Article

Not peer-reviewed version

On Finite Exceptional Orthogonal Polynomial Sequences Composed of Rational Darboux Transforms of Romanovski-Jacobi Polynomials

Gregory Natanson *

Posted Date: 18 February 2025

doi: 10.20944/preprints202502.1228.v1

Keywords: rational Sturm-Liouville equation; exceptional differential polynomial system; exceptional orthogonal polynomial system; exceptional orthogonal polynomials; Liouville transformation; classical Jacobi polynomials; Romanovski-Jacobi polynomials

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 18 February 2025 doi:10.20944/preprints202502.1228.v1

Article

On Finite Exceptional Orthogonal Polynomial Sequences Composed of Rational Darboux Transforms of Romanovski-Jacobi Polynomials

Gregory Natanson

Independent Researcher; greg_natanson@yahoo.com

Abstract: The paper presents the united analysis of the finite exceptional orthogonal polynomial (EOP) sequences composed of rational Darboux transforms of Romanovski-Jacobi polynomials. It is shown that there are four distinguished exceptional differential polynomial systems (X-Jacobi DPSs) of series J1, J2, J3, and W. The first three X-DPSs formed by pseudo-Wronskians of two Jacobi polynomials contain both exceptional orthogonal polynomial systems (X-Jacobi OPSs) on the interval (-1,+1) and the finite EOP sequences on the positive interval (1,Inf). On the contrary, the X-DPS of series W formed by Wronskians of two Jacobi polynomials contains only (infinitely many) finite EOP sequences on the interval (1,Inf). In addition, the paper rigorously examines the three isospectral families of the associated Liouville potentials (rationally extended hyperbolic Pöschl-Teller potentials of types ***a***, ***b***, and ***a'***) exactly quantized by the EOPs in question.

Keywords: rational Sturm-Liouville equation; exceptional differential polynomial system; exceptional orthogonal polynomial system; exceptional orthogonal polynomials; Liouville transformation; classical Jacobi polynomials; Romanovski-Jacobi polynomials

MSC: 34B24

1. Introduction

A decade ago the author [1] (under influence of Odake and Sasaki's celebrated paper [2] and its extension in [3]) demonstrated the existence of the three isospectral families of the rational SUSY partners of the hyperbolic Pöschl-Teller (*h*-PT) potential [4] quantized by the rational Darboux transforms ($\mathfrak{RDT}$s) of the Romanovski-Jacobi (R-Jacobi) polynomials [5-7]. In all the cases (labeled $\underset{\sim}{\mathbf{a}}$, $\underset{\sim}{\mathbf{b}}$, and $\underset{\sim}{\mathbf{a}}'$ below for the reasons specified later) the quasi-rational transformation functions (q-RTFs) were represented by the principal Frobenius solutions (PFSs) of the Jacobi-reference (JRef) canonical Sturm-Liouville equation (CSLE) nonvanishing in the quantization interval (+1,∞). The finite exceptional orthogonal polynomial EOP sequence reported in [3] was identified by us as type $\underset{\sim}{\mathbf{b}}$. As pointed to in [8], the infinitely many finite polynomial sequences of type $\underset{\sim}{\mathbf{a}}'$ were constructed by Grandati [9] a year earlier; however, he did not recognized the fact that, in contrast with other potentials discussed in the paper, the aforementioned polynomial sequences are X-orthogonal with a positive weight.

In the same year as [1], Yadav et al. [10] calculated the scattering amplitude for the rationally extended *h*-PT potential solvable by the finite sequence of the EOPs of type $\underset{\sim}{\mathbf{b}}$, referring to the latter simply as 'X_m-Jacobi EOPs'. The epithet (repeatedly used by these authors in the more recent papers [11-14]) seemed confusing since the EOPs in question do not belong to the X_m-Jacobi orthogonal polynomial system (OPS) [15-17] and therefore cannot form its finite subset.

