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A Unified Proof of the Extended, Generalized, and Grand Riemann Hypothesis

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Article

A Unified Proof of the Extended, Generalized, and Grand Riemann Hypothesis

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Abstract

The Extended Riemann Hypothesis (for Dedekind zeta functions), the Generalized Riemann Hypothesis (for Dirichlet L-functions), and the Grand Riemann Hypothesis (for all L-functions) are proved under a unified framework.

Keywords: Extended Riemann Hypothesis; Generalized Riemann Hypothesis; Grand Riemann Hypothesis

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1. Introduction

Inspired by Lemma 8 in Ref.[1],we first establish Theorems 1–4 progressively (the proofs do not depend on Lemma 8). Thereafter, we prove the Extended, Generalized, and Grand Riemann Hypotheses within a unified framework built upon Theorem 4. For convenience, Lemma 8 and other related lemmas of Ref.[1] are listed in the Appendix of this paper.

It should be noted that in this article and Ref.[1], 'j' is used to denote the imaginary unit ($j^2 = -1$), while 'i' serves as a natural number index.

In the following contents, we adopt the fact—guaranteed by Lemma 6 and Lemma 7 in the Appendix—that the unknown multiplicities of the zeros of any given entire function are finite, unique, and therefore invariant.

2. Four Theorems

As pointed out in Ref.[2], p.57, we can enumerate the non-trivial zeros of the zeta function in order of the increasing absolute value of their imaginary parts, where zeros whose imaginary parts have the same absolute value are arranged arbitrarily. Thus we remove the default ordering of $|\beta_i|$, $|\beta_1| \le |\beta_2| \le \cdots$, as condition hereafter for simplicity.

Theorem 1: Given an entire function

$$F(s) = \prod_{i=1}^{\infty}\left(1 + \frac{(s-\alpha_i)^2}{\beta_i^2}\right)^{m_i}, \quad s \in \mathbb{C}, \tag{1}$$

which converges absolutely on $\mathbb{C}$ and uniformly on every compact subset of $\mathbb{C}$. Here

- $\rho_i = \alpha_i + j\beta_i$ and $\bar{\rho}_i = \alpha_i - j\beta_i$ are the complex-conjugate zeros of $F(s)$,
- $0 < \alpha_i < k$, $\beta_i \ne 0$, $k > 0$ are real numbers,
- $m_i \ge 1$ is the common multiplicity of ρ_i and $\bar{\rho}_i$,
- $\sum_{i=1}^{\infty} \frac{1}{|\rho_i|^2} < \infty$.

Then

$$F(s) = F(k-s) \Longrightarrow \alpha_i = \frac{k}{2}\ (\forall i) \quad \text{and} \quad 0 < |\beta_1| < |\beta_2| < |\beta_3| < \cdots . \tag{2}$$

Remark: It should be noted that m_i is actually the multiplicity of quadruplets of zeros $(\rho_i, \bar{\rho}_i, k-\rho_i, k-\bar{\rho}_i)$ under the assumption $F(s) = F(k-s)$.

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Proof. Considering

$$\left(1+\frac{(s-\alpha_i)^2}{\beta_i^2}\right)^{m_i} = \underbrace{\left(1+\frac{(s-\alpha_i)^2}{\beta_i^2}\right)\cdots}_{m_i \text{ times}}$$

we have from Eq.(1)

$$F(s)=F(k-s) \Rightarrow \prod_{i=1}^{\infty}\left(1+\frac{(s-\alpha_i)^2}{\beta_i^2}\right) = \prod_{i=1}^{\infty}\left(1+\frac{(k-s-\alpha_i)^2}{\beta_i^2}\right) \tag{3}$$

where the i^{th} factor appears m_i times in any order.

By the definition of divisibility of entire functions [3][4] (or more specifically the definition that a polynomial divides an entire function expressed as infinite product of polynomial factors [1]), Eq.(3) implies that each polynomial factor on either side divides the infinite product on the opposite side, i.e.,

$$F(s)=F(k-s) \Rightarrow \begin{cases} \left(1+\frac{(s-\alpha_i)^2}{\beta_i^2}\right) \mid \prod_{i=1}^{\infty}(1+\frac{(k-s-\alpha_i)^2}{\beta_i^2}) \\ \left(1+\frac{(k-s-\alpha_i)^2}{\beta_i^2}\right) \mid \prod_{i=1}^{\infty}(1+\frac{(s-\alpha_i)^2}{\beta_i^2}) \end{cases} \tag{4}$$

where "|" is the divisible sign, $i \in \mathbb{N} = \{1,2,3,\cdots\}$.

Both polynomials $1+\frac{(s-\alpha_i)^2}{\beta_i^2}$ and $1+\frac{(k-s-\alpha_i)^2}{\beta_i^2}$ have discriminant $\Delta = -4\cdot\frac{1}{\beta_i^2} < 0$. Hence they are irreducible in $\mathbb{R}[x]$. By Lemma 5 (see Appendix for details), Eq.(4) yields:

$$\begin{cases} \left(1+\frac{(s-\alpha_i)^2}{\beta_i^2}\right) \mid \left(1+\frac{(k-s-\alpha_l)^2}{\beta_l^2}\right) \\ \left(1+\frac{(k-s-\alpha_i)^2}{\beta_i^2}\right) \mid \left(1+\frac{(s-\alpha_l)^2}{\beta_l^2}\right) \\ i,l \in \mathbb{N} \end{cases} \tag{5}$$

For the special kind of polynomials in Eq.(5), "divisible" means "equal", which can be verified by comparing the like terms in the following equation to get $k'=1$.

$$\left(1+\frac{(s-\alpha_i)^2}{\beta_i^2}\right) = k'\left(1+\frac{(k-s-\alpha_l)^2}{\beta_l^2}\right), k' \neq 0 \in \mathbb{R} \tag{6}$$

Further, due to the uniqueness of the multiplicity m_i, the only solution to Eq.(5) is $i=l$, otherwise, duplicated zeros (in quadruplets) with $\alpha_i+\alpha_l=k, \beta_i^2=\beta_l^2, l\neq i$ would be generated to change m_i. Therefore we have from Eq.(5):

$$\left(1+\frac{(s-\alpha_i)^2}{\beta_i^2}\right) = \left(1+\frac{(k-s-\alpha_i)^2}{\beta_i^2}\right), i \in \mathbb{N} \tag{7}$$

By comparing the like terms in Eq.(7), we obtain $\alpha_i = \frac{k}{2}(\forall i)$. Further, to ensure the uniqueness of m_i while $\alpha_i=\frac{k}{2}$, we need limit the β_i values to be distinct, i.e., $0<|\beta_1|<|\beta_2|<|\beta_3|<\cdots$.
That completes the proof of Theorem 1. □

Remark: Theorem 1 extends the critical strip from $0<\Re(s)<1$ to $0<\Re(s)<k$ ($k>0$ is a real constant).

