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Posted Date: 5 December 2025

doi: 10.20944/preprints202512.0286.v2

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On ABC Triples

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Abstract

In this paper, we study the properties of co-prime positive integers a, b, c such that $a + b = c$ in relation to their radicals. We establish an explicit upper bound for c in relation to $rad(abc)$.

Keywords: radical of an integer; abc conjecture; Diophantine equations

MSC: 11A25, 11D75, 11J25

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1. Introduction

The *radical* of a positive integer n, often denoted as $rad(n)$, is the product of all prime factors of n, that is, the largest square-free factor of n. The *abc conjecture* is a conjecture in number theory first proposed by Joseph Oesterlé and David Masser in 1985 that is related to the radicals of positive integers [4,5]. The statement of the abc conjecture is as follows:

Conjecture 1.1. *Let a, b, c be positive integers such that $c = a + b$ and $\gcd(a, b, c) = 1$, then for each positive real number $\varepsilon > 0$, there are only finite triples (a, b, c) such that $rad(abc)^{1+\varepsilon} < c$.*

A set of three positive integers (a, b, c) that is a subject of Conjecture 1 is called an *abc triple*. So far there is no widely accepted proof for the abc conjecture and the conjecture is still regarded as an open problem.

A proven upper bound for c in an abc triple is

$$\log c < K \cdot rad(abc)^{\frac{1}{3}}(\log rad(abc))^3 \tag{1}$$

with K being a constant independent of a, b and c. This upper bound was proven by Stewart and Yu in 2001 [6].

In addition, Stewart and Tijdeman proved that the following lower bound holds for infinitely many abc triples in 1986 [7]:

$$\log c > \log rad(abc) + k\frac{\sqrt{\log c}}{\log\log c} \tag{2}$$

for all $k < 4$, and van Frankenhuysen improved k to 6.068 in 2000.[8]

Moreover, Granville and Tucker ([2]) has proposed the following conjecture:

Conjecture 1.2. *Let a, b, c be positive integers such that $c = a + b$ and $\gcd(a, b, c) = 1$, then $c < rad(abc)^2$.*

Besides, there is a stronger conjecture proposed by Baker in 2004 [1], stating that

Conjecture 1.3. *Let a, b, c be positive integers such that $c = a + b$ and $\gcd(a, b, c) = 1$, then $c < \frac{6}{5}rad(abc)\frac{(\log rad(abc))^{\omega(abc)}}{(\omega(abc))!}$*

where $\omega(abc)$ is the number of distinct prime factors of abc.

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Since it has been shown by Laishram and Shorey (2011,[3]) that $\frac{6}{5}\frac{(\log rad(abc))^{\omega(abc)}}{(\omega(abc))!} < rad(abc)^{\frac{3}{4}}$, Conjecture 1.3 implies that $c < rad(abc)^{\frac{7}{4}}$.

In this paper, we intend to discuss some properties of triples involved in the abc conjecture, improving the upper bound for c in an abc triple. Unless otherwise specified, $rad(n) = \prod_{i=1}^{\omega(n)} p_i$ indicates the radical of a positive integer $n = \prod_{i=1}^{\omega(n)} p_i^{a_i}$ with $p_1, \cdots, p_{\omega(n)}$ being prime numbers and $1 \le a_1, \cdots, a_{\omega(n)}$ being positive integers, and $\log n$ indicates the natural logarithm of n.

2. Main Results

Definition 1. $R(n) = \frac{rad(n)}{n}$

Theorem 1. *If $c = a + b$ and $\gcd(a, b, c) = 1$, then $c < rad(abc)^2$*

Proof. W.l.o.g. assume that $a < b$. First, we cannot have $rad(abc)^2 = c$, because if $rad(a)^2 rad(b)^2 rad(c)^2 = rad(abc)^2 = c$, then we have $rad(a)rad(b) \mid c \implies \gcd(a, b, c) > 1$, which is a contradiction, thus we either have $rad(abc)^2 > c$ or $rad(abc)^2 < c$.

Claim: Assume that $z = \frac{1}{4rad(abc)}$ and $rad(abc)^2 < c$, then for all nonnegative integers $0 \le m$, $(1 + \frac{z}{2b})^m rad(abc)^2 < c$ implies that $(1 + \frac{z}{2b})^{m+1} rad(abc)^2 < c$.

Proof of the claim:

We prove the claim by mathematical induction.