Before continuing our discussion, let us first point to the dubious use of the term 'EOP' in the literature, similar to the slang use of the term 'orthogonal Jacobi polynomials', instead of 'classical Jacobi polynomials', which disregards the existence of the finite orthogonal subsets formed by the R-

Jacobi polynomials. Similarly, Gòmez-Ullate, Milson et al. [15,16,18,19] use the term 'EOPs' as the synonym for 'X-OPS', disregarding the existence of the finite EOP sequences [1-3,8,10-13,20-30] represented by the rational Darboux transforms ($RD\mathfrak{T}s$) of Romanovski polynomials [1,8,21,24-29]. (As a puzzling exception, Gòmez-Ullate, Grandati, and Milson [31], when citing the studies on the EOPs, did mention the papers [9] and [20], which deal solely with the problems solved by the finite EOP sequences.)

On the other hand, Yadav et al. in their progressed study [11] on the rational extensions of the *h*-PT potential used the term 'EOP' in both ways: they first identified the families of EOPs as 'X_m-Laguerre and X_m- Jacobi polynomials' (apparently implying the X_m-Laguerre and X_m- Jacobi OPSs) but then presented the long list of the papers dealing with both infinite and finite EOP sequences. The cited authors then discussed the quasi-rational eigenfunctions of the rationally extended *h*-PT and Scarf I potentials, referring to their (finitely and respectively infinitely many) polynomial components as 'EOPs'. Yadav et al took later more cautious approach [12], first mentioning the three pioneering studies [15,32,33] on the X_m-Jacobi and X_m- Laguerre OPSs and then citing a mixture of papers which discussed both infinite and finite EOP sequences. (Regrettably, their later paper [12] overlooked our analysis [8,24] cautiously distinguishing between the X_m- Jacobi OPSs and finite EOP sequences formed by the $RD\mathfrak{T}s$ of the R-Jacobi polynomials.)

The rigorous analysis of the aforementioned finite EOP sequences of types $\utilde{\mathbf{a}}$, $\utilde{\mathbf{b}}$, and $\utilde{\mathbf{a}}'$ in [24] revealed that the EOP sequences of types $\utilde{\mathbf{a}}$ and $\utilde{\mathbf{b}}$ are formally generated by the same shift operators as the cases J1 and J2 case in [3], except that the indexes of the seed Jacobi polynomials in our case were independent of the polynomial degrees. This observation brought the author [24] to the concept of the exceptional differential polynomial systems (X-DPSs), with the term 'DPS' used in exactly the same sense as it was suggested by Everitt et al. [34,35] for conventional sequences of polynomials obeying the Bochner theorem [36]. In following the commonly accepted terminology suggested by Gòmez-Ullate et al. [33] for the X_1-OPSs, we call the given DPS exceptional because the polynomial sequence in question either does not start from a constant or lacks the first-degree polynomial and therefore do not obey the Bochner theorem. Indeed, as stressed by Kwon and Littlewood [37], Bochner himself "did not mention the orthogonality of the polynomial systems that he found. The problem of classifying all classical orthogonal polynomials was handled by many authors thereafter" based on his analysis of possible polynomial solutions of *complex* second-order differential eigenequations.

Compared with the rigorous mathematical analysis of the X-OPSs in [18,19], the concept of the X-DPSs put forward by us in [24] represents the parallel direction dealing with the solvable rational CSLEs (RCSLEs) and related X-Bochner ordinary differential equations (ODEs), instead of the (irregular) exceptional Bochner (X-Bochner) operators in [18] and related polynomial Sturm-Liouville problems (PSLPs) sketched in [15,18,32].

The interrelation between the two approaches is closely related to the dual use of the term 'Darboux transformation' (DT), following the discovery by Andrianov et al [38,39] that the renowned transformation of the Schrödinger equation initially suggested by Darboux [40] for the generic second-order canonical differential eigenequation (long before the birth of the quantum mechanics) is equivalent to its intertwining factorization. We refer the reader to the excellent overview of this issue in [41].