Theorem 2: Given an entire function

$$G(s)=\prod_{i=1}^{\infty}\left(1-\frac{s}{\rho_i}\right), s\in\mathbb{C} \tag{8}$$

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which converges absolutely on $\mathbb{C}$ and uniformly on every compact subset of $\mathbb{C}$ in the form of

$$\begin{aligned} G(s) &= \prod_{i=1}^{\infty}\left(\frac{\beta_i^2}{\alpha_i^2+\beta_i^2}+\frac{(s-\alpha_i)^2}{\alpha_i^2+\beta_i^2}\right)^{m_i} \\ &= \prod_{i=1}^{\infty}\left(1-\frac{s}{\rho_i}\right)^{m_i}\left(1-\frac{s}{\bar{\rho}_i}\right)^{m_i} \end{aligned} \tag{9}$$

Here

- $\rho_i = \alpha_i + j\beta_i$ and $\bar{\rho}_i = \alpha_i - j\beta_i$ are the complex-conjugate zeros of $G(s)$,
- $0 < \alpha_i < k$, $\beta_i \neq 0$, $k > 0$ are real numbers,
- $m_i \geq 1$ is the common multiplicity of ρ_i and $\bar{\rho}_i$,
- $\sum_{i=1}^{\infty}\frac{1}{|\rho_i|^2} < \infty$.

Then

$$G(s) = G(k-s) \Longrightarrow \alpha_i = \frac{k}{2}\ (\forall i) \quad \text{and} \quad 0 < |\beta_1| < |\beta_2| < |\beta_3| < \cdots . \tag{10}$$

Proof. According to Theorem 1 and Lemma 9 (see Appendix), we have

$$\begin{aligned} & F(s) = F(k-s) \\ & \Leftrightarrow \\ & \prod_{i=1}^{\infty}\left(\frac{\beta_i^2}{\alpha_i^2+\beta_i^2}\right)^{m_i} F(s) = \prod_{i=1}^{\infty}\left(\frac{\beta_i^2}{\alpha_i^2+\beta_i^2}\right)^{m_i} F(k-s) \\ & \Leftrightarrow \\ & \prod_{i=1}^{\infty}\left(\frac{\beta_i^2}{\alpha_i^2+\beta_i^2}+\frac{(s-\alpha_i)^2}{\alpha_i^2+\beta_i^2}\right)^{m_i} = \prod_{i=1}^{\infty}\left(\frac{\beta_i^2}{\alpha_i^2+\beta_i^2}+\frac{(k-s-\alpha_i)^2}{\alpha_i^2+\beta_i^2}\right)^{m_i} \\ & \Leftrightarrow \\ & G(s) = G(k-s) \end{aligned} \tag{11}$$

Then we know that

$$G(s) = G(k-s) \Longrightarrow \alpha_i = \frac{k}{2}\ (\forall i) \quad \text{and} \quad 0 < |\beta_1| < |\beta_2| < |\beta_3| < \cdots . \tag{12}$$

That completes the proof of Theorem 2. □

Theorem 3: Given an entire function represented by its Hadamard product:

$$\Lambda(\lambda, s) = e^{A(\lambda)+B(\lambda)s}\prod_{i=1}^{\infty}\left(1-\frac{s}{\rho_i}\right)e^{\frac{s}{\rho_i}}, s \in \mathbb{C} \tag{13}$$

and that

$$\Lambda(\overline{\lambda}, k-s) = e^{A(\overline{\lambda})+B(\overline{\lambda})(k-s)}\prod_{i=1}^{\infty}\left(1-\frac{k-s}{\rho_i}\right)e^{\frac{k-s}{\rho_i}}, s \in \mathbb{C} \tag{14}$$

Here

- λ denotes a mathematical object, $\overline{\lambda}$ is the dual of λ,
- $\rho_i = \alpha_i + j\beta_i$ and $\bar{\rho}_i = \alpha_i - j\beta_i$ are the complex-conjugate zeros of $\Lambda(\lambda, s)$,
- $0 < \alpha_i < k$, $\beta_i \neq 0$, $k > 0$ are real numbers,
- $\sum_{i=1}^{\infty}\frac{1}{|\rho_i|^2} < \infty$.

Then

$$\Lambda(\lambda, s) = \varepsilon(\lambda)\Lambda(\overline{\lambda}, k-s) \Longrightarrow \alpha_i = \frac{k}{2}\ (\forall i) \quad \text{and} \quad 0 < |\beta_1| < |\beta_2| < |\beta_3| < \cdots . \tag{15}$$

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where $\varepsilon(\lambda)$ is a complex number of absolute value 1, called the "root number" of L-function $L(\lambda, s)$.

Remark: For more details about $\varepsilon(\lambda)$, see Ref.[5], p.94.

Proof. First, we have

$$\begin{aligned}\Lambda(\lambda, s) &= e^{A(\lambda)+B(\lambda)s} \prod_{i=1}^{\infty} \left(1 - \frac{s}{\rho_i}\right) e^{\frac{s}{\rho_i}} \\ &= e^{A(\lambda)+B(\lambda)s} \prod_{i=1}^{\infty} \left(1 - \frac{s}{\rho_i}\right)\left(1 - \frac{s}{\bar{\rho}_i}\right) e^{\frac{2\alpha_i s}{\alpha_i^2+\beta_i^2}} \\ &= e^{A(\lambda)+B(\lambda)s} e^{s \cdot \sum_{i=1}^{\infty} \frac{2\alpha_i}{\alpha_i^2+\beta_i^2}} \prod_{i=1}^{\infty} \left(1 - \frac{s}{\rho_i}\right)\left(1 - \frac{s}{\bar{\rho}_i}\right)\end{aligned} \tag{16}$$

Noticing that $\sum_{i=1}^{\infty} \frac{2\alpha_i}{\alpha_i^2+\beta_i^2} < 2k \cdot \sum_{i=1}^{\infty} \frac{1}{|\rho_i|^2} < \infty$, then we have $\sum_{i=1}^{\infty} \frac{2\alpha_i}{\alpha_i^2+\beta_i^2} = c, c \in \mathbb{R}, c \neq 0$. With further consideration of the multiplicity $m_i \geq 1$ of each zero, we obtain

$$\Lambda(\lambda, s) = e^{A(\lambda)+[B(\lambda)+c]s} \prod_{i=1}^{\infty} \left(1 - \frac{s}{\rho_i}\right)^{m_i} \left(1 - \frac{s}{\bar{\rho}_i}\right)^{m_i} \tag{17}$$

Accordingly

$$\Lambda(\bar{\lambda}, k-s) = e^{A(\bar{\lambda})+[B(\bar{\lambda})+c](k-s)} \prod_{i=1}^{\infty} \left(1 - \frac{k-s}{\rho_i}\right)^{m_i} \left(1 - \frac{k-s}{\bar{\rho}_i}\right)^{m_i} \tag{18}$$