First, since $rad(abc)^2 < c$ and since $c = a + b < 2b$, we have $rad(abc)^2 < c < 2b$

Therefore, we have

$$R(ab) = \frac{rad(ab)}{ab} < \frac{rad(ab)}{a(\frac{rad(abc)^2}{2})} = \frac{2rad(ab)}{a \cdot rad(abc)^2} \tag{3}$$

and

$$R(c) = \frac{rad(c)}{c} < \frac{rad(c)}{rad(abc)^2} \tag{4}$$

This implies that $R(ab) - \frac{2rad(ab)}{a \cdot rad(abc)^2} < 0$ and $R(c) - \frac{rad(c)}{rad(abc)^2} < 0$

Therefore we have

$$0 < (R(ab) - \frac{2rad(ab)}{a \cdot rad(abc)^2})(R(c) - \frac{rad(c)}{rad(abc)^2}) \tag{5}$$

Which implies that

$$\frac{2rad(ab)}{a \cdot rad(abc)^2} R(c) + \frac{rad(c)}{rad(abc)^2} R(ab) < R(abc) + (\frac{2}{a \cdot rad(abc)^3}) \tag{6}$$

Multiplying a on both sides, we have

$$\begin{aligned} &\frac{2}{c \cdot rad(abc)} + \frac{1}{b \cdot rad(abc)} \\ &< a \cdot R(abc) + (\frac{2}{rad(abc)^3}) = a \cdot \frac{rad(abc)}{abc} + (\frac{2}{rad(abc)^3}). \end{aligned} \tag{7}$$

Since $rad(abc)^2 < c$, (6) implies that

$$\frac{2}{c \cdot rad(abc)} + \frac{1}{b \cdot rad(abc)} < \frac{1}{b\sqrt{c}} + \frac{2}{rad(abc)^3} \tag{8}$$

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Let $\rho = \frac{1}{rad(abc)}$, then by multiplying bc on both sides, we have

$$(2b+c)\rho < \sqrt{c} + 2\rho^3 bc \tag{9}$$

Which implies that

$$\frac{2b+c-\frac{\sqrt{c}}{\rho}}{2bc} < \rho^2 \implies rad(abc)^2 < \frac{2b}{2b+c-\frac{\sqrt{c}}{\rho}}c \tag{10}$$

If $2b+c-\frac{\sqrt{c}}{\rho} \leq 2b+z$, then we have

$$\begin{aligned} c - \frac{\sqrt{c}}{\rho} - z \leq 0 &\implies \sqrt{c} \leq \frac{1}{2}(\frac{1}{\rho} + \sqrt{\frac{1}{\rho^2} + 4z}) \\ &\implies c \leq \frac{1}{4}(\frac{2}{\rho^2} + \frac{2}{\rho}\sqrt{\frac{1}{\rho^2} + 4z}) \leq \frac{1}{\rho^2} + z \end{aligned} \tag{11}$$

By Bernoulli's inequality, we have $\sqrt{\frac{1}{\rho^2} + 4z} \leq \frac{1}{\rho}(1+2z\rho^2) = \frac{1}{\rho} + 2z\rho$, therefore, we have

$$c \leq \frac{1}{4}(\frac{2}{\rho^2} + \frac{2}{\rho}\sqrt{\frac{1}{\rho^2} + 4z}) \leq \frac{1}{\rho^2} + z = rad(abc)^2 + \frac{1}{4rad(abc)} \tag{12}$$

but since $rad(abc)$ and c are positive integers with $6 \leq rad(abc)$, we have $\frac{1}{4rad(abc)} < 1$, this implies that $c \leq rad(abc)^2$, which contradicts with our assumption.

Therefore we have $2b+z < 2b+c-\frac{\sqrt{c}}{\rho}$, and by (10) we have

$$rad(abc)^2 < \frac{2b}{2b+c-\frac{\sqrt{c}}{\rho}}c < \frac{2b}{2b+z}c \implies (1+\frac{z}{2b})rad(abc)^2 < c \tag{13}$$

Thus the claim holds for $m = 0$.

Now assume that the proposition holds for all positive integers $1 \leq m \leq k-1$, then for the case $m = k$, since $(1+\frac{z}{2b})^k rad(abc)^2 < c$ and since $c = a+b < 2b$, we have $(1+\frac{z}{2b})^k rad(abc)^2 < c < 2b$. For the sake of brevity, we may have $\zeta = (1+\frac{z}{2b})^k$ in the following paragraphs.

