^{(\delta)}\big\}$ denotes the power set of $\mathscr{O}_{\mathfrak{g}}^{(\delta)}$ and:
  - (*i.*) $\mathscr{Q}\big(\mathscr{O}_{\mathfrak{g}}^{(\gamma)}\big) \stackrel{\text{def}}{=} \big\{\mathscr{U}_{\mathfrak{g}}^{(\gamma)} = \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}^{(\gamma)}(\mathscr{O}_{\mathfrak{g}}):\ \mathscr{O}_{\mathfrak{g}} \in \mathscr{P}\big(\mathscr{O}_{\mathfrak{g}}^{(\delta)}\big)\big\}$ denotes the class of all sets which are γ^{th} $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-coderived sets of members of $\mathscr{P}\big(\mathscr{O}_{\mathfrak{g}}^{(\delta)}\big)$;
  - (*ii.*) $\mathscr{Q}\big(\mathscr{O}_{\mathfrak{g}}^{(\gamma+\delta)}\big) \stackrel{\text{def}}{=} \big\{\mathscr{U}_{\mathfrak{g}}^{(\delta+\gamma)} = \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}^{(\delta)}\big(\mathscr{U}_{\mathfrak{g}}^{(\gamma)}\big):\ \mathscr{U}_{\mathfrak{g}}^{(\gamma)} \in \mathscr{Q}\big(\mathscr{O}_{\mathfrak{g}}^{(\gamma)}\big)\big\}$ denotes the class of all sets which are δ^{th} $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-coderived sets of members of $\mathscr{Q}\big(\mathscr{O}_{\mathfrak{g}}^{(\gamma)}\big)$.
- II. $\mathscr{P}\big(\mathscr{K}_{\mathfrak{g}}^{(\eta)}\big) \stackrel{\text{def}}{=} \big\{\mathscr{K}_{\mathfrak{g}}:\ \mathscr{K}_{\mathfrak{g}} \subseteq \mathscr{K}_{\mathfrak{g}}^{(\eta)}\big\}$ denotes the power set of $\mathscr{K}_{\mathfrak{g}}^{(\eta)}$ and:

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- (*i.*) $\mathscr{Q}\big(\mathscr{K}_{\mathfrak{g}}^{(\sigma)}\big) \stackrel{\text{def}}{=} \big\{\mathscr{V}_{\mathfrak{g}}^{(\sigma)} = \mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\sigma)}(\mathscr{K}_{\mathfrak{g}}) :\ \mathscr{K}_{\mathfrak{g}} \in \mathscr{P}\big(\mathscr{K}_{\mathfrak{g}}^{(\eta)}\big)\big\}$ denotes the class of all sets which are σ^{th} $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-derived sets of members of $\mathscr{P}\big(\mathscr{K}_{\mathfrak{g}}^{(\eta)}\big)$;
- (*ii.*) $\mathscr{Q}\big(\mathscr{K}_{\mathfrak{g}}^{(\sigma+\eta)}\big) \stackrel{\text{def}}{=} \big\{\mathscr{V}_{\mathfrak{g}}^{(\eta+\sigma)} = \mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\eta)}\big(\mathscr{V}_{\mathfrak{g}}^{(\sigma)}\big) :\ \mathscr{V}_{\mathfrak{g}}^{(\sigma)} \in \mathscr{Q}\big(\mathscr{K}_{\mathfrak{g}}^{(\sigma)}\big)\big\}$ denotes the class of all sets which are η^{th} $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-derived sets of members of $\mathscr{Q}\big(\mathscr{K}_{\mathfrak{g}}^{(\sigma)}\big)$.

In a $\mathscr{T}_{\mathfrak{g}}$-space, suppose given a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-open set of ranks $\delta_1,\ \delta_2,\ \ldots,\ \delta_\alpha$ and a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-closed set of ranks $\delta_1,\ \delta_2,\ \ldots,\ \delta_\beta$, and suppose $\delta = \gcd(\delta_\nu :\ \nu \in I_\alpha^*)$ and $\eta = \gcd\big(\eta_\nu :\ \nu \in I_\beta^*\big)$ be the greatest common divisors of $\delta_1,\ \delta_2,\ \ldots,\ \delta_\alpha$ and $\eta_1,\ \eta_2,\ \ldots,\ \eta_\beta$, respectively, there exists, in general, a pair (γ,σ), satisfying $(0,0) \prec (\gamma,\sigma)$, such that all members of $\mathscr{P}\big(\mathscr{O}_{\mathfrak{g}}^{(\delta)}\big)$ belong to $\mathscr{Q}\big(\mathscr{O}_{\mathfrak{g}}^{(\gamma)}\big) \cup \mathscr{Q}\big(\mathscr{O}_{\mathfrak{g}}^{(\gamma+\delta)}\big)$ and all members of $\mathscr{Q}\big(\mathscr{K}_{\mathfrak{g}}^{(\sigma)}\big) \cup \mathscr{Q}\big(\mathscr{K}_{\mathfrak{g}}^{(\sigma+\eta)}\big)$ belong to $\mathscr{P}\big(\mathscr{K}_{\mathfrak{g}}^{(\eta)}\big)$. The proof of this statement is contained in the following theorem.

THEOREM 3.67. *Let $\mathscr{O}_{\mathfrak{g}}^{(\delta)} \subset \mathfrak{T}_{\mathfrak{g}}$ be a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-open set of rank $\delta \in \{\delta_\nu :\ \nu \in I_\alpha^*\}$ and $\mathscr{K}_{\mathfrak{g}}^{(\eta)} \subset \mathfrak{T}_{\mathfrak{g}}$ be a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-closed set of rank $\eta \in \big\{\eta_\nu :\ \nu \in I_\beta^*\big\}$ in a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$. Suppose $\delta = \gcd(\delta_\nu :\ \nu \in I_\alpha^*)$ and $\eta = \gcd\big(\eta_\nu :\ \nu \in I_\beta^*\big)$ be the greatest common divisors of $\delta_1,\ \delta_2,\ \ldots,\ \delta_\alpha$ and $\eta_1,\ \eta_2,\ \ldots,\ \eta_\beta$, respectively, then:*

- I. $(\exists\gamma :\ 0 \prec \gamma)\Big[\bigvee_{\mu=\gamma,\gamma+\delta}\big(\mathscr{U}_{\mathfrak{g}}^{(\gamma)} \in \mathscr{Q}\big(\mathscr{O}_{\mathfrak{g}}^{(\mu)}\big)\big) \longleftarrow \mathscr{U}_{\mathfrak{g}}^{(\gamma)} \in \mathscr{P}\big(\mathscr{O}_{\mathfrak{g}}^{(\delta)}\big)\Big]$,
- II. $(\exists\sigma :\ 0 \prec \sigma)\Big[\bigvee_{\nu=\sigma,\sigma+\eta}\big(\mathscr{V}_{\mathfrak{g}}^{(\sigma)} \in \mathscr{Q}\big(\mathscr{K}_{\mathfrak{g}}^{(\nu)}\big)\big) \longrightarrow \mathscr{V}_{\mathfrak{g}}^{(\sigma)} \in \mathscr{P}\big(\mathscr{K}_{\mathfrak{g}}^{(\eta)}\big)\Big]$.

PROOF. Let $\mathscr{O}_{\mathfrak{g}}^{(\delta)} \subset \mathfrak{T}_{\mathfrak{g}}$ be a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-open set of rank $\delta \in \{\delta_\nu :\ \nu \in I_\alpha^*\}$ and $\mathscr{K}_{\mathfrak{g}}^{(\eta)} \subset \mathfrak{T}_{\mathfrak{g}}$ be a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-closed set of rank $\eta \in \big\{\eta_\nu :\ \nu \in I_\beta^*\big\}$ in a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$. Suppose $\delta = \gcd(\delta_\nu :\ \nu \in I_\alpha^*)$ and $\eta = \gcd\big(\eta_\nu :\ \nu \in I_\beta^*\big)$ be the greatest common divisors of $\delta_1,\ \delta_2,\ \ldots,\ \delta_\alpha$ and $\eta_1,\ \eta_2,\ \ldots,\ \eta_\beta$, respectively. Then:

I. Since non-zero negative and positive sets of integers $\big\{\theta_k \in \mathbb{Z}_-^* :\ k \in I_\nu^*\big\}$ and $\big\{\theta_k \in \mathbb{Z}_+^* :\ k \in I_\alpha^* \setminus I_\nu^*\big\}$, respectively, exist, in actual fact $(|\theta_1|, |\theta_2|, \ldots, |\theta_\nu|) \prec (\delta_1, \delta_2, \ldots, \delta_\nu)$ and $(\theta_{\nu+1}, \theta_{\nu+2}, \ldots, \theta_\alpha) \prec (\delta_{\nu+1}, \delta_{\nu+2}, \ldots, \delta_\alpha)$, such that

$$\begin{aligned}\delta = \gcd(\delta_\nu :\ \nu \in I_\alpha^*) = \sum_{\nu \in I_\alpha^*} \theta_\nu \delta_\nu &= \sum_{\mu \in I_\nu^*} \theta_\mu \delta_\mu + \sum_{\mu \in I_\alpha^* \setminus I_\nu^*} \theta_\mu \delta_\mu \\ &= -\Bigg(\sum_{\mu \in I_\nu^*} |\theta_\mu|\, \delta_\mu\Bigg) + \sum_{\mu \in I_\alpha^* \setminus I_\nu^*} \theta_\mu \delta_\mu.\end{aligned}$$

Set $\gamma = \sum_{\mu \in I_\nu^*} |\theta_\mu|\, \delta_\mu$. Accordingly, $\delta + \gamma = \gcd(\delta_\nu :\ \nu \in I_\alpha^*) + \gamma = \sum_{\mu \in I_\alpha^* \setminus I_\nu^*} \theta_\mu \delta_\mu$. Since a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-open set of rank $\delta \in \{\delta_\nu :\ \nu \in I_\alpha^*\}$ is also $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-open of rank $\delta + \gamma \in \{\delta_\nu + \gamma :\ \nu \in I_\alpha^*\}$, $0 \prec \gamma$, it follows that $\mathscr{U}_{\mathfrak{g}}^{(\gamma)} \in \mathscr{P}\big(\mathscr{O}_{\mathfrak{g}}^{(\delta)}\big)$ implies $\mathscr{U}_{\mathfrak{g}}^{(\gamma)} \in \mathscr{Q}\big(\mathscr{O}_{\mathfrak{g}}^{(\gamma)}\big)$, with $\mathscr{O}_{\mathfrak{g}}^{(\delta)} \subset \mathfrak{T}_{\mathfrak{g}}$ being $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-open of rank $\delta \in \{\delta_\nu :\ \nu \in I_\alpha^*\}$. Moreover, $\mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\delta)}\big(\mathscr{U}_{\mathfrak{g}}^{(\gamma)}\big) \in \mathscr{Q}\big(\mathscr{O}_{\mathfrak{g}}^{(\gamma+\delta)}\big)$ for any $\mathscr{U}_{\mathfrak{g}}^{(\gamma)} \in \mathscr{P}\big(\mathscr{O}_{\mathfrak{g}}^{(\delta)}\big)$. But $\mathscr{Q}\big(\mathscr{O}_{\mathfrak{g}}^{(\gamma)}\big) \subseteq \mathscr{Q}\big(\mathscr{O}_{\mathfrak{g}}^{(\gamma+\delta)}\big)$ and consequently, $\mathscr{U}_{\mathfrak{g}}^{(\gamma+\delta)} \in \mathscr{P}\big(\mathscr{O}_{\mathfrak{g}}^{(\delta)}\big)$ implies $\mathscr{U}_{\mathfrak{g}}^{(\gamma+\delta)} \in \mathscr{Q}\big(\mathscr{O}_{\mathfrak{g}}^{(\gamma+\delta)}\big)$.

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II. Since non-zero negative and positive sets of integers $\{\vartheta_k \in \mathbb{Z}_-^* : k \in I_\nu^*\}$ and $\{\vartheta_k \in \mathbb{Z}_+^* : k \in I_\beta^* \setminus I_\nu^*\}$, respectively, exist, in actual fact $(|\vartheta_1|, |\vartheta_2|, \dots, |\vartheta_\nu|) \prec (\eta_1, \eta_2, \dots, \eta_\nu)$ and $(\vartheta_{\nu+1}, \vartheta_{\nu+2}, \dots, \vartheta_\beta) \prec (\eta_{\nu+1}, \eta_{\nu+2}, \dots, \eta_\beta)$, such that

$$\eta = \gcd\big(\eta_\nu : \nu \in I_\beta^*\big) = \sum_{\nu \in I_\beta^*} \vartheta_\nu \eta_\nu = \sum_{\mu \in I_\nu^*} \vartheta_\mu \eta_\mu + \sum_{\mu \in I_\beta^* \setminus I_\nu^*} \vartheta_\mu \eta_\mu$$

$$= -\Bigg(\sum_{\mu \in I_\nu^*} |\vartheta_\mu| \, \eta_\mu\Bigg) + \sum_{\mu \in I_\beta^* \setminus I_\nu^*} \vartheta_\mu \eta_\mu.$$

Set $\sigma = \sum_{\mu \in I_\nu^*} |\vartheta_\mu| \, \eta_\mu$. Accordingly, $\eta + \sigma = \gcd\big(\eta_\nu : \nu \in I_\beta^*\big) + \sigma = \sum_{\mu \in I_\beta^* \setminus I_\nu^*} \vartheta_\mu \eta_\mu$. Since a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-closed set of rank $\eta \in \{\eta_\nu : \nu \in I_\beta^*\}$ is also $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-closed of rank $\eta + \sigma \in \{\eta_\nu + \sigma : \nu \in I_\beta^*\}$, $0 \prec \sigma$, it follows that $\mathscr{V}_{\mathfrak{g}}^{(\sigma)} \in \mathscr{Q}\big(\mathscr{K}_{\mathfrak{g}}^{(\sigma)}\big)$ implies $\mathscr{V}_{\mathfrak{g}}^{(\sigma)} \in \mathscr{P}\big(\mathscr{K}_{\mathfrak{g}}^{(\eta)}\big)$, with $\mathscr{K}_{\mathfrak{g}}^{(\eta)} \subset \mathfrak{T}_{\mathfrak{g}}$ being $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-closed of rank $\eta \in \{\eta_\nu : \nu \in I_\beta^*\}$. Moreover, $\mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\eta)}\big(\mathscr{V}_{\mathfrak{g}}^{(\sigma)}\big) \in \mathscr{Q}\big(\mathscr{K}_{\mathfrak{g}}^{(\sigma+\eta)}\big)$ for any $\mathscr{V}_{\mathfrak{g}}^{(\sigma)} \in \mathscr{Q}\big(\mathscr{K}_{\mathfrak{g}}^{(\sigma)}\big)$. But $\mathscr{Q}\big(\mathscr{K}_{\mathfrak{g}}^{(\sigma)}\big) \supseteq \mathscr{Q}\big(\mathscr{O}_{\mathfrak{g}}^{(\sigma+\eta)}\big)$ and consequently, $\mathscr{V}_{\mathfrak{g}}^{(\sigma+\eta)} \in \mathscr{Q}\big(\mathscr{K}_{\mathfrak{g}}^{(\sigma+\eta)}\big)$ implies $\mathscr{V}_{\mathfrak{g}}^{(\sigma+\eta)} \in \mathscr{P}\big(\mathscr{K}_{\mathfrak{g}}^{(\eta)}\big)$. The proof of the theorem is complete. Q.E.D.

In the event that $\delta = \gcd(\delta_\nu : \nu \in I_\alpha^*) = 1$ and $\eta = \gcd\big(\eta_\nu : \nu \in I_\beta^*\big) = 1$, the members of the collections $\{\delta_\nu : \nu \in I_\alpha^*\}$ and $\{\eta_\nu : \nu \in I_\beta^*\}$ are said to be *relatively prime*. Accordingly, the following corollary is an immediate consequence of the above theorem.

Corollary 3.68. *Let $\mathscr{O}_{\mathfrak{g}}^{(\delta)} \subset \mathfrak{T}_{\mathfrak{g}}$ be a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-open set of rank $\delta \in \{\delta_\nu : \nu \in I_\alpha^*\}$ and $\mathscr{K}_{\mathfrak{g}}^{(\eta)} \subset \mathfrak{T}_{\mathfrak{g}}$ be a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-closed set of rank $\eta \in \{\eta_\nu : \nu \in I_\beta^*\}$ in a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$. Suppose $\delta = \gcd(\delta_\nu : \nu \in I_\alpha^*)$ and $\eta = \gcd\big(\eta_\nu : \nu \in I_\beta^*\big)$ be the greatest common divisors of $\delta_1, \delta_2, \dots, \delta_\alpha$ and $\eta_1, \eta_2, \dots, \eta_\beta$, respectively, satisfying $(\delta, \eta) = (1, 1)$, then:*

- I. $(\exists \gamma : 0 \prec \gamma)\Big[\bigwedge_{\mu = \gamma, \gamma+1} \big(\mathscr{Q}\big(\mathscr{O}_{\mathfrak{g}}^{(\mu)}\big) \supseteq \mathscr{P}\big(\mathscr{O}_{\mathfrak{g}}^{(\delta)}\big)\big)\Big]$,
- II. $(\exists \sigma : 0 \prec \sigma)\Big[\bigwedge_{\mu = \sigma, \sigma+1} \big(\mathscr{Q}\big(\mathscr{K}_{\mathfrak{g}}^{(\mu)}\big) \subseteq \mathscr{P}\big(\mathscr{K}_{\mathfrak{g}}^{(\eta)}\big)\big)\Big]$.

In a $\mathscr{T}_{\mathfrak{g}}$-space, suppose given a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-open set of ranks $\delta_1, \delta_2, \dots, \delta_\alpha$ and a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-closed set of ranks $\delta_1, \delta_2, \dots, \delta_\beta$, and suppose $\gcd(\delta_\nu : \nu \in I_\alpha^*) = 1$ and $\gcd\big(\eta_\nu : \nu \in I_\beta^*\big) = 1$, then there exist, in general, supersets of the $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-open set whose $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-coderived sets contain the $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-open, and subsets of the $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-closed set whose $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-derived sets are contained in the $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-closed set, as proved in the following proposition.

Proposition 3.69. *Let $\mathscr{O}_{\mathfrak{g}}^{(\delta)} \subset \mathfrak{T}_{\mathfrak{g}}$ be a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-open set of rank $\delta \in \{\delta_\nu : \nu \in I_\alpha^*\}$ and $\mathscr{K}_{\mathfrak{g}}^{(\eta)} \subset \mathfrak{T}_{\mathfrak{g}}$ be a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-closed set of rank $\eta \in \{\eta_\nu : \nu \in I_\beta^*\}$ in a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$. Suppose $\delta = \gcd(\delta_\nu : \nu \in I_\alpha^*)$ and $\eta = \gcd\big(\eta_\nu : \nu \in I_\beta^*\big)$ be the greatest common divisors of $\delta_1, \delta_2, \dots, \delta_\alpha$ and $\eta_1, \eta_2, \dots, \eta_\beta$, respectively, satisfying $(\delta, \eta) = (1, 1)$, then:*

- I. $\big(\exists \mathscr{O}_{\mathfrak{g}} \supseteq \mathscr{O}_{\mathfrak{g}}^{(\delta)}\big)\big[\mathfrak{g}\text{-Cod}_{\mathfrak{g}}(\mathscr{O}_{\mathfrak{g}}) \supseteq \mathscr{O}_{\mathfrak{g}}^{(\delta)}\big]$,
- II. $\big(\exists \mathscr{K}_{\mathfrak{g}} \subseteq \mathscr{K}_{\mathfrak{g}}^{(\eta)}\big)\big[\mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{K}_{\mathfrak{g}}) \subseteq \mathscr{K}_{\mathfrak{g}}^{(\eta)}\big]$.

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Proof. Let $\mathscr{O}_{\mathfrak{g}}^{(\delta)} \subset \mathfrak{T}_{\mathfrak{g}}$ be a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-open set of rank $\delta \in \{\delta_\nu : \nu \in I_\alpha^*\}$ and $\mathscr{K}_{\mathfrak{g}}^{(\eta)} \subset \mathfrak{T}_{\mathfrak{g}}$ be a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-closed set of rank $\eta \in \{\eta_\nu : \nu \in I_\beta^*\}$ in a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$. Suppose $\delta = \gcd(\delta_\nu : \nu \in I_\alpha^*)$ and $\eta = \gcd(\eta_\nu : \nu \in I_\beta^*)$ be the greatest common divisors of $\delta_1, \delta_2, \ldots, \delta_\alpha$ and $\eta_1, \eta_2, \ldots, \eta_\beta$, respectively, satisfying $(\delta, \eta) = (1, 1)$. Then:

I. Since $\gcd(\delta_\nu : \nu \in I_\alpha^*) = 1$, non-zero negative and positive sets of integers $\{\theta_k \in \mathbb{Z}_-^* : k \in I_\nu^*\}$ and $\{\theta_k \in \mathbb{Z}_+^* : k \in I_\alpha^* \setminus I_\nu^*\}$, respectively, exist such that

$$\gcd(\delta_\nu : \nu \in I_\alpha^*) = \sum_{\nu \in I_\alpha^*} \theta_\nu \delta_\nu = -\left(\sum_{\mu \in I_\nu^*} |\theta_\mu| \, \delta_\mu\right) + \sum_{\mu \in I_\alpha^* \setminus I_\nu^*} \theta_\mu \delta_\mu = 1.$$

Set $\gamma = \sum_{\mu \in I_\nu^*} |\theta_\mu| \, \delta_\mu$. Accordingly, $\gcd(\delta_\nu : \nu \in I_\alpha^*) + \gamma = 1 + \gamma = \sum_{\mu \in I_\alpha^* \setminus I_\nu^*} \theta_\mu \delta_\mu$. Since a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-open set of rank $\delta \in \{\delta_\nu : \nu \in I_\alpha^*\}$ is also $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-open of rank $\delta + \gamma \in \{\delta_\nu + \gamma : \nu \in I_\alpha^*\}$, $0 \prec \gamma$, it follows that $\mathscr{U}_{\mathfrak{g}}^{(\gamma)} \in \mathscr{P}(\mathscr{O}_{\mathfrak{g}}^{(\delta)})$ implies $\mathscr{U}_{\mathfrak{g}}^{(\gamma)} \in \mathscr{Q}(\mathscr{O}_{\mathfrak{g}}^{(\gamma)})$, with $\mathscr{O}_{\mathfrak{g}}^{(\delta)} \subset \mathfrak{T}_{\mathfrak{g}}$ being $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-open of rank $\delta \in \{\delta_\nu : \nu \in I_\alpha^*\}$. Moreover, $\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}(\mathscr{U}_{\mathfrak{g}}^{(\gamma)}) = \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}^{(1)}(\mathscr{U}_{\mathfrak{g}}^{(\gamma)}) \in \mathscr{Q}(\mathscr{O}_{\mathfrak{g}}^{(\gamma+1)})$ for any $\mathscr{U}_{\mathfrak{g}}^{(\gamma)} \in \mathscr{P}(\mathscr{O}_{\mathfrak{g}}^{(\delta)})$. But $\mathscr{Q}(\mathscr{O}_{\mathfrak{g}}^{(\gamma)}) \subseteq \mathscr{Q}(\mathscr{O}_{\mathfrak{g}}^{(\gamma+1)})$ and consequently, $\mathscr{U}_{\mathfrak{g}}^{(\gamma+1)} \in \mathscr{P}(\mathscr{O}_{\mathfrak{g}}^{(\delta)})$ implies $\mathscr{U}_{\mathfrak{g}}^{(\gamma+1)} \in \mathscr{Q}(\mathscr{O}_{\mathfrak{g}}^{(\gamma+1)})$. Set $\mathscr{O}_{\mathfrak{g}} = \mathscr{U}_{\mathfrak{g}}^{(\gamma)}$. Then,

$$\mathscr{U}_{\mathfrak{g}}^{(\gamma)} \cap \mathscr{U}_{\mathfrak{g}}^{(\gamma+1)} \longleftrightarrow \mathscr{U}_{\mathfrak{g}}^{(\gamma)} \cap \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}(\mathscr{U}_{\mathfrak{g}}^{(\gamma)}) \longleftrightarrow \mathscr{O}_{\mathfrak{g}} \cap \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}(\mathscr{O}_{\mathfrak{g}}) \supseteq \mathscr{O}_{\mathfrak{g}}^{(\delta)}.$$

Thus, $\mathscr{O}_{\mathfrak{g}} \supseteq \mathscr{O}_{\mathfrak{g}}^{(\delta)}$ exists such that $\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}(\mathscr{O}_{\mathfrak{g}}) \supseteq \mathscr{O}_{\mathfrak{g}}^{(\delta)}$.

II. Since $\gcd(\eta_\nu : \nu \in I_\beta^*) = 1$, non-zero negative and positive sets of integers $\{\vartheta_k \in \mathbb{Z}_-^* : k \in I_\nu^*\}$ and $\{\vartheta_k \in \mathbb{Z}_+^* : k \in I_\beta^* \setminus I_\nu^*\}$, respectively, exist such that

$$\gcd(\eta_\nu : \nu \in I_\beta^*) = \sum_{\nu \in I_\beta^*} \vartheta_\nu \eta_\nu = -\left(\sum_{\mu \in I_\nu^*} |\vartheta_\mu| \, \eta_\mu\right) + \sum_{\mu \in I_\beta^* \setminus I_\nu^*} \vartheta_\mu \eta_\mu = 1.$$

Set $\sigma = \sum_{\mu \in I_\nu^*} |\vartheta_\mu| \, \eta_\mu$. Accordingly, $\gcd(\eta_\nu : \nu \in I_\beta^*) + \sigma = 1 + \sigma = \sum_{\mu \in I_\beta^* \setminus I_\nu^*} \vartheta_\mu \eta_\mu$. Since a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-closed set of rank $\eta \in \{\eta_\nu : \nu \in I_\beta^*\}$ is also $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-closed of rank $\eta + \sigma \in \{\eta_\nu + \sigma : \nu \in I_\beta^*\}$, $0 \prec \sigma$, it follows that $\mathscr{V}_{\mathfrak{g}}^{(\sigma)} \in \mathscr{Q}(\mathscr{K}_{\mathfrak{g}}^{(\sigma)})$ implies $\mathscr{V}_{\mathfrak{g}}^{(\sigma)} \in \mathscr{P}(\mathscr{K}_{\mathfrak{g}}^{(\eta)})$, with $\mathscr{K}_{\mathfrak{g}}^{(\eta)} \subset \mathfrak{T}_{\mathfrak{g}}$ being $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-closed of rank $\eta \in \{\eta_\nu : \nu \in I_\beta^*\}$. Moreover, $\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}(\mathscr{V}_{\mathfrak{g}}^{(\sigma)}) = \mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}^{(1)}(\mathscr{V}_{\mathfrak{g}}^{(\sigma)}) \in \mathscr{Q}(\mathscr{K}_{\mathfrak{g}}^{(\sigma+1)})$ for any $\mathscr{V}_{\mathfrak{g}}^{(\sigma)} \in \mathscr{Q}(\mathscr{K}_{\mathfrak{g}}^{(\sigma)})$. But $\mathscr{Q}(\mathscr{K}_{\mathfrak{g}}^{(\sigma)}) \supseteq \mathscr{Q}(\mathscr{O}_{\mathfrak{g}}^{(\sigma+1)})$ and consequently, $\mathscr{V}_{\mathfrak{g}}^{(\sigma+1)} \in \mathscr{Q}(\mathscr{K}_{\mathfrak{g}}^{(\sigma+1)})$ implies $\mathscr{V}_{\mathfrak{g}}^{(\sigma+1)} \in \mathscr{P}(\mathscr{K}_{\mathfrak{g}}^{(\eta)})$. Set $\mathscr{K}_{\mathfrak{g}} = \mathscr{V}_{\mathfrak{g}}^{(\sigma)}$. Then,

$$\mathscr{V}_{\mathfrak{g}}^{(\sigma)} \cup \mathscr{V}_{\mathfrak{g}}^{(\sigma+1)} \longleftrightarrow \mathscr{V}_{\mathfrak{g}}^{(\sigma)} \cup \mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}(\mathscr{V}_{\mathfrak{g}}^{(\sigma)}) \longleftrightarrow \mathscr{K}_{\mathfrak{g}} \cup \mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}(\mathscr{K}_{\mathfrak{g}}) \subseteq \mathscr{K}_{\mathfrak{g}}^{(\eta)}.$$

Thus, $\mathscr{K}_{\mathfrak{g}} \subseteq \mathscr{K}_{\mathfrak{g}}^{(\eta)}$ exists such that $\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}(\mathscr{K}_{\mathfrak{g}}) \subseteq \mathscr{K}_{\mathfrak{g}}^{(\eta)}$. The proof of the proposition is complete. Q.E.D.

