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Article

On the Functions of Marcinkiewicz Integrals along Surfaces of Revolution on Product Domains via Extrapolation

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Abstract: In this paper, we establish certain L^p bounds for several classes of rough Marcinkiewicz integrals over surfaces of revolution on product spaces. By using these bounds and using an extrapolation argument, we obtain the L^p boundedness of these Marcinkiewicz integrals under very weak conditions on the kernel functions. Several previous results on Marcinkiewicz operators are essentially extended or improved. Our results represent natural extensions and improvements of several known results on Marcinkiewicz integrals over symmetric spaces.

Keywords: rough integrals; surfaces of revolution; product domains; Marcinkiewicz integrals; extrapolation

1. Introduction

Throughout this article, we assume that $\kappa \geq 2$ ($\kappa = m$ or n) and $\mathbb{R}^\kappa$ is the Euclidean space of dimension κ. Also, we assume that $\mathbb{S}^{\kappa-1}$ is the unit sphere in $\mathbb{R}^\kappa$ equipped with the normalized Lebesgue surface measure $d\mu_\kappa(\cdot) \equiv d\mu$.

For $\tau_1 = \alpha_1 + i\beta_1, \tau_2 = \alpha_2 + i\beta_2$ $(\alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathbb{R}$ *with* $\alpha_1, \alpha_2 > 0)$, we let

$$K_{\Omega,h}(v,u) = \frac{\Omega(v,u)h(|v|,|u|)}{|v|^{m-\tau_1}|u|^{n-\tau_2}},$$

where h is a measurable function defined on $\mathbb{R}_+ \times \mathbb{R}_+$ and Ω is a measurable function defined on $\mathbb{R}^m \times \mathbb{R}^n$, integrable over $\mathbb{S}^{m-1} \times \mathbb{S}^{n-1}$ and satisfies the following:

$$\Omega(rv, su) = \Omega(v,u), \ \forall r,s > 0, \tag{1}$$

$$\int_{\mathbb{S}^{m-1}} \Omega(v,.)d\mu(v) = \int_{\mathbb{S}^{n-1}} \Omega(.,u)d\mu(u) = 0. \tag{2}$$

For a suitable mapping $\Phi : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$, the parametric Marcinkiewicz integral operator $\mathcal{M}_{\Omega,\Phi,h}$ along the surface of revolution $\Gamma_\Phi(x,y) = (x,y,\Phi(|x|,|y|))$ is defined, initially for $f \in C_0^\infty(\mathbb{R}^m \times \mathbb{R}^n \times \mathbb{R})$, by

$$\mathcal{M}_{\Omega,\Phi,h}(f)(x,y,w) = \left(\iint_{\mathbb{R}_+ \times \mathbb{R}_+} |H_{r,s}(f)(x,y,w)|^2 \frac{drds}{rs}\right)^{1/2}, \tag{3}$$

where

$$H_{r,s}(f)(x,y,w) = \frac{1}{r^{\tau_1}s^{\tau_2}} \int_{|u|\leq s} \int_{|v|\leq r} f(x-v, y-u, w-\Phi(|v|,|u|))K_{\Omega,h}(v,u)dvdu.$$

We remark that the Marcinkiewicz operator is a natural generalization of the Marcinkiewicz operator $\mathcal{M}^\phi_{\Omega,h}$ along surface of revolution $\Gamma_\Phi(x) = (x, \Phi(|x|))$ in the one parameter setting which is given by

$$\mathcal{M}^{\phi}_{\Omega,h}(f)(x,x_{m+1}) = \left(\int_{\mathbb{R}_+}\left|\frac{1}{r^{\tau_1}}\int_{|v|\leq r} f(x-v,x_{m+1}-\phi(|v|))\frac{\Omega(v)h(|v|)}{|v|^{m-\tau_1}}dv\right|^2\frac{dr}{r}\right)^{1/2}. \quad (4)$$

The study of the L^p boundedeness of the operator $\mathcal{M}^{\phi}_{\Omega,h}$ under various conditions on h, Ω and ϕ has attracted the attention of many authors. For a sample of known results relevant to our study, the readers are referred to consult [1–5].

Our main focus in this paper is the operator $\mathcal{M}_{\Omega,\Phi,h}$. When $\Phi \equiv 0$ and $\tau_1 = 1 = \tau_2$, we denote the operator $\mathcal{M}_{\Omega,\Phi,h}$ by $\mathcal{M}_{\Omega,h}$. In addition, when $h \equiv 1$, then $\mathcal{M}_{\Omega,h}$ reduces to the classical Marcinkiewicz integral on product domains, which is denoted by $\mathcal{M}_{\Omega}$. The investigation of the L^p boundedness of the operator $\mathcal{M}_{\Omega}$ initiated in [6] in which the author proved the L^2 boundedness of $\mathcal{M}_{\Omega}$ under the condition $\Omega \in L(\log L)^2(\mathbb{S}^{m-1}\times\mathbb{S}^{n-1})$. Subsequently, the L^p boundedness of $\mathcal{M}_{\Omega}$ has attracted the attention of many authors. For instance, in [7] the authors proved the L^p $(1 < p < \infty)$ boundedness of $\mathcal{M}_{\Omega}$ if $\Omega \in L(\log L)(\mathbb{S}^{m-1}\times\mathbb{S}^{n-1})$. In addition, they pointed out that by adapting a similar argument as that used in [8] to the product space setting, the assumption $\Omega \in L(\log L)(\mathbb{S}^{m-1}\times\mathbb{S}^{n-1})$ is optimal in the sense that if we replace it by any weaker condition $\Omega \in L(\log L)^{\varepsilon}(\mathbb{S}^{m-1}\times\mathbb{S}^{n-1})$ with $0 < \varepsilon < 1$, then $\mathcal{M}_{\Omega}$ may lose the L^2 boundedness. On the other hand, under the assumption Ω belongs to $B^{(0,0)}_q(\mathbb{S}^{m-1}\times\mathbb{S}^{n-1})$ with $q > 1$, it was proved in [9] that $\mathcal{M}_{\Omega}$ is of type (p,p) for all $p \in (\infty)$ and that the condition $\Omega \in B^{(0,0)}_q(\mathbb{S}^{m-1}\times\mathbb{S}^{n-1})$ is optimal in the sense that we cannot replace it by $\Omega \in B^{(0,\varepsilon)}_q(\mathbb{S}^{m-1}\times\mathbb{S}^{n-1})$ with $\varepsilon \in (-1,0)$ so that $\mathcal{M}_{\Omega}$ is bounded on $L^2(\mathbb{R}^m\times\mathbb{R}^n)$. Here $B^{(0,\nu)}_q(\mathbb{S}^{m-1}\times\mathbb{S}^{n-1})$ is a special class of block spaces introduced in [10]. Later on, the authors of [11] employed Yano's extrapolation technique found in [12] to establish the L^p boundedness of $\mathcal{M}_{\Omega,h}$ for all $|1/p - 1/2| < \min\{1/2, 1/\gamma'\}$ provided that Ω belongs to either $L(\log L)(\mathbb{S}^{m-1}\times\mathbb{S}^{n-1})$ or to $B^{(0,0)}_q(\mathbb{S}^{m-1}\times\mathbb{S}^{n-1})$ and $h \in \Delta_{\gamma}(\mathbb{R}_+\times\mathbb{R}_+)$ for some $\gamma > 1$, where here $\Delta_{\gamma}(\mathbb{R}_+\times\mathbb{R}_+)$ (for $\gamma > 1$) denotes the collection of measurable functions h such that

$$\|h\|_{\Delta_{\gamma}(\mathbb{R}_+\times\mathbb{R}_+)} = \sup_{j,k\in\mathbb{Z}}\left(\int_{2^j}^{2^{j+1}}\int_{2^k}^{2^{k+1}}|h(r,s)|^{\gamma}\frac{drds}{rs}\right)^{1/\gamma} < \infty.$$

For a sample of past studies as well as more information about the applications and development of the operator $\mathcal{M}_{\Omega}$, we refer the readers to see [7,9,13–18] and the references therein.