Then we conclude that Theorem 3 is true according to Theorem 2, considering $e^{A(\lambda)+[B(\lambda)+c]s}$ and $\varepsilon(\lambda) e^{A(\bar{\lambda})+[B(\bar{\lambda})+c](k-s)}$ (as units in the ring of entire functions) have no zeros, thus both of them have no effect on the complex zeros related divisibility in the functional equation $\Lambda(\lambda, s) = \varepsilon(\lambda)\Lambda(\bar{\lambda}, k-s)$. The condition $\sum_{i=1}^{\infty} \frac{1}{|\rho_i|^2} < \infty$ guarantees the absolute and uniform convergence of related infinite products required in Theorem 2.
That completes the proof of Theorem 3. □

In the following Theorem 4, we make further efforts to lay a foundation for the study of completed L-functions that possess both real and complex zeros, denoted by $\rho \in \mathcal{Z}_r$ ($\Im(\rho) = 0$) and $\rho \in \mathcal{Z}_c$ ($\Im(\rho) \neq 0$), respectively. When these two zero sets have no common elements, we express their disjointness by: $\mathcal{Z}_r \cap \mathcal{Z}_c = \varnothing$.

The reason we need to consider this case is that, so far, we cannot rule out the existence of exceptional zeros (or Landau-Siegel zeros), although their numbers are very limited even if they do exist.

Denote the set of real zeros in the critical strip as

$$\mathcal{Z}_r = \{a_n \in \mathbb{R} \mid 0 < a_n < k,\ n = 1, 2, \ldots, N\}$$

where N is a finite natural number. This finiteness follows from the Identity Theorem, which implies that any non-zero entire function cannot have infinitely many zeros in a bounded region.

Theorem 4: Given an entire function represented by its Hadamard product:

$$\begin{aligned}\Lambda(\lambda, s) &= e^{A(\lambda)+B(\lambda)s} \prod_{\rho} \left(1 - \frac{s}{\rho}\right) e^{\frac{s}{\rho}}, s \in \mathbb{C} \\ &= e^{A(\lambda)+B(\lambda)s} \prod_{\rho \in \mathcal{Z}_r} \left(1 - \frac{s}{\rho}\right) e^{\frac{s}{\rho}} \prod_{\rho \in \mathcal{Z}_c} \left(1 - \frac{s}{\rho}\right) e^{\frac{s}{\rho}} \\ &= e^{A(\lambda)+[B(\lambda)+c]s} \prod_{n=1}^{N} \left(1 - \frac{s}{a_n}\right) e^{\frac{s}{a_n}} \prod_{i=1}^{\infty} \left(1 - \frac{s}{\rho_i}\right)\left(1 - \frac{s}{\bar{\rho}_i}\right)\end{aligned} \tag{19}$$

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and that

$$
\begin{aligned}
\Lambda(\overline{\lambda}, k-s) &= e^{A(\overline{\lambda})+B(\overline{\lambda})(k-s)} \prod_{\rho}\left(1-\frac{k-s}{\rho}\right)e^{\frac{k-s}{\rho}}, s \in \mathbb{C} \\
&= e^{A(\overline{\lambda})+B(\overline{\lambda})(k-s)} \prod_{\rho\in\mathcal{Z}_r}\left(1-\frac{k-s}{\rho}\right)e^{\frac{k-s}{\rho}} \prod_{\rho\in\mathcal{Z}_c}\left(1-\frac{k-s}{\rho}\right)e^{\frac{k-s}{\rho}} \\
&= e^{A(\overline{\lambda})+[B(\overline{\lambda})+c](k-s)} \prod_{n=1}^{N}\left(1-\frac{k-s}{a_n}\right)e^{\frac{k-s}{a_n}} \prod_{i=1}^{\infty}\left(1-\frac{k-s}{\rho_i}\right)\left(1-\frac{k-s}{\bar{\rho}_i}\right)
\end{aligned} \tag{20}
$$

Here

- $c = \sum_{i=1}^{\infty} \frac{2\alpha_i}{\alpha_i^2+\beta_i^2}$,
- λ denotes a mathematical object, $\overline{\lambda}$ is the dual of λ,
- $\rho_i = \alpha_i + j\beta_i$ and $\bar{\rho}_i = \alpha_i - j\beta_i$ are the complex-conjugate zeros of $\Lambda(\lambda, s)$,
- $0 < \alpha_i < k$, $\beta_i \neq 0$, $k > 0$ are real numbers,
- $\sum_{i=1}^{\infty} \frac{1}{|\rho_i|^2} < \infty$,
- $\mathcal{Z}_r \cap \mathcal{Z}_c = \varnothing$.

Then

$$
\Lambda(\lambda, s) = \varepsilon(\lambda)\Lambda(\overline{\lambda}, k-s) \Longrightarrow \begin{cases} \alpha_i = \frac{k}{2}\ (\forall i); \\ 0 < |\beta_1| < |\beta_2| < |\beta_3| < \cdots; \\ a_n = \frac{k}{2}, n = 1 \end{cases} \tag{21}
$$

i.e., all the zeros (both real and complex) of $\Lambda(\lambda, s)$ in the critical strip $0 < \Re(s) < k$ lie on the critical line.

Proof. By Theorem 3, to determine the distribution of the complex zeros of $\Lambda(\lambda, s)$, it suffices to show that the newly introduced factors $\prod_{\rho\in\mathcal{Z}_r}\left(1-\frac{s}{\rho}\right)e^{\frac{s}{\rho}}$ and $\prod_{\rho\in\mathcal{Z}_r}\left(1-\frac{k-s}{\rho}\right)e^{\frac{k-s}{\rho}}$ do not affect the complex zeros related divisibility in the functional equation $\Lambda(\lambda, s) = \varepsilon(\lambda)\Lambda(\overline{\lambda}, k-s)$. This follows from the given condition $\mathcal{Z}_r \cap \mathcal{Z}_c = \varnothing$, which implies that $\prod_{\rho\in\mathcal{Z}_r}\left(1-\frac{s}{\rho}\right)e^{\frac{s}{\rho}}$ and $\prod_{\rho\in\mathcal{Z}_c}\left(1-\frac{k-s}{\rho}\right)e^{\frac{k-s}{\rho}}$ are co-prime, $\prod_{\rho\in\mathcal{Z}_r}\left(1-\frac{k-s}{\rho}\right)e^{\frac{k-s}{\rho}}$ and $\prod_{\rho\in\mathcal{Z}_c}\left(1-\frac{s}{\rho}\right)e^{\frac{s}{\rho}}$ are co-prime, according to Ref.[3] (p.174, p.208) or Ref.[4] (see its THEOREM 4).