In a $\mathscr{T}_{\mathfrak{g}}$-space, suppose given a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-open set of rank δ and a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-closed set of rank η, then $\mathscr{P}(\mathscr{O}_{\mathfrak{g}}^{(\delta)})$, $\mathscr{Q}(\mathscr{O}_{\mathfrak{g}}^{(\delta)})$, $\mathscr{Q}(\mathscr{O}_{\mathfrak{g}}^{(2\delta)})$, …, $\mathscr{Q}(\mathscr{O}_{\mathfrak{g}}^{(\alpha\delta)})$, … forms a monotone increasing sequence of collections of sets and, $\mathscr{P}(\mathscr{K}_{\mathfrak{g}}^{(\eta)})$, $\mathscr{Q}(\mathscr{K}_{\mathfrak{g}}^{(\eta)})$, $\mathscr{Q}(\mathscr{K}_{\mathfrak{g}}^{(2\eta)})$, …, $\mathscr{Q}(\mathscr{K}_{\mathfrak{g}}^{(\beta\eta)})$, … forms a monotone decreasing sequences of collections of sets, as proved in the following theorem.

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THEOREM 3.70. *If $\mathscr{O}_{\mathfrak{g}}^{(\delta)} \subset \mathfrak{T}_{\mathfrak{g}}$ be a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-open set of rank δ and $\mathscr{K}_{\mathfrak{g}}^{(\eta)} \subset \mathfrak{T}_{\mathfrak{g}}$ be a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-closed set of rank η in a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$, then $\langle \mathscr{P}(\mathscr{O}_{\mathfrak{g}}^{(\delta)}), \mathscr{Q}(\mathscr{O}_{\mathfrak{g}}^{(\delta)}), \mathscr{Q}(\mathscr{O}_{\mathfrak{g}}^{(2\delta)}), \ldots, \mathscr{Q}(\mathscr{O}_{\mathfrak{g}}^{(\alpha\delta)}), \ldots \rangle$ and $\langle \mathscr{P}(\mathscr{K}_{\mathfrak{g}}^{(\eta)}), \mathscr{Q}(\mathscr{K}_{\mathfrak{g}}^{(\eta)}), \mathscr{Q}(\mathscr{K}_{\mathfrak{g}}^{(2\eta)}), \ldots, \mathscr{Q}(\mathscr{K}_{\mathfrak{g}}^{(\beta\eta)}), \ldots \rangle$, respectively, are monotone increasing and monotone decreasing sequences:*

- I. $\mathscr{P}(\mathscr{O}_{\mathfrak{g}}^{(\delta)}) \subseteq \mathscr{Q}(\mathscr{O}_{\mathfrak{g}}^{(\delta)}) \subseteq \mathscr{Q}(\mathscr{O}_{\mathfrak{g}}^{(2\delta)}) \subseteq \cdots \subseteq \mathscr{Q}(\mathscr{O}_{\mathfrak{g}}^{(\alpha\delta)}) \subseteq \cdots$,
- II. $\mathscr{P}(\mathscr{K}_{\mathfrak{g}}^{(\eta)}) \supseteq \mathscr{Q}(\mathscr{K}_{\mathfrak{g}}^{(\eta)}) \supseteq \mathscr{Q}(\mathscr{K}_{\mathfrak{g}}^{(2\eta)}) \supseteq \cdots \supseteq \mathscr{Q}(\mathscr{K}_{\mathfrak{g}}^{(\beta\eta)}) \supseteq \cdots$.

PROOF. Let $\mathscr{O}_{\mathfrak{g}}^{(\delta)} \subset \mathfrak{T}_{\mathfrak{g}}$ be a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-open set of rank δ and $\mathscr{K}_{\mathfrak{g}}^{(\eta)} \subset \mathfrak{T}_{\mathfrak{g}}$ be a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-closed set of rank η in a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$. Then:

I. Suppose $\mathscr{O}_{\mathfrak{g}} \in \mathscr{Q}(\mathscr{O}_{\mathfrak{g}}^{(\delta)})$ hold. Then, $\mathscr{U}_{\mathfrak{g}}^{(\delta)} = \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}^{(\delta)}(\mathscr{O}_{\mathfrak{g}}) \in \mathscr{Q}(\mathscr{O}_{\mathfrak{g}}^{(2\delta)})$ also holds. Because of $\mathscr{U}_{\mathfrak{g}}^{(\delta)} \in \mathscr{Q}(\mathscr{O}_{\mathfrak{g}}^{(2\delta)})$, the relation $\mathscr{O}_{\mathfrak{g}} \in \mathscr{P}(\mathscr{O}_{\mathfrak{g}}^{(\delta)})$ necessarily holds also and, moreover, each $\mathscr{O}_{\mathfrak{g}} \in \mathscr{P}(\mathscr{O}_{\mathfrak{g}}^{(\delta)})$ has a δ^{th} $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-coderived set $\mathscr{U}_{\mathfrak{g}}^{(2\delta)} = \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}^{(\delta)}(\mathscr{U}_{\mathfrak{g}}^{(\delta)}) = \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}^{(2\delta)}(\mathscr{O}_{\mathfrak{g}}) \in \mathscr{Q}(\mathscr{K}_{\mathfrak{g}}^{(2\delta)})$. Consequently, it results that $\mathscr{P}(\mathscr{O}_{\mathfrak{g}}^{(\delta)}) \subseteq \mathscr{Q}(\mathscr{O}_{\mathfrak{g}}^{(\delta)}) \subseteq \mathscr{Q}(\mathscr{O}_{\mathfrak{g}}^{(2\delta)})$ holds true. Furthermore, suppose $\mathscr{P}(\mathscr{O}_{\mathfrak{g}}^{(\delta)}) \subseteq \mathscr{Q}(\mathscr{O}_{\mathfrak{g}}^{(\delta)}) \subseteq \mathscr{Q}(\mathscr{O}_{\mathfrak{g}}^{(2\delta)}) \subseteq \cdots \subseteq \mathscr{Q}(\mathscr{O}_{\mathfrak{g}}^{(\alpha\delta)})$ holds true for some α such that $0 \prec \alpha$ and $\mathscr{O}_{\mathfrak{g}} \in \mathscr{Q}(\mathscr{O}_{\mathfrak{g}}^{((\alpha-1)\delta)})$, then the relation $\mathscr{U}_{\mathfrak{g}}^{(\delta)} = \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}^{(\delta)}(\mathscr{O}_{\mathfrak{g}}) \in \mathscr{Q}(\mathscr{O}_{\mathfrak{g}}^{(\alpha\delta)})$ also holds true because, $\mathscr{O}_{\mathfrak{g}} \in \mathscr{Q}(\mathscr{O}_{\mathfrak{g}}^{(\alpha\delta)})$ by inductive hypothesis and, $\mathscr{U}_{\mathfrak{g}}^{(\alpha\delta)} = \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}^{(\delta)}(\mathscr{U}_{\mathfrak{g}}^{((\alpha-1)\delta)}) = \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}^{(\alpha\delta)}(\mathscr{O}_{\mathfrak{g}}) \in \mathscr{Q}(\mathscr{O}_{\mathfrak{g}}^{(\alpha\delta)})$ holds true for every $\mathscr{U}_{\mathfrak{g}}^{((\alpha-1)\delta)} \in \mathscr{Q}(\mathscr{O}_{\mathfrak{g}}^{((\alpha-1)\delta)})$. Hence, $\mathscr{Q}(\mathscr{O}_{\mathfrak{g}}^{((\alpha-1)\delta)}) \subseteq \mathscr{Q}(\mathscr{O}_{\mathfrak{g}}^{(\alpha\delta)})$ and the induction is complete.

II. Suppose $\mathscr{K}_{\mathfrak{g}} \in \mathscr{Q}(\mathscr{K}_{\mathfrak{g}}^{(\eta)})$ hold. Then, $\mathscr{V}_{\mathfrak{g}}^{(\eta)} = \mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}^{(\eta)}(\mathscr{K}_{\mathfrak{g}}) \in \mathscr{Q}(\mathscr{K}_{\mathfrak{g}}^{(\eta)})$ also holds. Because of $\mathscr{V}_{\mathfrak{g}}^{(\eta)} \in \mathscr{Q}(\mathscr{K}_{\mathfrak{g}}^{(\eta)})$, the relation $\mathscr{K}_{\mathfrak{g}} \in \mathscr{P}(\mathscr{K}_{\mathfrak{g}}^{(\eta)})$ necessarily holds also and, moreover, each $\mathscr{K}_{\mathfrak{g}} \in \mathscr{P}(\mathscr{K}_{\mathfrak{g}}^{(\eta)})$ has a η^{th} $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-derived set $\mathscr{V}_{\mathfrak{g}}^{(2\eta)} = \mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}^{(\eta)}(\mathscr{V}_{\mathfrak{g}}^{(\eta)}) = \mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}^{(2\eta)}(\mathscr{K}_{\mathfrak{g}}) \in \mathscr{Q}(\mathscr{K}_{\mathfrak{g}}^{(\eta)})$. Consequently, it results that $\mathscr{P}(\mathscr{K}_{\mathfrak{g}}^{(\eta)}) \supseteq \mathscr{Q}(\mathscr{K}_{\mathfrak{g}}^{(\eta)}) \supseteq \mathscr{Q}(\mathscr{K}_{\mathfrak{g}}^{(2\eta)})$ holds true. Furthermore, suppose $\mathscr{P}(\mathscr{K}_{\mathfrak{g}}^{(\eta)}) \supseteq \mathscr{Q}(\mathscr{K}_{\mathfrak{g}}^{(\eta)}) \supseteq \mathscr{Q}(\mathscr{K}_{\mathfrak{g}}^{(2\eta)}) \supseteq \cdots \supseteq \mathscr{Q}(\mathscr{K}_{\mathfrak{g}}^{(\beta\eta)})$ holds true for some β such that $0 \prec \beta$ and $\mathscr{K}_{\mathfrak{g}} \in \mathscr{Q}(\mathscr{K}_{\mathfrak{g}}^{(\beta\eta)})$, then the relation $\mathscr{V}_{\mathfrak{g}}^{(\eta)} = \mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}^{(\eta)}(\mathscr{K}_{\mathfrak{g}}) \in \mathscr{Q}(\mathscr{K}_{\mathfrak{g}}^{(\beta\eta)})$ also holds true because, $\mathscr{K}_{\mathfrak{g}} \in \mathscr{Q}(\mathscr{K}_{\mathfrak{g}}^{((\beta-1)\eta)})$ by inductive hypothesis and, $\mathscr{V}_{\mathfrak{g}}^{(\beta\eta)} = \mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}^{(\eta)}(\mathscr{V}_{\mathfrak{g}}^{((\beta-1)\eta)}) = \mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}^{(\beta\eta)}(\mathscr{K}_{\mathfrak{g}}) \in \mathscr{Q}(\mathscr{K}_{\mathfrak{g}}^{(\beta\eta)})$ holds true for every $\mathscr{V}_{\mathfrak{g}}^{((\beta-1)\eta)} \in \mathscr{Q}(\mathscr{K}_{\mathfrak{g}}^{((\beta-1)\eta)})$. Hence, $\mathscr{Q}(\mathscr{K}_{\mathfrak{g}}^{((\beta-1)\eta)}) \supseteq \mathscr{Q}(\mathscr{K}_{\mathfrak{g}}^{(\beta\eta)})$ and the induction is complete. The proof of the theorem is therefore complete. Q.E.D.

On the essential properties of the concepts of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-open and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-closed sets of rank δ in $\mathscr{T}_{\mathfrak{g}}$-spaces, the discussion of the present section terminates here.

The categorical classifications of $\mathfrak{g}$-$\mathfrak{T}$-derived and $\mathfrak{g}$-$\mathfrak{T}$-coderived operators, their δ^{th}-iterates called, respectively, $\mathfrak{g}$-$\mathfrak{T}$-derived and $\mathfrak{g}$-$\mathfrak{T}$-coderived operators of order δ, and $\mathfrak{g}$-$\mathfrak{T}$-open and $\mathfrak{g}$-$\mathfrak{T}$-closed sets of rank δ in the $\mathscr{T}$-space $\mathfrak{T} \subset \mathfrak{T}_{\mathfrak{g}}$ and, $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-derived and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-coderived operators, their δ^{th}-iterates called, respectively, $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-derived and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-coderived operators of order δ, and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-open and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-closed sets of rank δ in the $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}}$ are discussed and diagrammed on this ground in the next sections.

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4. Discussion

4.1. Categorical Classifications. Having adopted a categorical approach in the classifications of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-derived and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-coderived operators, their δ^{th}-iterates called, respectively, $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-derived and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-coderived operators of order δ, and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-open and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-closed sets of rank δ, the twofold purposes here are to establish the various relationships between the classes of $\mathfrak{g}$-$\mathfrak{T}$-derived and $\mathfrak{g}$-$\mathfrak{T}$-coderived operators in the $\mathscr{T}$-space $\mathfrak{T}$ and the classes of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-derived and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-coderived operators in the $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}}$, the classes of their δ^{th}-iterates in their $\mathscr{T}$, $\mathscr{T}_{\mathfrak{g}}$-spaces $\mathfrak{T}$, $\mathfrak{T}_{\mathfrak{g}}$, respectively, the classes of $\mathfrak{g}$-$\mathfrak{T}$-open and $\mathfrak{g}$-$\mathfrak{T}$-closed sets of rank δ in the $\mathscr{T}$-space $\mathfrak{T}$ and the classes of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-open and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-closed sets of rank δ in the $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}}$, and to illustrate them through diagrams.

In a $\mathscr{T}$-space $\mathfrak{T}=(\Omega,\mathscr{T})$, the relation $\mathfrak{g}\text{-Int}_0(\mathscr{S}_{\mathfrak{g}})\subseteq\mathfrak{g}\text{-Int}_1(\mathscr{S}_{\mathfrak{g}})\subseteq\mathfrak{g}\text{-Int}_3(\mathscr{S}_{\mathfrak{g}})\supseteq\mathfrak{g}\text{-Int}_2(\mathscr{S}_{\mathfrak{g}})$ holds for any $\mathscr{S}_{\mathfrak{g}}\in\mathscr{P}(\Omega)$. Likewise, in a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}}=(\Omega,\mathscr{T}_{\mathfrak{g}})$, the relation $\mathfrak{g}\text{-Int}_{\mathfrak{g},0}(\mathscr{S}_{\mathfrak{g}})\subseteq\mathfrak{g}\text{-Int}_{\mathfrak{g},1}(\mathscr{S}_{\mathfrak{g}})\subseteq\mathfrak{g}\text{-Int}_{\mathfrak{g},3}(\mathscr{S}_{\mathfrak{g}})\supseteq\mathfrak{g}\text{-Int}_{\mathfrak{g},2}(\mathscr{S}_{\mathfrak{g}})$ holds for any $\mathscr{S}_{\mathfrak{g}}\in\mathscr{P}(\Omega)$. Moreover, the relation $\mathfrak{g}\text{-Int}_{\nu}(\mathscr{S}_{\mathfrak{g}})\subseteq\mathfrak{g}\text{-Int}_{\mathfrak{g},\nu}(\mathscr{S}_{\mathfrak{g}})$ also holds true for any $(\nu,\mathscr{S}_{\mathfrak{g}})\in I_3^0\times\mathfrak{T}_{\mathfrak{g}}$. But, for every $(\nu,\mathscr{S}_{\mathfrak{g}})\in I_3^0\times\mathfrak{T}_{\mathfrak{g}}$, the relations $\mathfrak{g}\text{-Int}_{\nu}(\mathscr{S}_{\mathfrak{g}})\longleftrightarrow\mathscr{S}_{\mathfrak{g}}\cap\mathfrak{g}\text{-Cod}_{\nu}(\mathscr{S}_{\mathfrak{g}})$, $\mathfrak{g}\text{-Int}_{\mathfrak{g},\nu}(\mathscr{S}_{\mathfrak{g}})\longleftrightarrow\mathscr{S}_{\mathfrak{g}}\cap\mathfrak{g}\text{-Cod}_{\mathfrak{g},\nu}(\mathscr{S}_{\mathfrak{g}})$ and $\left(\mathfrak{g}\text{-Cod}_{\nu}(\mathscr{S}_{\mathfrak{g}}),\mathfrak{g}\text{-Cod}_{\mathfrak{g},\nu}(\mathscr{S}_{\mathfrak{g}})\right)\supseteq\left(\text{cod}(\mathscr{S}_{\mathfrak{g}}),\text{cod}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})\right)$ hold true. Consequently, this diagram, which is to be read horizontally, from left to right and vertically, from top to bottom, follows:

$$\begin{array}{ccccccc}
\text{cod}(\mathscr{S}_{\mathfrak{g}}) & \subseteq & \text{cod}(\mathscr{S}_{\mathfrak{g}}) & \subseteq & \text{cod}(\mathscr{S}_{\mathfrak{g}}) & \supseteq & \text{cod}(\mathscr{S}_{\mathfrak{g}}) \\
\rotatebox{90}{$\supseteq$} & & \rotatebox{90}{$\supseteq$} & & \rotatebox{90}{$\supseteq$} & & \rotatebox{90}{$\supseteq$} \\
\mathfrak{g}\text{-Cod}_0(\mathscr{S}_{\mathfrak{g}}) & \subseteq & \mathfrak{g}\text{-Cod}_1(\mathscr{S}_{\mathfrak{g}}) & \subseteq & \mathfrak{g}\text{-Cod}_3(\mathscr{S}_{\mathfrak{g}}) & \supseteq & \mathfrak{g}\text{-Cod}_2(\mathscr{S}_{\mathfrak{g}}) \\
\rotatebox{90}{$\supseteq$} & & \rotatebox{90}{$\supseteq$} & & \rotatebox{90}{$\supseteq$} & & \rotatebox{90}{$\supseteq$} \\
\mathfrak{g}\text{-Cod}_{\mathfrak{g},0}(\mathscr{S}_{\mathfrak{g}}) & \subseteq & \mathfrak{g}\text{-Cod}_{\mathfrak{g},1}(\mathscr{S}_{\mathfrak{g}}) & \subseteq & \mathfrak{g}\text{-Cod}_{\mathfrak{g},3}(\mathscr{S}_{\mathfrak{g}}) & \supseteq & \mathfrak{g}\text{-Cod}_{\mathfrak{g},2}(\mathscr{S}_{\mathfrak{g}}) \\
\rotatebox{90}{$\subseteq$} & & \rotatebox{90}{$\subseteq$} & & \rotatebox{90}{$\subseteq$} & & \rotatebox{90}{$\subseteq$} \\
\text{cod}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) & \subseteq & \text{cod}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) & \subseteq & \text{cod}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) & \supseteq & \text{cod}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})
\end{array} \tag{4.1}$$

In Fig. 1, we present the relationships between the elements of the collections $\{\mathfrak{g}\text{-Cod}_{\nu}:\mathscr{S}_{\mathfrak{g}}\longmapsto\mathfrak{g}\text{-Cod}_{\nu}(\mathscr{S}_{\mathfrak{g}}):\ \nu\in I_3^0\}$ in the $\mathscr{T}$-space $\mathfrak{T}\subset\mathfrak{T}_{\mathfrak{g}}$ and $\{\mathfrak{g}\text{-Cod}_{\mathfrak{g},\nu}:\mathscr{S}_{\mathfrak{g}}\longmapsto\mathfrak{g}\text{-Cod}_{\mathfrak{g},\nu}(\mathscr{S}_{\mathfrak{g}}):\ \nu\in I_3^0\}$ in the $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}}\supset\mathfrak{T}$; Fig. 1 may well be called a $(\mathfrak{g}\text{-Cod},\mathfrak{g}\text{-Cod}_{\mathfrak{g}})$-*valued diagram.*

In a $\mathscr{T}$-space $\mathfrak{T}=(\Omega,\mathscr{T})$, the relation $\mathfrak{g}\text{-Cl}_0(\mathscr{S}_{\mathfrak{g}})\supseteq\mathfrak{g}\text{-Cl}_1(\mathscr{S}_{\mathfrak{g}})\supseteq\mathfrak{g}\text{-Cl}_3(\mathscr{S}_{\mathfrak{g}})\subseteq\mathfrak{g}\text{-Cl}_2(\mathscr{S}_{\mathfrak{g}})$ holds for any $\mathscr{S}_{\mathfrak{g}}\in\mathscr{P}(\Omega)$. Likewise, in a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}}=(\Omega,\mathscr{T}_{\mathfrak{g}})$, the relation $\mathfrak{g}\text{-Cl}_{\mathfrak{g},0}(\mathscr{S}_{\mathfrak{g}})\supseteq\mathfrak{g}\text{-Cl}_{\mathfrak{g},1}(\mathscr{S}_{\mathfrak{g}})\supseteq\mathfrak{g}\text{-Cl}_{\mathfrak{g},3}(\mathscr{S}_{\mathfrak{g}})\subseteq\mathfrak{g}\text{-Cl}_{\mathfrak{g},2}(\mathscr{S}_{\mathfrak{g}})$ holds for any $\mathscr{S}_{\mathfrak{g}}\in\mathscr{P}(\Omega)$. Moreover, the relation $\mathfrak{g}\text{-Cl}_{\nu}(\mathscr{S}_{\mathfrak{g}})\supseteq\mathfrak{g}\text{-Cl}_{\mathfrak{g},\nu}(\mathscr{S}_{\mathfrak{g}})$ also holds true for any $(\nu,\mathscr{S}_{\mathfrak{g}})\in I_3^0\times\mathfrak{T}_{\mathfrak{g}}$. But, for every $(\nu,\mathscr{S}_{\mathfrak{g}})\in I_3^0\times\mathfrak{T}_{\mathfrak{g}}$, the relations $\mathfrak{g}\text{-Cl}_{\mathfrak{g},\nu}(\mathscr{S}_{\mathfrak{g}})\longleftrightarrow\mathscr{S}_{\mathfrak{g}}\cup\mathfrak{g}\text{-Der}_{\mathfrak{g},\nu}(\mathscr{S}_{\mathfrak{g}})$, $\mathfrak{g}\text{-Cl}_{\mathfrak{g},\nu}(\mathscr{S}_{\mathfrak{g}})\longleftrightarrow\mathscr{S}_{\mathfrak{g}}\cup\mathfrak{g}\text{-Der}_{\mathfrak{g},\nu}(\mathscr{S}_{\mathfrak{g}})$ and $\left(\mathfrak{g}\text{-Der}_{\nu}(\mathscr{S}_{\mathfrak{g}}),\mathfrak{g}\text{-Der}_{\mathfrak{g},\nu}(\mathscr{S}_{\mathfrak{g}})\right)\subseteq\left(\text{der}(\mathscr{S}_{\mathfrak{g}}),\text{der}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})\right)$ hold true. Consequently, this diagram, which is to be read horizontally, from left to right and vertically,

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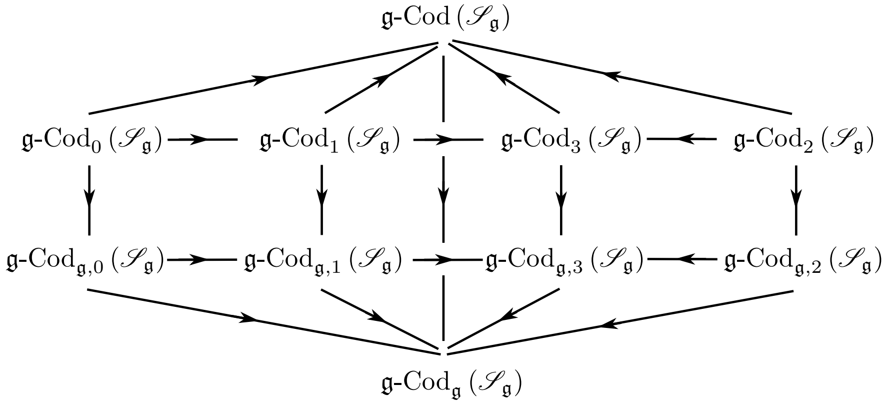

FIGURE 1. Relationships: $\mathfrak{g}$-$\mathfrak{T}$-coderived operators in $\mathscr{T}$-spaces and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-coderived operators in $\mathscr{T}_{\mathfrak{g}}$-spaces.

from top to bottom, follows:

$$
\begin{array}{ccccccc}
\operatorname{der}(\mathscr{S}_{\mathfrak{g}}) & \supseteq & \operatorname{der}(\mathscr{S}_{\mathfrak{g}}) & \supseteq & \operatorname{der}(\mathscr{S}_{\mathfrak{g}}) & \subseteq & \operatorname{der}(\mathscr{S}_{\mathfrak{g}}) \\
\cup\!| & & \cup\!| & & \cup\!| & & \cup\!| \\
\mathfrak{g}\text{-Der}_0(\mathscr{S}_{\mathfrak{g}}) & \supseteq & \mathfrak{g}\text{-Der}_1(\mathscr{S}_{\mathfrak{g}}) & \supseteq & \mathfrak{g}\text{-Der}_3(\mathscr{S}_{\mathfrak{g}}) & \subseteq & \mathfrak{g}\text{-Der}_2(\mathscr{S}_{\mathfrak{g}}) \\
\cup\!| & & \cup\!| & & \cup\!| & & \cup\!| \\
\mathfrak{g}\text{-Der}_{\mathfrak{g},0}(\mathscr{S}_{\mathfrak{g}}) & \supseteq & \mathfrak{g}\text{-Der}_{\mathfrak{g},1}(\mathscr{S}_{\mathfrak{g}}) & \supseteq & \mathfrak{g}\text{-Der}_{\mathfrak{g},3}(\mathscr{S}_{\mathfrak{g}}) & \subseteq & \mathfrak{g}\text{-Der}_{\mathfrak{g},2}(\mathscr{S}_{\mathfrak{g}}) \\
\cap\!| & & \cap\!| & & \cap\!| & & \cap\!| \\
\operatorname{der}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) & \supseteq & \operatorname{der}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) & \supseteq & \operatorname{der}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) & \subseteq & \operatorname{der}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})
\end{array}
$$

(4.2)

In FIG. 2, we present the relationships between the elements of the collections $\{\mathfrak{g}\text{-Der}_{\nu} : \mathscr{S}_{\mathfrak{g}} \longmapsto \mathfrak{g}\text{-Der}_{\nu}(\mathscr{S}_{\mathfrak{g}}) : \ \nu \in I_3^0\}$ in the $\mathscr{T}$-space $\mathfrak{T} \subset \mathfrak{T}_{\mathfrak{g}}$ and $\{\mathfrak{g}\text{-Der}_{\mathfrak{g},\nu} : \mathscr{S}_{\mathfrak{g}} \longmapsto \mathfrak{g}\text{-Der}_{\mathfrak{g},\nu}(\mathscr{S}_{\mathfrak{g}}) : \ \nu \in I_3^0\}$ in the $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} \supset \mathfrak{T}$; FIG. 2 may well be called a $(\mathfrak{g}\text{-Der}, \mathfrak{g}\text{-Der}_{\mathfrak{g}})$*-valued diagram.*

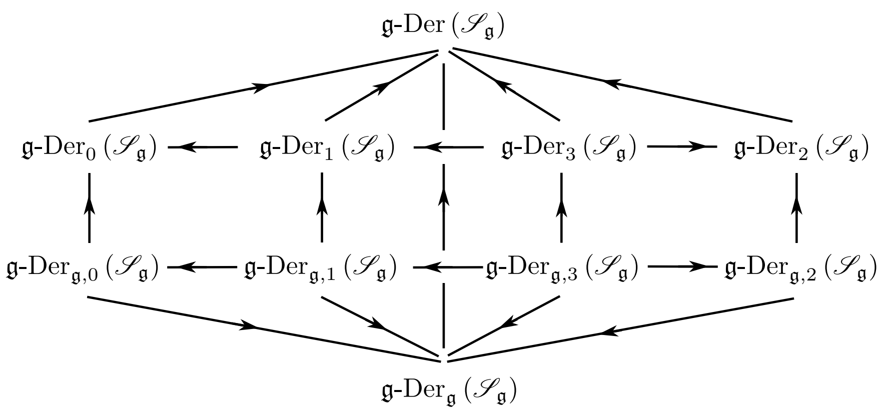

FIGURE 2. Relationships: $\mathfrak{g}$-$\mathfrak{T}$-derived operators in $\mathscr{T}$-spaces and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-derived operators in $\mathscr{T}_{\mathfrak{g}}$-spaces.