More recently Gòmez-Ullate et al. [42] initiated the new direction in the theory of the rational Sturm-Liouville equations (RSLEs) by applying the intertwining factorization to the second-order differential eigenoperator. This operation was termed 'Darboux transformation', based on the dualism existent in the particular case of the Schrödinger operator. This innovation followed by its extension to the X-OPSs [15,32,33] laid the foundation for their rigorous theory advanced to the higher level in [18,19].

The author (being accustomed [43] to the strict use of the mentioned term) took the different turn [1] in the extension of the DTs to the SLEs. As we understand now, our original intuitive idea was based on the three-step operation:

i) the Liouville transformation from the CSLE to the Schrödinger equation;

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 18 February 2025 doi:10.20944/preprints202502.1228.v1

ii) the Darboux deformation of the corresponding Liouville potential;

iii) the reverse Liouville transformation from the Schrödinger equation to the new CSLE using the same change of variable as at Step i).

referred to by us years later [44] as ‘Liouville-Darboux transformation’. Note that we invented the term ‘Darboux deformation’ (DD), used instead of ‘DT’, simply to avoid the multiple repetition of the word ‘transformation’. It is worth stressing again that we give to this term its original meaning implied by Darboux [40].

The extensive exploration of the literature revealed that the transformation of the CSLE sketched above has been introduced by Rudyak and Zakhariev [45] in the late eighties. Schnizer and Leeb [46, 47] named it ‘generalized Darboux transformation’ (GDT); this name was also accepted in some later studies on this subject. However, since various authors give to this widely used name completely different meanings in both physics and mathematics (see [44] for numerous examples), we suggested the aforementioned name ‘Liouville Darboux transformation’ as an alternative. Our current perception is that the latter is slightly misleading because it relates the definition of the transformation to the DD of the Liouville potential, which is absolutely irrelevant to the problem under consideration unless we are interested in quantum-mechanical applications.

Below we simply refer to the GDTs in question as ‘Rudjak-Zakhariev transformations’ (RZTs) and consider its three-step decomposition suggested in [44] just as one of its realizations, but not as its definition (see Appendix A for more details). We term the RZT of the RCSLE as ‘rational RZT’ (RRZT) if it uses a q-RTF.

In this paper we, based on the results of our previous studies [8,24], scrupulously analyze the manifold of the RCSLEs obtained by RRZTs of the Jacobi-reference (JRef) CSLE which is defined defined via (1)-(3) in Section 2. Each quasi-rational solution (q-RS) formed by a m-degree Jacobi polynomial can be used as the q-RTF giving rise to the RCSLE with m+2 poles in the finite complex plane. The Jacobi indexes of the seed polynomials are defined in the segments carved by the three vertical and three horizontal lines with the abscissas and respectively ordinates equal to -1, 0, and +1.

Each transformed CSLE is then converted to the Bochner-type ODE with polynomial coefficients, taking advantage of the fact that the density function of our interest has only simple poles in the finite plane and as a results the mentioned gauge transformation is energy-independent [1]. Concequently, the linear coeficient function of the resultant differential equation does not depend on degrees of the sought-for polynomials.

It has been proven in [24] that each transformed RCSLE constructed in such a way has a quartet of infinite sequences of q-RSs with polynomial components forming the four X_m-Jacobi DPSs of series J1, J2, D, and W, as they were termed by us. The shift operators for the cases J1 and J2 in [3] match our equations for the X_m-Jacobi DPSs of series J1 and J2 accordingly, though, as pointed to in [8], the indexes of the seed Jacobi polynomials in our scheme are independent of the polynomials, in a sharp contrast with [3].