By the work done in these cited papers, many mathematicians have been motivated to study Marcinkiewicz operator along surfaces of revolution on product spaces of the form

$$\mathcal{M}^{\phi,\psi}_{\Omega,h}(f)(x,x_{m+1},y,y_{n+1}) = \left(\iint_{\mathbb{R}_+\times\mathbb{R}_+}\left|H^{\phi,\psi}_{r,s}(f)\right|^2\frac{drds}{rs}\right)^{1/2}, \quad (5)$$

where

$$H^{\phi,\psi}_{r,s}(f) = \frac{1}{r^{\tau_1}s^{\tau_2}}\int_{|u|\leq s}\int_{|v|\leq r} f(x-v,x_{m+1}-\phi(|v|),y-u,y_{n+1}-\psi(|u|))K_{\Omega,h}(v,u)dvdu.$$

The L^p boundedness of the operator $\mathcal{M}^{\phi,\psi}_{\Omega,h}$ under different conditions on the functions ϕ, ψ, Ω, and h was discussed by many authors (one can consult [15,19,20].

Very recently, in [21] the authors studied the L^p boundedness of the singular integral operators $\mathcal{T}_{\Omega,\Phi,h}$ along surfaces of revolution on product domains which is defined by

$$\mathcal{T}_{\Omega,\Phi,h}(f)(x,y,w) = \iint_{\mathbb{R}^m\times\mathbb{R}^n} f(x-v, y-u, w-\Phi(|v|,|u|))\frac{\Omega(v,u)h(|v|,|u|)}{|v|^m|u|^n}dvdu, \tag{6}$$

where $\Phi : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$ is a suitable mapping. Under various conditions on Φ, the authors proved the L^p boundedness of $\mathcal{T}_{\Omega,\Phi,h}$ if Ω belongs to either $L(\log L)^2(\mathbb{S}^{m-1}\times\mathbb{S}^{n-1})$ or to $B_q^{(1,0)}(\mathbb{S}^{m-1}\times\mathbb{S}^{n-1})$.

In light of the results in [20] regarding the boundedness of Marcinkiewicz operator $\mathcal{M}_{\Omega,h}^{\phi,\psi}$ and of the results in [21] regarding the boundedness of singular integral $\mathcal{T}_{\Omega,\Phi,h}$, a question arises naturally is the following:

Question: Under the same conditions as those imposed on Φ in [21], is the operator $\mathcal{M}_{\Omega,\Phi,h}$ bounded whenever $h \in \Delta_\gamma(\mathbb{R}_+\times\mathbb{R}_+)$ for some $\gamma > 1$ and Ω lies in either the space $L(\log L)(\mathbb{S}^{m-1}\times\mathbb{S}^{n-1})$ or in the space $B_q^{(0,0)}(\mathbb{S}^{m-1}\times\mathbb{S}^{n-1})$ with $q > 1$?

In this article, we shall answer this question in affirmative. Indeed, we have the following:

Theorem 1. *Let $\Phi \in C^1(\mathbb{R}_+\times\mathbb{R}_+)$ such that for any fixed $t, \ell > 0$, we have $\Psi_{1,t}(.) = \Phi(t,.)$, $\Psi_{2,\ell}(.) = \Phi(.,\ell)$ are in $C^2(\mathbb{R}_+)$, increasing and convex functions with $\Psi_{1,t}(0) = \Psi_{2,\ell}(0) = 0$. Suppose that $h \in \Delta_\gamma(\mathbb{R}_+\times\mathbb{R}_+)$ for some $\gamma > 1$ and $\Omega \in L^q(\mathbb{S}^{m-1}\times\mathbb{S}^{n-1})$ for some $q \in (1,2]$. Then there is a constant C_p such that*

$$\|\mathcal{M}_{\Omega,\Phi,h}(f)\|_{L^p(\mathbb{R}^m\times\mathbb{R}^n\times\mathbb{R})} \le C_p\frac{\gamma}{(q-1)(\gamma-1)}\|\Omega\|_{L^q(\mathbb{S}^{m-1}\times\mathbb{S}^{n-1})}\|h\|_{\Delta_\gamma(\mathbb{R}_+\times\mathbb{R}_+)}\|f\|_{L^p(\mathbb{R}^m\times\mathbb{R}^n\times\mathbb{R})}$$

for all $|1/p-1/2| < \min\{1/2, 1/\gamma'\}$.

Theorem 2. *Let Ω and h be given as in Theorem 1. Suppose that $\Phi(t,\ell) = \sum_{i=1}^{d_1}\sum_{j=1}^{d_2} a_{j,i}t^{\alpha_i}\ell^{\beta_j}$ with $\alpha_i, \beta_j > 0$ is a generalized polynomial on $\mathbb{R}^2$. Then there is a constant C_p such that*

$$\|\mathcal{M}_{\Omega,\Phi,h}(f)\|_{L^p(\mathbb{R}^m\times\mathbb{R}^n\times\mathbb{R})} \le C_p\frac{\gamma}{(q-1)(\gamma-1)}\|\Omega\|_{L^q(\mathbb{S}^{m-1}\times\mathbb{S}^{n-1})}\|h\|_{\Delta_\gamma(\mathbb{R}_+\times\mathbb{R}_+)}\|f\|_{L^p(\mathbb{R}^m\times\mathbb{R}^n\times\mathbb{R})}$$

for all $|1/p-1/2| < \min\{1/2, 1/\gamma'\}$.

Theorem 3. *Let Ω and h be given as in Theorem 1. Suppose that $\Phi(t,\ell) = \phi(t)P(\ell)$, where $\phi(t)$ is in $C^2(\mathbb{R}_+)$, increasing and convex function with $\phi(0) = 0$ and P is a generalized polynomials given by $P(\ell) = \sum_{j=1}^{d_2} a_j\ell^{\beta_j}$ with $\beta_j > 0$. Then there is a constant C_p such that*

$$\|\mathcal{M}_{\Omega,\Phi,h}(f)\|_{L^p(\mathbb{R}^m\times\mathbb{R}^n\times\mathbb{R})} \le C_p\frac{\gamma}{(q-1)(\gamma-1)}\|\Omega\|_{L^q(\mathbb{S}^{m-1}\times\mathbb{S}^{n-1})}\|h\|_{\Delta_\gamma(\mathbb{R}_+\times\mathbb{R}_+)}\|f\|_{L^p(\mathbb{R}^m\times\mathbb{R}^n\times\mathbb{R})}$$

for all $|1/p-1/2| < \min\{1/2, 1/\gamma'\}$.

Theorem 4. *Let Ω and h be given as in Theorem 1. Suppose that $\Phi(t,\ell) = \phi_1(t) + \phi_2(t)$, where $\phi_j(t)$ ($j = 1,2$) is either a generalized polynomial or is in $C^2(\mathbb{R}_+)$, increasing and convex function with $\phi_j(0) = 0$. Then there is a constant C_p such that*

$$\|\mathcal{M}_{\Omega,\Phi,h}(f)\|_{L^p(\mathbb{R}^m\times\mathbb{R}^n\times\mathbb{R})} \le C_p\frac{\gamma}{(q-1)(\gamma-1)}\|\Omega\|_{L^q(\mathbb{S}^{m-1}\times\mathbb{S}^{n-1})}\|h\|_{\Delta_\gamma(\mathbb{R}_+\times\mathbb{R}_+)}\|f\|_{L^p(\mathbb{R}^m\times\mathbb{R}^n\times\mathbb{R})}$$

for all $|1/p-1/2| < \min\{1/2, 1/\gamma'\}$.

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By the conclusions from Theorems 1-4 along with the extrapolation argument found in [12,22], we obtain the following:

Theorem 5. *Let Ω satisfy the conditions (1)-(2). Suppose that h and Φ and ψ are given as in either Theorem 1, Theorem 2, Theorem 3, or Theorem 4.*
(i) If $\mho \in B_q^{(0,0)}(\mathbb{S}^{m-1}\times\mathbb{S}^{n-1})$ for some $q>1$, then the inequality

$$\begin{aligned}&\left\|\mathcal{M}_{\Omega,\Phi,h}(f)\right\|_{L^p(\mathbb{R}^m\times\mathbb{R}^n\times\mathbb{R})}\\ \leq\;& C_p\,\|h\|_{\Delta_\mu(\mathbb{R}_+\times\mathbb{R}_+)}\,\|f\|_{L^p(\mathbb{R}^m\times\mathbb{R}^n\times\mathbb{R})}\left(1+\|\Omega\|_{B_q^{(0,0)}(\mathbb{S}^{m-1}\times\mathbb{S}^{n-1})}\right)\end{aligned}$$

holds for all $|1/p-1/2|<\min\{1/2,1/\gamma'\}$;
(ii) If $\Omega\in L(\log L)(\mathbb{S}^{m-1}\times\mathbb{S}^{n-1})$, then the inequality

$$\begin{aligned}&\left\|\mathcal{M}_{\Omega,\Phi,h}(f)\right\|_{L^p(\mathbb{R}^m\times\mathbb{R}^n\times\mathbb{R})}\\ \leq\;& C_p\,\|h\|_{\Delta_\mu(\mathbb{R}_+\times\mathbb{R}_+)}\,\|f\|_{L^p(\mathbb{R}^m\times\mathbb{R}^n\times\mathbb{R})}\left(1+\|\Omega\|_{L(logL)(\mathbb{S}^{m-1}\times\mathbb{S}^{n-1})}\right)\end{aligned}$$

holds for all $|1/p-1/2|<\min\{1/2,1/\gamma'\}$.