The various relationships between the classes of $\mathfrak{g}$-$\mathfrak{T}$-derived and $\mathfrak{g}$-$\mathfrak{T}$-coderived operators in the $\mathscr{T}$-space $\mathfrak{T}$ and the classes of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-derived and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-coderived operators in the $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}}$ are therefore established. The various relationships between the classes of their δ^{th}-iterates in their $\mathscr{T}$, $\mathscr{T}_{\mathfrak{g}}$-spaces $\mathfrak{T}$, $\mathfrak{T}_{\mathfrak{g}}$, respectively, will now be established based on the notions of *coarseness* (or, *smallness*, *weakness*), or alternatively, *finness* (or, *largeness*, *strongness*).

Let it be granted some pair $(\nu,\mu)\in I_3^0\times I_3^0$ of categories and some pair of ordinals (δ,η) such that $(1,1)\preccurlyeq(\delta,\eta)\prec(\lambda,\lambda)$. Suppose the relations "$\mathfrak{g}\text{-}\mathrm{Cod}_{\nu}^{(\eta)}\precsim\mathfrak{g}\text{-}\mathrm{Cod}_{\mu}^{(\delta)}$" and "$\mathfrak{g}\text{-}\mathrm{Der}_{\nu}^{(\eta)}\succsim\mathfrak{g}\text{-}\mathrm{Der}_{\mu}^{(\delta)}$" stand for "$\mathfrak{g}\text{-}\mathrm{Cod}_{\nu}^{(\eta)}(\mathscr{S}_{\mathfrak{g}})\subseteq\mathfrak{g}\text{-}\mathrm{Cod}_{\mu}^{(\delta)}(\mathscr{S}_{\mathfrak{g}})$" and "$\mathfrak{g}\text{-}\mathrm{Der}_{\nu}^{(\eta)}(\mathscr{S}_{\mathfrak{g}})\supseteq\mathfrak{g}\text{-}\mathrm{Der}_{\mu}^{(\delta)}(\mathscr{S}_{\mathfrak{g}})$," or equivalently, "$\mathfrak{g}\text{-}\mathrm{Cod}_{\mu}^{(\delta)}\succsim\mathfrak{g}\text{-}\mathrm{Cod}_{\nu}^{(\eta)}$" and "$\mathfrak{g}\text{-}\mathrm{Der}_{\mu}^{(\delta)}\precsim\mathfrak{g}\text{-}\mathrm{Der}_{\nu}^{(\eta)}$" stand for the relations "$\mathfrak{g}\text{-}\mathrm{Cod}_{\mu}^{(\delta)}(\mathscr{S}_{\mathfrak{g}})\supseteq\mathfrak{g}\text{-}\mathrm{Cod}_{\nu}^{(\eta)}(\mathscr{S}_{\mathfrak{g}})$" and "$\mathfrak{g}\text{-}\mathrm{Der}_{\mu}^{(\delta)}(\mathscr{S}_{\mathfrak{g}})\subseteq\mathfrak{g}\text{-}\mathrm{Der}_{\nu}^{(\eta)}(\mathscr{S}_{\mathfrak{g}})$," respectively, in a $\mathscr{T}$-space $\mathfrak{T}=(\Omega,\mathscr{T})$; "$\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g},\nu}^{(\eta)}\precsim\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g},\mu}^{(\delta)}$" and "$\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g},\nu}^{(\eta)}\succsim\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g},\mu}^{(\delta)}$" stand for "$\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g},\nu}^{(\eta)}(\mathscr{S}_{\mathfrak{g}})\subseteq\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g},\mu}^{(\delta)}(\mathscr{S}_{\mathfrak{g}})$" and "$\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g},\nu}^{(\eta)}(\mathscr{S}_{\mathfrak{g}})\supseteq\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g},\mu}^{(\delta)}(\mathscr{S}_{\mathfrak{g}})$," or equivalently, "$\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g},\mu}^{(\delta)}\succsim\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g},\nu}^{(\eta)}$" and "$\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g},\mu}^{(\delta)}\precsim\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g},\nu}^{(\eta)}$" stand for "$\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g},\mu}^{(\delta)}(\mathscr{S}_{\mathfrak{g}})\supseteq\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g},\nu}^{(\eta)}(\mathscr{S}_{\mathfrak{g}})$" and "$\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g},\mu}^{(\delta)}(\mathscr{S}_{\mathfrak{g}})\subseteq\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g},\nu}^{(\eta)}(\mathscr{S}_{\mathfrak{g}})$," respectively, in a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}}=(\Omega,\mathscr{T}_{\mathfrak{g}})$. Then, $\mathfrak{g}\text{-}\mathrm{Cod}_{\nu}^{(\eta)}$, $\mathfrak{g}\text{-}\mathrm{Der}_{\mu}^{(\delta)}:\mathscr{P}(\Omega)\longrightarrow\mathscr{P}(\Omega)$ are *coarser* (or, *smaller*, *weaker*) than $\mathfrak{g}\text{-}\mathrm{Cod}_{\mu}^{(\delta)}$, $\mathfrak{g}\text{-}\mathrm{Der}_{\nu}^{(\eta)}:\mathscr{P}(\Omega)\longrightarrow\mathscr{P}(\Omega)$ or, $\mathfrak{g}\text{-}\mathrm{Cod}_{\mu}^{(\delta)}$, $\mathfrak{g}\text{-}\mathrm{Der}_{\nu}^{(\eta)}:\mathscr{P}(\Omega)\longrightarrow\mathscr{P}(\Omega)$ are *finer* (or, *larger*, *stronger*) than $\mathfrak{g}\text{-}\mathrm{Cod}_{\nu}^{(\eta)}$, $\mathfrak{g}\text{-}\mathrm{Der}_{\mu}^{(\delta)}:\mathscr{P}(\Omega)\longrightarrow\mathscr{P}(\Omega)$; $\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g},\mu}^{(\delta)}$, $\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g},\nu}^{(\eta)}:\mathscr{P}(\Omega)\longrightarrow\mathscr{P}(\Omega)$ are *finer* (or, *larger*, *stronger*) than $\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g},\nu}^{(\eta)}$,$\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g},\mu}^{(\delta)}:\mathscr{P}(\Omega)\longrightarrow\mathscr{P}(\Omega)$ or, $\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g},\nu}^{(\eta)}$,$\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g},\mu}^{(\delta)}:\mathscr{P}(\Omega)\longrightarrow\mathscr{P}(\Omega)$ are *coarser* (or, *smaller*, *weaker*) than $\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g},\mu}^{(\delta)}$, $\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g},\nu}^{(\eta)}:\mathscr{P}(\Omega)\longrightarrow\mathscr{P}(\Omega)$.

In view of the above descriptions, for any pair (δ,η) of ordinals such that $1\preccurlyeq\delta\prec\eta\prec\lambda$, this first diagram, which is to be read horizontally, from left to right and vertically, from top to bottom, follows:

$$
\begin{array}{ccccccc}
\mathrm{cod}^{(\eta)} & \precsim & \mathrm{cod}^{(\eta)} & \precsim & \mathrm{cod}^{(\eta)} & \succsim & \mathrm{cod}^{(\eta)} \\
\wr\curlywedge & & \wr\curlywedge & & \wr\curlywedge & & \wr\curlywedge \\
\mathfrak{g}\text{-}\mathrm{Cod}_{0}^{(\eta)} & \precsim & \mathfrak{g}\text{-}\mathrm{Cod}_{1}^{(\eta)} & \precsim & \mathfrak{g}\text{-}\mathrm{Cod}_{3}^{(\eta)} & \succsim & \mathfrak{g}\text{-}\mathrm{Cod}_{2}^{(\eta)} \\
\wr\curlyvee & & \wr\curlyvee & & \wr\curlyvee & & \wr\curlyvee \\
\mathfrak{g}\text{-}\mathrm{Cod}_{0}^{(\delta)} & \precsim & \mathfrak{g}\text{-}\mathrm{Cod}_{1}^{(\delta)} & \precsim & \mathfrak{g}\text{-}\mathrm{Cod}_{3}^{(\delta)} & \succsim & \mathfrak{g}\text{-}\mathrm{Cod}_{2}^{(\delta)} \\
\wr\curlywedge & & \wr\curlywedge & & \wr\curlywedge & & \wr\curlywedge \\
\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g},0}^{(\delta)} & \precsim & \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g},1}^{(\delta)} & \precsim & \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g},3}^{(\delta)} & \succsim & \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g},2}^{(\delta)} \\
\wr\curlywedge & & \wr\curlywedge & & \wr\curlywedge & & \wr\curlywedge \\
\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g},0}^{(\eta)} & \precsim & \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g},1}^{(\eta)} & \precsim & \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g},3}^{(\eta)} & \succsim & \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g},2}^{(\eta)} \\
\wr\curlyvee & & \wr\curlyvee & & \wr\curlyvee & & \wr\curlyvee \\
\mathrm{cod}_{\mathfrak{g}}^{(\eta)} & \precsim & \mathrm{cod}_{\mathfrak{g}}^{(\eta)} & \precsim & \mathrm{cod}_{\mathfrak{g}}^{(\eta)} & \succsim & \mathrm{cod}_{\mathfrak{g}}^{(\eta)}\,.
\end{array}
\tag{4.3}
$$

Accordingly, Eq. (4.3) may well be called a $(\delta,\eta)^{\text{th}}$-*iterate* $(\mathfrak{g}\text{-}\mathrm{Cod},\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}})$-*valued diagram*. For any pair (δ,η) of ordinals such that $1\preccurlyeq\delta\prec\eta\prec\lambda$, this second diagram, which is to be read horizontally, from left to right and vertically, from top to bottom, also follows:

$$
\begin{array}{ccccccc}
\operatorname{der}^{(\eta)} & \succsim & \operatorname{der}^{(\eta)} & \succsim & \operatorname{der}^{(\eta)} & \precsim & \operatorname{der}^{(\eta)} \\
\rotatebox{-90}{$\succsim$} & & \rotatebox{-90}{$\succsim$} & & \rotatebox{-90}{$\succsim$} & & \rotatebox{-90}{$\succsim$} \\
\mathfrak{g}\text{-}\mathrm{Der}_{0}^{(\eta)} & \succsim & \mathfrak{g}\text{-}\mathrm{Der}_{1}^{(\eta)} & \succsim & \mathfrak{g}\text{-}\mathrm{Der}_{3}^{(\eta)} & \precsim & \mathfrak{g}\text{-}\mathrm{Der}_{2}^{(\eta)} \\
\rotatebox{-90}{$\precsim$} & & \rotatebox{-90}{$\precsim$} & & \rotatebox{-90}{$\precsim$} & & \rotatebox{-90}{$\precsim$} \\
\mathfrak{g}\text{-}\mathrm{Der}_{0}^{(\delta)} & \succsim & \mathfrak{g}\text{-}\mathrm{Der}_{1}^{(\delta)} & \succsim & \mathfrak{g}\text{-}\mathrm{Der}_{3}^{(\delta)} & \precsim & \mathfrak{g}\text{-}\mathrm{Der}_{2}^{(\delta)} \\
\rotatebox{-90}{$\succsim$} & & \rotatebox{-90}{$\succsim$} & & \rotatebox{-90}{$\succsim$} & & \rotatebox{-90}{$\succsim$} \\
\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g},0}^{(\delta)} & \succsim & \mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g},1}^{(\delta)} & \succsim & \mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g},3}^{(\delta)} & \precsim & \mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g},2}^{(\delta)} \\
\rotatebox{-90}{$\succsim$} & & \rotatebox{-90}{$\succsim$} & & \rotatebox{-90}{$\succsim$} & & \rotatebox{-90}{$\succsim$} \\
\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g},0}^{(\eta)} & \succsim & \mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g},1}^{(\eta)} & \succsim & \mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g},3}^{(\eta)} & \precsim & \mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g},2}^{(\eta)} \\
\rotatebox{-90}{$\precsim$} & & \rotatebox{-90}{$\precsim$} & & \rotatebox{-90}{$\precsim$} & & \rotatebox{-90}{$\precsim$} \\
\operatorname{der}_{\mathfrak{g}}^{(\eta)} & \succsim & \operatorname{der}_{\mathfrak{g}}^{(\eta)} & \succsim & \operatorname{der}_{\mathfrak{g}}^{(\eta)} & \precsim & \operatorname{der}_{\mathfrak{g}}^{(\eta)} .
\end{array}
\tag{4.4}
$$

Accordingly, Eq. (4.4) may well be called a $(\delta,\eta)^{\mathrm{th}}$-*iterate* $\left(\mathfrak{g}\text{-}\mathrm{Der}, \mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}\right)$-*valued diagram*.

The various relationships between the classes of the δ^{th}-iterates of $\mathfrak{g}$-$\mathfrak{T}$-derived and $\mathfrak{g}$-$\mathfrak{T}$-coderived operators in the $\mathscr{T}$-space $\mathfrak{T}$ and the classes of the δ^{th}-iterates of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-derived and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-coderived operators in the $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}}$ are therefore established. The various relationships between the classes of $\mathfrak{g}$-$\mathfrak{T}$-open and $\mathfrak{g}$-$\mathfrak{T}$-closed sets of rank δ in the $\mathscr{T}$-space $\mathfrak{T}$ and the classes of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-open and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-closed sets of rank δ in the $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}}$ will now be established, again, based on the notions of *coarseness* (or, *smallness*, *weakness*), or alternatively, *finness* (or, *largeness*, *strongness*).

Let it be granted some pair $(\nu,\mu) \in I_3^0 \times I_3^0$ of categories and some pair of ordinals (δ,η) such that $(1,1) \preccurlyeq (\delta,\eta) \prec (\lambda,\lambda)$. Furthermore, let "$\mathfrak{g}$-$\nu$-$\mathfrak{T}^{(\eta)}$-O" and "$\mathfrak{g}$-$\mu$-$\mathfrak{T}^{(\delta)}$-K" stand for "$\mathfrak{g}$-$\mathfrak{T}$-openness of category ν and rank η" and "$\mathfrak{g}$-$\mathfrak{T}$-closedness of category μ and rank δ," respectively, in a $\mathscr{T}$-space $\mathfrak{T} = (\Omega, \mathscr{T})$; "$\mathfrak{g}$-$\nu$-$\mathfrak{T}_{\mathfrak{g}}^{(\eta)}$-O" and "$\mathfrak{g}$-$\mu$-$\mathfrak{T}_{\mathfrak{g}}^{(\delta)}$-K" stand for "$\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-openness of category ν and rank η" and "$\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-closedness of category μ and rank δ," respectively, in a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$.

Suppose "$\mathfrak{g}$-$\mathrm{Int}_{\nu}^{(\eta)} \succsim \mathfrak{g}$-$\mathrm{Int}_{\mu}^{(\delta)}$" and "$\mathfrak{g}$-$\mathrm{Cl}_{\nu}^{(\eta)} \precsim \mathfrak{g}$-$\mathrm{Cl}_{\mu}^{(\delta)}$" hold true, then the implications "$\mathfrak{g}$-ν-$\mathfrak{T}^{(\eta)}$-O $\longleftarrow$ $\mathfrak{g}$-μ-$\mathfrak{T}^{(\delta)}$-O" and "$\mathfrak{g}$-ν-$\mathfrak{T}^{(\eta)}$-K $\longrightarrow$ $\mathfrak{g}$-μ-$\mathfrak{T}^{(\delta)}$-K" also hold true, or equivalently, "$\mathfrak{g}$-$\mathrm{Int}_{\mu}^{(\delta)} \precsim \mathfrak{g}$-$\mathrm{Int}_{\nu}^{(\eta)}$" and "$\mathfrak{g}$-$\mathrm{Cl}_{\mu}^{(\delta)} \succsim \mathfrak{g}$-$\mathrm{Cl}_{\nu}^{(\eta)}$" hold true imply that "$\mathfrak{g}$-$\mu$-$\mathfrak{T}^{(\delta)}$-O $\longleftarrow$ $\mathfrak{g}$-ν-$\mathfrak{T}^{(\eta)}$-O" and "$\mathfrak{g}$-μ-$\mathfrak{T}^{(\delta)}$-K $\longrightarrow$ $\mathfrak{g}$-ν-$\mathfrak{T}^{(\eta)}$-K" also hold true, respectively, in a $\mathscr{T}$-space $\mathfrak{T} = (\Omega, \mathscr{T})$; "$\mathfrak{g}$-$\mathrm{Int}_{\mathfrak{g},\nu}^{(\eta)} \succsim \mathfrak{g}$-$\mathrm{Int}_{\mathfrak{g},\mu}^{(\delta)}$" and "$\mathfrak{g}$-$\mathrm{Cl}_{\mathfrak{g},\nu}^{(\eta)} \precsim \mathfrak{g}$-$\mathrm{Cl}_{\mathfrak{g},\mu}^{(\delta)}$" hold true, then the implications "$\mathfrak{g}$-ν-$\mathfrak{T}^{(\eta)}$-O $\longleftarrow$ $\mathfrak{g}$-μ-$\mathfrak{T}_{\mathfrak{g}}^{(\delta)}$-O" and "$\mathfrak{g}$-$\nu$-$\mathfrak{T}_{\mathfrak{g}}^{(\eta)}$-K $\longrightarrow$ $\mathfrak{g}$-μ-$\mathfrak{T}_{\mathfrak{g}}^{(\delta)}$-K" also hold true, or equivalently, "$\mathfrak{g}$-$\mathrm{Int}_{\mathfrak{g},\mu}^{(\delta)} \precsim \mathfrak{g}$-$\mathrm{Int}_{\mathfrak{g},\nu}^{(\eta)}$" and "$\mathfrak{g}$-$\mathrm{Cl}_{\mathfrak{g},\mu}^{(\delta)} \succsim \mathfrak{g}$-$\mathrm{Cl}_{\mathfrak{g},\nu}^{(\eta)}$" hold true imply that "$\mathfrak{g}$-$\mu$-$\mathfrak{T}_{\mathfrak{g}}^{(\delta)}$-O $\longleftarrow$ $\mathfrak{g}$-ν-$\mathfrak{T}_{\mathfrak{g}}^{(\eta)}$-O" and "$\mathfrak{g}$-$\mu$-$\mathfrak{T}_{\mathfrak{g}}^{(\delta)}$-K $\longrightarrow$ $\mathfrak{g}$-ν-$\mathfrak{T}_{\mathfrak{g}}^{(\eta)}$-K" also hold true, respectively, in a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$

In view of the above descriptions, for any pair (δ,η) of ordinals such that $1 \preccurlyeq \delta \prec \eta \prec \lambda$, this first diagram, which is to be read horizontally, from left to right and vertically, from top to bottom, follows:

$$\begin{array}{ccccccc}
\mathfrak{T}^{(\eta)}\text{-O} & \longrightarrow & \mathfrak{T}^{(\eta)}\text{-O} & \longrightarrow & \mathfrak{T}^{(\eta)}\text{-O} & \longleftarrow & \mathfrak{T}^{(\eta)}\text{-O} \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\mathfrak{g}\text{-0-}\mathfrak{T}^{(\eta)}\text{-O} & \longrightarrow & \mathfrak{g}\text{-1-}\mathfrak{T}^{(\eta)}\text{-O} & \longrightarrow & \mathfrak{g}\text{-3-}\mathfrak{T}^{(\eta)}\text{-O} & \longleftarrow & \mathfrak{g}\text{-2-}\mathfrak{T}^{(\eta)}\text{-O} \\
\uparrow & & \uparrow & & \uparrow & & \uparrow \\
\mathfrak{g}\text{-0-}\mathfrak{T}^{(\delta)}\text{-O} & \longrightarrow & \mathfrak{g}\text{-1-}\mathfrak{T}^{(\delta)}\text{-O} & \longrightarrow & \mathfrak{g}\text{-3-}\mathfrak{T}^{(\delta)}\text{-O} & \longleftarrow & \mathfrak{g}\text{-2-}\mathfrak{T}^{(\delta)}\text{-O} \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\mathfrak{g}\text{-0-}\mathfrak{T}_{\mathfrak{g}}^{(\delta)}\text{-O} & \longrightarrow & \mathfrak{g}\text{-1-}\mathfrak{T}_{\mathfrak{g}}^{(\delta)}\text{-O} & \longrightarrow & \mathfrak{g}\text{-3-}\mathfrak{T}_{\mathfrak{g}}^{(\delta)}\text{-O} & \longleftarrow & \mathfrak{g}\text{-2-}\mathfrak{T}_{\mathfrak{g}}^{(\delta)}\text{-O} \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\mathfrak{g}\text{-0-}\mathfrak{T}_{\mathfrak{g}}^{(\eta)}\text{-O} & \longrightarrow & \mathfrak{g}\text{-1-}\mathfrak{T}_{\mathfrak{g}}^{(\eta)}\text{-O} & \longrightarrow & \mathfrak{g}\text{-3-}\mathfrak{T}_{\mathfrak{g}}^{(\eta)}\text{-O} & \longleftarrow & \mathfrak{g}\text{-2-}\mathfrak{T}_{\mathfrak{g}}^{(\eta)}\text{-O} \\
\uparrow & & \uparrow & & \uparrow & & \uparrow \\
\mathfrak{T}_{\mathfrak{g}}^{(\eta)}\text{-O} & \longrightarrow & \mathfrak{T}_{\mathfrak{g}}^{(\eta)}\text{-O} & \longrightarrow & \mathfrak{T}_{\mathfrak{g}}^{(\eta)}\text{-O} & \longleftarrow & \mathfrak{T}_{\mathfrak{g}}^{(\eta)}\text{-O}\,.
\end{array} \tag{4.5}$$

Accordingly, Eq. (4.5) may well be called a $(\delta,\eta)^{\text{th}}$*-rank* $(\mathfrak{g}\text{-}\mathfrak{T},\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}})$*-openness diagram*. For any pair (δ,η) of ordinals such that $1 \preccurlyeq \delta \prec \eta \prec \lambda$, this second diagram, which is to be read horizontally, from left to right and vertically, from top to bottom, also follows:

$$\begin{array}{ccccccc}
\mathfrak{T}^{(\eta)}\text{-K} & \longleftarrow & \mathfrak{T}^{(\eta)}\text{-K} & \longleftarrow & \mathfrak{T}^{(\eta)}\text{-K} & \longrightarrow & \mathfrak{T}^{(\eta)}\text{-K} \\
\uparrow & & \uparrow & & \uparrow & & \uparrow \\
\mathfrak{g}\text{-0-}\mathfrak{T}^{(\eta)}\text{-K} & \longleftarrow & \mathfrak{g}\text{-1-}\mathfrak{T}^{(\eta)}\text{-K} & \longleftarrow & \mathfrak{g}\text{-3-}\mathfrak{T}^{(\eta)}\text{-K} & \longrightarrow & \mathfrak{g}\text{-2-}\mathfrak{T}^{(\eta)}\text{-K} \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\mathfrak{g}\text{-0-}\mathfrak{T}^{(\delta)}\text{-K} & \longleftarrow & \mathfrak{g}\text{-1-}\mathfrak{T}^{(\delta)}\text{-K} & \longleftarrow & \mathfrak{g}\text{-3-}\mathfrak{T}^{(\delta)}\text{-K} & \longrightarrow & \mathfrak{g}\text{-2-}\mathfrak{T}^{(\delta)}\text{-K} \\
\uparrow & & \uparrow & & \uparrow & & \uparrow \\
\mathfrak{g}\text{-0-}\mathfrak{T}_{\mathfrak{g}}^{(\delta)}\text{-K} & \longleftarrow & \mathfrak{g}\text{-1-}\mathfrak{T}_{\mathfrak{g}}^{(\delta)}\text{-K} & \longleftarrow & \mathfrak{g}\text{-3-}\mathfrak{T}_{\mathfrak{g}}^{(\delta)}\text{-K} & \longrightarrow & \mathfrak{g}\text{-2-}\mathfrak{T}_{\mathfrak{g}}^{(\delta)}\text{-K} \\
\uparrow & & \uparrow & & \uparrow & & \uparrow \\
\mathfrak{g}\text{-0-}\mathfrak{T}_{\mathfrak{g}}^{(\eta)}\text{-K} & \longleftarrow & \mathfrak{g}\text{-1-}\mathfrak{T}_{\mathfrak{g}}^{(\eta)}\text{-K} & \longleftarrow & \mathfrak{g}\text{-3-}\mathfrak{T}_{\mathfrak{g}}^{(\eta)}\text{-K} & \longrightarrow & \mathfrak{g}\text{-2-}\mathfrak{T}_{\mathfrak{g}}^{(\eta)}\text{-K} \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\mathfrak{T}_{\mathfrak{g}}^{(\eta)}\text{-K} & \longleftarrow & \mathfrak{T}_{\mathfrak{g}}^{(\eta)}\text{-K} & \longleftarrow & \mathfrak{T}_{\mathfrak{g}}^{(\eta)}\text{-K} & \longrightarrow & \mathfrak{T}_{\mathfrak{g}}^{(\eta)}\text{-K}\,.
\end{array} \tag{4.6}$$

Accordingly, Eq. (4.6) may well be called a $(\delta,\eta)^{\text{th}}$*-rank* $(\mathfrak{g}\text{-}\mathfrak{T},\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}})$*-closedness diagram*.