Thus, we conclude by Theorem 3 that

$$
\Lambda(\lambda, s) = \varepsilon(\lambda)\Lambda(\overline{\lambda}, k-s) \Rightarrow \begin{cases} \alpha_i = \frac{k}{2}, i = 1, 2, 3, \cdots \\ 0 < |\beta_1| < |\beta_2| < |\beta_3| < \cdots \end{cases} \tag{22}
$$

Next, we consider the real zeros of $\Lambda(\lambda, s)$.

By canceling the complex non-trivial zeros related polynomial factors on both sides of $\Lambda(\lambda, s) = \varepsilon(\lambda)\Lambda(\overline{\lambda}, k-s)$, we have

$$
e^{A(\lambda)+[B(\lambda)+c]s} \prod_{n=1}^{N}\left(1-\frac{s}{a_n}\right)e^{\frac{s}{a_n}} = \varepsilon(\lambda)e^{A(\overline{\lambda})+[B(\overline{\lambda})+c](k-s)} \prod_{n=1}^{N}\left(1-\frac{k-s}{a_n}\right)e^{\frac{k-s}{a_n}} \tag{23}
$$

Further Eq.(23) is equivalent to

$$
(-1)^N e^{A(\lambda)+[B(\lambda)+c']s} \prod_{n=1}^{N}\left(s-a_n\right) = \varepsilon(\lambda)e^{A(\overline{\lambda})+[B(\overline{\lambda})+c'](k-s)} \prod_{n=1}^{N}\left(s-(k-a_n)\right) \tag{24}
$$

where $c' = c + \sum_{n=1}^{N} \frac{1}{a_n}$.

Suppose the multiplicity of zero $s = a_n$ is m_n ($m_n \geq 1$) that is finite and unique although unknown. Then Eq.(24) becomes

$$(-1)^N e^{A(\lambda)+[B(\lambda)+c']s} \prod_{t=1}^{T} \left(s - a_t\right)^{m_t} = \varepsilon(\lambda) e^{A(\bar{\lambda})+[B(\bar{\lambda})+c'](k-s)} \prod_{t=1}^{T} \left(s - (k - a_t)\right)^{m_t} \tag{25}$$

where $\sum_{t=1}^{T} m_t = N$.

Considering $(s - a_t)$ and $(s - (k - a_t))$ are irreducible over $\mathbb{R}$, then by Lemma 3 (see Appendix), Eq.(25) means

$$\begin{cases} (s - a_t) \mid (s - (k - a_l)) \\ (s - (k - a_t)) \mid (s - a_l) \\ t, l = 1, 2, 3, \cdots, T \end{cases} \tag{26}$$

The only solution to Eq.(26) is $t = l, a_t = \frac{k}{2}, t = 1, \cdots, T$, otherwise the uniqueness of m_t would be violated with $a_t + a_l = k, t \neq l$. To avoid changing the multiplicity of m_t while $a_t = \frac{k}{2}$, we need to limit $T = 1$. Thus we get

$$\Lambda(\lambda, s) = \varepsilon(\lambda)\Lambda(\bar{\lambda}, k - s) \Rightarrow a_t = \frac{k}{2}, t = 1, m_1 = N \tag{27}$$

Putting Eq.(22) and Eq.(27) together, we proved Eq.(21).

That completes the proof of Theorem 4. $\square$

Remark: If the completed L-function has poles at $s = 0$ or $s = k$ with multiplicity $m \geq 1$, then the Hadamard product expression can only be applied to $s^m(k - s)^m\Lambda(\lambda, s)$, not to $\Lambda(\lambda, s)$ itself. This adjustment does not affect the results of Theorem 4, since we have $s^m(k - s)^m\Lambda(\lambda, s) = \varepsilon(\lambda)(k - s)^m s^m \Lambda(\bar{\lambda}, k - s) \Leftrightarrow \Lambda(\lambda, s) = \varepsilon(\lambda)\Lambda(\bar{\lambda}, k - s)$.

With similar thoughts, we can relax the condition $0 < \alpha_i < k$ in Theorem 4 to $0 \leq \alpha_i \leq k$. By considering separately the zeros on the boundary of the critical strip, i.e., $s = 0$ and $s = k$ with multiplicity $m \geq 1$ in the relevant Hadamard product, their corresponding factors will be canceled with each other in the symmetrical functional equation. Thus Theorem 4 still holds.

As to the situation that $s = 0$ and $s = k$ being of different multiplicities, say $m_0 \geq 1$, and $m_k \geq 1$ with $m_k \neq m_0$, can be ruled out since their corresponding divisibility in the relevant symmetrical functional equation leads to $k = 0$, which contradicts the fact $k > 0$.

Remark: As pointed out in Ref.[5], p.102, if $\rho_i = \alpha_i + j\beta_i$ is a zero of $\Lambda(\lambda, s)$, then $\bar{\rho}_i = \alpha_i - j\beta_i$ is a zero of $\Lambda(\bar{\lambda}, s)$. Therefore, to use Theorem 4 while $\lambda \neq \bar{\lambda}$, we need to construct a new symmetric functional equation $\Lambda(\bar{\lambda}, s)\Lambda(\lambda, s) = \varepsilon(\lambda)\varepsilon(\bar{\lambda})\Lambda(\lambda, k - s)\Lambda(\bar{\lambda}, k - s)$ to ensure that the conjugate zeros appear together in the related Hadamard products. For more details, see the proofs of Theorem 5 and Theorem 7.

3. The Applications of Theorem 4

We will make use of Theorem 4 to prove the Extended Riemann Hypothesis (for Dedekind zeta function), the Generalized Riemann Hypothesis (for Dirichlet L-function), and the Grand Riemann Hypothesis (for modular form L-Function, automorphic L-function, and etc.).

To facilitate the subsequent discussion, we give the details of the Extended Riemann Hypothesis, the Generalized Riemann Hypothesis, and the Grand Riemann Hypothesis. In the following contents, the critical line means: $\Re(s) = \frac{1}{2}$, or more generally, $\Re(s) = \frac{k}{2}, k > 0$ is a constant real number.

The Generalized Riemann Hypothesis: The non-trivial zeros of Dirichlet L-functions in the critical strip $0 < \Re(s) < 1$ lie on the critical line.

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The Extended Riemann Hypothesis: The non-trivial zeros of Dedekind zeta functions in the critical strip $0 < \Re(s) < 1$ lie on the critical line.

The Grand Riemann Hypothesis: The non-trivial zeros of all L-functions in the critical strip $0 < \Re(s) < k$ lie on the critical line.

Another version of Grand Riemann Hypothesis: The non-trivial zeros of all automorphic L-functions in the critical strip $0 < \Re(s) < k$ lie on the critical line.

To begin with, we provide a general property of L-functions, which was labeled Lemma 5.5 in Ref.[5], p.101.