Remarks

(1) The conditions on Ω in Theorem 5 are optimal. In fact, they are the weakest conditions in their particular classes, (see [7,9]).

(2) For the special cases $h\equiv 1$ and $\Phi\equiv 0$, the authors of [18] confirmed the L^p $(1<p<\infty)$ boundedness of $\mathcal{M}_{\Omega,\Phi,h}$ whenever $\Omega\in L^q\left(\mathbb{S}^{m-1}\times\mathbb{S}^{n-1}\right)$ for some $q>1$. This result is extended in Theorem 5 in which $\Omega\in L(\log L)(\mathbb{S}^{m-1}\times\mathbb{S}^{n-1})\cup B_q^{(0,0)}(\mathbb{S}^{m-1}\times\mathbb{S}^{n-1})\supset L^q\left(\mathbb{S}^{m-1}\times\mathbb{S}^{n-1}\right)$.

(3) For the special case $h\in\Delta_\gamma(\mathbb{R}_+\times\mathbb{R}_+)$ with $\gamma>2$, our results give the boundedness of $\mathcal{M}_{\Omega,\Phi,h}$ for all $p\in(1,\infty)$ which is the full range.

(4) For the special case $\Phi\equiv 0$, Theorem 5 gives that $\mathcal{M}_{\Omega,\Phi,h}$ is bounded on $L^p(\mathbb{R}^m\times\mathbb{R}^n)$ for all $|1/p-1/2|<\min\{1/2,1/\gamma'\}$, which is the result established in [11]. Hence, our results essentially improve the main results in [11].

(5) The surfaces of revolutions $\Gamma_\Phi(x,y)=(x,y,\Phi(|x|,|y|))$ considered in our Theorems 1–5 cover several important natural classical surfaces. For instance, our theorems allow surfaces of the type Γ_Φ with $\Phi(t,s)=s^2t^2(e^{-1/s}+e^{-1/t}),(s,t>0)$, $\Phi(t,s)=t^\alpha s^\beta$ with $\alpha,\beta>0$, $\Phi(t,s)=P(s,t)$ is a polynomial, $\Phi(t,s)=\phi_1(t)\phi_2(s)$, where each $\phi_i\in C^2[0,\infty)$ is a convex increasing function with $\phi_i(0)=0$.

Henceforward, the constant C denotes a positive real constant which not necessary be the same at each occurrence but independent of all the essential variables.

2. Preliminary Lemmas

We devote this section to introducing some notations and establishing some auxiliary lemmas. For $\theta\geq 2$ and a suitable mapping $\Phi(r,s)$ on $\mathbb{R}_+\times\mathbb{R}_+$, we define the family of measures $\{\lambda_{\Omega,\Phi,h,r,s}:=\lambda_{r,s}:r,s\in\mathbb{R}_+\}$ and its concerning maximal operators λ_h^* and $\mathrm{M}_{h,\theta}$ on $\mathbb{R}^n\times\mathbb{R}^m\times\mathbb{R}$ by

$$\iiint_{\mathbb{R}^m\times\mathbb{R}^n\times\mathbb{R}} f\,d\lambda_{r,s} = \frac{1}{r^{\tau_1}s^{\tau_2}}\int_{1/2s\leq|u|\leq s}\int_{1/2r\leq|v|\leq r} f(v,u,\Phi(|v|,|u|))K_{\Omega,h}(v,u)\,dvdu,$$

$$\lambda_h^*(f)(x,y,w) = \sup_{r,s\in\mathbb{R}_+}\left||\lambda_{r,s}|*f(x,y,w)\right|,$$

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and

$$\mathrm{M}_{h,\theta}(f)(x,y,w) = \sup_{j,k\in\mathbb{Z}} \int_{\theta^j}^{\theta^{j+1}} \int_{\theta^k}^{\theta^{k+1}} ||\lambda_{r,s}| * f(x,y,w)| \frac{drds}{rs},$$

where $|\lambda_{r,s}|$ is defined in the same way as $\lambda_{r,s}$ but with replacing Ωh by $|\Omega h|$.

Lemma 1. *Let $\Omega \in L^q\left(\mathbb{S}^{m-1} \times \mathbb{S}^{n-1}\right)$ with $1 < q \leq 2$ and satisfy the conditions (1)-(2). Suppose that $\Phi \in C^1(\mathbb{R}_+ \times \mathbb{R}_+)$. For $r, s > 0$, let*

$$\mathcal{G}(t,\ell) = \iint_{\mathbb{S}^{n-1}\times\mathbb{S}^{m-1}} e^{-i\{trv\cdot\xi+\ell su\cdot\zeta+\eta\Phi(r,s)\}}\,\Omega\left(v,u\right)d\mu\left(v\right)d\mu\left(u\right).$$

Then there are constants $C > 0$ and δ with $0 < \delta < \frac{1}{2q'}$ such that for $(\xi,\zeta,\eta) \in \mathbb{R}^m \times \mathbb{R}^n \times \mathbb{R}_+$, we have

$$\int_{1/2}^{1}\int_{1/2}^{1} |\mathcal{G}(t,\ell)|^2 \frac{dtd\ell}{t\ell} \leq C\,\|\Omega\|^2_{L^q(\mathbb{S}^{m-1}\times\mathbb{S}^{m-1})} |r\xi|^{\pm\frac{\delta}{q'}} |s\zeta|^{\pm\frac{\delta}{q'}}, \tag{7}$$

where $a^{\pm b} = \min\{a^b, a^{-b}\}$.

Proof. By Schwartz inequality, we get that

$$\int_{1/2}^{1}\int_{1/2}^{1} |\mathcal{G}(t,\ell)|^2 \frac{dtd\ell}{t\ell} \leq C\int_{\mathbb{S}^{n-1}} \left(\iint_{\mathbb{S}^{m-1}\times\mathbb{S}^{m-1}} \mathcal{F}(\xi,v,x) \right.$$
$$\left. \times \quad \Omega\left(v,u\right)\overline{\Omega\left(x,u\right)}d\mu\left(v\right)d\mu\left(x\right)\right) d\mu\left(u\right),$$

where $\mathcal{F}(\xi,v,x) = \int_1^2 e^{-i\frac{t}{2}r\xi\cdot(v-x)}\frac{dt}{t}$. Let $\rho = \xi/|\ \xi\ |$. Then by Van der Corput's lemma, we get

$$|\ \mathcal{F}(\xi,v,x)\ | \leq C\ |\ r\xi\cdot(v-x)\ |^{-1} \leq C\ |\ r\xi\ |^{-1}|\ r\rho\cdot(v-x)\ |^{-1},$$

which when combined with the trivial estimate $|\ \mathcal{F}(\xi,v,x)\ | \leq C$, we deduce that

$$|\ \mathcal{F}(\xi,v,x)\ | \leq C\ |\ r\xi\ |^{-\delta}|\ r\rho\cdot(v-x)\ |^{-\delta}, \tag{8}$$

where $0 < \delta < 1$. Hence, by Hölder's inequality we obtain that

$$\int_{1/2}^{1}\int_{1/2}^{1} |\mathcal{G}(t,\ell)|^2 \frac{dtd\ell}{t\ell} \leq C\,|r\xi|^{-\frac{\delta}{q'}}\,\|\Omega\|^2_{L^q(\mathbb{S}^{m-1}\times\mathbb{S}^{n-1})}$$
$$\times \left(\iint_{\mathbb{S}^{m-1}\times\mathbb{S}^{m-1}} |\rho\cdot(v-x)|^{-\delta q'}\,d\mu(v)d\mu(x)\right)^{1/q'}.$$

By choosing δ so that $0 < \delta < \frac{1}{2q'}$, we get that the last integral is finite. Thus,

$$\int_{1/2}^{1}\int_{1/2}^{1} |\mathcal{G}(t,\ell)|^2 \frac{dtd\ell}{t\ell} \leq C\,\|\Omega\|^2_{L^q(\mathbb{S}^{m-1}\times\mathbb{S}^{n-1})} |r\xi|^{-\frac{\delta}{q'}}. \tag{9}$$