The various relationships between the classes of $\mathfrak{g}$-$\mathfrak{T}$-open and $\mathfrak{g}$-$\mathfrak{T}$-closed sets of rank δ in the $\mathscr{T}$-space $\mathfrak{T}$ and the classes of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-open and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-closed sets of rank δ in the $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}}$ are therefore established.

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As in the works of other authors [CJS05, Don97, JJLL08, TC16], the manner we have positioned the arrows is solely to stress that, in general, the implications in Figs 1, 2 and Eqs (4.5), (4.6) are irreversible.

At this stage, a nice application is worth considering, and is presented in the following section.

4.2. A Nice Application. Focusing on essential concepts from the standpoint of the theory of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-derived and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-coderived operators in an attempt to shed lights on some essential properties of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-derived and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-coderived operators, their δ^{th}-order derivative $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-derived and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-coderived operators defined by transfinite recursion on the class of successor ordinals and, $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-open and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-closed sets of rank δ in $\mathscr{T}_{\mathfrak{g}}$-spaces, established in their earlier sections, we shall now present a nice application comprising of some interesting cases.

Let the 7-point set $\Omega = \{\xi_{\nu} : \nu \in I_7^*\}$ denotes the underlying set and consider the $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$, where Ω is 5-element topologized by the choice:

$$\begin{aligned} (4.7)\qquad \mathscr{T}_{\mathfrak{g}}(\Omega) &= \big\{\emptyset,\ \{\xi_1\},\ \{\xi_1,\xi_3,\xi_5\},\ \{\xi_1,\xi_3,\xi_4,\xi_5,\xi_7\}\big\}\\ &= \big\{\mathscr{O}_{\mathfrak{g},1},\ \mathscr{O}_{\mathfrak{g},2},\ \mathscr{O}_{\mathfrak{g},3},\ \mathscr{O}_{\mathfrak{g},4}\big\};\\ (4.8)\qquad \neg\mathscr{T}_{\mathfrak{g}}(\Omega) &= \big\{\Omega,\ \{\xi_2,\xi_3,\xi_4,\xi_5,\xi_6,\xi_7\},\ \{\xi_2,\xi_4,\xi_6,\xi_7\},\ \{\xi_2,\xi_6\}\big\}\\ &= \big\{\mathscr{K}_{\mathfrak{g},1},\ \mathscr{K}_{\mathfrak{g},2},\ \mathscr{K}_{\mathfrak{g},3},\ \mathscr{K}_{\mathfrak{g},4}\big\}. \end{aligned}$$

Evidently, the set-valued set maps $\mathscr{T}_{\mathfrak{g}}, \neg\mathscr{T}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\{\xi_{\nu} : \nu \in I_7^*\})$ establish the classes of $\mathscr{T}_{\mathfrak{g}}$-open and $\mathscr{T}_{\mathfrak{g}}$-closed sets, respectively. Since conditions $\mathscr{T}_{\mathfrak{g}}(\emptyset) = \emptyset$, $\mathscr{T}_{\mathfrak{g}}(\mathscr{O}_{\mathfrak{g},\nu}) \subseteq \mathscr{O}_{\mathfrak{g},\nu}$ for every $\nu \in I_4^*$, and $\mathscr{T}_{\mathfrak{g}}\big(\bigcup_{\nu\in I_4^*}\mathscr{O}_{\mathfrak{g},\nu}\big) = \bigcup_{\nu\in I_4^*}\mathscr{T}_{\mathfrak{g}}(\mathscr{O}_{\mathfrak{g},\nu})$ are satisfied, it is clear that the one-valued map $\mathscr{T}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\{\xi_{\nu} : \nu \in I_7^*\})$ is a $\mathfrak{g}$-topology and hence, $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$ is a $\mathscr{T}_{\mathfrak{g}}$-space. Moreover, it is easily checked that $(\mathscr{O}_{\mathfrak{g},\mu}, \mathscr{K}_{\mathfrak{g},\mu}) \in \mathfrak{g}\text{-}\nu\text{-O}[\mathfrak{T}] \times \mathfrak{g}\text{-}\nu\text{-K}[\mathfrak{T}]$ for each $(\nu,\mu) \in I_3^0 \times I_4^*$. Thus, the $\mathscr{T}_{\mathfrak{g}}$-open sets forming the $\mathfrak{g}$-topology $\mathscr{T}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\{\xi_{\nu} : \nu \in I_7^*\})$ and the $\mathscr{T}_{\mathfrak{g}}$-closed sets forming the complement $\mathfrak{g}$-topology $\neg\mathscr{T}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\{\xi_{\nu} : \nu \in I_7^*\})$ of the $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$ are, respectively, $\mathfrak{g}$-$\mathfrak{T}$-open and $\mathfrak{g}$-$\mathfrak{T}$-closed sets relative to the $\mathscr{T}$-space $\mathfrak{T} = (\Omega, \mathscr{T}) = (\Omega, \mathscr{T}_{\mathfrak{g}} \cup \{\Omega\})$.

After calculations, the classes $\mathfrak{g}\text{-}\nu\text{-O}[\mathfrak{T}_{\mathfrak{g}}]$ and $\mathfrak{g}\text{-}\nu\text{-K}[\mathfrak{T}_{\mathfrak{g}}]$, respectively, of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-open and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-closed sets of categories $\nu \in \{0,2\}$ then take the following forms:

$$\begin{aligned} \mathfrak{g}\text{-}\nu\text{-O}[\mathfrak{T}_{\mathfrak{g}}] &= \big\{\mathscr{S}_{\mathfrak{g}} \subset \mathfrak{T}_{\mathfrak{g}} : \mathscr{S}_{\mathfrak{g}} \subseteq \mathscr{O}_{\mathfrak{g},4}\big\};\\ (4.9)\qquad \mathfrak{g}\text{-}\nu\text{-K}[\mathfrak{T}_{\mathfrak{g}}] &= \big\{\mathscr{S}_{\mathfrak{g}} \subset \mathfrak{T}_{\mathfrak{g}} : \mathscr{S}_{\mathfrak{g}} \supseteq \mathscr{K}_{\mathfrak{g},4}\big\} \quad \forall\nu \in \{0,2\}. \end{aligned}$$

On the other hand, those of categories $\nu \in \{1,3\}$ take the following forms:

$$\begin{aligned} \mathfrak{g}\text{-}\nu\text{-O}[\mathfrak{T}_{\mathfrak{g}}] &= \big\{\mathscr{S}_{\mathfrak{g}} \subset \mathfrak{T}_{\mathfrak{g}} : \mathscr{S}_{\mathfrak{g}} \subseteq \mathscr{K}_{\mathfrak{g},1}\big\};\\ (4.10)\qquad \mathfrak{g}\text{-}\nu\text{-K}[\mathfrak{T}_{\mathfrak{g}}] &= \big\{\mathscr{S}_{\mathfrak{g}} \subset \mathfrak{T}_{\mathfrak{g}} : \mathscr{S}_{\mathfrak{g}} \supseteq \mathscr{O}_{\mathfrak{g},1}\big\} \quad \forall\nu \in \{1,3\}. \end{aligned}$$

Based on the $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-sets in $\mathfrak{g}\text{-0-O}[\mathfrak{T}_{\mathfrak{g}}]$, $\mathfrak{g}\text{-0-K}[\mathfrak{T}_{\mathfrak{g}}]$, …, $\mathfrak{g}\text{-3-O}[\mathfrak{T}_{\mathfrak{g}}]$, $\mathfrak{g}\text{-3-K}[\mathfrak{T}_{\mathfrak{g}}]$, these three interesting cases follow:

Case I. *On $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-Derived, $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-Coderived Operators.* Introduce the $\mathfrak{T}_{\mathfrak{g}}$-sets $\mathscr{R}_{\mathfrak{g}} = \{\xi_1,\xi_2,\xi_4\}$, $\mathscr{S}_{\mathfrak{g}} = \mathscr{R}_{\mathfrak{g}} \cup \{\xi_7\}$, $\mathscr{U}_{\mathfrak{g}} = \{\xi_3,\xi_5,\xi_6,\xi_7\}$, and $\mathscr{V}_{\mathfrak{g}} = \mathscr{U}_{\mathfrak{g}} \setminus \{\xi_3\}$;

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thus, $\big(\mathscr{S}_{\mathfrak{g}},\mathscr{U}_{\mathfrak{g}}\big)\supseteq\big(\mathscr{R}_{\mathfrak{g}},\mathscr{V}_{\mathfrak{g}}\big)$. Then, for each $\mathscr{W}_{\mathfrak{g}}\in\{\mathscr{R}_{\mathfrak{g}},\mathscr{S}_{\mathfrak{g}}\}$ and $\mathscr{Y}_{\mathfrak{g}}\in\{\mathscr{U}_{\mathfrak{g}},\mathscr{V}_{\mathfrak{g}}\}$, the following results present themselves:

$$\begin{aligned} \mathfrak{g}\text{-Cl}_{\mathfrak{g},\nu}\big(\mathscr{W}_{\mathfrak{g}}\cap\mathfrak{g}\text{-Op}_{\mathfrak{g}}(\{\xi_\mu\})\big) &= \big(\mathscr{W}_{\mathfrak{g}}\setminus\{\xi_\mu\}\big)\cup\mathscr{K}_{\mathfrak{g},4}\quad\forall(\mu,\nu)\in I_7^*\times\{0,2\},\\ \mathfrak{g}\text{-Cl}_{\mathfrak{g},\nu}\big(\mathscr{W}_{\mathfrak{g}}\cap\mathfrak{g}\text{-Op}_{\mathfrak{g}}(\{\xi_\mu\})\big) &= \mathscr{W}_{\mathfrak{g}}\setminus\{\xi_\mu\}\quad\forall(\mu,\nu)\in I_7^*\times\{1,3\},\\ \mathfrak{g}\text{-Int}_{\mathfrak{g},\nu}\big(\mathscr{Y}_{\mathfrak{g}}\cup\{\xi_\mu\}\big) &= \big(\mathscr{Y}_{\mathfrak{g}}\cup\{\xi_\mu\}\big)\setminus\mathscr{K}_{\mathfrak{g},4}\quad\forall(\mu,\nu)\in I_7^*\times\{0,2\},\\ \mathfrak{g}\text{-Int}_{\mathfrak{g},\nu}\big(\mathscr{Y}_{\mathfrak{g}}\cup\{\xi_\mu\}\big) &= \mathscr{Y}_{\mathfrak{g}}\cup\{\xi_\mu\}\quad\forall(\mu,\nu)\in I_7^*\times\{1,3\}. \end{aligned}\tag{4.11}$$

For each $\mathscr{W}_{\mathfrak{g}}\in\{\mathscr{R}_{\mathfrak{g}},\mathscr{S}_{\mathfrak{g}}\}$ and $\mathscr{Y}_{\mathfrak{g}}\in\{\mathscr{U}_{\mathfrak{g}},\mathscr{V}_{\mathfrak{g}}\}$, the following results also present themselves:

$$\begin{aligned} \text{cl}_{\mathfrak{g}}\big(\mathscr{W}_{\mathfrak{g}}\cap\mathfrak{g}\text{-Op}_{\mathfrak{g}}(\{\xi_\mu\})\big) &= \mathscr{K}_{\mathfrak{g},3}\quad\forall(\mu,\mathscr{W}_{\mathfrak{g}})\in I_1^*\times\{\mathscr{R}_{\mathfrak{g}},\mathscr{S}_{\mathfrak{g}}\},\\ \text{cl}_{\mathfrak{g}}\big(\mathscr{W}_{\mathfrak{g}}\cap\mathfrak{g}\text{-Op}_{\mathfrak{g}}(\{\xi_\mu\})\big) &= \mathscr{K}_{\mathfrak{g},1}\quad\forall(\mu,\mathscr{W}_{\mathfrak{g}})\in(I_7^*\setminus I_1^*)\times\{\mathscr{R}_{\mathfrak{g}},\mathscr{S}_{\mathfrak{g}}\},\\ \text{int}_{\mathfrak{g}}\big(\mathscr{Y}_{\mathfrak{g}}\cup\{\xi_\mu\}\big) &= \mathscr{O}_{\mathfrak{g},2}\setminus\{\xi_\mu\}\quad\forall(\mu,\mathscr{W}_{\mathfrak{g}})\in I_7^*\times\{\mathscr{U}_{\mathfrak{g}},\mathscr{V}_{\mathfrak{g}}\}. \end{aligned}\tag{4.12}$$

Thus, for each $\mathscr{W}_{\mathfrak{g}}\in\{\mathscr{R}_{\mathfrak{g}},\mathscr{S}_{\mathfrak{g}}\}$ and $\mathscr{Y}_{\mathfrak{g}}\in\{\mathscr{U}_{\mathfrak{g}},\mathscr{V}_{\mathfrak{g}}\}$, it follows that:

$$\xi_\mu\in\begin{cases} \mathfrak{g}\text{-Cl}_{\mathfrak{g},\nu}\big(\mathscr{W}_{\mathfrak{g}}\cap\mathfrak{g}\text{-Op}_{\mathfrak{g}}(\{\xi_\mu\})\big) & \forall(\mu,\nu)\in\{2,6\}\times\{0,2\},\\ \mathfrak{g}\text{-Op}_{\mathfrak{g}}\circ\mathfrak{g}\text{-Cl}_{\mathfrak{g},\nu}\big(\mathscr{W}_{\mathfrak{g}}\cap\mathfrak{g}\text{-Op}_{\mathfrak{g}}(\{\xi_\mu\})\big) & \forall(\mu,\nu)\in I_7^*\times\{1,3\},\\ \mathfrak{g}\text{-Int}_{\mathfrak{g},\nu}\big(\mathscr{Y}_{\mathfrak{g}}\cup\{\xi_\mu\}\big) & \forall(\mu,\nu)\in I_7^*\times\{1,3\},\\ \mathfrak{g}\text{-Op}_{\mathfrak{g}}\circ\mathfrak{g}\text{-Int}_{\mathfrak{g},\nu}\big(\mathscr{Y}_{\mathfrak{g}}\cup\{\xi_\mu\}\big) & \forall(\mu,\nu)\in\{2,6\}\times\{0,2\}. \end{cases}\tag{4.13}$$

On the other hand, it also follows that:

$$\begin{cases} \xi_\mu\in\text{cl}_{\mathfrak{g}}\big(\mathscr{W}_{\mathfrak{g}}\cap\mathfrak{g}\text{-Op}_{\mathfrak{g}}(\{\xi_\mu\})\big) & \forall(\mu,\mathscr{W}_{\mathfrak{g}})\in(I_7^*\setminus I_1^*)\times\{\mathscr{R}_{\mathfrak{g}},\mathscr{S}_{\mathfrak{g}}\},\\ \xi_\mu\notin\text{int}_{\mathfrak{g}}\big(\mathscr{Y}_{\mathfrak{g}}\cup\{\xi_\mu\}\big) & \forall(\mu,\mathscr{W}_{\mathfrak{g}})\in(I_7^*\setminus I_1^*)\times\{\mathscr{U}_{\mathfrak{g}},\mathscr{V}_{\mathfrak{g}}\}. \end{cases}\tag{4.14}$$

Taking the above results into account, the $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-derived operation of $\mathfrak{g}\text{-Der}_{\mathfrak{g},\nu}:\mathscr{P}(\Omega)\longrightarrow\mathscr{P}(\Omega)$ on the $\mathfrak{T}_{\mathfrak{g}}$-sets $\mathscr{R}_{\mathfrak{g}}$, $\mathscr{S}_{\mathfrak{g}}\subset\mathfrak{T}_{\mathfrak{g}}$, and the $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-coderived operation of $\mathfrak{g}\text{-Cod}_{\mathfrak{g},\nu}:\mathscr{P}(\Omega)\longrightarrow\mathscr{P}(\Omega)$ on the $\mathfrak{T}_{\mathfrak{g}}$-sets $\mathscr{U}_{\mathfrak{g}}$, $\mathscr{V}_{\mathfrak{g}}\subset\mathfrak{T}_{\mathfrak{g}}$, for all $\nu\in I_3^0$, then, produce the following results:

$$\begin{cases} \mathfrak{g}\text{-Der}_{\mathfrak{g},\nu}(\mathscr{W}_{\mathfrak{g}})=\mathscr{K}_{\mathfrak{g},4} & \forall(\nu,\mathscr{W}_{\mathfrak{g}})\in\{0,2\}\times\{\mathscr{R}_{\mathfrak{g}},\mathscr{S}_{\mathfrak{g}}\},\\ \mathfrak{g}\text{-Der}_{\mathfrak{g},\nu}(\mathscr{W}_{\mathfrak{g}})=\mathscr{O}_{\mathfrak{g},1} & \forall(\nu,\mathscr{W}_{\mathfrak{g}})\in\{1,3\}\times\{\mathscr{R}_{\mathfrak{g}},\mathscr{S}_{\mathfrak{g}}\},\\ \mathfrak{g}\text{-Cod}_{\mathfrak{g},\nu}(\mathscr{Y}_{\mathfrak{g}})=\mathscr{O}_{\mathfrak{g},4} & \forall(\nu,\mathscr{Y}_{\mathfrak{g}})\in\{0,2\}\times\{\mathscr{U}_{\mathfrak{g}},\mathscr{V}_{\mathfrak{g}}\},\\ \mathfrak{g}\text{-Cod}_{\mathfrak{g},\nu}(\mathscr{Y}_{\mathfrak{g}})=\mathscr{K}_{\mathfrak{g},1} & \forall(\nu,\mathscr{Y}_{\mathfrak{g}})\in\{1,3\}\times\{\mathscr{U}_{\mathfrak{g}},\mathscr{V}_{\mathfrak{g}}\}. \end{cases}\tag{4.15}$$

Likewise, taking the above results into account, the $\mathfrak{T}_{\mathfrak{g}}$-derived operation of $\text{der}_{\mathfrak{g}}:\mathscr{P}(\Omega)\longrightarrow\mathscr{P}(\Omega)$ on the $\mathfrak{T}_{\mathfrak{g}}$-sets $\mathscr{R}_{\mathfrak{g}}$, $\mathscr{S}_{\mathfrak{g}}\subset\mathfrak{T}_{\mathfrak{g}}$, and the $\mathfrak{T}_{\mathfrak{g}}$-coderived operation of $\text{cod}_{\mathfrak{g}}:\mathscr{P}(\Omega)\longrightarrow\mathscr{P}(\Omega)$ on the $\mathfrak{T}_{\mathfrak{g}}$-sets $\mathscr{U}_{\mathfrak{g}}$, $\mathscr{V}_{\mathfrak{g}}\subset\mathfrak{T}_{\mathfrak{g}}$, then, also produce the following results:

$$\begin{cases} \text{der}_{\mathfrak{g}}(\mathscr{W}_{\mathfrak{g}})=\mathscr{K}_{\mathfrak{g},2} & \forall\mathscr{W}_{\mathfrak{g}}\in\{\mathscr{R}_{\mathfrak{g}},\mathscr{S}_{\mathfrak{g}}\},\\ \text{cod}_{\mathfrak{g}}(\mathscr{Y}_{\mathfrak{g}})=\mathscr{O}_{\mathfrak{g},2} & \forall\mathscr{Y}_{\mathfrak{g}}\in\{\mathscr{U}_{\mathfrak{g}},\mathscr{V}_{\mathfrak{g}}\}. \end{cases}\tag{4.16}$$

Hence, for each $\mathscr{W}_{\mathfrak{g}}\in\{\mathscr{R}_{\mathfrak{g}},\mathscr{S}_{\mathfrak{g}}\}$ and $\mathscr{Y}_{\mathfrak{g}}\in\{\mathscr{U}_{\mathfrak{g}},\mathscr{V}_{\mathfrak{g}}\}$, it results that:

$$\begin{cases} \mathfrak{g}\text{-Der}_{\mathfrak{g},0}(\mathscr{W}_{\mathfrak{g}})\supseteq\mathfrak{g}\text{-Der}_{\mathfrak{g},1}(\mathscr{W}_{\mathfrak{g}})\supseteq\mathfrak{g}\text{-Der}_{\mathfrak{g},3}(\mathscr{W}_{\mathfrak{g}})\subseteq\mathfrak{g}\text{-Der}_{\mathfrak{g},2}(\mathscr{W}_{\mathfrak{g}}),\\ \mathfrak{g}\text{-Cod}_{\mathfrak{g},0}(\mathscr{Y}_{\mathfrak{g}})\subseteq\mathfrak{g}\text{-Cod}_{\mathfrak{g},1}(\mathscr{Y}_{\mathfrak{g}})\subseteq\mathfrak{g}\text{-Cod}_{\mathfrak{g},3}(\mathscr{Y}_{\mathfrak{g}})\supseteq\mathfrak{g}\text{-Cod}_{\mathfrak{g},2}(\mathscr{Y}_{\mathfrak{g}}). \end{cases}\tag{4.17}$$

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The $(\precsim, \succsim)$-relations $\mathfrak{g}\text{-Der}_{\mathfrak{g},0} \succsim \mathfrak{g}\text{-Der}_{\mathfrak{g},1} \succsim \mathfrak{g}\text{-Der}_{\mathfrak{g},3} \precsim \mathfrak{g}\text{-Der}_{\mathfrak{g},2}$ and $\mathfrak{g}\text{-Cod}_{\mathfrak{g},0} \precsim \mathfrak{g}\text{-Cod}_{\mathfrak{g},1} \precsim \mathfrak{g}\text{-Cod}_{\mathfrak{g},3} \succsim \mathfrak{g}\text{-Cod}_{\mathfrak{g},2}$ are thus verified. Clearly, the following results also hold true:

$$\begin{cases} \mathfrak{g}\text{-Der}_{\mathfrak{g},\nu}(\mathscr{W}_{\mathfrak{g}}) \subseteq \text{der}_{\mathfrak{g}}(\mathscr{W}_{\mathfrak{g}}) & \forall (\nu, \mathscr{W}_{\mathfrak{g}}) \in I_3^0 \times \{\mathscr{R}_{\mathfrak{g}}, \mathscr{S}_{\mathfrak{g}}\}, \\ \mathfrak{g}\text{-Cod}_{\mathfrak{g},\nu}(\mathscr{Y}_{\mathfrak{g}}) \supseteq \text{cod}_{\mathfrak{g}}(\mathscr{W}_{\mathfrak{g}}) & \forall (\nu, \mathscr{Y}_{\mathfrak{g}}) \in I_3^0 \times \{\mathscr{U}_{\mathfrak{g}}, \mathscr{V}_{\mathfrak{g}}\}. \end{cases} \tag{4.18}$$

Thus, the $(\precsim, \succsim)$-relations $\mathfrak{g}\text{-Der}_{\mathfrak{g},\nu} \precsim \text{der}_{\mathfrak{g}}$ and $\mathfrak{g}\text{-Cod}_{\mathfrak{g},\nu} \succsim \text{cod}_{\mathfrak{g}}$, for all $\nu \in I_3^0$, are also verified.

The case in which are presented some essential properties of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-derived and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-coderived operators in $\mathscr{T}_{\mathfrak{g}}$-spaces are therefore accomplished and ends here.

Case II. *On δ^{th}-Order $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-Derived, $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-Coderived Operators.* Consider again the $\mathfrak{T}_{\mathfrak{g}}$-sets $\mathscr{R}_{\mathfrak{g}} = \{\xi_1, \xi_2, \xi_4\}$, $\mathscr{S}_{\mathfrak{g}} = \mathscr{R}_{\mathfrak{g}} \cup \{\xi_7\}$, $\mathscr{U}_{\mathfrak{g}} = \{\xi_3, \xi_5, \xi_6, \xi_7\}$, and $\mathscr{V}_{\mathfrak{g}} = \mathscr{U}_{\mathfrak{g}} \setminus \{\xi_3\}$. Then, for any δ such that $1 \preccurlyeq \delta \prec \lambda$, the δ^{th}-order $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-derived operation of $\mathfrak{g}\text{-Der}_{\mathfrak{g},\nu}^{(\delta)} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ on the $\mathfrak{T}_{\mathfrak{g}}$-sets $\mathscr{R}_{\mathfrak{g}}, \mathscr{S}_{\mathfrak{g}} \subset \mathfrak{T}_{\mathfrak{g}}$, and the δ^{th}-order $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-coderived operation of $\mathfrak{g}\text{-Cod}_{\mathfrak{g},\nu}^{(\delta)} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ on the $\mathfrak{T}_{\mathfrak{g}}$-sets $\mathscr{U}_{\mathfrak{g}}, \mathscr{V}_{\mathfrak{g}} \subset \mathfrak{T}_{\mathfrak{g}}$, for all $\nu \in I_3^0$, produce the following results:

$$\begin{cases} \mathfrak{g}\text{-Der}_{\mathfrak{g},\nu}^{(\delta)}(\mathscr{W}_{\mathfrak{g}}) = \mathscr{K}_{\mathfrak{g},4} & \forall (\nu, \mathscr{W}_{\mathfrak{g}}) \in \{0,2\} \times \{\mathscr{R}_{\mathfrak{g}}, \mathscr{S}_{\mathfrak{g}}\}, \\ \mathfrak{g}\text{-Der}_{\mathfrak{g},\nu}^{(\delta)}(\mathscr{W}_{\mathfrak{g}}) = \mathscr{O}_{\mathfrak{g},1} & \forall (\nu, \mathscr{W}_{\mathfrak{g}}) \in \{1,3\} \times \{\mathscr{R}_{\mathfrak{g}}, \mathscr{S}_{\mathfrak{g}}\}, \\ \mathfrak{g}\text{-Cod}_{\mathfrak{g},\nu}^{(\delta)}(\mathscr{Y}_{\mathfrak{g}}) = \mathscr{O}_{\mathfrak{g},4} & \forall (\nu, \mathscr{Y}_{\mathfrak{g}}) \in \{0,2\} \times \{\mathscr{U}_{\mathfrak{g}}, \mathscr{V}_{\mathfrak{g}}\}, \\ \mathfrak{g}\text{-Cod}_{\mathfrak{g},\nu}^{(\delta)}(\mathscr{Y}_{\mathfrak{g}}) = \mathscr{K}_{\mathfrak{g},1} & \forall (\nu, \mathscr{Y}_{\mathfrak{g}}) \in \{1,3\} \times \{\mathscr{U}_{\mathfrak{g}}, \mathscr{V}_{\mathfrak{g}}\}. \end{cases} \tag{4.19}$$

Likewise, for any δ such that $1 \preccurlyeq \delta \prec \lambda$, the δ^{th}-order $\mathfrak{T}_{\mathfrak{g}}$-derived operation of $\text{der}_{\mathfrak{g},\nu}^{(\delta)} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ on the $\mathfrak{T}_{\mathfrak{g}}$-sets $\mathscr{R}_{\mathfrak{g}}, \mathscr{S}_{\mathfrak{g}} \subset \mathfrak{T}_{\mathfrak{g}}$, and the δ^{th}-order $\mathfrak{T}_{\mathfrak{g}}$-coderived operation of $\text{cod}_{\mathfrak{g},\nu}^{(\delta)} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ on the $\mathfrak{T}_{\mathfrak{g}}$-sets $\mathscr{U}_{\mathfrak{g}}, \mathscr{V}_{\mathfrak{g}} \subset \mathfrak{T}_{\mathfrak{g}}$, for all $\nu \in I_3^0$, also produce the following results:

$$\begin{cases} \text{der}_{\mathfrak{g}}^{(\delta)}(\mathscr{W}_{\mathfrak{g}}) = \mathscr{K}_{\mathfrak{g},2} & \forall \mathscr{W}_{\mathfrak{g}} \in \{\mathscr{R}_{\mathfrak{g}}, \mathscr{S}_{\mathfrak{g}}\}, \\ \text{cod}_{\mathfrak{g}}^{(\delta)}(\mathscr{Y}_{\mathfrak{g}}) = \mathscr{O}_{\mathfrak{g},2} & \forall \mathscr{Y}_{\mathfrak{g}} \in \{\mathscr{U}_{\mathfrak{g}}, \mathscr{V}_{\mathfrak{g}}\}. \end{cases} \tag{4.20}$$