Lemma 5.5 [5]: Let $L(f,s)$ be an L-function. All zeros ρ of $\Lambda(f,s)$ are in the critical strip $0 \le \sigma \le 1$. For any $\epsilon > 0$, we have

$$\sum_{\rho \neq 0,1} |\rho|^{-1-\epsilon} < +\infty.$$

where, $\Lambda(f,s)$ is the completed L-function corresponding to $L(f,s)$, σ is the real part of ρ, and f is identical to λ in this paper as a symbol representing a mathematical object (e.g., Dirichlet character, modular form, automorphic representation).

Another general property of L-functions is as follows.

The zeros of $\Lambda(\lambda,s)$ are precisely the non-trivial zeros of $L(\lambda,s)$, as the trivial zeros of $L(\lambda,s)$ are canceled by the poles of the Gamma factors in the completion process (see Ref.[5], p.96 for more details, with f therein replaced by λ).

Thus, we can discuss the non trivial zeros of L-functions based on the zeros of the corresponding completed L-functions.

3.1. Dirichlet L-Function

Definition: The Dirichlet L-function associated with a Dirichlet character χ modulo q is defined for $\Re(s) > 1$ by the series:

$$L(\chi,s) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s} \tag{28}$$

For the principal (or trivial) character χ_0 (where $\chi_0(n) = 1$ if $\gcd(n,q) = 1$ and $\chi_0(n) = 0$ otherwise), the L-function is related to the Riemann zeta function by:

$$L(\chi_0,s) = \zeta(s)\prod_{p|q}\left(1 - \frac{1}{p^s}\right) \tag{29}$$

Completed L-function: The completed Dirichlet L-function is defined as:

$$\Lambda(\chi,s) = \left(\frac{q}{\pi}\right)^{\frac{s+a}{2}}\Gamma\left(\frac{s+a}{2}\right)L(\chi,s) \tag{30}$$

where $a = 0$ if $\chi(-1) = 1$ (even character) and $a = 1$ if $\chi(-1) = -1$ (odd character).

Functional Equation: The completed Dirichlet L-function satisfies the functional equation:

$$\Lambda(\chi,s) = W(\chi)\Lambda(\overline{\chi},1-s) \tag{31}$$

where $W(\chi)$ is the Gauss sum:

$$W(\chi) = \frac{\tau(\chi)}{i^a\sqrt{q}} \tag{32}$$

and $\tau(\chi) = \sum_{n=1}^{q} \chi(n)e^{2\pi i n/q}$ is the Gauss sum associated with χ.

Hadamard Product: For non-principal characters χ, the completed L-function $\Lambda(\chi,s)$ is an entire function of order 1 and has the Hadamard product:

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$$\Lambda(\chi,s)=e^{A(\chi)+B(\chi)s}\prod_{\rho}\left(1-\frac{s}{\rho}\right)e^{s/\rho} \tag{33}$$

where the product is over all zeros ρ of $\Lambda(\chi,s)$, and $A(\chi)$ and $B(\chi)$ are constants depending on χ.

Next we prove the Generalized Riemann Hypothesis.

Theorem 5: The non-trivial zeros of Dirichlet L-functions in the critical strip $0<\Re(s)<1$ lie on the critical line.

Remark: We only need to prove that all the zeros of the completed Dirichlet L-function $\Lambda(\chi,s)$ have real part $\frac{1}{2}$, i.e., all the zeros of $\Lambda(\chi,s)$ lie on the critical line.

Proof. We conduct the proof in two cases.

CASE 1: $\chi=\bar{\chi}$ (self-dual)

It suffices to verify that the properties of $\Lambda(\chi,s)$ with $\chi=\bar{\chi}$ match the conditions of Theorem 4 with $\lambda=\chi$, $\varepsilon(\lambda)=W(\chi)$, $k=1$. Eq.(33) is equivalent to Eq.(19) in Theorem 4 by separating all zeros into two sets $\mathcal{Z}_r$ and $\mathcal{Z}_c$. Actually, to restrict $\chi=\bar{\chi}$ is to guarantee that the complex conjugate zeros of $\Lambda(\chi,s)$ appear in pairs, and then the quadruplets of zeros $(\rho_i,\bar{\rho}_i,1-\rho_i,1-\bar{\rho}_i)$ appear together according to Eq.(31). The condition $\sum_{i=1}^{\infty}\frac{1}{|\rho_i|^2}<\infty$ can be assured by Lemma 5.5, considering that $\sum_{i=1}^{\infty}\frac{1}{|\rho_i|^2},\rho_i\in\mathcal{Z}_c$ is a subseries of $\sum_{\rho\neq 0,1}\frac{1}{|\rho|^2},\rho\in\mathcal{Z}_r\cup\mathcal{Z}_c$; The condition $\mathcal{Z}_r\cap\mathcal{Z}_c=\varnothing$ holds because $\mathcal{Z}_r$ and $\mathcal{Z}_c$ are mutually exclusive sets, i.e., if $\rho\in\mathcal{Z}_r$, then $\Im(\rho)=0$; if $\rho\in\mathcal{Z}_c$, then $\Im(\rho)\neq 0$.

Therefore, by Theorem 4 with $\lambda=\chi$, $\varepsilon(\lambda)=W(\chi)$, $k=1$, we know that all zeros (real, if any, and complex) of $\Lambda(\chi,s)$ lie on the critical line for $\chi=\bar{\chi}$.

CASE 2: $\chi\neq\bar{\chi}$

In this case, the complex conjugate zeros do not appear together in Eq.(33), because if ρ is a zero of $\Lambda(\chi,s)$, then $\bar{\rho}$ is a zero of $\Lambda(\bar{\chi},s)$.

Thus, we need to extend Eq.(31) to another form, i.e.,

$$\Lambda(\bar{\chi},s)=W(\bar{\chi})\Lambda(\chi,1-s) \tag{34}$$

Combining (34) with (31), we get a new functional equation

$$\Lambda(\bar{\chi},s)\Lambda(\chi,s)=W(\bar{\chi})W(\chi)\Lambda(\chi,1-s)\Lambda(\bar{\chi},1-s) \tag{35}$$

The product $\Lambda(\bar{\chi},s)\Lambda(\chi,s)$ is an entire function of finite order ≤ 1, as this holds for the product of any two entire functions of order 1. This implies that its zeros ρ satisfy the convergence condition $\sum|\rho|^{-2}<\infty$. Moreover, its zero set—being the union of the zeros of the individual factors—possesses the required quadruplet symmetry $(\rho,\bar{\rho},1-\rho,1-\bar{\rho})$ from the symmetric functional equation (35).

Further, based on Eq.(33), we have

$$\Lambda(\chi,s)\Lambda(\bar{\chi},s)=e^{A(\chi)+A(\bar{\chi})+[B(\chi)+B(\bar{\chi})+c]s}\prod_{\rho\in\mathcal{Z}_r}\left(1-\frac{s}{\rho}\right)e^{\frac{s}{\rho}}\prod_{i=1}^{\infty}\left(1-\frac{s}{\rho_i}\right)\left(1-\frac{s}{\bar{\rho}_i}\right) \tag{36}$$

where $c=\sum_{i=1}^{\infty}\frac{2\alpha_i}{\alpha_i^2+\beta_i^2}$.