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Similarly we have

$$\int_{1/2}^{1}\int_{1/2}^{1}|\mathcal{G}(t,\ell)|^2\frac{dtd\ell}{t\ell} \leq C\|\Omega\|^2_{L^q(\mathbb{S}^{m-1}\times\mathbb{S}^{n-1})}|s\zeta|^{-\frac{\delta}{q'}}. \tag{10}$$

Also, by the conditions (1)-(2) and a simple change of variable we have

$$\begin{aligned}\int_{1/2}^{1}\int_{1/2}^{1}|\mathcal{G}(t,\ell)|^2\frac{dtd\ell}{t\ell} &\leq C\int_{1/2}^{1}\int_{1/2}^{1}\left(\iint_{\mathbb{S}^{m-1}\times\mathbb{S}^{n-1}}|e^{-itr\xi\cdot v}-1||\Omega(v,u)|d\mu(v)\,d\mu(u)\right)^2\frac{dtd\ell}{t\ell}\\ &\leq C\|\Omega\|^2_{L^1(\mathbb{S}^{m-1}\times\mathbb{S}^{n-1})}|r\xi|^2.\end{aligned}$$

By combine the last estimate with the trivial estimate $\int_{1/2}^{1}\int_{1/2}^{1}|\mathcal{G}(t,\ell)|^2\frac{dtd\ell}{t\ell}\leq C\|\Omega\|^2_{L^1(\mathbb{S}^{m-1}\times\mathbb{S}^{n-1})}$, we get

$$\int_{1/2}^{1}\int_{1/2}^{1}|\mathcal{G}(t,\ell)|^2\frac{dtd\ell}{t\ell} \leq C\|\Omega\|^2_{L^q(\mathbb{S}^{m-1}\times\mathbb{S}^{n-1})}|r\xi|^{\frac{\delta}{q'}}. \tag{11}$$

Similarly, we have

$$\int_{1/2}^{1}\int_{1/2}^{1}|\mathcal{G}(t,\ell)|^2\frac{dtd\ell}{t\ell} \leq C\|\Omega\|^2_{L^q(\mathbb{S}^{m-1}\times\mathbb{S}^{n-1})}|s\zeta|^{\frac{\delta}{q'}}. \tag{12}$$

Therefore, by combining the estimates (9)-(12), we get (7) which ends the proof of this lemma. □

Lemma 2. *Suppose that $\Omega \in L^q\left(\mathbb{S}^{m-1}\times\mathbb{S}^{n-1}\right)$ with $q>1$ satisfies the conditions (1)-(2), $h\in\Delta_\gamma(\mathbb{R}_+\times\mathbb{R}_+)$ with $\gamma>1$, $\Phi\in C^1(\mathbb{R}_+\times\mathbb{R}_+)$, and $\theta\geq 2$. Then there is a real number $C>$ such that the estimates*

$$\|\lambda_{r,s}\| \leq C\|\Omega\|_{L^q(\mathbb{S}^{m-1}\times\mathbb{S}^{n-1})}\|h\|_{\Delta_\gamma(\mathbb{R}_+\times\mathbb{R}_+)}, \tag{13}$$

$$\begin{aligned}\int_{\theta^j}^{\theta^{j+1}}\int_{\theta^k}^{\theta^{k+1}}\left|\hat{\lambda}_{r,s}(\xi,\zeta,\eta)\right|^2\frac{drds}{rs} &\leq C\ln^2(\theta)\|\Omega\|^2_{L^q(\mathbb{S}^{m-1}\times\mathbb{S}^{n-1})}\|h\|^2_{\Delta_\gamma(\mathbb{R}_+\times\mathbb{R}_+)}\\ &\times\left|\theta^k\xi\right|^{\pm\frac{2\delta}{q'\epsilon'}}\left|\theta^j\zeta\right|^{\pm\frac{2\delta}{q'\epsilon'}}\end{aligned} \tag{14}$$

hold for all $j,k\in\mathbb{Z}$, where δ is the same as in Lemma 1, $\epsilon=\max\{2,\gamma'\}$ and $\|\lambda_{r,s}\|$ indicates to the total variation of $\lambda_{r,s}$.

Proof. It is clear that the estimate (13) is obtained by the definition of $\lambda_{r,s}$. Thanks to Hölder's inequality, we have

$$\begin{aligned}\left|\hat{\lambda}_{r,s}(\xi,\zeta,\eta)\right| &\leq C\int_{\frac{1}{2}s}^{s}\int_{\frac{1}{2}r}^{r}|h(t,\ell)|\left|\iint_{\mathbb{S}^{m-1}\times\mathbb{S}^{n-1}}e^{-i\{tv\cdot\xi+\ell u\cdot\zeta+\Phi(t,\ell)\eta\}}\right.\\ &\times\left.\Omega(v,u)\,d\mu(v)\,d\mu(u)\right|\frac{dtd\ell}{t\ell}\\ &\leq C\|h\|_{\Delta_\gamma(\mathbb{R}_+\times\mathbb{R}_+)}\left(\int_{1/2}^{1}\int_{1/2}^{1}|\mathcal{G}(t,\ell)|^{\gamma'}\frac{dtd\ell}{t\ell}\right)^{1/\gamma'}.\end{aligned}$$

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For the case $\gamma \in (1,2]$, we deduce that

$$\left|\hat{\lambda}_{r,s}(\xi,\zeta,\eta)\right| \leq \|h\|_{\Delta_\gamma(\mathbb{R}_+\times\mathbb{R}_+)} \|\Omega\|^{(1-2/\gamma')}_{L^1(\mathbb{S}^{m-1}\times\mathbb{S}^{n-1})} \left(\int\limits_{1/2}^{1}\int\limits_{1/2}^{1} |\mathcal{G}(t,\ell)|^2 \frac{dtd\ell}{t\ell}\right)^{1/\gamma'}.$$

However, for the case $\gamma > 2$, by using Hölder's inequality we get that

$$\left|\hat{\lambda}_{r,s}(\xi,\zeta,\eta)\right| \leq \|h\|_{\Delta_\gamma(\mathbb{R}_+\times\mathbb{R}_+)} \left(\int\limits_{1/2}^{1}\int\limits_{1/2}^{1} |\mathcal{G}(t,\ell)|^2 \frac{dtd\ell}{t\ell}\right)^{1/2}.$$

Therefore, for either case of γ we have

$$\left|\hat{\lambda}_{r,s}(\xi,\zeta,\eta)\right| \leq C\,\|\Omega\|^{(\epsilon-2)/\gamma'}_{L^1(\mathbb{S}^{m-1}\times\mathbb{S}^{n-1})} \|h\|_{\Delta_\gamma(\mathbb{R}_+\times\mathbb{R}_+)} \left(\int_{1/2}^{1}\int_{1/2}^{1} |\mathcal{G}(t,\ell)|^2 \frac{dtd\ell}{t\ell}\right)^{1/\epsilon'},$$

where $\epsilon = \max\{2,\gamma'\}$. Hence, Lemma 1 leads to

$$\left|\hat{\lambda}_{r,s}(\xi,\zeta,\eta)\right|^2 \leq C\,\|\Omega\|^2_{L^q(\mathbb{S}^{m-1}\times\mathbb{S}^{n-1})} \|h\|^2_{\Delta_\gamma(\mathbb{R}_+\times\mathbb{R}_+)} |r\xi|^{\pm\frac{2\delta}{\epsilon' q'}} |s\zeta|^{\pm\frac{2\delta}{\epsilon' q'}}.$$

As $\theta^k \leq r \leq \theta^{k+1}$ and $\theta^j \leq s \leq \theta^{j+1}$, we get that

$$\left|\hat{\lambda}_{r,s}(\xi,\zeta,\eta)\right|^2 \leq C\,\|\Omega\|^2_{L^q(\mathbb{S}^{m-1}\times\mathbb{S}^{n-1})} \|h\|^2_{\Delta_\gamma(\mathbb{R}_+\times\mathbb{R}_+)} \left|\theta^k\xi\right|^{\pm\frac{2\delta}{\epsilon' q'}} \left|\theta^j\zeta\right|^{\pm\frac{2\delta}{\epsilon' q'}}. \tag{15}$$

Consequently,

$$\begin{aligned}\int_{\theta^j}^{\theta^{j+1}}\int_{\theta^k}^{\theta^{k+1}} \left|\hat{\lambda}_{r,s}(\xi,\zeta,\eta)\right|^2 \frac{drds}{rs} &\leq C\ln^2(\theta)\,\|\Omega\|^2_{L^q(\mathbb{S}^{m-1}\times\mathbb{S}^{n-1})} \|h\|^2_{\Delta_\gamma(\mathbb{R}_+\times\mathbb{R}_+)} \\ &\times \left|\theta^k\xi\right|^{\pm\frac{2\delta}{q'\epsilon'}} \left|\theta^j\zeta\right|^{\pm\frac{2\delta}{q'\epsilon'}}.\end{aligned}$$

The proof is complete. □

The following lemmas play a key role in proving our main results.