By virtue of the above results, it follows, on the one hand, that

$$\mathfrak{g}\text{-Der}_{\mathfrak{g},\nu}^{(\lambda)}(\mathscr{W}_{\mathfrak{g}}) = \bigcap_{\delta \prec \lambda} \mathfrak{g}\text{-Der}_{\mathfrak{g},\nu}^{(\delta)}(\mathscr{W}_{\mathfrak{g}}) = \emptyset \quad \forall (\nu, \mathscr{W}_{\mathfrak{g}}) \in \{1,3\} \times \{\mathscr{R}_{\mathfrak{g}}, \mathscr{S}_{\mathfrak{g}}\}, \tag{4.21}$$

and, on the other hand, that

$$\begin{cases} \mathfrak{g}\text{-Der}_{\mathfrak{g},\nu}^{(\lambda)}(\mathscr{W}_{\mathfrak{g}}) = \bigcap_{\delta \prec \lambda} \mathfrak{g}\text{-Der}_{\mathfrak{g},\nu}^{(\delta)}(\mathscr{W}_{\mathfrak{g}}) \neq \emptyset & \forall (\nu, \mathscr{W}_{\mathfrak{g}}) \in \{0,2\} \times \{\mathscr{R}_{\mathfrak{g}}, \mathscr{S}_{\mathfrak{g}}\}, \\ \mathfrak{g}\text{-Cod}_{\mathfrak{g},\nu}^{(\lambda)}(\mathscr{Y}_{\mathfrak{g}}) = \bigcap_{\delta \prec \lambda} \mathfrak{g}\text{-Cod}_{\mathfrak{g},\nu}^{(\delta)}(\mathscr{Y}_{\mathfrak{g}}) \neq \emptyset & \forall (\nu, \mathscr{Y}_{\mathfrak{g}}) \in I_3^0 \times \{\mathscr{U}_{\mathfrak{g}}, \mathscr{V}_{\mathfrak{g}}\}, \\ \text{der}_{\mathfrak{g}}^{(\lambda)}(\mathscr{W}_{\mathfrak{g}}) = \bigcap_{\delta \prec \lambda} \text{der}_{\mathfrak{g}}^{(\delta)}(\mathscr{W}_{\mathfrak{g}}) \neq \emptyset & \forall (\nu, \mathscr{W}_{\mathfrak{g}}) \in I_3^0 \times \{\mathscr{R}_{\mathfrak{g}}, \mathscr{S}_{\mathfrak{g}}\}, \\ \text{cod}_{\mathfrak{g}}^{(\lambda)}(\mathscr{Y}_{\mathfrak{g}}) = \bigcap_{\delta \prec \lambda} \text{cod}_{\mathfrak{g}}^{(\delta)}(\mathscr{Y}_{\mathfrak{g}}) \neq \emptyset & \forall (\nu, \mathscr{Y}_{\mathfrak{g}}) \in I_3^0 \times \{\mathscr{U}_{\mathfrak{g}}, \mathscr{V}_{\mathfrak{g}}\}. \end{cases} \tag{4.22}$$

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Hence, for any δ such that $1 \preccurlyeq \delta \prec \lambda$, it results that the following results hold true for each $\mathscr{W}_{\mathfrak{g}} \in \{\mathscr{R}_{\mathfrak{g}}, \mathscr{S}_{\mathfrak{g}}\}$ and $\mathscr{Y}_{\mathfrak{g}} \in \{\mathscr{U}_{\mathfrak{g}}, \mathscr{V}_{\mathfrak{g}}\}$:

$$\begin{cases} \mathfrak{g}\text{-Der}_{\mathfrak{g},0}^{(\delta)}(\mathscr{W}_{\mathfrak{g}}) \supseteq \mathfrak{g}\text{-Der}_{\mathfrak{g},1}^{(\delta)}(\mathscr{W}_{\mathfrak{g}}) \supseteq \mathfrak{g}\text{-Der}_{\mathfrak{g},3}^{(\delta)}(\mathscr{W}_{\mathfrak{g}}) \subseteq \mathfrak{g}\text{-Der}_{\mathfrak{g},2}^{(\delta)}(\mathscr{W}_{\mathfrak{g}}), \\ \mathfrak{g}\text{-Cod}_{\mathfrak{g},0}^{(\delta)}(\mathscr{Y}_{\mathfrak{g}}) \subseteq \mathfrak{g}\text{-Cod}_{\mathfrak{g},1}^{(\delta)}(\mathscr{Y}_{\mathfrak{g}}) \subseteq \mathfrak{g}\text{-Cod}_{\mathfrak{g},3}^{(\delta)}(\mathscr{Y}_{\mathfrak{g}}) \supseteq \mathfrak{g}\text{-Cod}_{\mathfrak{g},2}^{(\delta)}(\mathscr{Y}_{\mathfrak{g}}). \end{cases} \tag{4.23}$$

For any δ such that $1 \preccurlyeq \delta \prec \lambda$, the $(\precsim,\succsim)$-relations $\mathfrak{g}\text{-Der}_{\mathfrak{g},0}^{(\delta)} \succsim \mathfrak{g}\text{-Der}_{\mathfrak{g},1}^{(\delta)} \succsim \mathfrak{g}\text{-Der}_{\mathfrak{g},3}^{(\delta)} \precsim \mathfrak{g}\text{-Der}_{\mathfrak{g},2}^{(\delta)}$ and $\mathfrak{g}\text{-Cod}_{\mathfrak{g},0}^{(\delta)} \precsim \mathfrak{g}\text{-Cod}_{\mathfrak{g},1}^{(\delta)} \precsim \mathfrak{g}\text{-Cod}_{\mathfrak{g},3}^{(\delta)} \succsim \mathfrak{g}\text{-Cod}_{\mathfrak{g},2}^{(\delta)}$ are thus verified. Clearly, for any δ such that $1 \preccurlyeq \delta \prec \lambda$, the following results also hold true:

$$\begin{cases} \mathfrak{g}\text{-Der}_{\mathfrak{g},\nu}^{(\delta)}(\mathscr{W}_{\mathfrak{g}}) \subseteq \text{der}_{\mathfrak{g}}^{(\delta)}(\mathscr{W}_{\mathfrak{g}}) & \forall (\nu,\mathscr{W}_{\mathfrak{g}}) \in I_3^0 \times \{\mathscr{R}_{\mathfrak{g}}, \mathscr{S}_{\mathfrak{g}}\}, \\ \mathfrak{g}\text{-Cod}_{\mathfrak{g},\nu}^{(\delta)}(\mathscr{Y}_{\mathfrak{g}}) \supseteq \text{cod}_{\mathfrak{g}}^{(\delta)}(\mathscr{W}_{\mathfrak{g}}) & \forall (\nu,\mathscr{Y}_{\mathfrak{g}}) \in I_3^0 \times \{\mathscr{U}_{\mathfrak{g}}, \mathscr{V}_{\mathfrak{g}}\}, \end{cases} \tag{4.24}$$

Thus, the $(\precsim,\succsim)$-relations $\mathfrak{g}\text{-Der}_{\mathfrak{g},\nu}^{(\delta)} \precsim \text{der}_{\mathfrak{g}}^{(\delta)}$ and $\mathfrak{g}\text{-Cod}_{\mathfrak{g},\nu}^{(\delta)} \succsim \text{cod}_{\mathfrak{g}}^{(\delta)}$, for all $\nu \in I_3^0$, are also verified.

The case in which are presented some essential properties of the δ^{th}-order derivative $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-derived and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-coderived operators, defined by transfinite recursion on the class of successor ordinals, of their $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-derived and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-coderived operators in $\mathscr{T}_{\mathfrak{g}}$-spaces are therefore accomplished and ends here.

Case III. *On δ^{th}-Rank $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-Open, $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-Closed Sets.* Consider again the $\mathfrak{T}_{\mathfrak{g}}$-sets $\mathscr{R}_{\mathfrak{g}} = \{\xi_1, \xi_2, \xi_4\}$, $\mathscr{S}_{\mathfrak{g}} = \mathscr{R}_{\mathfrak{g}} \cup \{\xi_7\}$, $\mathscr{U}_{\mathfrak{g}} = \{\xi_3, \xi_5, \xi_6, \xi_7\}$, and $\mathscr{V}_{\mathfrak{g}} = \mathscr{U}_{\mathfrak{g}} \setminus \{\xi_3\}$.

After calculations, the classes $\mathfrak{g}$-ν-$\text{DE}^{(\delta)}[\mathscr{W}_{\mathfrak{g}};\mathfrak{T}_{\mathfrak{g}}]$ and $\mathfrak{g}$-ν-$\text{CD}^{(\delta)}[\mathscr{Y}_{\mathfrak{g}};\mathfrak{T}_{\mathfrak{g}}]$, respectively, of δ^{th}-rank $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-open and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-closed sets of categories $\nu \in \{0,2\}$, for any δ such that $1 \preccurlyeq \delta \prec \lambda$, then take the following forms:

$$\begin{aligned} \mathfrak{g}\text{-}\nu\text{-DE}^{(\delta)}[\mathscr{W}_{\mathfrak{g}};\mathfrak{T}_{\mathfrak{g}}] &= \emptyset \quad \forall (\nu,\mathscr{W}_{\mathfrak{g}}) \in \{0,2\} \times \{\mathscr{R}_{\mathfrak{g}}, \mathscr{S}_{\mathfrak{g}}\}; \\ \mathfrak{g}\text{-}\nu\text{-CD}^{(\delta)}[\mathscr{Y}_{\mathfrak{g}};\mathfrak{T}_{\mathfrak{g}}] &= \emptyset \quad \forall (\nu,\mathscr{Y}_{\mathfrak{g}}) \in \{0,2\} \times \{\mathscr{U}_{\mathfrak{g}}, \mathscr{V}_{\mathfrak{g}}\}. \end{aligned} \tag{4.25}$$

On the other hand, those of categories $\nu \in \{1,3\}$, for any δ such that $1 \preccurlyeq \delta \prec \lambda$, take the following forms:

$$\begin{aligned} \mathfrak{g}\text{-}\nu\text{-DE}^{(\delta)}[\mathscr{W}_{\mathfrak{g}};\mathfrak{T}_{\mathfrak{g}}] &= \{\mathscr{O}_{\mathfrak{g},1}\} \quad \forall (\nu,\mathscr{W}_{\mathfrak{g}}) \in \{1,3\} \times \{\mathscr{R}_{\mathfrak{g}}, \mathscr{S}_{\mathfrak{g}}\}; \\ \mathfrak{g}\text{-}\nu\text{-CD}^{(\delta)}[\mathscr{Y}_{\mathfrak{g}};\mathfrak{T}_{\mathfrak{g}}] &= \{\mathscr{K}_{\mathfrak{g},1}\} \quad \forall (\nu,\mathscr{Y}_{\mathfrak{g}}) \in \{1,3\} \times \{\mathscr{U}_{\mathfrak{g}}, \mathscr{V}_{\mathfrak{g}}\}. \end{aligned} \tag{4.26}$$

For any δ such that $1 \preccurlyeq \delta \prec \lambda$, it follows, on the one hand, that

$$\mathscr{W}_{\mathfrak{g}} \supseteq \mathfrak{g}\text{-Cl}_{\mathfrak{g},\nu}^{(\delta)}(\mathscr{W}_{\mathfrak{g}}) = \begin{cases} \mathscr{K}_{\mathfrak{g},1} & \forall (\nu,\mathscr{W}_{\mathfrak{g}}) \in \{0,2\} \times \{\mathscr{R}_{\mathfrak{g}}, \mathscr{S}_{\mathfrak{g}}\}, \\ \mathscr{O}_{\mathfrak{g},1} & \forall (\nu,\mathscr{W}_{\mathfrak{g}}) \in \{1,3\} \times \{\mathscr{R}_{\mathfrak{g}}, \mathscr{S}_{\mathfrak{g}}\}, \end{cases} \tag{4.27}$$

and, on the other hand, that

$$\mathscr{Y}_{\mathfrak{g}} \subseteq \mathfrak{g}\text{-Int}_{\mathfrak{g},\nu}^{(\delta)}(\mathscr{Y}_{\mathfrak{g}}) = \begin{cases} \mathscr{O}_{\mathfrak{g},1} & \forall (\nu,\mathscr{Y}_{\mathfrak{g}}) \in \{0,2\} \times \{\mathscr{U}_{\mathfrak{g}}, \mathscr{V}_{\mathfrak{g}}\}, \\ \mathscr{K}_{\mathfrak{g},1} & \forall (\nu,\mathscr{Y}_{\mathfrak{g}}) \in \{1,3\} \times \{\mathscr{U}_{\mathfrak{g}}, \mathscr{V}_{\mathfrak{g}}\}. \end{cases} \tag{4.28}$$

Hence, for any δ such that $1 \preccurlyeq \delta \prec \lambda$, each $\mathscr{W}_{\mathfrak{g}} \in \{\mathscr{R}_{\mathfrak{g}}, \mathscr{S}_{\mathfrak{g}}\}$ is δ^{th}-rank $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-closed only in the event that it is of categories $\nu \in \{1,3\}$ because, in the event that it is of categories $\nu \in \{0,2\}$, the $\supseteq$-relation $\mathscr{W}_{\mathfrak{g}} \supseteq \mathfrak{g}\text{-Cl}_{\mathfrak{g},\nu}^{(\delta)}(\mathscr{W}_{\mathfrak{g}})$ is an untrue statement in $\mathscr{W}_{\mathfrak{g}}$ since $\mathscr{W}_{\mathfrak{g}} \not\supseteq \mathscr{K}_{\mathfrak{g},1}$. Similarly, for any δ such that $1 \preccurlyeq \delta \prec \lambda$, each $\mathscr{Y}_{\mathfrak{g}} \in \{\mathscr{U}_{\mathfrak{g}}, \mathscr{V}_{\mathfrak{g}}\}$ is δ^{th}-rank $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-open only in the event that it is of categories $\nu \in \{1,3\}$ because,

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in the event that it is of categories $\nu \in \{0,2\}$, the $\subseteq$-relation $\mathscr{Y}_{\mathfrak{g}} \subseteq \mathfrak{g}\text{-Int}^{(\delta)}_{\mathfrak{g},\nu}(\mathscr{Y}_{\mathfrak{g}})$ is an untrue statement in $\mathscr{Y}_{\mathfrak{g}}$ since $\mathscr{Y}_{\mathfrak{g}} \nsubseteq \mathscr{O}_{\mathfrak{g},1}$.

Moreover, for any δ such that $1 \preccurlyeq \delta \prec \lambda$, it results that the following results hold true for each $\mathscr{W}_{\mathfrak{g}} \in \{\mathscr{R}_{\mathfrak{g}}, \mathscr{S}_{\mathfrak{g}}\}$ and $\mathscr{Y}_{\mathfrak{g}} \in \{\mathscr{U}_{\mathfrak{g}}, \mathscr{V}_{\mathfrak{g}}\}$:

$$\left\{\begin{array}{l} \mathfrak{g}\text{-Cl}^{(\delta)}_{\mathfrak{g},0}(\mathscr{W}_{\mathfrak{g}}) \supseteq \mathfrak{g}\text{-Cl}^{(\delta)}_{\mathfrak{g},1}(\mathscr{W}_{\mathfrak{g}}) \supseteq \mathfrak{g}\text{-Cl}^{(\delta)}_{\mathfrak{g},3}(\mathscr{W}_{\mathfrak{g}}) \subseteq \mathfrak{g}\text{-Cl}^{(\delta)}_{\mathfrak{g},2}(\mathscr{W}_{\mathfrak{g}}), \\ \mathfrak{g}\text{-Int}^{(\delta)}_{\mathfrak{g},0}(\mathscr{Y}_{\mathfrak{g}}) \subseteq \mathfrak{g}\text{-Int}^{(\delta)}_{\mathfrak{g},1}(\mathscr{Y}_{\mathfrak{g}}) \subseteq \mathfrak{g}\text{-Int}^{(\delta)}_{\mathfrak{g},3}(\mathscr{Y}_{\mathfrak{g}}) \supseteq \mathfrak{g}\text{-Int}^{(\delta)}_{\mathfrak{g},2}(\mathscr{Y}_{\mathfrak{g}}). \end{array}\right. \tag{4.29}$$

For any δ such that $1 \preccurlyeq \delta \prec \lambda$, the $(\precsim, \succsim)$-relations $\mathfrak{g}\text{-Cl}^{(\delta)}_{\mathfrak{g},0} \succsim \mathfrak{g}\text{-Cl}^{(\delta)}_{\mathfrak{g},1} \succsim \mathfrak{g}\text{-Cl}^{(\delta)}_{\mathfrak{g},3} \precsim \mathfrak{g}\text{-Cl}^{(\delta)}_{\mathfrak{g},2}$ and $\mathfrak{g}\text{-Int}^{(\delta)}_{\mathfrak{g},0} \precsim \mathfrak{g}\text{-Int}^{(\delta)}_{\mathfrak{g},1} \precsim \mathfrak{g}\text{-Int}^{(\delta)}_{\mathfrak{g},3} \succsim \mathfrak{g}\text{-Int}^{(\delta)}_{\mathfrak{g},2}$ are thus verified. Accordingly, the $(\longleftarrow, \longrightarrow)$-relations $\mathfrak{T}^{(\delta)}_{\mathfrak{g}}\text{-K} \longleftarrow \mathfrak{T}^{(\delta)}_{\mathfrak{g}}\text{-K} \longleftarrow \mathfrak{T}^{(\delta)}_{\mathfrak{g}}\text{-K} \longrightarrow \mathfrak{T}^{(\delta)}_{\mathfrak{g}}\text{-K}$ and $\mathfrak{T}^{(\delta)}_{\mathfrak{g}}\text{-O} \longrightarrow \mathfrak{T}^{(\delta)}_{\mathfrak{g}}\text{-O} \longrightarrow \mathfrak{T}^{(\delta)}_{\mathfrak{g}}\text{-O} \longleftarrow \mathfrak{T}^{(\delta)}_{\mathfrak{g}}\text{-O}$ hold true for any δ such that $1 \preccurlyeq \delta \prec \lambda$.

The case in which are presented some essential properties of δ^{th}-rank $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-open and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-closed sets in $\mathscr{T}_{\mathfrak{g}}$-spaces are therefore accomplished and ends here.

If the discussions of this nice application be explore a step further, other interesting conclusions can be drawn from the study of, firstly, the essential properties of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-derived and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-coderived operators, secondly, the essential properties of their δ^{th}-order derivative $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-derived and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-coderived operators, and thirdly, the essential properties of δ^{th}-rank $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-open and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-closed sets in $\mathscr{T}_{\mathfrak{g}}$-spaces. The next section provides concluding remarks and future directions of the theory of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-derived and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-coderived operators discussed in the preceding sections.

4.3. Concluding Remarks. In this Pure Mathematical manuscript titled *Theory of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-Derived and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-Coderived Operators* and subtitled *Definitions, Essential Properties, Iterations, and Ranks*, a new theory has been developed with the threefold objectives of, firstly, presenting the definitions and the essential properties of a new class of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-derived and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-coderived operators in $\mathscr{T}_{\mathfrak{g}}$-spaces, secondly, presenting the definitions and the essential properties of the concepts of δ^{th}-order derivative $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-derived and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-coderived operators in $\mathscr{T}_{\mathfrak{g}}$-spaces and, thirdly, presenting the definitions and the essential properties of the concepts of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-open and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-closed sets of rank δ in $\mathscr{T}_{\mathfrak{g}}$-spaces.

Concisely, the definitions of the concepts of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-derived and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-coderived operators, their δ^{th}-order derivative $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-derived and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-coderived operators, and δ^{th}-rank $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-open and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-closed sets in $\mathscr{T}_{\mathfrak{g}}$-spaces were presented in as general and unified a manner as possible such that the passage from these concepts to $\mathfrak{g}$-$\mathfrak{T}$-derived and $\mathfrak{g}$-$\mathfrak{T}$-coderived operators, their δ^{th}-order derivative $\mathfrak{g}$-$\mathfrak{T}$-derived and $\mathfrak{g}$-$\mathfrak{T}$-coderived operators, and δ^{th}-rank $\mathfrak{g}$-$\mathfrak{T}$-open and $\mathfrak{g}$-$\mathfrak{T}$-closed sets in $\mathscr{T}_{\mathfrak{g}}$-spaces, and also to $\mathfrak{T}$-derived and $\mathfrak{T}_{\mathfrak{g}}$-coderived operators, their δ^{th}-order derivative $\mathfrak{T}_{\mathfrak{g}}$-derived and $\mathfrak{T}_{\mathfrak{g}}$-coderived operators, and δ^{th}-rank $\mathfrak{T}_{\mathfrak{g}}$-open and $\mathfrak{T}_{\mathfrak{g}}$-closed sets in $\mathscr{T}$-spaces, is not impossible [Subsect. 2.1: Def. 2.1 & Rem. 2.4; Def. 2.11 & Rem. 2.12; Def. 2.13]; the essential properties of such novel types of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-derived and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-coderived operators, their δ^{th}-order derivative $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-derived and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-coderived operators, and δ^{th}-rank $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-open and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-closed sets in $\mathscr{T}_{\mathfrak{g}}$-spaces were discussed in such a manner as to show that much of the fundamental structure of $\mathscr{T}_{\mathfrak{g}}$-spaces is better considered for $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-derived and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-coderived operators $\mathfrak{g}\text{-Der}_{\mathfrak{g}}$, $\mathfrak{g}\text{-Cod}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$, their δ^{th}-order derivative $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-derived and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-coderived operators $\mathfrak{g}\text{-Der}^{(\delta)}_{\mathfrak{g}}$, $\mathfrak{g}\text{-Cod}^{(\delta)}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$, and the δ^{th}-rank $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-closure and

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$\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-interior operators $\mathfrak{g}\text{-Cl}_{\mathfrak{g}}^{(\delta)}$, $\mathfrak{g}\text{-Int}_{\mathfrak{g}}^{(\delta)} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ than for the $\mathfrak{T}_{\mathfrak{g}}$-derived and $\mathfrak{T}_{\mathfrak{g}}$-coderived operators $\text{der}_{\mathfrak{g}}$, $\text{cod}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$, their δ^{th}-order derivative $\mathfrak{T}_{\mathfrak{g}}$-derived and $\mathfrak{T}_{\mathfrak{g}}$-coderived operators $\text{der}_{\mathfrak{g}}^{(\delta)}$, $\text{cod}_{\mathfrak{g}}^{(\delta)} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$, and the δ^{th}-rank $\mathfrak{T}_{\mathfrak{g}}$-closure and $\mathfrak{T}_{\mathfrak{g}}$-interior operators $\text{cl}_{\mathfrak{g}}^{(\delta)}$, $\text{int}_{\mathfrak{g}}^{(\delta)} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$, respectively [Subsect. 3.1: Props 3.3–3.33, Thms 3.1–3.36, Cors 3.8–3.34 & Lems 3.4–3.35; Subsect. 3.2: Props 3.39–3.49, Thms 3.37–3.53 & Cors 3.38–3.56; Subsect. 3.3: Props 3.63–3.69, Thms 3.60–3.70, Cors 3.62–3.68 & Lem. 3.59]; the axiomatic definitions of the concepts of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-derived and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-coderived operators, their δ^{th}-order derivative $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-derived and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-coderived operators in $\mathscr{T}_{\mathfrak{g}}$-spaces were then presented from a purely mathematical or abstract point of view [Subsect. 3.2: Defs 3.27 & 3.31; Subsect. 3.3: Defs 3.57 & 3.58].

Precisely, the outstanding facts are:

- I. *On $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-Derived, $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-Coderived Operators.*
  - I. If the definitions of $\mathfrak{g}\text{-Der}_{\mathfrak{g}}$, $\mathfrak{g}\text{-Cod}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ are based on $\text{cl}_{\mathfrak{g}}$, $\text{int}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ instead of $\mathfrak{g}\text{-Cl}_{\mathfrak{g}}$, $\mathfrak{g}\text{-Int}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$, then $\left(\mathfrak{g}\text{-Der}_{\mathfrak{g}}, \mathfrak{g}\text{-Cod}_{\mathfrak{g}}\right) \stackrel{\text{def}}{=} \left(\text{der}_{\mathfrak{g}}, \text{cod}_{\mathfrak{g}}\right)$, and therefore, $\text{der}_{\mathfrak{g}}$, $\text{cod}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ are called, respectively, a $\mathfrak{T}_{\mathfrak{g}}$-derived and a $\mathfrak{T}_{\mathfrak{g}}$-coderived operators in a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$; if the definitions of $\mathfrak{g}\text{-Der}_{\mathfrak{g}}$, $\mathfrak{g}\text{-Cod}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ are based on $\mathfrak{g}\text{-Cl}$, $\mathfrak{g}\text{-Int} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ instead of $\mathfrak{g}\text{-Cl}_{\mathfrak{g}}$, $\mathfrak{g}\text{-Int}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$, then $\left(\mathfrak{g}\text{-Der}_{\mathfrak{g}}, \mathfrak{g}\text{-Cod}_{\mathfrak{g}}\right) \stackrel{\text{def}}{=} \left(\mathfrak{g}\text{-Der}, \mathfrak{g}\text{-Cod}\right)$, and therefore, $\mathfrak{g}\text{-Der}$, $\mathfrak{g}\text{-Cod} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ are called, respectively, a $\mathfrak{g}$-$\mathfrak{T}$-derived and a $\mathfrak{g}$-$\mathfrak{T}$-coderived operators in a $\mathscr{T}$-space $\mathfrak{T} = (\Omega, \mathscr{T})$; if the definitions of $\mathfrak{g}\text{-Der}_{\mathfrak{g}}$, $\mathfrak{g}\text{-Cod}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ are based on cl, $\text{int} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ instead of $\mathfrak{g}\text{-Cl}_{\mathfrak{g}}$, $\mathfrak{g}\text{-Int}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$, then the equality $\left(\mathfrak{g}\text{-Der}_{\mathfrak{g}}, \mathfrak{g}\text{-Cod}_{\mathfrak{g}}\right) \stackrel{\text{def}}{=} \left(\text{der}, \text{cod}\right)$ holds, and der, $\text{cod} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ are called, respectively, a $\mathfrak{T}$-derived and a $\mathfrak{T}$-coderived operators in a $\mathscr{T}$-space $\mathfrak{T} = (\Omega, \mathscr{T})$.
  - II. If "$\mathfrak{g}\text{-Der}_{\mathfrak{g}} \precsim \text{der}_{\mathfrak{g}}$" stands for "$\mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \subseteq \text{der}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$" and "$\mathfrak{g}\text{-Cod}_{\mathfrak{g}} \succsim \text{cod}_{\mathfrak{g}}$," for "$\mathfrak{g}\text{-Cod}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \supseteq \text{cod}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$," then it follows that: $\mathfrak{g}\text{-Der}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ is *coarser* (or, *smaller*, *weaker*) than $\text{der}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ or, $\text{der}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ is *finer* (or, *larger*, *stronger*) than $\mathfrak{g}\text{-Der}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$; $\mathfrak{g}\text{-Cod}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ is *finer* (or, *larger*, *stronger*) than $\text{cod}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ or, $\text{cod}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ is *coarser* (or, *smaller*, *weaker*) than $\mathfrak{g}\text{-Cod}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$.
  - III. A necessary and sufficient condition for the set-valued map $\mathfrak{g}\text{-Der}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ to be a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-derived operator in a strong $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$ is that, for every $\left(\{\xi\}, \mathscr{R}_{\mathfrak{g}}, \mathscr{S}_{\mathfrak{g}}\right) \in \times_{\alpha \in I_3^*} \mathscr{P}(\Omega)$ such that $\{\xi\} \subset \mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}})$, it satisfies:
    - (*i.*) $\mathfrak{g}\text{-Der}_{\mathfrak{g}}(\emptyset) = \emptyset$,
    - (*ii.*) $\mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}) = \mathfrak{g}\text{-Der}_{\mathfrak{g}}\left(\mathscr{R}_{\mathfrak{g}} \cap \mathfrak{g}\text{-Op}_{\mathfrak{g}}(\{\xi\})\right)$,
    - (*iii.*) $\mathfrak{g}\text{-Der}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}) \subseteq \mathscr{R}_{\mathfrak{g}} \cup \mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}})$,
    - (*iv.*) $\mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}} \cup \mathscr{S}_{\mathfrak{g}}) = \bigcup_{\mathscr{U}_{\mathfrak{g}} = \mathscr{R}_{\mathfrak{g}}, \mathscr{S}_{\mathfrak{g}}} \mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{U}_{\mathfrak{g}})$.