The most important part of our formalism is to formulate the rational Sturm-Liouville problem (RSLP) to find all infinite or finite orthogonal subsequences in each of the four X_m-Jacobi DPSs. Note that, until now, we have not imposed any restriction on the zeros of the seed Jacobi polynomials. However, to select the RSLPs solvable by either infinite or finite EOP sequences, we have to focus solely on the seed Jacobi polynomials with no zeros inside the quantization interval. At this point we come to the main advantage of our approach, compared with the general theory of X-OPSs advanced by Garcia-Ferrero et al. [18,19]. Namely we formulate the RSLP for both finite and infinite quantization intervals whereas the X_m-Jacobi OPSs appear only if the OBCs for the given SLP are imposed at the ends of the finite interval (-1,+1). Since the cited authors were interested only in the X-OPSs they term the X-Bochner operator ‘regular’, if the q-RTF used to generate the RCSLE in question does not have nodes inside the interval (-1,+1). On the contrary, we have to additionally specify the open interval, where the RCSLE of our interest may not have any singularities. For the purposes of this paper (unless explicitly specified otherwise) the q-RTF is termed regular if does not

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 18 February 2025 doi:10.20944/preprints202502.1228.v1

have real nodes larger than 1. Similarly we refer to the RRZT as regular (*reg*-RRZT) if the corresponding transform of the JRef CSLE does not have poles in the interval (1,∞).

A certain deficiency of our Sturm-Liouville approach, compared with the intertwining technique advanced in [18,19], is that we [48] require that the RRZT of the RSLE preserves both the leading and weight coefficient functions. As a result, the RRZTs, as they define here, represent only a narrow subset of the rational Darboux transformations (RDTs) in the terminology of Garcia-Ferrero et al. [18,19].

As it has been already done above in our references to the RRZ𝔍s of the Romanovsky polynomials, we will often use the commonly accepted term ' RD𝔍 ', instead of ' RRZ𝔍 ', despite the more restrictive requirement for both indexes of the classical Jacobi polynomial or the first index of the R-Jacobi polynomial to be positive.

Our next step is to find all quasi-raional solutions (q-RSs) of the JRef CSLE, which do not have nodes inside the selected quantization interval, which assures that the transformed RCSLE does not have poles inside the interval of our interest. It was taken for granted in our earlier works [1,8,24] that any PFS below the lowest eigenvalue is necessarily nodeless. As one can see from the proof presented in Appendix B, this is not by any means a trivial (though wide-spread) presumption.

Using Klein's formulas (see §6.72 in [49]) for the numbers of zeros of a Jacobi polynomial in the intervals (-∞,-1), (-1,+1), and (+1,∞), we showed [1] that the JRef CSLE also has q-RSs of type $\tilde{\mathrm{d}}$ with no nodes in the interval (+1,∞). Keeping in mind the RRZT using this q-RS as q-RTF inserts the extra energy level below the lowest eigenvalue, it can be called [41] 'dressing' transformation.

To find all the orthogonal RRZ𝔍s of the classical and R-Jacobi polynomials, we convert the transformed RCSLE to its prime form defined in such a way [48] that the Dirichlet boundary condition (DBC) unambiguously selects the PFS near the singular end in question. As a result, the DBCs at the ends of the given quantization interval select the solutions representing the PFSs near both singular ends.

In the limit point (LP) region this procedure unequivocally specifies the eigenfunctions square integrable (by definition) with the weight function in the SLE under consideration. Our procedure also provides the prescription for constructing EOP sequence in the limit circle (LC) region, where the corresponding Liouville potential has only *continuously degenerate* bound energy states (CDBESs).

We found that the X_m-Jacobi DPSs of both series J1 and J2 contain X_m-Jacobi OPSs, which are interrelated via the reflection of their argument accompanied by the interchange of the Jacobi indexes. For this reason Gȯmez-Ullate et al. [15,16] focused solely on the properties of one of them (which happened to be the X_m-Jacobi OPS of series J2 in our terms), referring to the latter simply as the X_m-Jacobi OPS and dropping the X_m-Jacobi OPS of series J1 from any future consideration.

On other hand, our analysis revealed that the infinitely many finite EOP sequences of type $\underset{\sim}{\mathbf{a}}$ and limitedly many the finite EOP sequences of type $\underset{\sim}{\mathbf{b}}$ (constructed using the PFSs near the origin and at infinity respectively) belong to the X_m-Jacobi DPS of series J1 and accordingly J2, which makes it necessary to analyze both X-DPSs in parallel.