Lemma 3. *Let $h \in \Delta_\gamma(\mathbb{R}_+\times\mathbb{R}_+)$ with $\gamma > 1$ and $\Omega \in L^q(\mathbb{S}^{m-1}\times\mathbb{S}^{n-1})$ for some $1 < q \leq 2$. Assume that $\Phi \in C^1(\mathbb{R}_+\times\mathbb{R}_+)$ such that for any fixed $t,\ell > 0$, we have $\Psi_{1,t}(.) = \Phi(t,.)$, $\Psi_{2,\ell}(.) = \Phi(.,\ell)$ are in $C^2(\mathbb{R}_+)$, increasing and convex functions with $\Psi_{1,t}(0) = \Psi_{2,\ell}(0) = 0$. Then for $f \in L^p(\mathbb{R}^m\times\mathbb{R}^n\times\mathbb{R})$ with $p \in (\gamma',\infty)$ there exists $C_p > 0$ such that*

$$\|\lambda^*_h(f)\|_{L^p(\mathbb{R}^m\times\mathbb{R}^n\times\mathbb{R})} \leq C_p\,\|\Omega\|_{L^q(\mathbb{S}^{m-1}\times\mathbb{S}^{n-1})} \|h\|_{\Delta_\gamma(\mathbb{R}_+\times\mathbb{R}_+)} \|f\|_{L^p(\mathbb{R}^m\times\mathbb{R}^n\times\mathbb{R})} \tag{16}$$

and

$$\|\mathrm{M}_{h,\theta}(f)\|_{L^p(\mathbb{R}^m\times\mathbb{R}^n\times\mathbb{R})} \leq C_p\ln^2(\theta)\,\|\Omega\|_{L^q(\mathbb{S}^{m-1}\times\mathbb{S}^{n-1})} \|h\|_{\Delta_\gamma(\mathbb{R}_+\times\mathbb{R}_+)} \|f\|_{L^p(\mathbb{R}^m\times\mathbb{R}^n\times\mathbb{R})}. \tag{17}$$

Proof. Thanks to Hölder's inequality, we get that

$$\begin{aligned}||\lambda_{r,s}| * f(x,y,w)| &\leq C\,\|\Omega\|^{1/\gamma'}_{L^1(\mathbb{S}^{m-1}\times\mathbb{S}^{n-1})} \|h\|_{\Delta_\gamma(\mathbb{R}_+\times\mathbb{R}_+)} \left(\frac{1}{rs}\int\limits_{s/2}^{s}\int\limits_{r/2}^{r}\int_{\mathbb{S}^{m-1}\times\mathbb{S}^{n-1}} |\Omega(v,u)|\right. \\ &\times \left. |f(x-tv,y-\ell u,w-\Phi(t,\ell))|^{\gamma'} d\mu(v)d\mu(u)dtd\ell\right)^{1/\gamma'}.\end{aligned}$$

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Hence, by Minkowski's inequality for integrals and Lemma 2.4 in [21], we deduce

$$\begin{aligned}\|\lambda_h^*(f)\|_{L^p(\mathbb{R}^m\times\mathbb{R}^n\times\mathbb{R})} &\leq C\,\|\Omega\|^{1/\gamma'}_{L^1(\mathbb{S}^{m-1}\times\mathbb{S}^{n-1})}\,\|h\|_{\Delta_\gamma(\mathbb{R}_+\times\mathbb{R}_+)}\left(\|\sigma_\Phi^*(|f|^{\gamma'})\|_{L^{(p/\gamma')}(\mathbb{R}^m\times\mathbb{R}^n\times\mathbb{R})}\right)^{1/\gamma'}\\ &\leq C\,\|\Omega\|_{L^q(\mathbb{S}^{m-1}\times\mathbb{S}^{n-1})}\,\|h\|_{\Delta_\gamma(\mathbb{R}_+\times\mathbb{R}_+)}\|f\|_{L^p(\mathbb{R}^m\times\mathbb{R}^n\times\mathbb{R})},\end{aligned}$$

where

$$\iiint_{\mathbb{R}^m\times\mathbb{R}^n\times\mathbb{R}} f\,d\sigma_{r,s} = \frac{1}{r^{\tau_1}s^{\tau_2}}\int_{1/2s\leq|u|\leq s}\int_{1/2r\leq|v|\leq r} f(v,u,\Phi(|v|,|u|))\frac{\Omega(v,u)}{|v|^{m-\tau_1}|u|^{n-\tau_2}}dvdu$$

and

$$\sigma_\Phi^*(f)(x,y,w) = \sup_{r,s\in\mathbb{R}_+}||\sigma_{r,s}|*f(x,y,w)|.$$

It is easy to see that the inequality (17) can be obtained from inequality (16). □

Similarly, by Lemmas 2.5-2.7 in [21], we get; respectively, the following results.

Lemma 4. *Let h and Ω be given as in Lemma 3. Assume that $\Phi(t,\ell)=\sum_{i=1}^{d_1}\sum_{j=1}^{d_2}a_{j,i}t^{\alpha_i}\ell^{\beta_j}$ with $\alpha_i,\beta_j>0$ is a generalized polynomial on $\mathbb{R}^2$. Then for $f\in L^p(\mathbb{R}^m\times\mathbb{R}^n\times\mathbb{R})$ with $p\in(\gamma',\infty)$, there exists $C_p>0$ such that*

$$\|\lambda_h^*(f)\|_{L^p(\mathbb{R}^m\times\mathbb{R}^n\times\mathbb{R})}\leq C_p\,\|\Omega\|_{L^q(\mathbb{S}^{m-1}\times\mathbb{S}^{n-1})}\,\|h\|_{\Delta_\gamma(\mathbb{R}_+\times\mathbb{R}_+)}\,\|f\|_{L^p(\mathbb{R}^m\times\mathbb{R}^n\times\mathbb{R})}$$

and

$$\|\mathrm{M}_{h,\theta}(f)\|_{L^p(\mathbb{R}^m\times\mathbb{R}^n\times\mathbb{R})}\leq C_p\ln^2(\theta)\,\|\Omega\|_{L^q(\mathbb{S}^{m-1}\times\mathbb{S}^{n-1})}\,\|h\|_{\Delta_\gamma(\mathbb{R}_+\times\mathbb{R}_+)}\,\|f\|_{L^p(\mathbb{R}^m\times\mathbb{R}^n\times\mathbb{R})}.$$

Lemma 5. *Let h and Ω be given as in Lemma 3. Assume that $\Phi(t,\ell)=\phi(t)P(\ell)$, where $\phi(t)$ is in $C^2(\mathbb{R}_+)$, increasing and convex function with $\phi(0)=0$ and P is a generalized polynomial given by $P(\ell)=\sum_{j=1}^{d_2}a_j\ell^{\beta_j}$ with $\beta_j>0$. Then for $f\in L^p(\mathbb{R}^m\times\mathbb{R}^n\times\mathbb{R})$ with $p\in(\gamma',\infty)$ there exists $C_p>0$ such that*

$$\|\lambda_h^*(f)\|_{L^p(\mathbb{R}^m\times\mathbb{R}^n\times\mathbb{R})}\leq C_p\,\|\Omega\|_{L^q(\mathbb{S}^{m-1}\times\mathbb{S}^{n-1})}\,\|h\|_{\Delta_\gamma(\mathbb{R}_+\times\mathbb{R}_+)}\,\|f\|_{L^p(\mathbb{R}^m\times\mathbb{R}^n\times\mathbb{R})}$$

and

$$\|\mathrm{M}_{h,\theta}(f)\|_{L^p(\mathbb{R}^m\times\mathbb{R}^n\times\mathbb{R})}\leq C_p\ln^2(\theta)\,\|\Omega\|_{L^q(\mathbb{S}^{m-1}\times\mathbb{S}^{n-1})}\,\|h\|_{\Delta_\gamma(\mathbb{R}_+\times\mathbb{R}_+)}\,\|f\|_{L^p(\mathbb{R}^m\times\mathbb{R}^n\times\mathbb{R})}.$$