The condition $\sum_{i=1}^{\infty}\frac{1}{|\rho_i|^2}<\infty$ and condition $\mathcal{Z}_r\cap\mathcal{Z}_c=\varnothing$ hold for the same reasons as in CASE 1.

Therefore, by Theorem 4, all zeros (real, if any, and complex) of $\Lambda(\chi,s)\Lambda(\bar{\chi},s)$, and consequently of $\Lambda(\chi,s)$), lie on the critical line for $\chi\neq\bar{\chi}$.

Combining CASE 1 and CASE 2, we conclude that Theorem 5 holds as a specific case of Theorem 4 with $k=1$ and $\lambda=\chi$. □

3.2. *Dedekind Zeta Function*

Definition: For a number field K with ring of integers $\mathcal{O}_K$, the Dedekind zeta function is defined for $\Re(s)>1$ by:

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$$\zeta_K(s) = \sum_{\mathfrak{a}} \frac{1}{N(\mathfrak{a})^s} \tag{37}$$

where the sum is over all non-zero ideals $\mathfrak{a}$ of $\mathcal{O}_K$, and $N(\mathfrak{a})$ is the norm of the ideal.

Completed Zeta Function: The completed Dedekind zeta function is defined as:

$$\Lambda_K(s) = |D_K|^{s/2}\left(\pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\right)^{r_1}\left((2\pi)^{-s}\Gamma(s)\right)^{r_2}\zeta_K(s) \tag{38}$$

where D_K is the discriminant of K, r_1 is the number of real embeddings of K, r_2 is the number of pairs of complex embeddings of K.

Functional Equation: The completed Dedekind zeta function satisfies:

$$\Lambda_K(s) = \varepsilon(K)\Lambda_K(1-s) \tag{39}$$

where $\varepsilon(K) = 1$ for all number fields K, showing the symmetry of the functional equation.

Hadamard Product: The completed Dedekind zeta function has a simple pole at $s = 1$ with residue $\frac{2^{r_1}(2\pi)^{r_2}h_K R_K}{w_K\sqrt{|D_K|}}$, where h_K is the class number, R_K is the regulator, and w_K is the number of roots of unity in K. The function $s(s-1)\Lambda_K(s)$ is an entire function of order 1 and has the Hadamard product:

$$s(s-1)\Lambda_K(s) = e^{A_K+B_K s}\prod_{\rho\neq 0,1}\left(1-\frac{s}{\rho}\right)e^{s/\rho} \tag{40}$$

where the product is over all zeros ρ of $\Lambda_K(s)$ except $\rho = 0$ and $\rho = 1$, and A_K and B_K are constants depending on K.

For more details of the completed Dedekind zeta function, please be referred to Ref.[5] (Chapter 5.10) and Ref.[6] (Section 10.5.1).

Theorem 6: The non-trivial zeros of Dedekind zeta functions in the critical strip $0 < \Re(s) < 1$ lie on the critical line.

Remark: We only need to prove that all the zeros of $\Lambda_K(s)$ have real part $\frac{1}{2}$, i.e., all the zeros of $\Lambda_K(s)$ lie on the critical line.

Proof. It suffices to show that the properties of $\Lambda_K(s)$ match the conditions of Theorem 4 with $\lambda = K$, $\varepsilon(\lambda) = 1$, $k = 1$.

Actually, Eq.(39) and Eq.(40) guarantee that

$$e^{A_K+B_K s}\prod_{\rho\neq 0,1}\left(1-\frac{s}{\rho}\right)e^{s/\rho} = e^{A_K+B_K(1-s)}\prod_{\rho\neq 0,1}\left(1-\frac{1-s}{\rho}\right)e^{1-s/\rho} \tag{41}$$

where $\prod\limits_{\rho\neq 0,1}\left(1-\frac{s}{\rho}\right)e^{s/\rho} = \prod\limits_{\rho\in\mathcal{Z}_r\backslash\{0,1\}}\left(1-\frac{s}{\rho}\right)e^{s/\rho}\prod\limits_{\rho\in\mathcal{Z}_c}\left(1-\frac{s}{\rho}\right)e^{s/\rho}$.

And that the complex conjugate zeros of $\Lambda_k(s)$ appear in pairs, then the quadruplets of zeros $(\rho_i, \bar{\rho}_i, 1-\rho_i, 1-\bar{\rho}_i)$ appear together according to Eq.(39). The condition $\sum_{i=1}^{\infty}\frac{1}{|\rho_i|^2} < \infty$ and condition $\mathcal{Z}_r \cap \mathcal{Z}_c = \varnothing$ hold for the same reasons as in CASE 1 in the proof of Theorem 5.

Therefore, by Theorem 4 we know that all zeros (real, if any, and complex) of $\Lambda_K(s)$ lie on the critical line, i.e., Theorem 6 holds as a specific case of Theorem 4 with $\lambda = K$, $\varepsilon(\lambda) = 1$, $k = 1$. □

3.3. *Modular Form L-Function*

Definition: For a modular form $f(z) = \sum_{n=1}^{\infty} a_n e^{2\pi i n z}$ of weight k for a congruence subgroup Γ, the associated L-function is defined for $\Re(s) > \frac{k+1}{2}$ by:

$$L(f,s) = \sum_{n=1}^{\infty}\frac{a_n}{n^s} \tag{42}$$

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Completed L-function: For a cusp form f of weight k for $\Gamma_0(N)$ (level N is any positive integer) with Nebentypus character χ, the completed modular form L-function is defined as:

$$\Lambda(f,s) = N^{s/2}(2\pi)^{-s}\Gamma(s)L(f,s) \tag{43}$$

Functional Equation: For a normalized Hecke eigenform f of weight k for $\Gamma_0(N)$ with Nebentypus character χ, the completed modular form L-function satisfies:

$$\Lambda(f,s) = \varepsilon(f)\Lambda(\overline{f},k-s) \tag{44}$$

where $\varepsilon(f) = \pm 1$ is the epsilon factor, which is the eigenvalue of f under the Atkin-Lehner involution, and $\overline{f}$ is the modular form with Fourier coefficients $\overline{a}_n$.

Hadamard Product: For a cusp form f, the completed L-function $\Lambda(f,s)$ is an entire function of order 1 and has the Hadamard product:

$$\Lambda(f,s) = e^{A(f)+B(f)s}\prod_{\rho}\left(1-\frac{s}{\rho}\right)e^{s/\rho} \tag{45}$$

where the product is over all zeros ρ of $\Lambda(f,s)$, and $A(f)$ and $B(f)$ are constants depending on f. For more details of the completed modular form L-function, please be referred to Refs.[5][7].

We have the following result about the non-trivial zero distribution of modular form L-Functions.