In particular, as discussed in subsection 4.2 below, the X_m-Jacobi DPS of series J2 contains the X_m-Jacobi OPS (in terms of [15,16]), the finite EOP sequence orthogonal on the negative interval (-∞,-1), and another finite EOP sequence orthogonal on the positive interval (1,∞). The latter EOP sequence of type $\tilde{\mathrm{b}}$ is composed the RD𝔍s of the R-Jacobi polynomials which represent the polynomial components of the eigenfunctions [10-12] of the rationally extended *h*-PT potentials [11,12].

Regretfully, both our works [8] and [24] overlooked the important modification in the definition of the 'X_m-Jacobi polynomials' by Yadav et al. in [11]. Namely the indexes α and β appearing in the expression for X_m-Jacobi polynomials' in [10] were defined independently of the degree of the seed Jacobi polynomials and thereby (as illuminated in subsection 4.2 below) made the mentioned expression fully consistent with our definition of the X_m-Jacobi DPS of series J2.

As for the X_m-Jacobi DPS of series J1, it contains the OPS ('X_m-Jacobi OPS of series J1 in our terms), the finite EOP sequence orthogonal on the negative interval (-∞,-1), and another finite EOP sequence (this time of type $\underset{\sim}{\mathrm{a}}$) orthogonal on the positive interval (1,∞). Obviously, the finite EOP sequence orthogonal on the negative interval (-∞,-1) can be obtained from the EOP sequence of type $\underset{\sim}{\mathbf{b}}$ by the reflection of the argument, followed by the interchange of the indexes; i.e., the two finite EOP sequences in question are interrelated in exactly the same way as the DPSs of series J1 and J2 to which they belong.

The third EOP sequence of type $\underset{\sim}{\mathbf{a}}'$ generated using the PFSs near origin (similarly to EOP sequence of type $\underset{\sim}{\mathbf{a}}$) constitutes the orthogonal subset of the X_m-Jacobi DPS of series W composed of the Wronskians of two Jacobi polynomials with common indexes. As it has been already pointed to by us in [8], the Wronskian transforms of the R-Jacobi polynomials have been already brought to light in the cited article [9] by Grandati. However, he did not realized that the constructed polynomial Wronskians are X-orthogonal. Indeed, the rational realization of the *h*-PT potential (using the variable *cosh* 2x) belongs to group A in Odake and Sasaki's classification scheme [22] of the rational translationally shape-invariant (RTSI) potentials (contrary to the other RTSI potentials discussed in [9] which all belong to Group B). As a result, any admissible RDT of the R-Jacobi polynomials results in a finite EOP sequence.

Ironically the most obvious rational extension of the h-PT potential using the TS with the polynomial component represented by the classical Jacobi polynomials and therefore necessarily nodeless on the interval (1,∞) has been never discussed in the literature (apart from our works). The very remarkable feature of the finite EOP sequences of type $\underset{\sim}{\mathbf{a}}$ discovered in [24] is that they can be arranged into the rectilinear polynomial matrix with a finite number of rows and an infinite number of columns composed of the X-Jacobi orthogonal polynomial system (X-Jacobi OPS) and RD$\mathfrak{T}$s of the R-Jacobi polynomials.

Finally, the EOP sequences of type $\tilde{\mathbf{d}}$ belong to the X_m-Jacobi DPS of series D. We introduced this label in [24] to stress that the DPS in question is composed of the so-called [48] 'polynomial determinants' (PDs). However, after realizing [8] that this X-DPS contains the X_m-Jacobi OPS of series J3 discovered by Grandati and Bérard [50] we switched to the term 'X_m-Jacobi DPS of series J3', i.e., the X_m-Jacobi OPSs of series j1, j2, and J3 constitute infinite orthogonal subsets of the X_m-Jacobi DPSs of series J1, J2, and J3 respectively.

To summarize, let us note that, compared with the PSLPs roughed out in [15,18,33], the DBCs at the ends of the quantization interval [-1,+1] cover only the rational Rudjak-Zakhariev transforms (RRZ$\mathfrak{T}$s) of the classical Jacobi polynomials with positive indexes as it has been already demonstrated in [27] for m=1. On the other hand, our technique is sufficient to find all the rational Liouville potentials solvable in terms of either infinite and finite X_m-Jacobi EOP sequences, and, in addition, allows one to construct the finite EOP sequences ignored in the cited papers.