Therefore, we have

$$R(ab) = \frac{rad(ab)}{ab} < \frac{rad(ab)}{a(\zeta\frac{rad(abc)^2}{2})} = \frac{2rad(ab)}{\zeta a \cdot rad(abc)^2} \tag{14}$$

and

$$R(c) = \frac{rad(c)}{c} < \frac{rad(c)}{\zeta rad(abc)^2} \tag{15}$$

This implies that $R(ab) - \frac{2rad(ab)}{\zeta a \cdot rad(abc)^2} < 0$ and $R(c) - \frac{rad(c)}{\zeta rad(abc)^2} < 0$

Therefore we have

$$0 < (R(ab) - \frac{2rad(ab)}{\zeta a \cdot rad(abc)^2})(R(c) - \frac{rad(c)}{\zeta rad(abc)^2}) \tag{16}$$

Which implies that

$$\frac{2rad(ab)}{\zeta a \cdot rad(abc)^2}R(c) + \frac{rad(c)}{\zeta rad(abc)^2}R(ab) < R(abc) + (\frac{2}{\zeta^2 a \cdot rad(abc)^3}) \tag{17}$$

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Since $\zeta rad(abc)^2 < c$, (17) implies that

$$\begin{aligned} &\frac{2rad(abc)}{\zeta ac \cdot rad(abc)^2} + \frac{rad(abc)}{\zeta ab \cdot rad(abc)^2} = \frac{2rad(ab)}{\zeta a \cdot rad(abc)^2} R(c) + \frac{rad(c)}{\zeta rad(abc)^2} R(ab) \\ &< R(abc) + (\frac{2}{\zeta^2 a \cdot rad(abc)^3}). \end{aligned} \tag{18}$$

Multiplying a on both sides, we have

$$\begin{aligned} &\frac{2}{\zeta c \cdot rad(abc)} + \frac{1}{\zeta b \cdot rad(abc)} \\ &< a \cdot R(abc) + (\frac{2}{\zeta^2 rad(abc)^3}) = a \cdot \frac{rad(abc)}{abc} + (\frac{2}{\zeta^2 rad(abc)^3}). \end{aligned} \tag{19}$$

Since $\zeta rad(abc)^2 < c$, we have

$$\begin{aligned} &\frac{2}{\zeta c \cdot rad(abc)} + \frac{1}{\zeta b \cdot rad(abc)} < \frac{rad(abc)}{bc} + (\frac{2}{\zeta^2 rad(abc)^3}) \\ &< \frac{\sqrt{\frac{c}{\zeta}}}{bc} + \frac{2}{\zeta^2 rad(abc)^3} = \frac{1}{b\sqrt{\zeta c}} + \frac{2}{\zeta^2 rad(abc)^3} \end{aligned} \tag{20}$$

Let $\rho = \frac{1}{\sqrt{\zeta} rad(abc)}$, then by multiplying $\sqrt{\zeta} bc$ on both sides, we have

$$(2b + c)\rho < \sqrt{c} + 2\rho^3 bc \tag{21}$$

Which implies that

$$\frac{2b + c - \frac{\sqrt{c}}{\rho}}{2bc} < \rho^2 \implies \zeta rad(abc)^2 < \frac{2b}{2b + c - \frac{\sqrt{c}}{\rho}} c \tag{22}$$

If $2b + c - \frac{\sqrt{c}}{\rho} \leq 2b + z$, then since $\lfloor \zeta rad(abc)^2 \rfloor + 1 \leq c$, $2b + c - \frac{\sqrt{c}}{\rho} \leq 2b + z$ imply that

$$\begin{aligned} &c \leq (\sqrt{\zeta} rad(abc))\sqrt{c} + z \\ &\implies \sqrt{c}(\sqrt{c} - \sqrt{\zeta} rad(abc)) \leq z = \frac{1}{4rad(abc)} \\ &\implies \sqrt{c} \leq \sqrt{\zeta} rad(abc) + \frac{1}{4rad(abc)\sqrt{c}}. \end{aligned} \tag{23}$$

which implies that

$$\begin{aligned} c &\leq \zeta rad(abc)^2 + \frac{2\sqrt{\zeta} rad(abc)}{4rad(abc)\sqrt{c}} + \frac{1}{16c \cdot rad(abc)^2} \\ &\leq \zeta rad(abc)^2 + \frac{2\sqrt{\zeta} rad(abc)}{4rad(abc)\sqrt{\zeta} rad(abc)} + \frac{1}{16c \cdot rad(abc)^2} \\ &= \zeta rad(abc)^2 + \frac{1}{2rad(abc)} + \frac{1}{16c \cdot rad(abc)^2} \\ &\leq \left\lfloor \zeta rad(abc)^2 \right\rfloor + 1 + \frac{1}{2rad(abc)} + \frac{1}{16c \cdot rad(abc)^2} \end{aligned} \tag{24}$$

But since $3 \leq c$ is a positive integer and since $\gcd(a, b, c) = 1$ and $a + b = c$, we have $6 \leq rad(abc)$, thus $\frac{1}{2rad(abc)} + \frac{1}{16c \cdot rad(abc)^2} < \frac{1}{12} + \frac{1}{16 \times 6^2} < 1$, therefore, (24) implies that $\lfloor \zeta rad(abc)^2 \rfloor + 1 \leq c \leq \lfloor \zeta rad(abc)^2 \rfloor + 1$, which indicates that $c = \lfloor \zeta rad(abc)^2 \rfloor + 1$.