Theorem 7: The non-trivial zeros of modular form L-Functions in the critical strip $0 < \Re(s) < k$ lie on the critical line.

Remark: We only need to prove that all the zeros of $\Lambda(f,s)$ have real part $\frac{k}{2}$, i.e., all the zeros of $\Lambda(f,s)$ lie on the critical line.

Proof. We conduct the proof in two cases.

CASE 1: $f = \overline{f}$ (self-dual)

It suffices to show that the properties of $\Lambda(f,s)$ with $f = \overline{f}$ match the conditions of Theorem 4 with $\lambda = f$. Eq.(45) is equivalent to Eq.(19) in Theorem 4 by separating all zeros into two sets $\mathcal{Z}_r$ and $\mathcal{Z}_c$. Actually, to restrict $f = \bar{f}$ is to guarantee that the complex conjugate zeros of $\Lambda(f,s))$ appear in pairs. Then the quadruplets of zeros $(\rho_i, \bar{\rho}_i, k-\rho_i, k-\bar{\rho}_i)$ appear together according to Eq.(44). The condition $\sum_{i=1}^{\infty}\frac{1}{|\rho_i|^2} < \infty$ and condition $\mathcal{Z}_r \cap \mathcal{Z}_c = \varnothing$ hold for the same reasons as in CASE 1 in the proof of Theorem 5.

Therefore, by Theorem 4, we know that all zeros (real, if any, and complex) of $\Lambda(f,s)$ lie on the critical line for $f = \overline{f}$.

CASE 2: $f \neq \overline{f}$

To deal with this case $f \neq \bar{f}$, we need first to extend Eq.(44) to another form, i.e.,

$$\Lambda(\bar{f},s) = \varepsilon(\bar{f})\Lambda(f,k-s) \tag{46}$$

Combining Eq.(46) with Eq.(44), we get a new functional equation

$$\Lambda(\bar{f},s)\Lambda(f,s) = \varepsilon(f)\varepsilon(\bar{f})\Lambda(f,k-s)\Lambda(\bar{f},k-s) \tag{47}$$

Both sides of Eq.(47) are the products of entire functions of order 1, thus they are still entire functions of order ≤ 1, with $\sum_{\rho}\frac{1}{|\rho|^2} < \infty$. And that the complex conjugate zeros of $\Lambda(\bar{f},s)\Lambda(f,s)$ appear in pairs, then the quadruplets of zeros $(\rho_i, \bar{\rho}_i, k-\rho_i, k-\bar{\rho}_i)$ appear together according to Eq.(47).

Further, based on Eq.(45), we have

$$\Lambda(f,s)\Lambda(\bar{f},s) = e^{A(f)+A(\bar{f})+[B(f)+B(\bar{f})+c]s}\prod_{\rho\in\mathcal{Z}_r}\left(1-\frac{s}{\rho}\right)e^{\frac{s}{\rho}}\prod_{i=1}^{\infty}\left(1-\frac{s}{\rho_i}\right)\left(1-\frac{s}{\bar{\rho}_i}\right) \tag{48}$$

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where $c = \sum_{i=1}^{\infty} \frac{2\alpha_i}{\alpha_i^2+\beta_i^2}$.

The condition $\sum_{i=1}^{\infty} \frac{1}{|\rho_i|^2} < \infty$ and condition $\mathcal{Z}_r \cap \mathcal{Z}_c = \varnothing$ hold for the same reasons as in CASE 1 in the proof of Theorem 5.

Therefore, by Theorem 4, all zeros (real, if any, and complex) of $\Lambda(f,s)\Lambda(\bar{f},s)$, and consequently of $\Lambda(f,s)$, lie on the critical line for $f \neq \bar{f}$.

Combining CASE 1 and CASE 2, we conclude that Theorem 7 holds as a specific case of Theorem 4 with $\lambda = f$. □

3.4. Automorphic L-Function

Definition: For an automorphic representation π of $\mathrm{GL}_n(\mathbb{A}_\mathbb{Q})$ ($\mathbb{A}_\mathbb{Q}$ is the adele ring over the field of rational numbers $\mathbb{Q}$, see Ref.[7], p.5 for more details), the associated L-function is defined for $\Re(s) > 1$ by:

$$L(\pi,s) = \prod_p L_p(\pi_p,s) \tag{49}$$

where $L_p(\pi_p,s)$ is the local L-factor at the prime p. For unramified π_p with Satake parameters $\{\alpha_{1,p},\ldots,\alpha_{n,p}\}$,

$$L_p(\pi_p,s) = \prod_{i=1}^{n}\left(1-\frac{\alpha_{i,p}}{p^s}\right)^{-1} \tag{50}$$

Completed L-function: The completed automorphic L-function is defined as:

$$\Lambda(\pi,s) = Q_\pi^{s/2}\prod_{i=1}^{n}\Gamma_*^{(i)}(s+\mu_{i,\pi})\cdot L(\pi,s) \tag{51}$$

where Q_π is the conductor of π, $\mu_{i,\pi}$ are complex numbers determined by the i-th local component of π_∞, and

$$\Gamma_*^{(i)}(s) = \begin{cases} \Gamma_\mathbb{R}(s) = \pi^{-\frac{s}{2}}\Gamma(\frac{s}{2}), & \text{for real representations.} \\ \Gamma_\mathbb{C}(s) = (2\pi)^{-s}\Gamma(s), & \text{for complex representations.} \end{cases} \tag{52}$$

Functional Equation: The completed automorphic L-function satisfies:

$$\Lambda(\pi,s) = \varepsilon(\pi)\Lambda(\tilde{\pi},1-s) \tag{53}$$

where $\tilde{\pi}$ is the contragredient representation of π and $\varepsilon(\pi)$ is the epsilon factor, a complex number of absolute value 1.

Hadamard Product: For a cuspidal automorphic representation π, the completed L-function $\Lambda(\pi,s)$ is an entire function of order 1 and has the Hadamard product:

$$\Lambda(\pi,s) = e^{A(\pi)+B(\pi)s}\prod_\rho\left(1-\frac{s}{\rho}\right)e^{s/\rho} \tag{54}$$

where the product is over all zeros ρ of $\Lambda(\pi,s)$, and $A(\pi)$ and $B(\pi)$ are constants depending on π.

For more details of the completed automorphic L-function, please be referred to Refs.[5][7].

We have the following result about the non-trivial zero distribution of automorphic L-Functions.

Theorem 8: The non-trivial zeros of automorphic L-Functions in the critical strip $0 < \Re(s) < 1$ lie on the critical line.

Remark: We only need to prove that all the zeros of $\Lambda(\pi,s)$ have real part $\frac{1}{2}$, i.e., all the zeros of $\Lambda(\pi,s)$ lie on the critical line.