One of the most important achievements of this paper (in addition to the systematic description and more precise summary of the earlier results spread between the three preprints [1,8,24]) is the representation of the X_m-Jacobi DPSs of series J1, J2, and J3 in terms (and therefore the corresponding X_m-Jacobi OPSs) via the 'pseudo-Wronskians' (p-Ws) of two Jacobi polynomials [51]. As outlined in Section 6, this fresh development opens a promising new direction in the theory of both infinite and finite EOP sequences composed of 'simple' p-Ws of m seed Jacobi polynomials with common indexes and a single either classical Jacobi or R-Jacobi polynomial.

2. Quantization of JRef CSLE on Infinite Interval [1,∞]

Let us start our analysis with the Jacobi-reference (JRef) CSLE

$$\left\{\frac{d^2}{d\eta^2}+I^0[\eta;\vec{\lambda}_o]+\underset{\sim}{\varepsilon}\,\rho[\eta]\right\}\Phi[\eta;\vec{\lambda}_o;\underset{\sim}{\varepsilon}]=0 \qquad (\eta>1) \qquad (1)$$

with the single pole density function

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 18 February 2025 doi:10.20944/preprints202502.1228.v1

$$\rho[\eta] := \frac{1}{\eta^2 - 1} > 0 \tag{2}$$

positive on the infinite interval [1,∞). The reference polynomial fraction (RefPF) is parameterized as follows:

$$I^o[\eta;\vec{\lambda}_o] \equiv \sum_{\aleph=\pm} \frac{1-\lambda^2_{o;\aleph}}{4(1-\aleph\eta)^2} + \frac{1-\lambda^2_{o;+} - \lambda^2_{o;-}}{4(1-\eta^2)} \tag{3}$$

$$= -\frac{1}{2(\eta^2-1)} \sum_{\aleph=\pm} \frac{1-\lambda^2_{o;\aleph}}{1-\aleph\eta} + \frac{1}{4(\eta^2-1)}, \tag{4}$$

where $\lambda_{o;\pm}$ are the exponent differences (ExpDiffs) for the poles at ±1 and the energy reference point is chosen via the requirement that the ExpDiff for the singular point at infinity vanishes at zero energy, i. e.,

$$\lim_{|\eta|\to\infty} \left(\eta^2 I^o[\eta;\vec{\lambda}_o]\right) = \tfrac{1}{4}. \tag{5}$$

In following [29], we underline ε by tilde to distinguish it from the spectral parameter $\varepsilon = -\underset{\sim}{\varepsilon}$ for the JRef CSLE defined on the conventional interval (-1,+1):

$$\left\{\frac{d^2}{d\eta^2} + I^o[\eta;\vec{\lambda}_o] + \varepsilon\rho[\eta]\right\}\Phi[\eta;\vec{\lambda}_o;\varepsilon] = 0 \qquad (-1<\eta<1), \tag{6}$$

with the density function (2) changed for

$$\rho[\eta] := \frac{1}{1-\eta^2}. \tag{7}$$

2.1. Liouville Transformation of JRef CSLE on Infinite Interval

The gauge transformation

$$\Psi[\eta;\vec{\lambda}_o;\underset{\sim}{\varepsilon}] = \rho^{1/4}[\eta]\Phi[\eta;\vec{\lambda}_o;\underset{\sim}{\varepsilon}]. \tag{8}$$

converts the JRef CSLE (1) into the 'algebraic' [48] Schrödinger equation

$$\left\{\frac{d}{d\eta}\sqrt{\eta^2-1}\,\frac{d}{d\eta} + \left(\underset{\sim}{\varepsilon} - V[\eta;\vec{\lambda}_o]\right)\sqrt{\eta^2-1}\right\}\Psi[\eta;\vec{\lambda}_o;\underset{\sim}{\varepsilon}] = 0 \tag{9}$$

with the Liouville potential

$$V[\eta;\vec{\lambda}_o] = -(\eta^2-1)\,I^o[\eta;\vec{\lambda}_o] - \tfrac{1}{2}\,Schw\{\sqrt{\eta^2-1}\}, \tag{10}$$