By (21) and $c = \lfloor \zeta rad(abc)^2 \rfloor + 1$, we have

 doi:10.20944/preprints202512.0286.v2

$$2b < \frac{\sqrt{c}}{\rho} + (2\rho^2 b - 1)c = \sqrt{\frac{\lfloor \zeta rad(abc)^2 \rfloor + 1}{\zeta rad(abc)^2}} + (\frac{2b}{\zeta rad(abc)^2} - 1)(\left\lfloor \zeta rad(abc)^2 \right\rfloor + 1) \tag{25}$$

Here we must have $\frac{2b}{\zeta rad(abc)^2} - 1 \leq 1$, this is because if $1 < \frac{2b}{\zeta rad(abc)^2} - 1$, then we have $\zeta rad(abc)^2 < b \leq (c-1)$, which implies that $\zeta rad(abc)^2 + 1 < c \leq \left\lfloor \zeta rad(abc)^2 \right\rfloor + 1$, a contradiction.

Also, we must have $b - a = 1$, this is because if $2 \leq b - a$, then we have

$$2 \leq b - a \implies a + 2 \leq b \implies a \leq b - 2 \implies c = a + b \leq 2b - 2 \implies c + 2 \leq 2b \tag{26}$$

Therefore, (25) implies that

$$\begin{aligned} &\left\lfloor \zeta rad(abc)^2 \right\rfloor + 3 = c + 2 \leq 2b < \left\lfloor \zeta rad(abc)^2 \right\rfloor + 1 + \sqrt{\frac{\lfloor \zeta rad(abc)^2 \rfloor + 1}{\zeta rad(abc)^2}} \\ &\implies 4 < \frac{\lfloor \zeta rad(abc)^2 \rfloor + 1}{\zeta rad(abc)^2} \implies 4\zeta rad(abc)^2 < \left\lfloor \zeta rad(abc)^2 \right\rfloor + 1 \end{aligned} \tag{27}$$

However, since $\left\lfloor \zeta rad(abc)^2 \right\rfloor \leq \zeta rad(abc)^2$, (27) implies that $4\zeta rad(abc)^2 < \zeta rad(abc)^2 + 1$, which further implies that $3rad(abc)^2 \leq 3\zeta rad(abc)^2 < 1$, a contradiction.

Therefore, we have $b - a = 1$, and by $b - a = 1$ we have $a = b - 1$, which implies that $c = a + b = 2b - 1$.

Again, by (21), we have

$$\begin{aligned} &2b = c + 1 < \frac{\sqrt{c}}{\rho} + (2\rho^2 b - 1)c \\ &\implies 2c + 1 < \frac{\sqrt{c}}{\rho} + \rho^2(c+1) \\ &\implies 2c + 1 < c + \frac{\lfloor \zeta rad(abc)^2 \rfloor + 2}{\zeta rad(abc)^2} \leq c + 1 + \frac{2}{\zeta rad(abc)^2} \\ &\implies c < \frac{2}{\zeta rad(abc)^2} \end{aligned} \tag{28}$$

But since $6 \leq rad(abc)$ and $1 < \zeta = (1 + \frac{z}{2b})^k$, we have $\frac{2}{\zeta rad(abc)^2} < 1$, therefore, (28) implies that $c < 1$, which is a contradiction.

Therefore we have $2b + z < 2b + c - \frac{\sqrt{c}}{\rho}$, and by $2b + z < 2b + c - \frac{\sqrt{c}}{\rho}$ and (22), we have

$$\begin{aligned} &(1 + \frac{z}{2b})^k rad(abc)^2 < \frac{2b}{2b + c - \frac{\sqrt{c}}{\rho}} c < \frac{2b}{2b + z} c \\ &\implies (1 + \frac{z}{2b})^{k+1} rad(abc)^2 = (1 + \frac{z}{2b})^k (1 + \frac{z}{2b}) rad(abc)^2 < c \end{aligned} \tag{29}$$

Thus the claim holds for $m = k$ as well.

Therefore, by mathematical induction, the claim holds for all $0 \leq m$, which implies that if $rad(abc)^2 < c$, then $(1 + \frac{z}{2b})^m rad(abc)^2 < c$ holds for all non-negative integers $0 \leq m$.

However, since c is finite and $1 < 1 + \frac{z}{2b}$, by the Archimedean property, there is a positive integer τ such that $c < (1 + \frac{z}{2b})^\tau rad(abc)^2$, therefore, the claim leads to a contradiction, thus $rad(abc)^2 < c$ is false.

Therefore $c < rad(abc)^2$. $\square$

Conflicts of Interest: The research received no external funding.

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