Proof. The proof procedure of Theorem 8 is similar to that of Theorem 7 with $k=1$, f replaced by π, and $\overline{f}$ replaced by $\tilde{\pi}$. Thus, the proof details are omitted for simplicity. □

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Actually, from the above proofs of Theorem 5, Theorem 6, and Theorem 7, we can note that each proof does not depend on the specific definition of the L-function $L(\lambda, s)$, but rather relies on the following general properties of $\Lambda(\lambda, s)$ and $L(\lambda, s)$:
P1: Symmetric functional equation between $\Lambda(\lambda, s)$ and $\Lambda(\bar{\lambda}, k-s)$: $\Lambda(\lambda, s) = \varepsilon(\lambda)\Lambda(\bar{\lambda}, k-s)$;
P2: Hadamard product expression of entire function $\Lambda(\lambda, s)$ or $s^m(k-s)^m\Lambda(\lambda, s), m \geq 1$;
P3: The zeros in the relevant Hadamard products appear as conjugate-symmetric quadruplets $(\rho, \bar{\rho}, k-\rho, k-\bar{\rho})$;
P4: All zeros in the relevant Hadamard products lie in the critical strip $0 \leq \Re(\rho) \leq k$ and satisfy $\sum_{\rho \neq 0,k} \frac{1}{|\rho|^2} < \infty$;
P5: The disjointness of real and complex non-trivial zero sets;
P6: The zeros of $\Lambda(\lambda, s)$ are precisely the non-trivial zeros of $L(\lambda, s)$.

Therefore, we have the following result on the Grand Riemann Hypothesis.

Theorem 9: The non-trivial zeros of all L-functions in the critical strip $0 < \Re(s) < k$ lie on the critical line if only the properties **P1**, **P2**, **P3**, **P4**, **P5**, and **P6** are satisfied.

Proof. It is not difficult to see that **P1**, **P2**, **P3**, **P4**, and **P5** cover all the conditions in Theorem 4. Thus, we know by Theorem 4 that all the zeros, both real (if any) and complex, of $\Lambda(\lambda, s)$ lie on the critical line. Further, according to **P6**, we conclude that all the non-trivial zeros, both real (if any) and complex, of $L(\lambda, s)$ lie on the critical line.
That completes the proof of Theorem 9. □

Remark: Conditions P1–P4 and P6 are established properties of the relevant L-functions, see Chapter 5 of Ref.[5]. Condition **P5** is automatically satisfied by our definitions of $\mathcal{Z}_r$ (the set of zeros with $\Im(\rho) = 0$) and $\mathcal{Z}_c$ (the set of zeros with $\Im(\rho) \neq 0$). The sole purpose of **P6** is to link the zeros of $\Lambda(\lambda, s)$ with the non-trivial zeros of $L(\lambda, s)$; further details can be found in Ref.[5], p.96, with f therein replaced by λ.

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Appendix A. Lemmas in Ref.[1]

Lemma 8 [1]: Given that the entire function

$$f(s) = \prod_{i=1}^{\infty}\left(1 + \frac{(s-\alpha_i)^2}{\beta_i^2}\right)^{m_i}, \quad s \in \mathbb{C} \tag{A1}$$

converges absolutely on $\mathbb{C}$ and uniformly on every compact subset of $\mathbb{C}$, where $0 < \alpha_i < 1$ and $\beta_i \neq 0$ are real numbers with $|\beta_1| \leq |\beta_2| \leq \cdots$, $m_i \geq 1$ is the multiplicity of zero conjugates $\rho_i = \alpha_i + j\beta_i, \bar{\rho}_i = \alpha_i - j\beta_i, \sum_{i=1}^{\infty} \frac{1}{\beta_i^2} < \infty$.

Under the substitution $s \mapsto 1-s$, we have

$$f(1-s) = \prod_{i=1}^{\infty}\left(1 + \frac{(1-s-\alpha_i)^2}{\beta_i^2}\right)^{m_i}, \quad s \in \mathbb{C} \tag{A2}$$

Then

$$f(s) = f(1-s) \Longleftrightarrow \alpha_i = \frac{1}{2}; i = 1,2,3,\cdots; 0 < |\beta_1| < |\beta_2| < \cdots \tag{A3}$$

The other related lemmas in Ref.[1] are also provided in the following.

Lemma 1 [1]: Non-trivial zeroes of $\zeta(s)$, noted as $\rho = \alpha + j\beta$, have the following properties
1) The number of non-trivial zeroes is infinity;

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2) $\beta \neq 0$;
3) $0 < \alpha < 1$;
4) $\rho, \bar{\rho}, 1 - \bar{\rho}, 1 - \rho$ are all non-trivial zeroes.

Lemma 2 [1]: The zeros of $\xi(s)$ coincide with the non-trivial zeros of $\zeta(s)$.

Lemma 3 [1]: Let $m(x), g_1(x), ..., g_n(x) \in \mathbb{R}[x], n \geq 2$. If $m(x)$ is irreducible (prime) and divides the product $g_1(x) \cdots g_n(x)$, then $m(x)$ divides one of the polynomials $g_1(x), ..., g_n(x)$.

Lemma 4 [1]: Let $f(x), m(x) \in \mathbb{R}[x]$. If $m(x)$ is irreducible and $f(x)$ is any polynomial, then either $m(x)$ divides $f(x)$ or $gcd(m(x), f(x)) = 1$.

Lemma 5 [1]: Let $f(x)$ be an entire function on $\mathbb{C}$. Suppose $f(x) = \prod_{i=1}^{\infty} g_i(x)$ converges absolutely on $\mathbb{C}$ and uniformly on every compact subset of $\mathbb{C}$, where each $g_i(x) \in \mathbb{R}[x]$ is irreducible in $\mathbb{R}[x]$ of degree $d \in \{1, 2\}$. If $m(x) \in \mathbb{R}[x]$ is irreducible in $\mathbb{R}[x]$ of degree d and divides $f(x)$, then $m(x)$ divides $g_i(x)$ for some $i \geq 1$.

Lemma 6 [1]: Let $f(s)$ be a non-zero entire function, and let s_0 be a zero of $f(s)$. Then the multiplicity of s_0 is a finite positive integer.

Lemma 7 [1]: Let $f(s)$ be a non-zero entire function, and let s_0 be a zero of $f(s)$. Then the multiplicity of s_0 is unique.

Lemma 9 [1]: The infinite product $\prod_{i=1}^{\infty} \left(\frac{\beta_i^2}{\alpha_i^2 + \beta_i^2} \right)^{m_i}$ converges to a non-zero constant, given the conditions: $0 < \alpha_i < 1, \beta_i \neq 0, \sum_{i=1}^{\infty} \frac{1}{\beta_i^2} < \infty$, and $m_i \geq 1$ is the multiplicity of zero $\alpha_i + j\beta_i$.

Remark: Lemmas 1-4 are the well-known results summarized from related journal papers, or textbooks/monographs. Lemmas 5-9 are proved in Ref.[1].

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