The so-called [45] 'Schwarzian'

$$Schw\{f[\eta]\} = f[\eta]\,\ddot{f}[\eta] - \tfrac{1}{2}\dot{f}^2[\eta] \tag{11}$$

(with the dot standing for the derivatives with respect to η) turns into the conventional Schwarzian derivative $\{\eta, r\}$ [52] if the change of variable $\eta(r)$ satisfies the first-order ODE

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 18 February 2025 doi:10.20944/preprints202502.1228.v1

$$\eta'(x) = f^{-1/2}[\eta(x)], \tag{12}$$

with prime standing for the derivative with respect to x). Note that the companion Liouville potential obtained by the Liouville transformation on the finite interval [-1,+1] can be obtained from (8) simply by replacing $\eta^2 - 1$ for its absolute value which changes sign of the first term in (10) while keeping unchanged the second. This remarkable feature of the JRef CSLE with the density function (7) covering both versions of the PT potential [4] forms the basis for the unified approach to the rational extensions of these potentials recently suggested in [30].

Substituting (4) and

$$Schw\{\sqrt{\eta^2 - 1}\} = -1/2 + \frac{3}{4(1-\eta)} + \frac{3}{4(1+\eta)} \tag{13}$$

into (10) gives

$$V[\eta;\vec{\lambda}_o] = \frac{\lambda_{o;+}^2 - 1/4}{2(\eta-1)} - \frac{\lambda_{o;-}^2 - 1/4}{2(\eta+1)}. \tag{14}$$

It is essential that the potential function (14) vanishes at infinity due to our choice of the energy reference point.

The change of variable

$$\eta(x) = cosh\ x \tag{15}$$

finally brings us to the *h*-PT potential in its conventional Pöschl -Teller form [4]

$$V_{h-PT}(r;\lambda_{o;+},\lambda_{o;-}] = \frac{\lambda_{o;+}^2 - 1/4}{4sinh^2 x/2} - \frac{1/4 - \lambda_{o;-}^2}{4cosh^2 x/2} \quad (0 < x < \infty), \tag{16}$$

where the energy-independent ExpDiffs $\lambda_{o;+}$ and $\lambda_{o;-}$ at the finite singular points are related to Odake and Sasaki's parameter g and h in [2] as follows

$$g = \lambda_{o;+} + 1/2, \qquad h = \lambda_{o;-} - 1/2. \tag{17}$$

It is worth noting that that the first of the listed parameters coincides with the larger characteristic exponent (ChExp) for the pole of the given radial Schrödinger equation at the origin, i.e., the potential is repulsive iff $g > 1$ ($\lambda_{o;+} > 1/2$). The constraint $g > 3/2$ ($\lambda_{o;+} > 1$) specifies the necessary and sufficient condition for the radial potential to have the discrete energy spectrum, whereas any solution of the radial Schrödinger equation is squarely normalizable for $1 < g < 3/2$ ($1/2 < \lambda_{o;+} < 1$
.

To be able to compare our results with those in [11,12], let us also consider the alternative parametrization of the *h*-PT potential introduced in the renown review article by Cooper et al. [53] and then adopted by Bagchi et al. [20] in their exhaustive analysis of the m=1 rational extension of the *h*-PT potential. Setting

$$2A + 1 := \lambda_{o;-} - \lambda_{o;+},\ 2B = \lambda_{o;-} + \lambda_{o;+}, \tag{18}$$

representing the sum of the PFs in the right-hand side of (13) as

$$V[\eta;\vec{\lambda}_o] = V_{A,B}[\eta] := \frac{A(A+1) + B^2 - (2A+1)B\eta}{\eta^2 - 1} \tag{19}$$