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– IV. A necessary and sufficient condition for the set-valued map $\mathfrak{g}\text{-Cod}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ to be a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-coderived operator in a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$ is that, for each $(\{\zeta\}, \mathscr{U}_{\mathfrak{g}}, \mathscr{V}_{\mathfrak{g}}) \in \times_{\alpha \in I_3^*} \mathscr{P}(\Omega)$ such that $\{\zeta\} \subset \mathfrak{g}\text{-Cod}_{\mathfrak{g}}(\mathscr{U}_{\mathfrak{g}})$, it satisfies:
  - ∗ *(i.)* $\mathfrak{g}\text{-Cod}_{\mathfrak{g}}(\Omega) = \Omega$,
  - ∗ *(ii.)* $\mathfrak{g}\text{-Cod}_{\mathfrak{g}}(\mathscr{U}_{\mathfrak{g}}) = \mathfrak{g}\text{-Cod}_{\mathfrak{g}}(\mathscr{U}_{\mathfrak{g}} \cup \{\zeta\})$,
  - ∗ *(iii.)* $\mathfrak{g}\text{-Cod}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Cod}_{\mathfrak{g}}(\mathscr{U}_{\mathfrak{g}}) \supseteq \mathscr{U}_{\mathfrak{g}} \cap \mathfrak{g}\text{-Cod}_{\mathfrak{g}}(\mathscr{U}_{\mathfrak{g}})$,
  - ∗ *(iv.)* $\mathfrak{g}\text{-Cod}_{\mathfrak{g}}(\mathscr{U}_{\mathfrak{g}} \cap \mathscr{V}_{\mathfrak{g}}) = \bigcap_{\mathscr{W}_{\mathfrak{g}} = \mathscr{U}_{\mathfrak{g}}, \mathscr{V}_{\mathfrak{g}}} \mathfrak{g}\text{-Cod}_{\mathfrak{g}}(\mathscr{W}_{\mathfrak{g}})$.

- II. *On δ^{th}-Order $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-Derived, $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-Coderived Operators.*
  - I. The substitution $(\mathfrak{g}\text{-Der}_{\mathfrak{g}}, \mathfrak{g}\text{-Cod}_{\mathfrak{g}}) \stackrel{\text{def}}{=} (\text{der}_{\mathfrak{g}}, \text{cod}_{\mathfrak{g}})$ into the δ^{th}-iterates $\mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta)}$, $\mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\delta)} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ of $\mathfrak{g}\text{-Der}_{\mathfrak{g}}$, $\mathfrak{g}\text{-Cod}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ defines the δ^{th}-iterates $\text{der}_{\mathfrak{g}}^{(\delta)}$, $\text{cod}_{\mathfrak{g}}^{(\delta)} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ of $\text{der}_{\mathfrak{g}}$, $\text{cod}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$, respectively, in a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$; the substitution $(\mathfrak{g}\text{-Der}_{\mathfrak{g}}, \mathfrak{g}\text{-Cod}_{\mathfrak{g}}) \stackrel{\text{def}}{=} (\mathfrak{g}\text{-Der}, \mathfrak{g}\text{-Cod})$ defines the δ^{th}-iterates $\mathfrak{g}\text{-Der}^{(\delta)}$, $\mathfrak{g}\text{-Cod}^{(\delta)} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ of $\mathfrak{g}\text{-Der}^{(\delta)}$, $\mathfrak{g}\text{-Cod}^{(\delta)} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$, respectively, in a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T} = (\Omega, \mathscr{T})$; finally, $(\mathfrak{g}\text{-Der}_{\mathfrak{g}}, \mathfrak{g}\text{-Cod}_{\mathfrak{g}}) \stackrel{\text{def}}{=} (\text{der}, \text{cod})$ defines the δ^{th}-iterates $\text{der}^{(\delta)}$, $\text{cod}^{(\delta)} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ of $\text{der}^{(\delta)}$, $\text{cod}^{(\delta)} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$, respectively, in a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T} = (\Omega, \mathscr{T})$.
  - II. A necessary and sufficient condition for the δ^{th}-iterate $\mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta)} : \mathscr{S}_{\mathfrak{g}} \in \mathscr{P}(\Omega) \longmapsto \mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta)}(\mathscr{S}_{\mathfrak{g}})$ of $\mathfrak{g}\text{-Der}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ to be a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-derived operator in a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$ is that, for every $(\{\xi\}, \mathscr{R}_{\mathfrak{g}}, \mathscr{S}_{\mathfrak{g}}) \in \times_{\alpha \in I_3^*} \mathscr{P}(\Omega)$ and δ such that $1 \preccurlyeq \delta \prec \lambda$, it satisfies:
    - ∗ *(i.)* $\mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta)}(\emptyset) = \emptyset$,
    - ∗ *(ii.)* $\mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta)}(\mathscr{R}_{\mathfrak{g}}) = \mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta)}(\mathscr{R}_{\mathfrak{g}} \cap \mathfrak{g}\text{-Op}_{\mathfrak{g}}(\{\xi\}))$,
    - ∗ *(iii.)* $\mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta)} \circ \mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta)}(\mathscr{R}_{\mathfrak{g}}) \subseteq \mathscr{R}_{\mathfrak{g}} \cup \mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}})$,
    - ∗ *(iv.)* $\mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta)}(\mathscr{R}_{\mathfrak{g}} \cup \mathscr{S}_{\mathfrak{g}}) = \bigcup_{\mathscr{U}_{\mathfrak{g}} = \mathscr{R}_{\mathfrak{g}}, \mathscr{S}_{\mathfrak{g}}} \mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta)}(\mathscr{U}_{\mathfrak{g}})$.
  - III. A necessary and sufficient condition for the δ^{th}-iterate $\mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\delta)} : \mathscr{S}_{\mathfrak{g}} \in \mathscr{P}(\Omega) \longmapsto \mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\delta)}(\mathscr{S}_{\mathfrak{g}})$ of $\mathfrak{g}\text{-Cod}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ to be a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-coderived operator in a strong $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$ is that, for every $(\{\xi\}, \mathscr{R}_{\mathfrak{g}}, \mathscr{S}_{\mathfrak{g}}) \in \times_{\alpha \in I_3^*} \mathscr{P}(\Omega)$ and δ such that $1 \preccurlyeq \delta \prec \lambda$, it satisfies:
    - ∗ *(i.)* $\mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\delta)}(\Omega) = \Omega$,
    - ∗ *(ii.)* $\mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\delta)}(\mathscr{R}_{\mathfrak{g}}) = \text{cod}_{\mathfrak{g}}^{(\delta)}(\mathscr{R}_{\mathfrak{g}} \cup \{\xi\})$,
    - ∗ *(iii.)* $\mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\delta)} \circ \mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\delta)}(\mathscr{R}_{\mathfrak{g}}) \supseteq \mathscr{R}_{\mathfrak{g}} \cap \mathfrak{g}\text{-Cod}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}})$,
    - ∗ *(iv.)* $\mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\delta)}(\mathscr{R}_{\mathfrak{g}} \cap \mathscr{S}_{\mathfrak{g}}) = \bigcap_{\mathscr{U}_{\mathfrak{g}} = \mathscr{R}_{\mathfrak{g}}, \mathscr{S}_{\mathfrak{g}}} \mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\delta)}(\mathscr{U}_{\mathfrak{g}})$.

- III. *On δ^{th}-Rank $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-Open, $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-Closed Sets.*
  - I. If the definitions of δ^{th}-rank $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-open and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-closed sets are based on $\text{int}_{\mathfrak{g}}^{(\delta)}$, $\text{cl}_{\mathfrak{g}}^{(\delta)} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ instead of $\mathfrak{g}\text{-Int}_{\mathfrak{g}}^{(\delta)}$, $\mathfrak{g}\text{-Cl}_{\mathfrak{g}}^{(\delta)} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$, the definitions of δ^{th}-rank $\mathfrak{T}_{\mathfrak{g}}$-open and $\mathfrak{T}_{\mathfrak{g}}$-closed

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sets, respectively, in a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$ follow; if the definitions of δ^{th}-rank $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-open and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-closed sets are based on $\mathfrak{g}$-Int, $\mathfrak{g}$-Cl : $\mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ instead of $\mathfrak{g}$-$\text{Int}_{\mathfrak{g}}$, $\mathfrak{g}$-$\text{Cl}_{\mathfrak{g}}$: $\mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$, $\mathfrak{g}$-$\text{Cl}_{\mathfrak{g}}^{(\delta)}$: $\mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$, the definitions of δ^{th}-rank $\mathfrak{g}$-$\mathfrak{T}$-open and $\mathfrak{g}$-$\mathfrak{T}$-closed sets, respectively, in a $\mathscr{T}$-space $\mathfrak{T} = (\Omega, \mathscr{T})$ follow; if the definitions of δ^{th}-rank $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-open and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-closed sets are based on int, cl : $\mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ instead of $\mathfrak{g}$-$\text{Int}_{\mathfrak{g}}$, $\mathfrak{g}$-$\text{Cl}_{\mathfrak{g}}$: $\mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$, the definitions of δ^{th}-rank $\mathfrak{T}$-open and $\mathfrak{T}$-closed sets, respectively, in a $\mathscr{T}$-space $\mathfrak{T} = (\Omega, \mathscr{T})$ follow.

- II. In a $\mathscr{T}_{\mathfrak{g}}$-space, δ^{th}-rank $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-openness and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-closedness are preserved under $\cup$, $\cap$-operations and, for any η such that $0 \prec \eta$, imply $\delta\eta^{\text{th}}$-rank $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-openness and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-closedness, respectively.
- III. In a $\mathscr{T}_{\mathfrak{g}}$-space, δ^{th}-rank $\mathfrak{T}_{\mathfrak{g}}$-openness and $\mathfrak{T}_{\mathfrak{g}}$-closedness imply δ^{th}-rank $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-openness and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-closedness, respectively.
- IV. In a $\mathscr{T}_{\mathfrak{g}}$-space, $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-open sets of ranks δ_1, δ_2, …, δ_α under $\cup$-operation and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-closed sets of ranks $\delta_1, \delta_2, \ldots, \delta_\beta$ under $\cap$-operation are, in general, both *not rank-preserving*; they become $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-open of rank $\delta = \text{lcm}\,(\delta_\nu : \nu \in I_\alpha^*)$ and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-closed of rank $\delta = \text{lcm}\big(\delta_\nu : \nu \in I_\beta^*\big)$.
- V. In a $\mathscr{T}_{\mathfrak{g}}$-space, suppose given a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-open set of ranks δ_1, δ_2, …, δ_α and a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-closed set of ranks δ_1, δ_2, …, δ_β, and suppose $\text{gcd}\,(\delta_\nu : \nu \in I_\alpha^*) = 1$ and $\text{gcd}\big(\eta_\nu : \nu \in I_\beta^*\big) = 1$, then there exist, in general, supersets of the $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-open set whose $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-coderived sets contain the $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-open, and subsets of the $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-closed set whose $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-derived sets are contained in the $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-closed set.

Hence, the proposed theory, in its own rights, has several advantages. Indeed, in relation to ITEMS I. I., II. I. and III. I., the theory offers very nice features, firstly, for the passage from $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-derived and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-coderived operators to $\mathfrak{T}_{\mathfrak{g}}$-derived and $\mathfrak{T}_{\mathfrak{g}}$-coderived operators, $\mathfrak{g}$-$\mathfrak{T}$-derived and $\mathfrak{g}$-$\mathfrak{T}$-coderived operators and $\mathfrak{T}$-derived and $\mathfrak{T}$-coderived operators, respectively (ITEM I. I.); secondly, for the passage from δ^{th}-order $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-derived and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-coderived operators to δ^{th}-order $\mathfrak{T}_{\mathfrak{g}}$-derived and $\mathfrak{T}_{\mathfrak{g}}$-coderived operators, δ^{th}-order $\mathfrak{g}$-$\mathfrak{T}$-derived and $\mathfrak{g}$-$\mathfrak{T}$-coderived operators and δ^{th}-order $\mathfrak{T}$-derived and $\mathfrak{T}$-coderived operators, respectively (ITEM II. I.); thirdly, for the passage from δ^{th}-rank $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-open and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-closed sets to δ^{th}-rank $\mathfrak{T}_{\mathfrak{g}}$-open and $\mathfrak{T}_{\mathfrak{g}}$-closed sets, δ^{th}-rank $\mathfrak{g}$-$\mathfrak{T}$-open and $\mathfrak{g}$-$\mathfrak{T}$-closed sets and δ^{th}-rank $\mathfrak{T}$-open and $\mathfrak{T}$-closed sets, respectively (ITEM III. I.). Thus, the theory holds equally well when $(\Omega, \mathscr{T}_{\mathfrak{g}}) = (\Omega, \mathscr{T})$ and other features adapted on this basis, in which case it might be called *Theory of* $\mathfrak{g}$-$\mathfrak{T}$*-Derived and* $\mathfrak{g}$-$\mathfrak{T}$*-Coderived Operators*. Therefore, in a $\mathscr{T}_{\mathfrak{g}}$-space the theoretical framework categorises the pair $\big(\mathfrak{g}\text{-Der}_{\mathfrak{g},0}, \mathfrak{g}\text{-Cod}_{\mathfrak{g},0}\big)$ of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$*-(derived, coderived) operators*, the pair $\big(\mathfrak{g}\text{-Der}_{\mathfrak{g},1}, \mathfrak{g}\text{-Cod}_{\mathfrak{g},1}\big)$ of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$*-semi-(derived, coderived) operators*, the pair $\big(\mathfrak{g}\text{-Der}_{\mathfrak{g},2}, \mathfrak{g}\text{-Cod}_{\mathfrak{g},2}\big)$ of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$*-pre-(derived, coderived) operators* and the pair $\big(\mathfrak{g}\text{-Der}_{\mathfrak{g},3}, \mathfrak{g}\text{-Cod}_{\mathfrak{g},3}\big)$ of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$*-semi-pre-(derived, coderived) operators* as pairs of $\mathfrak{g}$-$\mathfrak{T}$*-(derived, coderived) operators of categories* 0*,* 1*,* 2*, and* 3, respectively, and theorises the concepts in a unified way.

In relation to ITEM I. II., the theory offers $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-derived structures as $\big(\Omega, \mathfrak{g}\text{-Der}_{\mathfrak{g}}\big)$ which are *coarser* (or, *smaller*, *weaker*) than $\mathfrak{T}_{\mathfrak{g}}$-derived structures as $(\Omega, \text{der}_{\mathfrak{g}})$ or, $(\Omega, \text{der}_{\mathfrak{g}})$ is *finer* (or, *larger*, *stronger*) than $\big(\Omega, \mathfrak{g}\text{-Der}_{\mathfrak{g}}\big)$; $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-coderived structures

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as $\left(\Omega, \mathfrak{g}\text{-Cod}_{\mathfrak{g}}\right)$ which are *finer* (or, *larger*, *stronger*) than $\mathfrak{T}_{\mathfrak{g}}$-coderived structures as $(\Omega, \text{cod}_{\mathfrak{g}})$ or, $(\Omega, \text{cod}_{\mathfrak{g}})$ is *coarser* (or, *smaller*, *weaker*) than $\left(\Omega, \mathfrak{g}\text{-Cod}_{\mathfrak{g}}\right)$. Hence, such $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-structures can be considered as a means of handling certain problems in Functional Analysis. In relation to ITEMS I. III.–IV. (*i.*)–(*iv.*) and ITEMS II. II.–III. (*i.*)–(*iv.*), respectively, the theory contains the necessary and sufficient conditions for set-valued maps and their δ^{th}-iterates to be characterized as $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-derived and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-coderived operators, and δ^{th}-order $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-derived and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-coderived operators. Therefore, the theory also offers $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-(derived, coderived) structures from which many other novel propositions can be deduced by means of these conditions by purely logical processes. Thus, the construction of a purely *deductive theory* of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-derived and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-coderived operators a step further is made possible. Finally, in relation to III. II.–V., the theory contains various properties of δ^{th}-rank $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-open and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-closed sets, firstly, under $\cup$, $\cap$-operations, secondly, with respect to themselves and various ranks, and thirdly, with respect to δ^{th}-rank $\mathfrak{T}_{\mathfrak{g}}$-open and $\mathfrak{T}_{\mathfrak{g}}$-closed sets, respectively.

In view of the foregoing facts, making the theorization of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-derived and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-coderived operators of mixed categories in $\mathscr{T}_{\mathfrak{g}}$-spaces by purely formal processes a prime subject of inquiry for future research may therefore be not without interest. More precisely, for some pair $(\nu, \mu) \in I_3^0 \times I_3^0$ such that $\nu \neq \mu$, to develop a purely deductive theory of $\mathfrak{g}$-(ν, μ)-$\mathfrak{T}_{\mathfrak{g}}$-derived and $\mathfrak{g}$-(ν, μ)-$\mathfrak{T}_{\mathfrak{g}}$-coderived operators $\mathfrak{g}\text{-Der}_{\mathfrak{g},\nu\mu}$, $\mathfrak{g}\text{-Cod}_{\mathfrak{g},\nu\mu} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$, respectively, in $\mathscr{T}_{\mathfrak{g}}$-spaces, where $\mathfrak{g}\text{-Der}_{\mathfrak{g},\nu\mu} : \mathscr{W}_{\mathfrak{g}} \longmapsto \mathfrak{g}\text{-Der}_{\mathfrak{g},\nu\mu}(\mathscr{W}_{\mathfrak{g}})$ describes a type of collection of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-derived points of $\mathscr{W}_{\mathfrak{g}}$ and $\mathfrak{g}\text{-Cod}_{\mathfrak{g},\nu\mu} : \mathscr{W}_{\mathfrak{g}} \longmapsto \mathfrak{g}\text{-Cod}_{\mathfrak{g},\nu\mu}(\mathscr{W}_{\mathfrak{g}})$ describes another type of collection of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-coderived points of $\mathscr{W}_{\mathfrak{g}}$, and *derivedness* is characterized by $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-closed sets belonging to the class $\left\{\mathscr{K}_{\mathfrak{g}} = \mathscr{K}_{\mathfrak{g},\nu} \cap \mathscr{K}_{\mathfrak{g},\mu} : \ (\mathscr{K}_{\mathfrak{g},\nu}, \mathscr{K}_{\mathfrak{g},\mu}) \in \mathfrak{g}\text{-}\nu\text{-K}\left[\mathfrak{T}_{\mathfrak{g}}\right] \times \mathfrak{g}\text{-}\mu\text{-K}\left[\mathfrak{T}_{\mathfrak{g}}\right]\right\}$ and *coderivedness* is characterized by $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-open sets belonging to the class $\left\{\mathscr{O}_{\mathfrak{g}} = \mathscr{O}_{\mathfrak{g},\nu} \cup \mathscr{O}_{\mathfrak{g},\mu} : \ (\mathscr{O}_{\mathfrak{g},\nu}, \mathscr{O}_{\mathfrak{g},\mu}) \in \mathfrak{g}\text{-}\nu\text{-O}\left[\mathfrak{T}_{\mathfrak{g}}\right] \times \mathfrak{g}\text{-}\mu\text{-O}\left[\mathfrak{T}_{\mathfrak{g}}\right]\right\}$. Such an interestingly promising theory is what the present authors thought would certainly be worth considering, and the discussion of this paper ends here.

APPENDIX A. PRE-PRELIMINARIES

In this pre-preliminaries section, the elements accompanying the foregoing preliminary section are presented below. In actual fact, they are the elements extracted from the pre-preliminaries and preliminaries sections of the previous Pure Mathematical manuscript of the authors entitled *Theory of* $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-*Interior and* $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-*Closure Operators*.

As in all the previous Pure Mathematical manuscripts of the authors (See, *Theories of* $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-*Sets*, $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-*Maps*, $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-*Connectedness*, $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-*Separation Axioms*, $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-*Compactness*, $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-*Interior and* $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-*Closure Operators* and, $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-*Exterior and* $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-*Frontier Operators*), $\mathfrak{U}$ is the *universe* of discourse, fixed within the framework of the theory of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-derived and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-coderived operators and containing as elements all sets (briefly, Ω, Γ-sets; $\mathscr{T}$, $\mathfrak{g}$-$\mathscr{T}$, $\mathfrak{T}$, $\mathfrak{g}$-$\mathfrak{T}$-sets; $\mathscr{T}_{\mathfrak{g}}$, $\mathfrak{g}$-$\mathscr{T}_{\mathfrak{g}}$, $\mathfrak{T}_{\mathfrak{g}}$, $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-sets) considered in this theory, and $I_n^0 \stackrel{\text{def}}{=} \left\{\nu \in \mathbb{N}^0 : \ \nu \leq n\right\}$; index sets I_∞^0, I_n^*, I_∞^* are defined similarly. A set $\Gamma \subset \mathfrak{U}$ is a subset of the set $\Omega \subset \mathfrak{U}$ and, for some $\mathscr{T}_{\mathfrak{g}}$-open set $\mathscr{O}_{\mathfrak{g}} \in \mathscr{T} \cup \mathfrak{g}\text{-}\mathscr{T} \cup \mathscr{T}_{\mathfrak{g}} \cup \mathfrak{g}\text{-}\mathscr{T}_{\mathfrak{g}}$, these implications hold:

$$\mathscr{O}_{\mathfrak{g}} \in \mathscr{T} \Rightarrow \mathscr{O}_{\mathfrak{g}} \in \mathfrak{g}\text{-}\mathscr{T} \Rightarrow \mathscr{O}_{\mathfrak{g}} \in \mathscr{T}_{\mathfrak{g}} \Rightarrow \mathscr{O}_{\mathfrak{g}} \in \mathfrak{g}\text{-}\mathscr{T}_{\mathfrak{g}} \Rightarrow \mathscr{O}_{\mathfrak{g}} \subset \Omega \subset \mathfrak{U}. \tag{A.1}$$

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In a natural way, a monotonic map $\mathscr{T}_{\mathfrak{g}}:\mathscr{P}(\Omega)\longrightarrow\mathscr{P}(\Omega)$ from the power set $\mathscr{P}(\Omega)$ of Ω into itself can be associated to a given mapping $\pi_{\mathfrak{g}}:\Omega\longrightarrow\Omega$, thereby inducing a $\mathfrak{g}$-topology $\mathscr{T}_{\mathfrak{g}}\subset\mathscr{P}(\Omega)$ on the underlying set $\Omega\subset\mathfrak{U}$ [PC12]. When some further axioms [LR15] is specified for $\mathscr{T}_{\mathfrak{g}}:\mathscr{P}(\Omega)\longrightarrow\mathscr{P}(\Omega)$ with no separation axioms assumed unless otherwise stated, the notion of a $\mathscr{T}_{\mathfrak{g}}$-space follows.

Definition A.1 ($\mathscr{T}_{\mathfrak{g}}$-Space). Let $\Omega\subset\mathfrak{U}$ be a given set and let $\mathscr{P}(\Omega)\overset{\text{def}}{=}\{\mathscr{O}_{\mathfrak{g},\nu}\subseteq\Omega:\ \nu\in I_{\infty}^{*}\}$ be the family of all subsets $\mathscr{O}_{\mathfrak{g},1},\mathscr{O}_{\mathfrak{g},2},\ldots,$ of Ω. Then every one-valued map of the type $\mathscr{T}_{\mathfrak{g}}:\mathscr{P}(\Omega)\longrightarrow\mathscr{P}(\Omega)$ satisfying the following axioms:

- Ax. i. $\mathscr{T}_{\mathfrak{g}}(\emptyset)=\emptyset$,
- Ax. ii. $\mathscr{T}_{\mathfrak{g}}(\mathscr{O}_{\mathfrak{g}})\subseteq\mathscr{O}_{\mathfrak{g}}$,
- Ax. iii. $\mathscr{T}_{\mathfrak{g}}\big(\bigcup_{\nu\in I_{\infty}^{*}}\mathscr{O}_{\mathfrak{g},\nu}\big)=\bigcup_{\nu\in I_{\infty}^{*}}\mathscr{T}_{\mathfrak{g}}(\mathscr{O}_{\mathfrak{g},\nu})$,

is called a "$\mathfrak{g}$-topology on Ω," and the structure $\mathfrak{T}_{\mathfrak{g}}\overset{\text{def}}{=}(\Omega,\mathscr{T}_{\mathfrak{g}})$ is called a "$\mathscr{T}_{\mathfrak{g}}$-space."

In Def. A.1, by Ax. i., Ax. ii. and Ax. iii., respectively, are meant that the unary operation $\mathscr{T}_{\mathfrak{g}}:\mathscr{P}(\Omega)\longrightarrow\mathscr{P}(\Omega)$ preserves nullary union, is contracting and preserves binary union. Any element $\mathscr{O}_{\mathfrak{g}}\in\mathscr{T}_{\mathfrak{g}}\overset{\text{def}}{=}\{\mathscr{O}_{\mathfrak{g}}:\ \mathscr{O}_{\mathfrak{g}}\in\mathscr{T}_{\mathfrak{g}}\}$ of the $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}}$ is called a $\mathscr{T}_{\mathfrak{g}}$-open set and its complement element $\complement_{\Omega}(\mathscr{O}_{\mathfrak{g}})=\mathscr{K}_{\mathfrak{g}}\in\neg\mathscr{T}_{\mathfrak{g}}\overset{\text{def}}{=}\{\mathscr{K}_{\mathfrak{g}}:\ \complement(\mathscr{K}_{\mathfrak{g}})\in\mathscr{T}_{\mathfrak{g}}\}$, a $\mathscr{T}_{\mathfrak{g}}$-closed set; by convention, $\mathscr{T}_{\mathfrak{g}}$ and $\neg\mathscr{T}_{\mathfrak{g}}$, respectively, stand for the classes of all $\mathscr{T}_{\mathfrak{g}}$-open and $\mathscr{T}_{\mathfrak{g}}$-closed sets relative to the $\mathfrak{g}$-topology $\mathscr{T}_{\mathfrak{g}}$. If there exists a $\nu\in I_{\infty}^{*}$ such that $\mathscr{O}_{\mathfrak{g},\nu}=\Omega$, then $\mathfrak{T}_{\mathfrak{g}}$ is called a strong $\mathscr{T}_{\mathfrak{g}}$-space [Cs5, PC12]. Moreover, if $\mathscr{T}_{\mathfrak{g}}\big(\bigcap_{\nu\in I_{n}^{*}}\mathscr{O}_{\mathfrak{g},\nu}\big)=\bigcap_{\nu\in I_{n}^{*}}\mathscr{T}_{\mathfrak{g}}(\mathscr{O}_{\mathfrak{g},\nu})$ holds for any index set $I_{n}^{*}\subset I_{\infty}^{*}$ such that $n<\infty$, then $\mathfrak{T}_{\mathfrak{g}}$ is called a quasi $\mathscr{T}_{\mathfrak{g}}$-space [Cs8].