Lemma 6. *Let h and Ω be given as in Lemma 3. Assume that $\Phi(t,\ell)=\phi_1(t)+\phi_2(t)$, where $\phi_j(t)$ $(j=1,2)$ is either a generalized polynomial or is in $C^2(\mathbb{R}_+)$, increasing and convex function with $\phi_j(0)=0$. Then for $f\in L^p(\mathbb{R}^m\times\mathbb{R}^n\times\mathbb{R})$ with $p\in(\gamma',\infty)$ there exists $C_p>0$ such that*

$$\|\lambda_h^*(f)\|_{L^p(\mathbb{R}^m\times\mathbb{R}^n\times\mathbb{R})}\leq C_p\,\|\Omega\|_{L^q(\mathbb{S}^{m-1}\times\mathbb{S}^{n-1})}\,\|h\|_{\Delta_\gamma(\mathbb{R}_+\times\mathbb{R}_+)}\,\|f\|_{L^p(\mathbb{R}^m\times\mathbb{R}^n\times\mathbb{R})}$$

and

$$\|\mathrm{M}_{h,\theta}(f)\|_{L^p(\mathbb{R}^m\times\mathbb{R}^n\times\mathbb{R})}\leq C_p\ln^2(\theta)\,\|\Omega\|_{L^q(\mathbb{S}^{m-1}\times\mathbb{S}^{n-1})}\,\|h\|_{\Delta_\gamma(\mathbb{R}_+\times\mathbb{R}_+)}\,\|f\|_{L^p(\mathbb{R}^m\times\mathbb{R}^n\times\mathbb{R})}.$$

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Lemma 7. *Let $\theta \geq 2$, $h \in \Delta_\gamma(\mathbb{R}_+ \times \mathbb{R}_+)$ with $\gamma > 1$, $\Omega \in L^q(\mathbb{S}^{m-1} \times \mathbb{S}^{n-1})$ with $1 < q \leq 2$, and Φ be given as in either Theorem 1, Theorem 2, Theorem 3, or Theorem 4. Then, for arbitrary set of functions $\{\mathcal{F}_{k,j}(\cdot,\cdot,\cdot),\ j,k \in \mathbb{Z}\}$ defined on $\mathbb{R}^m \times \mathbb{R}^n \times \mathbb{R}$, a constant $C_p > 0$ exists such that the inequality*

$$\left\| \left(\sum_{j,k\in\mathbb{Z}} \int_{\theta^j}^{\theta^{j+1}} \int_{\theta^k}^{\theta^{k+1}} \left|\sigma_{r,s} * \mathcal{F}_{j,k}\right|^2 \frac{drds}{rs} \right)^{1/2} \right\|_{L^p(\mathbb{R}^m\times\mathbb{R}^n\times\mathbb{R})}$$
$$\leq C_p \ln^2(\theta) \|\Omega\|_{L^q(\mathbb{S}^{m-1}\times\mathbb{S}^{n-1})} \|h\|_{\Delta_\gamma(\mathbb{R}_+\times\mathbb{R}_+)} \left\| \left(\sum_{j,k\in\mathbb{Z}} \left|\mathcal{F}_{j,k}\right|^2 \right)^{1/2} \right\|_{L^p(\mathbb{R}^m\times\mathbb{R}^n\times\mathbb{R})} \tag{18}$$

holds for all $|1/p - 1/2| < \min\{1/\gamma', 1/2\}$.

Proof. We will follow a similar argument as in [16]. We point out here that we shall prove this lemma only whenever Φ is given as in Theorem 1 since the proofs for the other cases follow the the same method except that we invoke Lemmas 4-6 instead of invoking Lemma 3. Also, we shall prove this lemma only for the case $1 < \gamma \leq 2$ since $\Delta_\gamma(\mathbb{R}_+ \times \mathbb{R}_+) \subseteq \Delta_2(\mathbb{R}_+ \times \mathbb{R}_+)$ for all $\gamma \geq 2$. In this case, we have $|1/p - 1/2| < 1/\gamma'$ which gives that $\frac{2\gamma}{3\gamma-2} < p < \frac{2\gamma}{2-\gamma}$. We need to consider two cases.

Case 1. $2 \leq p < \frac{2\gamma}{2-\gamma}$. By duality there exists a non-negative function $X \in L^{(p/2)'}(\mathbb{R}^m \times \mathbb{R}^n \times \mathbb{R})$ such that $\|X\|_{L^{(p/2)'}(\mathbb{R}^m\times\mathbb{R}^n\times\mathbb{R})} \leq 1$ and

$$\left\| \left(\sum_{j,k\in\mathbb{Z}} \int_{\theta^j}^{\theta^{j+1}} \int_{\theta^k}^{\theta^{k+1}} \left|\lambda_{r,s} * \mathcal{F}_{j,k}\right|^2 \frac{drds}{rs} \right)^{1/2} \right\|^2_{L^p(\mathbb{R}^m\times\mathbb{R}^n\times\mathbb{R})}$$
$$= \iiint_{\mathbb{R}^m\times\mathbb{R}^n\times\mathbb{R}} \sum_{j,k\in\mathbb{Z}} \int_{\theta^j}^{\theta^{j+1}} \int_{\theta^k}^{\theta^{k+1}} \left|\lambda_{r,s} * \mathcal{F}_{j,k}(x,y,w)\right|^2 \frac{drds}{rs} X(x,y,w)dxdydw.$$

By Schwartz's inequality we have

$$\begin{aligned} \left|\lambda_{r,s} * \mathcal{F}_{j,k}(x,y,w)\right|^2 \leq\ & C\|\Omega\|_{L^q(\mathbb{S}^{m-1}\times\mathbb{S}^{n-1})} \|h\|^\gamma_{\Delta_\gamma(\mathbb{R}_+\times\mathbb{R}_+)} \left(\int_{\frac{1}{2}s}^{s} \int_{\frac{1}{2}r}^{r} \iint_{\mathbb{S}^{m-1}\times\mathbb{S}^{n-1}} \right. \\ & \times \left|\mathcal{F}_{j,k}(x - tv, y - \ell u, w - \Phi(t,\ell))\right|^2 \\ & \times \left. |h(t,\ell)|^{2-\gamma} |\Omega(v,u)| \, d\sigma(v) d\sigma(u) \frac{dtd\ell}{t\ell} \right). \end{aligned}$$

Hence, we have

$$
\begin{aligned}
&\left\| \left(\sum_{j,k\in\mathbb{Z}} \int_{\theta^j}^{\theta^{j+1}} \int_{\theta^k}^{\theta^{k+1}} \left|\lambda_{r,s} * \mathcal{F}_{j,k}\right|^2 \frac{drds}{rs} \right)^{1/2} \right\|^2_{L^p(\mathbb{R}^m\times\mathbb{R}^n\times\mathbb{R})} \qquad \leq C\, \|h\|^{\gamma}_{\Delta_\gamma(\mathbb{R}_+\times\mathbb{R}_+)} \\
\times \quad & \|\Omega\|_{L^q(\mathbb{S}^{m-1}\times\mathbf{S}^{n-1})} \iiint_{\mathbb{R}^m\times\mathbb{R}^n\times\mathbb{R}} \left(\sum_{j,k\in\mathbb{Z}} \left|\mathcal{F}_{j,k}(x,y,w)\right|^2 \right) \mathrm{M}_{|h|^{2-\gamma},\theta} \widetilde{\mathrm{X}}(-x,-y,-w)\,dxdydw \\
\leq \quad & C\, \|\Omega\|_{L^q(\mathbb{S}^{m-1}\times\mathbb{S}^{n-1})}\, \|h\|^{\gamma}_{\Delta_\gamma(\mathbb{R}_+\times\mathbb{R}_+)} \left\| \sum_{j,k\in\mathbb{Z}} \left|\mathcal{F}_{j,k}\right|^2 \right\|_{L^{(p/2)}(\mathbb{R}^m\times\mathbb{R}^n\times\mathbb{R})} \\
\times \quad & \left\| \mathrm{M}_{|h|^{2-\gamma},\theta}(\widetilde{G}) \right\|_{L^{(p/2)'}(\mathbb{R}^m\times\mathbb{R}^n\times\mathbb{R})},
\end{aligned}
$$

where $\widetilde{\mathrm{X}}(-x,-y,-w) = X(x,y,w)$. Notice that, since $h \in \Delta_\gamma(\mathbb{R}_+\times\mathbb{R}_+)$, then $|h|^{2-\gamma} \in \Delta_{\frac{\gamma}{2-\gamma}}(\mathbb{R}_+\times\mathbb{R}_+)$. Thus, by Lemma 3 and Hölder's inequality,