In the $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}}$, the operator $\operatorname{int}_{\mathfrak{g}}:\mathscr{P}(\Omega)\longrightarrow\mathscr{P}(\Omega)$ carrying each $\mathfrak{T}_{\mathfrak{g}}$-set $\mathscr{S}_{\mathfrak{g}}\subset\mathfrak{T}_{\mathfrak{g}}$ into its interior $\operatorname{int}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})=\Omega-\operatorname{cl}_{\mathfrak{g}}(\Omega\setminus\mathscr{S}_{\mathfrak{g}})\subset\mathfrak{T}_{\mathfrak{g}}$ is called a "$\mathfrak{T}_{\mathfrak{g}}$-interior operator;" the operator $\operatorname{cl}_{\mathfrak{g}}:\mathscr{P}(\Omega)\longrightarrow\mathscr{P}(\Omega)$ carrying each $\mathfrak{T}_{\mathfrak{g}}$-set $\mathscr{S}_{\mathfrak{g}}\subset\mathfrak{T}_{\mathfrak{g}}$ into its closure $\operatorname{cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})=\Omega-\operatorname{int}_{\mathfrak{g}}(\Omega\setminus\mathscr{S}_{\mathfrak{g}})\subset\mathfrak{T}_{\mathfrak{g}}$ is called a "$\mathfrak{T}_{\mathfrak{g}}$-closure operator." The classes $\mathrm{C}_{\mathscr{T}_{\mathfrak{g}}}^{\mathrm{sub}}[\mathscr{S}_{\mathfrak{g}}]\overset{\text{def}}{=}\{\mathscr{O}_{\mathfrak{g}}\in\mathscr{T}_{\mathfrak{g}}:\ \mathscr{O}_{\mathfrak{g}}\subseteq\mathscr{S}_{\mathfrak{g}}\}$ and $\mathrm{C}_{\neg\mathscr{T}_{\mathfrak{g}}}^{\mathrm{sup}}[\mathscr{S}_{\mathfrak{g}}]\overset{\text{def}}{=}\{\mathscr{K}_{\mathfrak{g}}\in\neg\mathscr{T}_{\mathfrak{g}}:\ \mathscr{K}_{\mathfrak{g}}\supseteq\mathscr{S}_{\mathfrak{g}}\}$, respectively, denote the classes of $\mathscr{T}_{\mathfrak{g}}$-open subsets and $\mathscr{T}_{\mathfrak{g}}$-closed supersets of the $\mathfrak{T}_{\mathfrak{g}}$-set $\mathscr{S}_{\mathfrak{g}}\subset\mathfrak{T}_{\mathfrak{g}}$ relative to the $\mathfrak{g}$-topology $\mathscr{T}_{\mathfrak{g}}$. That $\mathrm{C}_{\mathscr{T}_{\mathfrak{g}}}^{\mathrm{sub}}[\mathscr{S}_{\mathfrak{g}}]\subseteq\mathscr{T}_{\mathfrak{g}}(\Omega)$ and $\neg\mathscr{T}_{\mathfrak{g}}(\Omega)\supseteq\mathrm{C}_{\neg\mathscr{T}_{\mathfrak{g}}}^{\mathrm{sup}}[\mathscr{S}_{\mathfrak{g}}]$ are true for the $\mathfrak{T}_{\mathfrak{g}}$-set $\mathscr{S}_{\mathfrak{g}}\subset\mathfrak{T}_{\mathfrak{g}}$ in question are clear from the context. To this end, the $\mathfrak{T}_{\mathfrak{g}}$-closure and the $\mathfrak{T}_{\mathfrak{g}}$-interior of a $\mathfrak{T}_{\mathfrak{g}}$-set $\mathscr{S}_{\mathfrak{g}}\subset\mathfrak{T}_{\mathfrak{g}}$ in a $\mathscr{T}_{\mathfrak{g}}$-space define themselves as

$$\text{(A.2)}\qquad \operatorname{int}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})\overset{\text{def}}{=}\bigcup_{\mathscr{O}_{\mathfrak{g}}\in\mathrm{C}_{\mathscr{T}_{\mathfrak{g}}}^{\mathrm{sub}}[\mathscr{S}_{\mathfrak{g}}]}\mathscr{O}_{\mathfrak{g}},\quad \operatorname{cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})\overset{\text{def}}{=}\bigcap_{\mathscr{K}_{\mathfrak{g}}\in\mathrm{C}_{\neg\mathscr{T}_{\mathfrak{g}}}^{\mathrm{sup}}[\mathscr{S}_{\mathfrak{g}}]}\mathscr{K}_{\mathfrak{g}},$$

respectively. We note in passing that, $\operatorname{cl}_{\mathfrak{g}}(\cdot)\neq\operatorname{cl}(\cdot)$ and $\operatorname{int}_{\mathfrak{g}}(\cdot)\neq\operatorname{int}(\cdot)$ in general, because the resulting sets obtained from the intersection of all $\mathscr{T}_{\mathfrak{g}}$-closed supersets and the union of all $\mathscr{T}_{\mathfrak{g}}$-open subsets, respectively, relative to the $\mathfrak{g}$-topology $\mathscr{T}_{\mathfrak{g}}$ are not necessarily equal to those which would be obtained from the intersection of all $\mathscr{T}$-closed supersets and the union of all $\mathscr{T}$-open subsets relative to the topology $\mathscr{T}$ [BKR13]. Throughout this work, by $\operatorname{cl}_{\mathfrak{g}}\circ\operatorname{int}_{\mathfrak{g}}(\cdot)$, $\operatorname{int}_{\mathfrak{g}}\circ\operatorname{cl}_{\mathfrak{g}}(\cdot)$, and $\operatorname{cl}_{\mathfrak{g}}\circ\operatorname{int}_{\mathfrak{g}}\circ\operatorname{cl}_{\mathfrak{g}}(\cdot)$, respectively, are meant $\operatorname{cl}_{\mathfrak{g}}(\operatorname{int}_{\mathfrak{g}}(\cdot))$, $\operatorname{int}_{\mathfrak{g}}(\operatorname{cl}_{\mathfrak{g}}(\cdot))$, and $\operatorname{cl}_{\mathfrak{g}}(\operatorname{int}_{\mathfrak{g}}(\operatorname{cl}_{\mathfrak{g}}(\cdot)))$; other composition operators are defined in a similar way. Also, the backslash $\Omega\setminus\mathscr{S}_{\mathfrak{g}}$ refers to the set-theoretic difference $\Omega-\mathscr{S}_{\mathfrak{g}}$. Finally, for convenience of notation, let $\mathscr{P}^{*}(\Omega)=\mathscr{P}(\Omega)\setminus\{\emptyset\}$, $\mathscr{T}_{\mathfrak{g}}^{*}=\mathscr{T}_{\mathfrak{g}}\setminus\{\emptyset\}$, and $\neg\mathscr{T}_{\mathfrak{g}}^{*}=\neg\mathscr{T}_{\mathfrak{g}}\setminus\{\emptyset\}$.

DEFINITION A.2 ($\mathfrak{g}$-Operation). Let $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$ be a $\mathscr{T}_{\mathfrak{g}}$-space. Then, a mapping $\operatorname{op}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ on $\mathscr{P}(\Omega)$ ranging in $\mathscr{P}(\Omega)$ is called a "$\mathfrak{g}$-operation" if and only if the following statements hold:

$$\text{(A.3)}\ \left(\forall \mathscr{S}_{\mathfrak{g}} \in \mathscr{P}^{*}(\Omega)\right)\left(\exists \left(\mathscr{O}_{\mathfrak{g}}, \mathscr{K}_{\mathfrak{g}}\right) \in \mathscr{T}_{\mathfrak{g}}^{*} \times \neg \mathscr{T}_{\mathfrak{g}}^{*}\right)\left[\left(\operatorname{op}_{\mathfrak{g}}(\emptyset) = \emptyset\right) \vee \left(\neg \operatorname{op}_{\mathfrak{g}}(\emptyset) = \emptyset\right) \vee \left(\mathscr{S}_{\mathfrak{g}} \subseteq \operatorname{op}_{\mathfrak{g}}(\mathscr{O}_{\mathfrak{g}})\right) \vee \left(\mathscr{S}_{\mathfrak{g}} \supseteq \neg \operatorname{op}_{\mathfrak{g}}(\mathscr{K}_{\mathfrak{g}})\right)\right],$$

where $\neg \operatorname{op}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ is called the "complementary $\mathfrak{g}$-operation" on $\mathscr{P}(\Omega)$ ranging in $\mathscr{P}(\Omega)$ and, for all $(\mathscr{S}_{\mathfrak{g}}, \mathscr{U}_{\mathfrak{g},\mu}, \mathscr{V}_{\mathfrak{g},\nu}) \in \times_{\alpha \in I_3^*} \mathscr{P}^{*}(\Omega)$ such that $\mathscr{W}_{\mathfrak{g}} = \mathscr{U}_{\mathfrak{g},\mu} \cup \mathscr{V}_{\mathfrak{g},\nu}$ and $\left(\hat{\mathscr{W}}_{\mathfrak{g}}, \neg \hat{\mathscr{W}}_{\mathfrak{g}}\right) = \left(\operatorname{op}_{\mathfrak{g}}(\mathscr{W}_{\mathfrak{g}}), \neg \operatorname{op}_{\mathfrak{g}}(\mathscr{W}_{\mathfrak{g}})\right)$, the following axioms are satisfied:

- AX. I. $\left(\mathscr{S}_{\mathfrak{g}} \subseteq \operatorname{op}_{\mathfrak{g}}(\mathscr{O}_{\mathfrak{g}})\right) \vee \left(\mathscr{S}_{\mathfrak{g}} \supseteq \neg \operatorname{op}_{\mathfrak{g}}(\mathscr{K}_{\mathfrak{g}})\right)$,
- AX. II. $\left(\operatorname{op}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \subseteq \operatorname{op}_{\mathfrak{g}} \circ \operatorname{op}_{\mathfrak{g}}(\mathscr{O}_{\mathfrak{g}})\right) \vee \left(\neg \operatorname{op}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \supseteq \neg \operatorname{op}_{\mathfrak{g}} \circ \neg \operatorname{op}_{\mathfrak{g}}(\mathscr{K}_{\mathfrak{g}})\right)$,
- AX. III. $\left(\hat{\mathscr{W}}_{\mathfrak{g}} \subseteq \bigcup_{\sigma=\mu,\nu} \operatorname{op}_{\mathfrak{g}}(\mathscr{O}_{\mathfrak{g},\sigma})\right) \bigvee \left(\neg \hat{\mathscr{W}}_{\mathfrak{g}} \supseteq \bigcup_{\sigma=\mu,\nu} \neg \operatorname{op}_{\mathfrak{g}}(\mathscr{K}_{\mathfrak{g},\sigma})\right)$,
- AX. IV. $\left(\mathscr{U}_{\mathfrak{g},\mu} \subseteq \mathscr{V}_{\mathfrak{g},\nu} \longrightarrow \operatorname{op}_{\mathfrak{g}}(\mathscr{O}_{\mathfrak{g},\mu}) \subseteq \operatorname{op}_{\mathfrak{g}}(\mathscr{O}_{\mathfrak{g},\nu})\right) \vee \left(\mathscr{U}_{\mathfrak{g},\mu} \supseteq \mathscr{V}_{\mathfrak{g},\nu} \longleftarrow \neg \operatorname{op}_{\mathfrak{g}}(\mathscr{K}_{\mathfrak{g},\mu}) \supseteq \neg \operatorname{op}_{\mathfrak{g}}(\mathscr{K}_{\mathfrak{g},\nu})\right)$,

for some $(\mathscr{O}_{\mathfrak{g}}, \mathscr{O}_{\mathfrak{g},\mu}, \mathscr{O}_{\mathfrak{g},\nu}) \in \times_{\alpha \in I_3^*} \mathscr{T}_{\mathfrak{g}}^{*}$ and $(\mathscr{K}_{\mathfrak{g}}, \mathscr{K}_{\mathfrak{g},\mu}, \mathscr{K}_{\mathfrak{g},\nu}) \in \times_{\alpha \in I_3^*} \neg \mathscr{T}_{\mathfrak{g}}^{*}$.

The formulation of DEF. A.2 is based on the axioms of the Čech closure operator [Boo11] and the various axioms used by many mathematicians to define closure operators [MHD83].

DEFINITION A.3 ($\mathbf{op}_{\mathfrak{g}}$-Elements). Let $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$ be a $\mathscr{T}_{\mathfrak{g}}$-space. Then, the class $\mathscr{L}_{\mathfrak{g}}[\Omega] \stackrel{\text{def}}{=} \left\{\mathbf{op}_{\mathfrak{g},\nu} = \left(\operatorname{op}_{\mathfrak{g},\nu}, \neg \operatorname{op}_{\mathfrak{g},\nu}\right) : \ \nu \in I_3^0\right\} \subseteq \mathscr{L}_{\mathfrak{g}}^{\omega}[\Omega] \times \mathscr{L}_{\mathfrak{g}}^{\kappa}[\Omega]$, where

$$\text{(A.4)} \qquad \operatorname{op}_{\mathfrak{g}} \in \mathscr{L}_{\mathfrak{g}}^{\omega}[\Omega] \stackrel{\text{def}}{=} \left\{\operatorname{op}_{\mathfrak{g},0},\ \operatorname{op}_{\mathfrak{g},1},\ \operatorname{op}_{\mathfrak{g},2},\ \operatorname{op}_{\mathfrak{g},3}\right\} = \left\{\operatorname{int}_{\mathfrak{g}},\ \operatorname{cl}_{\mathfrak{g}} \circ \operatorname{int}_{\mathfrak{g}},\ \operatorname{int}_{\mathfrak{g}} \circ \operatorname{cl}_{\mathfrak{g}},\ \operatorname{cl}_{\mathfrak{g}} \circ \operatorname{int}_{\mathfrak{g}} \circ \operatorname{cl}_{\mathfrak{g}}\right\},$$

$$\text{(A.5)} \qquad \neg \operatorname{op}_{\mathfrak{g}} \in \mathscr{L}_{\mathfrak{g}}^{\kappa}[\Omega] \stackrel{\text{def}}{=} \left\{\neg \operatorname{op}_{\mathfrak{g},0},\ \neg \operatorname{op}_{\mathfrak{g},1},\ \neg \operatorname{op}_{\mathfrak{g},2},\ \neg \operatorname{op}_{\mathfrak{g},3}\right\} = \left\{\operatorname{cl}_{\mathfrak{g}},\ \operatorname{int}_{\mathfrak{g}} \circ \operatorname{cl}_{\mathfrak{g}},\ \operatorname{cl}_{\mathfrak{g}} \circ \operatorname{int}_{\mathfrak{g}},\ \operatorname{int}_{\mathfrak{g}} \circ \operatorname{cl}_{\mathfrak{g}} \circ \operatorname{int}_{\mathfrak{g}}\right\},$$

stands for the class of all possible pairs of $\mathfrak{g}$-operators and its complementary $\mathfrak{g}$-operators in the $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}}$.

The use of $\mathbf{op}_{\mathfrak{g}} = \left(\operatorname{op}_{\mathfrak{g}}, \neg \operatorname{op}_{\mathfrak{g}}\right) \in \mathscr{L}_{\mathfrak{g}}[\Omega]$ on a class of $\mathfrak{T}_{\mathfrak{g}}$-sets will construct a new class of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-sets, just as the use of $\mathscr{L}[\Omega] \stackrel{\text{def}}{=} \left\{\mathbf{op}_{\nu} = \left(\operatorname{op}_{\nu}, \neg \operatorname{op}_{\nu}\right) : \ \nu \in I_3^0\right\}$ on the class of $\mathfrak{T}$-sets have constructed the new class of $\mathfrak{g}$-$\mathfrak{T}$-sets. But since $\operatorname{cl}_{\mathfrak{g}} \neq \operatorname{cl}$ and $\operatorname{int}_{\mathfrak{g}} \neq \operatorname{int}$, in general, it follows that $\mathbf{op}_{\mathfrak{g},\nu} \neq \mathbf{op}_{\nu}$ for some $\nu \in I_3^0$ and therefore, the new class of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-sets that will be obtained from the first construction will, in general, differ from the new class of $\mathfrak{g}$-$\mathfrak{T}$-sets that had been obtained from the second construction. Employing the set-builder notations, the notion of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-set of category ν may then be defined as thus:

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Definition A.4. Let $(\mathscr{S}_{\mathfrak{g}}, \mathscr{O}_{\mathfrak{g}}, \mathscr{K}_{\mathfrak{g}}) \in \mathfrak{T}_{\mathfrak{g}} \times \mathscr{T}_{\mathfrak{g}} \times \neg\mathscr{T}_{\mathfrak{g}}$ and let $\mathbf{op}_{\mathfrak{g},\nu} \in \mathscr{L}_{\mathfrak{g}}[\Omega]$ be a $\mathfrak{g}$-operator in a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$. Suppose the predicates

$$
\begin{aligned}
\mathrm{P}_{\mathfrak{g}}\big(\mathscr{S}_{\mathfrak{g}}, \mathscr{O}_{\mathfrak{g}}, \mathscr{K}_{\mathfrak{g}}; \mathbf{op}_{\mathfrak{g},\nu}; \subseteq, \supseteq\big) &\stackrel{\text{def}}{=} \mathrm{P}_{\mathfrak{g}}\big(\mathscr{S}_{\mathfrak{g}}, \mathscr{O}_{\mathfrak{g}}; \mathbf{op}_{\mathfrak{g},\nu}; \subseteq\big) \vee \mathrm{P}_{\mathfrak{g}}\big(\mathscr{S}_{\mathfrak{g}}, \mathscr{K}_{\mathfrak{g}}; \mathbf{op}_{\mathfrak{g},\nu}; \supseteq\big), \\
\mathrm{P}_{\mathfrak{g}}\big(\mathscr{S}_{\mathfrak{g}}, \mathscr{O}_{\mathfrak{g}}; \mathbf{op}_{\mathfrak{g},\nu}; \subseteq\big) &\stackrel{\text{def}}{=} \big(\exists\,(\mathscr{O}_{\mathfrak{g}}, \mathrm{op}_{\mathfrak{g},\nu}) \in \mathscr{T}_{\mathfrak{g}} \times \mathscr{L}^{\omega}_{\mathfrak{g}}[\Omega]\big) \\
&\qquad \big[\mathscr{S}_{\mathfrak{g}} \subseteq \mathrm{op}_{\mathfrak{g},\nu}(\mathscr{O}_{\mathfrak{g}})\big], \\
\mathrm{P}_{\mathfrak{g}}\big(\mathscr{S}_{\mathfrak{g}}, \mathscr{K}_{\mathfrak{g}}; \mathbf{op}_{\mathfrak{g},\nu}; \supseteq\big) &\stackrel{\text{def}}{=} \big(\exists\,(\mathscr{K}_{\mathfrak{g}}, \neg\,\mathrm{op}_{\mathfrak{g},\nu}) \in \neg\mathscr{T}_{\mathfrak{g}} \times \mathscr{L}^{\kappa}_{\mathfrak{g}}[\Omega]\big) \\
&\qquad \big[\mathscr{S}_{\mathfrak{g}} \supseteq \neg\,\mathrm{op}_{\mathfrak{g},\nu}(\mathscr{K}_{\mathfrak{g}})\big]
\end{aligned} \tag{A.6}
$$

be "Boolean-valued functions" on $\mathfrak{T}_{\mathfrak{g}} \times (\mathscr{T}_{\mathfrak{g}} \cup \neg\mathscr{T}_{\mathfrak{g}}) \times \mathscr{L}_{\mathfrak{g}}[\Omega] \times \{\subseteq, \supseteq\}$, then

$$
\begin{aligned}
\mathfrak{g}\text{-}\nu\text{-}\mathrm{S}[\mathfrak{T}_{\mathfrak{g}}] &\stackrel{\text{def}}{=} \big\{\mathscr{S}_{\mathfrak{g}} \subset \mathfrak{T}_{\mathfrak{g}} :\ \mathrm{P}_{\mathfrak{g}}\big(\mathscr{S}_{\mathfrak{g}}, \mathscr{O}_{\mathfrak{g}}, \mathscr{K}_{\mathfrak{g}}; \mathbf{op}_{\mathfrak{g},\nu}; \subseteq, \supseteq\big)\big\}, \\
\mathfrak{g}\text{-}\nu\text{-}\mathrm{O}[\mathfrak{T}_{\mathfrak{g}}] &\stackrel{\text{def}}{=} \big\{\mathscr{S}_{\mathfrak{g}} \subset \mathfrak{T}_{\mathfrak{g}} :\ \mathrm{P}_{\mathfrak{g}}\big(\mathscr{S}_{\mathfrak{g}}, \mathscr{O}_{\mathfrak{g}}; \mathbf{op}_{\mathfrak{g},\nu}; \subseteq\big)\big\}, \\
\mathfrak{g}\text{-}\nu\text{-}\mathrm{K}[\mathfrak{T}_{\mathfrak{g}}] &\stackrel{\text{def}}{=} \big\{\mathscr{S}_{\mathfrak{g}} \subset \mathfrak{T}_{\mathfrak{g}} :\ \mathrm{P}_{\mathfrak{g}}\big(\mathscr{S}_{\mathfrak{g}}, \mathscr{K}_{\mathfrak{g}}; \mathbf{op}_{\mathfrak{g},\nu}; \supseteq\big)\big\},
\end{aligned} \tag{A.7}
$$

respectively, are called the classes of all $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-sets, $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-open sets and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-closed sets of category ν in $\mathfrak{T}_{\mathfrak{g}}$.

Thus, $\mathscr{S}_{\mathfrak{g}} \subset \mathfrak{T}_{\mathfrak{g}}$ is called a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-set of category ν if and only if there exists a pair $(\mathscr{O}_{\mathfrak{g}}, \mathscr{K}_{\mathfrak{g}}) \in \mathscr{T}_{\mathfrak{g}} \times \neg\mathscr{T}_{\mathfrak{g}}$ of $\mathscr{T}_{\mathfrak{g}}$-open and $\mathscr{T}_{\mathfrak{g}}$-closed sets and a $\mathfrak{g}$-operator $\mathbf{op}_{\mathfrak{g},\nu} \in \mathscr{L}_{\mathfrak{g}}[\Omega]$ of category ν such that the following statement holds:

$$
(\exists \xi)\,\big[(\xi \in \mathscr{S}_{\mathfrak{g}}) \wedge \big(\big(\mathscr{S}_{\mathfrak{g}} \subseteq \mathrm{op}_{\mathfrak{g},\nu}(\mathscr{O}_{\mathfrak{g}})\big) \vee \big(\mathscr{S}_{\mathfrak{g}} \supseteq \neg\,\mathrm{op}_{\mathfrak{g},\nu}(\mathscr{K}_{\mathfrak{g}})\big)\big)\big].
$$

Clearly,

$$
\begin{aligned}
\mathfrak{g}\text{-}\mathrm{S}[\mathfrak{T}_{\mathfrak{g}}] \stackrel{\text{def}}{=} \textstyle\bigcup_{\nu \in I_3^0} \mathfrak{g}\text{-}\nu\text{-}\mathrm{S}[\mathfrak{T}_{\mathfrak{g}}] &= \bigcup_{\nu \in I_3^0} \big(\mathfrak{g}\text{-}\nu\text{-}\mathrm{O}[\mathfrak{T}_{\mathfrak{g}}] \cup \mathfrak{g}\text{-}\nu\text{-}\mathrm{K}[\mathfrak{T}_{\mathfrak{g}}]\big) \\
&= \Big(\bigcup_{\nu \in I_3^0} \mathfrak{g}\text{-}\nu\text{-}\mathrm{O}[\mathfrak{T}_{\mathfrak{g}}]\Big) \cup \Big(\bigcup_{\nu \in I_3^0} \mathfrak{g}\text{-}\nu\text{-}\mathrm{K}[\mathfrak{T}_{\mathfrak{g}}]\Big) \\
&\stackrel{\text{def}}{=} \mathfrak{g}\text{-}\mathrm{O}[\mathfrak{T}_{\mathfrak{g}}] \cup \mathfrak{g}\text{-}\mathrm{K}[\mathfrak{T}_{\mathfrak{g}}],
\end{aligned}
$$

then, defines the class of all $\mathfrak{g}$-ν-$\mathfrak{T}_{\mathfrak{g}}$-sets as the union of the classes of all $\mathfrak{g}$-ν-$\mathfrak{T}_{\mathfrak{g}}$-open and $\mathfrak{g}$-ν-$\mathfrak{T}_{\mathfrak{g}}$-closed sets, defined by $\mathfrak{g}$-$\mathrm{O}[\mathfrak{T}_{\mathfrak{g}}]$ and $\mathfrak{g}$-$\mathrm{K}[\mathfrak{T}_{\mathfrak{g}}]$ respectively. It is interesting to view the concepts of open, semi-open, preopen, semi-preopen sets [And86, And84, CM64, Lev63, MEMED82, Nj5] as $\mathfrak{g}$-$\mathfrak{T}$-open sets of categories 0, 1, 2, and 3, respectively; likewise, to view the concepts of closed, semi-closed, preclosed, semi-preclosed sets [And96] as $\mathfrak{g}$-$\mathfrak{T}$-closed sets of categories 0, 1, 2, and 3, respectively. These can be realised by omitting the subscript "$\mathfrak{g}$" in all symbols of the above definitions. The remark follows.