$$
\begin{aligned}
&\left\| \left(\sum_{j,k\in\mathbb{Z}} \int_{\theta^j}^{\theta^{j+1}} \int_{\theta^k}^{\theta^{k+1}} \left|\lambda_{r,s} * \mathcal{F}_{j,k}\right|^2 \frac{drds}{rs} \right)^{1/2} \right\|^2_{L^p(\mathbb{R}^m\times\mathbb{R}^n\times\mathbb{R})} \\
\leq \quad & C \ln^2(\theta)\, \|\Omega\|_{L^q(\mathbb{S}^{m-1}\times\mathbb{S}^{n-1})}\, \|h\|^{\gamma}_{\Delta_\gamma(\mathbb{R}_+\times\mathbb{R}_+)} \left\| \left(\sum_{j,k\in\mathbb{Z}} \left|\mathcal{F}_{j,k}\right|^2 \right)^{1/2} \right\|^2_{L^p(\mathbb{R}^m\times\mathbb{R}^n\times\mathbb{R})} \\
\times \quad & \left\| \lambda^*_{|h|^{2-\mu}}(\widetilde{\mathrm{X}}) \right\|_{L^{(p/2)'}(\mathbb{R}^m\times\mathbb{R}^n\times\mathbb{R})} \\
\leq \quad & C_p \ln^2(\theta)\, \|\Omega\|^2_{L^q(\mathbb{S}^{m-1}\times\mathbb{S}^{n-1})}\, \|h\|^2_{\Delta_\gamma(\mathbb{R}_+\times\mathbb{R}_+)} \left\| \left(\sum_{j,k\in\mathbb{Z}} \left|\mathcal{F}_{j,k}\right|^2 \right)^{1/2} \right\|^2_{L^p(\mathbb{R}^m\times\mathbb{R}^n\times\mathbb{R})}.
\end{aligned}
$$

Case 2. $\frac{2\gamma}{3\gamma-2} < p < 2$. By duality there esits a collection of functions $\mathrm{Y} = \mathrm{Y}_{j,k}(x,y,w,r,s)$ defined on $\mathbb{R}^m \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}_+ \times \mathbb{R}_+$ such that

$$
\left\| \left\| \|\mathrm{Y}_{j,k}\|_{L^2([\theta^k,\theta^{k+1}]\times[\theta^j,\theta^{j+1}],\frac{drds}{rs})} \right\|_{l^2} \right\|_{L^{p'}(\mathbb{R}^m\times\mathbb{R}^n\times\mathbb{R})} \leq 1
$$

and

$$
\begin{aligned}
&\left\| \left(\sum_{j,k\in\mathbb{Z}} \int_{\theta^j}^{\theta^{j+1}} \int_{\theta^k}^{\theta^{k+1}} \left|\lambda_{r,s} * \mathcal{F}_{j,k}\right|^2 \frac{drds}{rs} \right)^{1/2} \right\|_{L^p(\mathbb{R}^m\times\mathbb{R}^n\times\mathbb{R})} \\
= \quad & \iiint_{\mathbb{R}^m\times\mathbb{R}^n\times\mathbb{R}} \sum_{j,k\in\mathbb{Z}} \int_{\theta^j}^{\theta^{j+1}} \int_{\theta^k}^{\theta^{k+1}} \left(\lambda_{r,s} * \mathcal{F}_{j,k}(x,y,w)\right) \mathrm{Y}_{j,k}(x,y,w,r,s) \frac{drds}{rs} dxdydw \\
\leq \quad & C_p \ln(\theta) \left\| (\Theta(\mathrm{Y}))^{1/2} \right\|_{L^{p'}(\mathbb{R}^m\times\mathbb{R}^n\times\mathbb{R})} \left\| \left(\sum_{j,k\in\mathbb{Z}} \left|\mathcal{F}_{j,k}\right|^2 \right)^{1/2} \right\|_{L^p(\mathbb{R}^m\times\mathbb{R}^n\times\mathbb{R})}, \qquad (19)
\end{aligned}
$$

where

$$\Theta(\Upsilon)(x,y,w)=\sum_{j,k\in\mathbb{Z}}\int_{\theta^j}^{\theta^{j+1}}\int_{\theta^k}^{\theta^{k+1}}\left|\lambda_{r,s}*\Upsilon_{j,k}(x,y,w,r,s)\right|^2\frac{drds}{rs}.$$

Thanks to the duality, we deduce that a function $W\in L^{(p'/2)'}(\mathbb{R}^m\times\mathbb{R}^n\times\mathbb{R})$ exists such that $\|W\|_{L^{(p'/2)'}(\mathbb{R}^m\times\mathbb{R}^n\times\mathbb{R})}\leq 1$ and

$$\begin{aligned}
&\left\|(\Theta(\Upsilon))^{1/2}\right\|^2_{L^{p'}(\mathbb{R}^m\times\mathbb{R}^n\times\mathbb{R})}\\
=&\sum_{j,k\in\mathbb{Z}}\iiint_{\mathbb{R}^m\times\mathbb{R}^n\times\mathbb{R}}\int_{\theta^j}^{\theta^{j+1}}\int_{\theta^k}^{\theta^{k+1}}\left|\lambda_{r,s}*\Upsilon_{j,k}(x,y,w,r,s)\right|^2\frac{drds}{rs}W(x,y,w)dxdydw\\
\leq& C\,\|\Omega\|_{L^q(\mathbb{S}^{m-1}\times\mathbb{S}^{n-1})}\,\|h\|^{\gamma}_{\Delta_\gamma(\mathbb{R}_+\times\mathbb{R}_+)}\left\|\lambda^*{}_{|h|^{2-\gamma}}(W)\right\|_{L^{(p'/2)'}(\mathbb{R}^m\times\mathbb{R}^n\times\mathbb{R})}\\
\times&\left\|\left(\sum_{j,k\in\mathbb{Z}}\int_{\theta^j}^{\theta^{j+1}}\int_{\theta^k}^{\theta^{k+1}}\left|\Upsilon_{j,k}(\cdot,\cdot,r,s)\right|^2\frac{drds}{rs}\right)\right\|_{L^{(p'/2)}(\mathbb{R}^m\times\mathbb{R}^n\times\mathbb{R})}\\
\leq& C\,\|\Omega\|^2_{L^q(\mathbb{S}^{m-1}\times\mathbb{S}^{n-1})}\,\|h\|^2_{\Delta_\gamma(\mathbb{R}_+\times\mathbb{R}_+)}
\end{aligned}$$

which gives along with (19) the inequality (18) holds for the case $\frac{2\gamma}{3\gamma-2}<p<2$. This finishes the proof of this lemma. □

3. Proof of main theorems

Assume that $h\in\Delta_\gamma(\mathbb{R}_+\times\mathbb{R}_+)$ for some $\gamma>1$, $\Omega\in L^q\left(\mathbb{S}^{m-1}\times\mathbb{S}^{n-1}\right)$ for some $1<q\leq 2$ and $\theta=2^{q'\gamma'}$. It is clear that Minkowski's inequality leads to

$$\begin{aligned}
\mathcal{M}_{\Omega,\Phi,h}(f)(x,y,w) &= \left(\iint_{\mathbb{R}_+\times\mathbb{R}_+}\left|\sum_{j,k=0}^{\infty}\frac{1}{s^{\tau_1}r^{\tau_2}}\int_{2^{-j-1}s<|u|\leq 2^{-j}s}\int_{2^{-k-1}r<|v|\leq 2^{-k}r}K_{\Omega,h}(v,u)\right.\right.\\
&\times\left.\left. f(x-v,y-u,w-\Phi(|v|,|u|))dvdu\right|^2\frac{drds}{rs}\right)^{1/2}\\
&\leq\sum_{j,k=0}^{\infty}\left(\iint_{\mathbb{R}_+\times\mathbb{R}_+}\left|\frac{1}{s^{\tau_1}r^{\tau_2}}\int_{2^{-j-1}s<|u|\leq 2^{-j}s}\int_{2^{-k-1}r<|v|\leq 2^{-k}r}K_{\Omega,h}(v,u)\right.\right.\\
&\times\left.\left. f(x-v,y-u,w-\Phi(|v|,|u|))dvdu\right|^2\frac{drds}{rs}\right)^{1/2}\\
&\leq\frac{2^{\alpha_1+\alpha_2}}{(2^{\alpha_1}-1)(2^{\alpha_2}-1)}\left(\iint_{\mathbb{R}_+\times\mathbb{R}_+}|\lambda_{r,s}*f(x,y,w)|^2\frac{drds}{rs}\right)^{1/2}. \qquad (20)
\end{aligned}$$

For $j\in\mathbb{Z}$, choose a set of smooth partition of unity $\{\Gamma_j\}$ defined on $(0,\infty)$ and adapted to the interval $[\theta^{-1-j},\theta^{1-j}]\equiv-\mathcal{I}_j$ with the following properties:

$$\begin{aligned}
\Gamma_j &\in C^\infty,\ 0\leq\Gamma_j\leq 1,\ \sum_{j\in\mathbb{Z}}\Gamma_j(t)=1,\\
\text{supp}\,(\Gamma_j) &\subseteq \mathcal{I}_j\ \ and\ \ \left|\frac{d^\mu\Gamma_j(t)}{dt^\mu}\right|\leq\frac{C_\mu}{t^\mu},
\end{aligned}$$

where C_μ is independent of the lacunary sequence $\{\theta^j;j\in\mathbb{Z}\}$.

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Define the multiplier operators $\{T_{j,k}\}$ on $\mathbb{R}^m \times \mathbb{R}^n \times \mathbb{R}$ by $(\widehat{T_{j,k}(f)})(\xi,\zeta,\eta) = \Gamma_j(|\xi|)\Gamma_k(|\zeta|)\hat{f}(\xi,\zeta,\eta)$. Hence, for any $f \in C_0^\infty(\mathbb{R}^m \times \mathbb{R}^n \times \mathbb{R})$, we have $f(x,y,w) = \sum\limits_{j,k\in\mathbb{Z}} (T_{j+a_2,k+a_1}(f))(x,y,w)$, which gives by Minkowski's inequality that

$$\left(\iint_{\mathbb{R}_+\times\mathbb{R}_+} |\lambda_{r,s} * f(x,y,w)|^2 \frac{drds}{rs}\right)^{1/2} \leq C \sum_{a_1,a_2\in\mathbb{Z}} \mathcal{A}_{a_2,a_1}(f)(x,y,w), \tag{21}$$

where

$$\mathcal{A}_{a_2,a_1}(f)(x,y,w) = \left(\iint_{\mathbb{R}_+\times\mathbb{R}_+} |\mathcal{B}_{a_2,a_1}(f)(x,y,w,r,s)|^2 \frac{drds}{rs}\right)^{1/2},$$

$$\mathcal{B}_{a_2,a_1}(g)(x,y,w,r,s) = \sum_{j,k\in\mathbb{Z}} \lambda_{r,s} * T_{j+a_2,k+a_1} * f(x,y,w)\chi_{[\theta^k,\theta^{k+1})\times[\theta^j,\theta^{j+1})}(r,s).$$

Therefore, to prove Theorem 1, it suffices to prove that for any p satisfying $|1/2-1/p| < \min\{1/\gamma',1/2\}$, there exists $\varepsilon > 0$ such that

$$\begin{aligned} &\|\mathcal{A}_{a_2,a_1}(f)\|_{L^p(\mathbb{R}^m\times\mathbb{R}^n\times\mathbb{R})} \\ \leq\ & C_p \ln(\theta)\, 2^{-\frac{\varepsilon}{2}(|a_1|+|a_2|)} \|\Omega\|_{L^q(\mathbb{S}^{m-1}\times\mathbb{S}^{n-1})} \|h\|_{\Delta_\gamma(\mathbb{R}_+\times\mathbb{R}_+)} \|f\|_{L^p(\mathbb{R}^m\times\mathbb{R}^n\times\mathbb{R})}. \end{aligned} \tag{22}$$

Let us first estimate the L^2-norm for $\mathcal{A}_{a_2,a_1}(f)$. By Plancherel's Theorem, Fubini's Theorem, Lemma 2, we deduce

$$\begin{aligned} &\|\mathcal{A}_{a_2,a_1}(f)\|^2_{L^2(\mathbb{R}^m\times\mathbb{R}^n\times\mathbb{R})} \\ \leq\ & \sum_{j,k\in\mathbb{Z}} \iiint_{\mho_{j+a_2,k+a_1}} \left(\int_{\theta^j}^{\theta^{j+1}}\int_{\theta^k}^{\theta^{k+1}} |\hat{\lambda}_{r,s}(\xi,\zeta,\eta)|^2 \frac{drds}{rs}\right) \left|\hat{f}(\xi,\zeta,\eta)\right|^2 d\xi d\zeta d\eta \\ \leq\ & C_p \ln^2(\theta)\, \|\Omega\|^2_{L^q(\mathbb{S}^{m-1}\times\mathbb{S}^{n-1})} \|h\|^2_{\Delta_\gamma(\mathbb{R}_+\times\mathbb{R}_+)} \\ \times\ & \sum_{j,k\in\mathbb{Z}} \iiint_{\mho_{j+a_2,k+a_1}} \left|\theta^k\xi\right|^{\pm\frac{2\delta}{q'\epsilon'}} \left|\theta^j\zeta\right|^{\pm\frac{2\delta}{q'\epsilon'}} \left|\hat{f}(\xi,\zeta,\eta)\right|^2 d\xi d\zeta d\eta \\ \leq\ & C_p \ln^2(\theta)\, 2^{-\varepsilon(|a_1|+|a_2|)} \|\mho\|^2_{L^q(\mathbb{S}^{m-1}\times\mathbb{S}^{n-1})} \|h\|^2_{\Delta_\gamma(\mathbb{R}_+\times\mathbb{R}_+)} \sum_{j,k\in\mathbb{Z}} \iiint_{\mho_{j+a_2,k+a_1}} \left|\hat{f}(\xi,\zeta,\eta)\right|^2 d\xi d\zeta d\eta \\ \leq\ & C_p \ln^2(\theta)\, 2^{-\varepsilon(|a_1|+|a_2|)} \|\mho\|^2_{L^q(\mathbb{S}^{m-1}\times\mathbf{S}^{n-1})} \|h\|^2_{\Delta_\gamma(\mathbb{R}_+\times\mathbb{R}_+)} \|f\|^2_{L^2(\mathbb{R}^m\times\mathbb{R}^n\times\mathbb{R})}, \end{aligned} \tag{23}$$

where $\mho_{j,k} = \{(\xi,\zeta,\eta) \in \mathbb{R}^m\times\mathbb{R}^n\times\mathbb{R} : (|\xi|,|\zeta|)) \in \mathcal{I}_j \times \mathcal{I}_k\}$ and $\varepsilon \in (0,1)$.

Next, we estimate the L^p-norm of $\mathcal{A}_{a_2,a_1}(f)$ as follows: By employing a similar argument as that used in [23] along with the Littlewood-Paley theory and Lemma 7, we get

$$\begin{aligned} &\|\mathcal{A}_{a_2,a_1}(g)\|_{L^p(\mathbb{R}^{m+1}\times\mathbb{R}^{n+1})} \\ \leq\ & C \left\| \left(\sum_{j,k\in\mathbb{Z}} \int_{\theta^j}^{\theta^{j+1}} \int_{\theta^k}^{\theta^{k+1}} \left(\left|\lambda_{r,s} * T_{j+a_2,k+a_1} * f\right|\right)^2 \frac{drds}{rs}\right)^{1/2} \right\|_{L^p(\mathbb{R}^m\times\mathbb{R}^n\times\mathbb{R})} \\ \leq\ & C_p \ln(\theta)\, \|\Omega\|_{L^q(\mathbb{S}^{m-1}\times\mathbb{S}^{n-1})} \|h\|_{\Delta_\gamma(\mathbb{R}_+\times\mathbb{R}_+)} \left\| \left(\sum_{j,k\in\mathbb{Z}} \left|T_{j+a_2,k+a_1} * f\right|^2 \right)^{1/2} \right\|_{L^p(\mathbb{R}^m\times\mathbb{R}^n\times\mathbb{R})} \\ \leq\ & C_p \frac{\gamma}{(\gamma-1)(q-1)} \|\Omega\|_{L^q(\mathbb{S}^{m-1}\times\mathbb{S}^{n-1})} \|h\|_{\Delta_\gamma(\mathbb{R}_+\times\mathbb{R}_+)} \|f\|_{L^p(\mathbb{R}^m\times\mathbb{R}^n\times\mathbb{R})}. \end{aligned} \tag{24}$$

Finally, by interpolating between (23) and (24), we obtain (22), which in turn finishes the proof of Theorem 1.

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