Remark A.5. Observing that, for every $\nu \in I_3^*$, the first and second components of the $\mathfrak{g}$-vector operator $\mathbf{op}_{\mathfrak{g},\nu} = \big(\mathrm{op}_{\mathfrak{g},\nu}, \neg\,\mathrm{op}_{\mathfrak{g},\nu}\big) \in \mathscr{L}_{\mathfrak{g}}[\Omega]$ are based on $\mathscr{T}_{\mathfrak{g}} \times \neg\mathscr{T}_{\mathfrak{g}}$, respectively, it follows that $\mathbf{op}_{\mathfrak{g},\nu} = \mathbf{op}_{\nu} \stackrel{\text{def}}{=} (\mathrm{op}_{\nu}, \neg\,\mathrm{op}_{\nu}) \in \mathscr{L}[\Omega]$ if based on $\mathscr{T} \times \neg\mathscr{T}$, respectively. In this way, $\mathbf{op} : \mathscr{P}(\Omega) \times \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega) \times \mathscr{P}(\Omega)$ is

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called a $\mathfrak{g}$-vector operator in a $\mathscr{T}$-space $\mathfrak{T} = (\Omega, \mathscr{T})$. Accordingly,

$$\text{(A.8)}\qquad \begin{aligned} \mathrm{op} \in \mathscr{L}^{\omega}[\Omega] &\stackrel{\text{def}}{=} \{\mathrm{op}_0,\ \mathrm{op}_1,\ \mathrm{op}_2,\ \mathrm{op}_3\} \\ &= \{\mathrm{int},\ \mathrm{cl}\circ\mathrm{int},\ \mathrm{int}\circ\mathrm{cl},\ \mathrm{cl}\circ\mathrm{int}\circ\mathrm{cl}\}, \end{aligned}$$

$$\text{(A.9)}\qquad \begin{aligned} \neg\,\mathrm{op} \in \mathscr{L}^{\kappa}[\Omega] &\stackrel{\text{def}}{=} \{\neg\,\mathrm{op}_0,\ \neg\,\mathrm{op}_1,\ \neg\,\mathrm{op}_2,\ \neg\,\mathrm{op}_3\} \\ &= \{\mathrm{cl},\ \mathrm{int}\circ\mathrm{cl},\ \mathrm{cl}\circ\mathrm{int},\ \mathrm{int}\circ\mathrm{cl}\circ\mathrm{int}\}, \end{aligned}$$

and, $\mathscr{L}_{\mathfrak{g}}[\Omega] \stackrel{\text{def}}{=} \{\mathbf{op}_{\mathfrak{g},\nu} = (\mathrm{op}_{\mathfrak{g},\nu}, \neg\,\mathrm{op}_{\mathfrak{g},\nu}) :\ \nu \in I_3^0\} \subseteq \mathscr{L}_{\mathfrak{g}}^{\omega}[\Omega] \times \mathscr{L}_{\mathfrak{g}}^{\kappa}[\Omega]$ stands for the class of all possible pairs of $\mathfrak{g}$-operators and its complementary $\mathfrak{g}$-operators in the $\mathscr{T}$-space $\mathfrak{T} = (\Omega, \mathscr{T})$.

By virtue of the above remark, if $(\mathscr{S}, \mathscr{O}, \mathscr{K}) \in \mathfrak{T} \times \mathscr{T} \times \neg\mathscr{T}$ and $\mathbf{op}_{\nu} \in \mathscr{L}[\Omega]$ in a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$, then the predicates

$$\begin{aligned} \mathrm{P}(\mathscr{S}, \mathscr{O}, \mathscr{K}; \mathbf{op}_{\nu}; \subseteq, \supseteq) &\stackrel{\text{def}}{=} \mathrm{P}(\mathscr{S}, \mathscr{O}; \mathbf{op}_{\nu}; \subseteq) \vee \mathrm{P}(\mathscr{S}, \mathscr{K}; \mathbf{op}_{\nu}; \supseteq), \\ \mathrm{P}(\mathscr{S}, \mathscr{O}; \mathbf{op}_{\nu}; \subseteq) &\stackrel{\text{def}}{=} (\exists\,(\mathscr{O}, \mathrm{op}_{\nu}) \in \mathscr{T} \times \mathscr{L}^{\omega}[\Omega])\,[\mathscr{S} \subseteq \mathrm{op}_{\nu}(\mathscr{O})], \\ \text{(A.10)}\ \mathrm{P}(\mathscr{S}, \mathscr{K}; \mathbf{op}_{\nu}; \supseteq) &\stackrel{\text{def}}{=} (\exists\,(\mathscr{K}, \neg\,\mathrm{op}_{\nu}) \in \neg\mathscr{T} \times \mathscr{L}^{\kappa}[\Omega])\,[\mathscr{S} \supseteq \neg\,\mathrm{op}_{\nu}(\mathscr{K})] \end{aligned}$$

are obviously "Boolean-valued functions" on $\mathfrak{T} \times (\mathscr{T} \cup \neg\mathscr{T}) \times \mathscr{L}[\Omega] \times \{\subseteq, \supseteq\}$ and,

$$\text{(A.11)}\qquad \begin{aligned} \mathfrak{g}\text{-}\nu\text{-S}[\mathfrak{T}] &\stackrel{\text{def}}{=} \{\mathscr{S} \subset \mathfrak{T} :\ \mathrm{P}(\mathscr{S}, \mathscr{O}, \mathscr{K}; \mathbf{op}_{\nu}; \subseteq, \supseteq)\}, \\ \mathfrak{g}\text{-}\nu\text{-O}[\mathfrak{T}] &\stackrel{\text{def}}{=} \{\mathscr{S} \subset \mathfrak{T} :\ \mathrm{P}(\mathscr{S}, \mathscr{O}; \mathbf{op}_{\nu}; \subseteq)\}, \\ \mathfrak{g}\text{-}\nu\text{-K}[\mathfrak{T}] &\stackrel{\text{def}}{=} \{\mathscr{S} \subset \mathfrak{T} :\ \mathrm{P}(\mathscr{S}, \mathscr{K}; \mathbf{op}_{\nu}; \supseteq)\}, \end{aligned}$$

respectively, are called the classes of all $\mathfrak{g}$-$\mathfrak{T}$-sets, $\mathfrak{g}$-$\mathfrak{T}$-open sets and $\mathfrak{g}$-$\mathfrak{T}$-closed sets of category ν in $\mathfrak{T}$. Therefore, $\mathscr{S} \subset \mathfrak{T}$ is called a $\mathfrak{g}$-$\mathfrak{T}$-set of category ν if and only if there exist a pair $(\mathscr{O}, \mathscr{K}) \in \mathscr{T} \times \neg\mathscr{T}$ of $\mathscr{T}$-open and $\mathscr{T}$-closed sets and a $\mathfrak{g}$-operator $\mathbf{op}_{\nu} \in \mathscr{L}[\Omega]$ of category ν such that the following statement holds:

$$(\exists\xi)\,[(\xi \in \mathscr{S}) \wedge ((\mathscr{S} \subseteq \mathrm{op}_{\nu}(\mathscr{O})) \vee (\mathscr{S} \supseteq \neg\,\mathrm{op}_{\nu}(\mathscr{K})))]\,.$$

Evidently,

$$\begin{aligned} \mathfrak{g}\text{-S}[\mathfrak{T}] \stackrel{\text{def}}{=} \textstyle\bigcup_{\nu\in I_3^0} \mathfrak{g}\text{-}\nu\text{-S}[\mathfrak{T}] &= \bigcup_{\nu\in I_3^0} (\mathfrak{g}\text{-}\nu\text{-O}[\mathfrak{T}] \cup \mathfrak{g}\text{-}\nu\text{-K}[\mathfrak{T}]) \\ &= \Bigg(\bigcup_{\nu\in I_3^0} \mathfrak{g}\text{-}\nu\text{-O}[\mathfrak{T}]\Bigg) \cup \Bigg(\bigcup_{\nu\in I_3^0} \mathfrak{g}\text{-}\nu\text{-K}[\mathfrak{T}]\Bigg) \\ &\stackrel{\text{def}}{=} \mathfrak{g}\text{-O}[\mathfrak{T}] \cup \mathfrak{g}\text{-K}[\mathfrak{T}]\,, \end{aligned}$$

then, defines the class of all $\mathfrak{g}$-ν-$\mathfrak{T}$-sets as the union of the classes of all $\mathfrak{g}$-ν-$\mathfrak{T}$-open and $\mathfrak{g}$-ν-$\mathfrak{T}$-closed sets, defined by $\mathfrak{g}\text{-O}[\mathfrak{T}]$ and $\mathfrak{g}\text{-K}[\mathfrak{T}]$ respectively.

Similar to the definitions of $\mathfrak{g}\text{-S}[\mathfrak{T}_{\mathfrak{g}}] = \mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}] \cup \mathfrak{g}\text{-K}[\mathfrak{T}_{\mathfrak{g}}]$ in $\mathfrak{T}_{\mathfrak{g}}$ and $\mathfrak{g}\text{-S}[\mathfrak{T}_{\mathfrak{g}}] = \mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}] \cup \mathfrak{g}\text{-K}[\mathfrak{T}_{\mathfrak{g}}]$ in $\mathfrak{T}$, those standing for $\mathrm{S}[\mathfrak{T}_{\mathfrak{g}}] = \mathrm{O}[\mathfrak{T}_{\mathfrak{g}}] \cup \mathrm{K}[\mathfrak{T}_{\mathfrak{g}}]$ in $\mathfrak{T}_{\mathfrak{g}}$ and $\mathrm{S}[\mathfrak{T}] = \mathrm{O}[\mathfrak{T}] \cup \mathrm{K}[\mathfrak{T}]$ in $\mathfrak{T}$ are defined as thus:

Definition A.6. If $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$ be a $\mathscr{T}_{\mathfrak{g}}$-space and $\mathfrak{T} = (\Omega, \mathscr{T})$ be a $\mathscr{T}$-space, then:

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- I. $\mathrm{O}[\mathfrak{T}_{\mathfrak{g}}] \overset{\text{def}}{=} \{\mathscr{S}_{\mathfrak{g}} \subset \mathfrak{T}_{\mathfrak{g}} : \mathrm{P}_{\mathfrak{g}}(\mathscr{S}, \mathscr{S}_{\mathfrak{g}}; \mathbf{op}_{\mathfrak{g},0}; =)\}$ and $\mathrm{K}[\mathfrak{T}_{\mathfrak{g}}] \overset{\text{def}}{=} \{\mathscr{S}_{\mathfrak{g}} \subset \mathfrak{T}_{\mathfrak{g}} : \mathrm{P}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}, \mathscr{S}_{\mathfrak{g}}; \mathbf{op}_{\mathfrak{g},0}; =)\}$ denote the classes of all $\mathfrak{T}_{\mathfrak{g}}$-open and $\mathfrak{T}_{\mathfrak{g}}$-closed sets, respectively, in $\mathfrak{T}_{\mathfrak{g}}$, with $\mathrm{S}[\mathfrak{T}_{\mathfrak{g}}] = \mathrm{O}[\mathfrak{T}_{\mathfrak{g}}] \cup \mathrm{K}[\mathfrak{T}_{\mathfrak{g}}]$;
- II. $\mathrm{O}[\mathfrak{T}] \overset{\text{def}}{=} \{\mathscr{S} \subset \mathfrak{T} : \mathrm{P}(\mathscr{S}, \mathscr{S}; \mathbf{op}_0; =)\}$ and $\mathrm{K}[\mathfrak{T}] \overset{\text{def}}{=} \{\mathscr{S} \subset \mathfrak{T} : \mathrm{P}(\mathscr{S}, \mathscr{S}; \mathbf{op}_0; =)\}$ denote the classes of all $\mathfrak{T}$-open and $\mathfrak{T}$-closed sets, respectively, in $\mathfrak{T}$, with $\mathrm{S}[\mathfrak{T}] = \mathrm{O}[\mathfrak{T}] \cup \mathrm{K}[\mathfrak{T}]$.

Remark A.7. Since

$$\mathrm{P}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}, \mathscr{S}_{\mathfrak{g}}, \mathscr{S}_{\mathfrak{g}}; \mathbf{op}_{\mathfrak{g},0}; =, =) \overset{\text{def}}{=} \mathrm{P}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}, \mathscr{S}_{\mathfrak{g}}; \mathbf{op}_{\mathfrak{g},0}; =) \vee \mathrm{P}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}, \mathscr{S}_{\mathfrak{g}}; \mathbf{op}_{\mathfrak{g},0}; =),$$

it is plain that $\mathrm{S}[\mathfrak{T}_{\mathfrak{g}}] \overset{\text{def}}{=} \{\mathscr{S}_{\mathfrak{g}} \subset \mathfrak{T}_{\mathfrak{g}} : \mathrm{P}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}, \mathscr{S}_{\mathfrak{g}}, \mathscr{S}_{\mathfrak{g}}; \mathbf{op}_{\mathfrak{g},0}; =, =)\}$; likewise, since

$$\mathrm{P}(\mathscr{S}, \mathscr{S}, \mathscr{S}; \mathbf{op}_{\mathfrak{g},0}; =, =) \overset{\text{def}}{=} \mathrm{P}(\mathscr{S}, \mathscr{S}; \mathbf{op}_0; =) \vee \mathrm{P}(\mathscr{S}, \mathscr{S}; \mathbf{op}_0; =),$$

it follows that $\mathrm{S}[\mathfrak{T}] \overset{\text{def}}{=} \{\mathscr{S} \subset \mathfrak{T} : \mathrm{P}(\mathscr{S}, \mathscr{S}, \mathscr{S}_{\mathfrak{g}}; \mathbf{op}_{\mathfrak{g},0}; =, =)\}$.

Given the $\mathfrak{T}_{\mathfrak{g}}$-sets $\mathscr{R}_{\mathfrak{g}}$, $\mathscr{S}_{\mathfrak{g}} \subset \mathfrak{T}_{\mathfrak{g}}$, $\mathscr{R}_{\mathfrak{g}}$ is said to be *equivalent* to $\mathscr{S}_{\mathfrak{g}}$, written $\mathscr{R}_{\mathfrak{g}} \sim \mathscr{S}_{\mathfrak{g}}$, if and only if, there exists a $\mathfrak{T}_{\mathfrak{g}}$-map $\pi_{\mathfrak{g}} : \mathscr{R}_{\mathfrak{g}} \longrightarrow \mathscr{S}_{\mathfrak{g}}$ which is bijective; the relation $\mathscr{R}_{\mathfrak{g}} \not\sim \mathscr{S}_{\mathfrak{g}}$, then, holds whenever $\mathscr{R}_{\mathfrak{g}}$ is *not equivalent* to $\mathscr{S}_{\mathfrak{g}}$.

The definitions of the notions of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-closure and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-interior operators of category ν in $\mathscr{T}_{\mathfrak{g}}$-spaces are now given.

Definition A.8 ($\mathfrak{g}$-ν-$\mathfrak{T}_{\mathfrak{g}}$-Interior, $\mathfrak{g}$-ν-$\mathfrak{T}_{\mathfrak{g}}$-Closure Operators). Let $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$ be a $\mathscr{T}_{\mathfrak{g}}$-space, let $\mathrm{C}^{\mathrm{sub}}_{\mathfrak{g}\text{-}\nu\text{-}\mathrm{O}[\mathfrak{T}_{\mathfrak{g}}]}[\mathscr{S}_{\mathfrak{g}}] \overset{\text{def}}{=} \{\mathscr{O}_{\mathfrak{g}} \in \mathfrak{g}\text{-}\nu\text{-}\mathrm{O}[\mathfrak{T}_{\mathfrak{g}}] : \mathscr{O}_{\mathfrak{g}} \subseteq \mathscr{S}_{\mathfrak{g}}\}$ be the family of all $\mathfrak{g}$-ν-$\mathfrak{T}_{\mathfrak{g}}$-open subsets of $\mathscr{S}_{\mathfrak{g}} \in \mathscr{P}(\Omega)$ relative to the class $\mathfrak{g}$-ν-$\mathrm{O}[\mathfrak{T}_{\mathfrak{g}}]$ of $\mathfrak{g}$-ν-$\mathfrak{T}_{\mathfrak{g}}$-open sets, and let $\mathrm{C}^{\mathrm{sup}}_{\mathfrak{g}\text{-}\nu\text{-}\mathrm{K}[\mathfrak{T}_{\mathfrak{g}}]}[\mathscr{S}_{\mathfrak{g}}] \overset{\text{def}}{=} \{\mathscr{K}_{\mathfrak{g}} \in \mathfrak{g}\text{-}\nu\text{-}\mathrm{K}[\mathfrak{T}_{\mathfrak{g}}] : \mathscr{K}_{\mathfrak{g}} \supseteq \mathscr{S}_{\mathfrak{g}}\}$ be the family of all $\mathfrak{g}$-ν-$\mathfrak{T}_{\mathfrak{g}}$-closed supersets of $\mathscr{S}_{\mathfrak{g}} \in \mathscr{P}(\Omega)$ relative to the class $\mathfrak{g}$-ν-$\mathrm{K}[\mathfrak{T}_{\mathfrak{g}}]$ of $\mathfrak{g}$-ν-$\mathfrak{T}_{\mathfrak{g}}$-closed sets. Then, the one-valued maps of the types

$$\text{(A.12)} \quad \begin{aligned} \mathfrak{g}\text{-}\mathrm{Int}_{\mathfrak{g},\nu} : \mathscr{P}(\Omega) &\longrightarrow \mathscr{P}(\Omega) \overset{\text{def}}{=} \{\mathscr{S}_{\mathfrak{g},\mu} \subseteq \Omega : \mu \in I^*_\infty\} \\ \mathscr{S}_{\mathfrak{g}} &\longmapsto \bigcup_{\mathscr{O}_{\mathfrak{g}} \in \mathrm{C}^{\mathrm{sub}}_{\mathfrak{g}\text{-}\nu\text{-}\mathrm{O}[\mathfrak{T}_{\mathfrak{g}}]}[\mathscr{S}_{\mathfrak{g}}]} \mathscr{O}_{\mathfrak{g}}, \end{aligned}$$

$$\text{(A.13)} \quad \begin{aligned} \mathfrak{g}\text{-}\mathrm{Cl}_{\mathfrak{g},\nu} : \mathscr{P}(\Omega) &\longrightarrow \mathscr{P}(\Omega) \overset{\text{def}}{=} \{\mathscr{S}_{\mathfrak{g},\mu} \subseteq \Omega : \mu \in I^*_\infty\} \\ \mathscr{S}_{\mathfrak{g}} &\longmapsto \bigcap_{\mathscr{K}_{\mathfrak{g}} \in \mathrm{C}^{\mathrm{sup}}_{\mathfrak{g}\text{-}\nu\text{-}\mathrm{K}[\mathfrak{T}_{\mathfrak{g}}]}[\mathscr{S}_{\mathfrak{g}}]} \mathscr{K}_{\mathfrak{g}} \end{aligned}$$

on $\mathscr{P}(\Omega)$ ranging in $\mathscr{P}(\Omega)$ are called, respectively, a "$\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-interior operator of category ν" and a "$\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-closure operator of category ν." The classes $\mathfrak{g}\text{-}\mathrm{I}[\mathfrak{T}_{\mathfrak{g}}] \overset{\text{def}}{=} \{\mathfrak{g}\text{-}\mathrm{Int}_{\mathfrak{g},\nu} : \nu \in I^0_3\}$ and $\mathfrak{g}\text{-}\mathrm{C}[\mathfrak{T}_{\mathfrak{g}}] \overset{\text{def}}{=} \{\mathfrak{g}\text{-}\mathrm{Cl}_{\mathfrak{g},\nu} : \nu \in I^0_3\}$, respectively, are called the classes of all $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-interior and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-closure operators.

Remark A.9. According to their definitions, $\mathfrak{g}\text{-}\mathrm{Int}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ is the *dual* of $\mathfrak{g}\text{-}\mathrm{Cl}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$, and conversely. For, the definition of the first rests on such concepts as $\cup$, $\subseteq$, $\mathscr{O}_{\mathfrak{g},1}$, $\mathscr{O}_{\mathfrak{g},2}$, ... whereas the second, on $\cap$, $\supseteq$, $\mathscr{K}_{\mathfrak{g},1}$, $\mathscr{K}_{\mathfrak{g},2}$, ..., which are dual concepts to $\cup$, $\subseteq$, $\mathscr{O}_{\mathfrak{g},1}$, $\mathscr{O}_{\mathfrak{g},2}$, ..., respectively.

It is interesting to view $\mathfrak{g}\text{-}\mathrm{Int}_{\mathfrak{g}}$, $\mathfrak{g}\text{-}\mathrm{Cl}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ as the components of some so-called $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-vector operator.

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Definition A.10 ($\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-Vector Operator). Let $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$ be a $\mathscr{T}_{\mathfrak{g}}$-space. Then, an operator of the type

$$\text{(A.14)} \qquad \begin{aligned} \mathfrak{g}\text{-}\mathbf{Ic}_{\mathfrak{g},\nu} : \mathscr{P}(\Omega) \times \mathscr{P}(\Omega) &\longrightarrow \mathscr{P}(\Omega) \times \mathscr{P}(\Omega) \\ (\mathscr{R}_{\mathfrak{g}}, \mathscr{S}_{\mathfrak{g}}) &\longmapsto \left(\mathfrak{g}\text{-}\mathrm{Int}_{\mathfrak{g},\nu}(\mathscr{R}_{\mathfrak{g}}), \mathfrak{g}\text{-}\mathrm{Cl}_{\mathfrak{g},\nu}(\mathscr{S}_{\mathfrak{g}})\right) \end{aligned}$$

on $\mathscr{P}(\Omega) \times \mathscr{P}(\Omega)$ ranging in $\mathscr{P}(\Omega) \times \mathscr{P}(\Omega)$ is called a "$\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-vector operator of category ν." Then, $\mathfrak{g}\text{-IC}[\mathfrak{T}_{\mathfrak{g}}] \stackrel{\text{def}}{=} \left\{\mathfrak{g}\text{-}\mathbf{Ic}_{\mathfrak{g},\nu} = \left(\mathfrak{g}\text{-}\mathrm{Int}_{\mathfrak{g},\nu}, \mathfrak{g}\text{-}\mathrm{Cl}_{\mathfrak{g},\nu}\right) : \ \nu \in I_3^0\right\}$ is called the class of all $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-vector operators.

The following remark is an immediate consequence of the above definition.

Remark A.11. Observing that, for every $\nu \in I_3^*$, the first and second components of the $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-vector operator $\mathfrak{g}\text{-}\mathbf{Ic}_{\mathfrak{g},\nu} = \left(\mathfrak{g}\text{-}\mathrm{Int}_{\mathfrak{g},\nu}, \mathfrak{g}\text{-}\mathrm{Cl}_{\mathfrak{g},\nu}\right)$ are based on $\mathfrak{g}$-ν-O$[\mathfrak{T}_{\mathfrak{g}}]$ and $\mathfrak{g}$-ν-K$[\mathfrak{T}_{\mathfrak{g}}]$, respectively, it follows that:

- I. $\mathfrak{g}\text{-}\mathbf{Ic}_{\mathfrak{g},\nu} = \mathbf{ic}_{\mathfrak{g}} \stackrel{\text{def}}{=} \left(\mathrm{int}_{\mathfrak{g}}, \mathrm{cl}_{\mathfrak{g}}\right)$ if based on $\mathrm{O}[\mathfrak{T}_{\mathfrak{g}}]$ and $\mathrm{K}[\mathfrak{T}_{\mathfrak{g}}]$;
- II. $\mathfrak{g}\text{-}\mathbf{Ic}_{\mathfrak{g},\nu} = \mathfrak{g}\text{-}\mathbf{Ic}_{\nu} \stackrel{\text{def}}{=} \left(\mathfrak{g}\text{-}\mathrm{Int}_{\nu}, \mathfrak{g}\text{-}\mathrm{Cl}_{\nu}\right)$ if based on $\mathfrak{g}$-ν-O$[\mathfrak{T}]$ and $\mathfrak{g}$-ν-K$[\mathfrak{T}]$;
- III. $\mathfrak{g}\text{-}\mathbf{Ic}_{\mathfrak{g},\nu} = \mathbf{ic} \stackrel{\text{def}}{=} \left(\mathrm{int}, \mathrm{cl}\right)$ if based on $\mathrm{O}[\mathfrak{T}]$ and $\mathrm{K}[\mathfrak{T}]$.

In this way, $\mathbf{ic}_{\mathfrak{g}}$, $\mathfrak{g}\text{-}\mathbf{Ic}_{\nu}$, $\mathbf{ic} : \mathscr{P}(\Omega) \times \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega) \times \mathscr{P}(\Omega)$ are called a $\mathfrak{T}_{\mathfrak{g}}$-vector operator in a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$, a $\mathfrak{g}$-$\mathfrak{T}$-vector operator of category ν in a $\mathscr{T}$-space $\mathfrak{T} = (\Omega, \mathscr{T})$ and a $\mathfrak{T}$-vector operator in a $\mathscr{T}$-space $\mathfrak{T} = (\Omega, \mathscr{T})$, respectively. Accordingly,

$$\begin{aligned} \mathfrak{g}\text{-IC}[\mathfrak{T}] &\stackrel{\text{def}}{=} \left\{\mathfrak{g}\text{-}\mathbf{Ic}_{\nu} = \left(\mathfrak{g}\text{-}\mathrm{Int}_{\nu}, \mathfrak{g}\text{-}\mathrm{Cl}_{\nu}\right) : \ \nu \in I_3^0\right\} \\ \text{(A.15)} \qquad &\subseteq \left\{\mathfrak{g}\text{-}\mathrm{Int}_{\nu} : \ \nu \in I_3^0\right\} \times \left\{\mathfrak{g}\text{-}\mathrm{Cl}_{\nu} : \ \nu \in I_3^0\right\} \stackrel{\text{def}}{=} \mathfrak{g}\text{-I}[\mathfrak{T}] \times \mathfrak{g}\text{-C}[\mathfrak{T}]. \end{aligned}$$

Then, $\mathfrak{g}$-IC$[\mathfrak{T}]$ denotes the class of all $\mathfrak{g}$-$\mathfrak{T}$-vector operators in the $\mathscr{T}$-space $\mathfrak{T} = (\Omega, \mathscr{T})$; $\mathfrak{g}$-I$[\mathfrak{T}]$ denotes the class of all $\mathfrak{g}$-$\mathfrak{T}$-interior operators while $\mathfrak{g}$-C$[\mathfrak{T}]$ denotes the class of all $\mathfrak{g}$-$\mathfrak{T}$-closure operators in the $\mathscr{T}$-space $\mathfrak{T} = (\Omega, \mathscr{T})$.

Definition A.12 (Complement $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-Operator). Let $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$ be a $\mathscr{T}_{\mathfrak{g}}$-space. Then, the one-valued map

$$\text{(A.16)} \qquad \begin{aligned} \mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g},\mathscr{R}_{\mathfrak{g}}} : \mathscr{P}(\Omega) &\longrightarrow \mathscr{P}(\Omega) \\ \mathscr{S}_{\mathfrak{g}} &\longmapsto \complement_{\mathscr{R}_{\mathfrak{g}}}(\mathscr{S}_{\mathfrak{g}}), \end{aligned}$$

where $\complement_{\mathscr{R}_{\mathfrak{g}}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ denotes the relative complement of its operand with respect to $\mathscr{R}_{\mathfrak{g}} \in \mathfrak{g}\text{-S}[\mathfrak{T}_{\mathfrak{g}}]$, is called a "natural complement $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-operator" on $\mathscr{P}(\Omega)$.

For clarity, the notation $\mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g},\mathscr{R}_{\mathfrak{g}}} = \mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}$ is employed whenever $\mathscr{R}_{\mathfrak{g}} = \Omega$ or $\mathscr{R}_{\mathfrak{g}}$ is understood from the context. When $\mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g},\mathscr{R}_{\mathfrak{g}}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ is with respect to $\mathscr{R}_{\mathfrak{g}} \in \mathrm{S}[\mathfrak{T}_{\mathfrak{g}}]$, $\mathscr{R}_{\mathfrak{g}} \in \mathfrak{g}\text{-S}[\mathfrak{T}]$ and $\mathscr{R}_{\mathfrak{g}} \in \mathrm{S}[\mathfrak{T}]$, the terms natural complement $\mathfrak{T}_{\mathfrak{g}}$-operator, natural complement $\mathfrak{g}$-$\mathfrak{T}$-operator and natural complement $\mathfrak{T}$-operator are employed and these terms stand for $\mathrm{Op}_{\mathfrak{g},\mathscr{R}_{\mathfrak{g}}}$, $\mathfrak{g}\text{-}\mathrm{Op}_{\mathscr{R}_{\mathfrak{g}}}$, $\mathrm{Op}_{\mathscr{R}_{\mathfrak{g}}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$, respectively.

The pre-preliminaries section ends here.

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