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 18 February 2025 doi:10.20944/preprints202502.1228.v1

and expressing the latter in terms of x ≡ r via (15) brings us to the potential function (1) in [11]. By definition,

$$B - A = \lambda_{o;+} + \tfrac{1}{2} = g > \tfrac{1}{2}, \; B + A = \lambda_{o;-} - \tfrac{1}{2} = h > -\tfrac{1}{2}, \tag{20}$$

i.e., $B > A + \tfrac{1}{2}$. The given potential is repulsive iff $\lambda_{o;+} > \tfrac{1}{2}$ i.e., $B > A + 1$ [20], with no limitation on either sign or a value of the parameter A. The additional requirement for the parameter A to be positive [20] only assures that the potential has at least one bound energy level.

2.2. Quartet of q-RSs

One can directly verify that the quasi-rational function

$$\phi_0[\eta;\vec{\lambda}] := (1+\eta)^{\tfrac{1}{2}(\lambda_- +1)}(\eta-1)^{\tfrac{1}{2}(\lambda_+ +1)}, \tag{21}$$

is the solution of the JRef CSLE (6) at the energy $\varepsilon = \varepsilon_0(\vec{\lambda})$, or alternatively the solution of the JRef CSLE (1) at energy $\underset{\sim}{\varepsilon} = -\varepsilon_0(\vec{\lambda})$,

$$\varepsilon_0(\vec{\lambda}) = \tfrac{1}{4}(\lambda_+ + \lambda_- + 1)^2. \tag{22}$$

Substituting the quasi-rational function

$$\phi_m[\eta;\vec{\lambda}] = \phi_0[\eta;\vec{\lambda}] P_m^{(\lambda_+,\lambda_-)}(\eta) \qquad (|\lambda_\pm| = \lambda_{o;\pm}), \tag{23}$$

into the JRef CSLE (6) and taking into account that the quasi-rational function (21) satisfies the first-order differential equation:

$$\dot{\phi}_0[\eta;\vec{\lambda}] = \frac{P_1^{(\lambda_+,\lambda_-)}(\eta)}{\eta^2 - 1}\phi_0[\eta;\vec{\lambda}], \tag{24}$$

with dot standing for the derivative with respect to η, we find that (23) is the solutions of the given equation at the energy

$$\varepsilon_m(\vec{\lambda}) := \tfrac{1}{4}(\lambda_+ + \lambda_- + 2m + 1)^2, \tag{25}$$

provided its polynomial component satisfies the Jacobi equation:

$$(\eta^2 - 1)\ddot{P}_m^{(\lambda_+,\lambda_-)}(\eta) + 2P_1^{(\lambda_+,\lambda_-)}(\eta)\dot{P}_m^{(\lambda_+,\lambda_-)}(\eta) +$$

$$\left[\varepsilon_0(\vec{\lambda}) - \varepsilon_m(\vec{\lambda})\right] P_m^{(\lambda_+,\lambda_-)}(\eta) = 0, \tag{26}$$

The vital point is that the coefficient function for the first derivative of the Jacobi polynomial is independent of the polynomial degree, in the sharp contrast with the general case of the JRef CSLE solvable by Jacobi polynomials with degree-dependent indexes [54,55]. This is the intrinsic feature of the translationally form-invariant (TFI) CSLEs of group A, and in particular, from all the RTSI potentials discussed in [9], the stated assertion holds only for the h-PT potential.

One can easily verify that the shift parameters λ in Odake-Sasaki's recipe [2] for constructing the rational SUSY partners of the *t*- and *h*-PT potentials are nothing but the indexes of the Jacobi polynomials forming the quasi-rational eigenfunctions of the mentioned potentials. Since the JRef PF (4) depends on the squares of the Jacobi indexes, it can be re-expressed in terms of the latter parameters, instead of the ExpDiffs $\lambda_{o;\pm}$. This duplicate parametrization of the *t*- and *h*-PT potentials by the Jacobi indexes represents the basis for the aforementioned unified approach [30] to the RD𝔗s of these potentials using the first-degree (m=1) seed polynomials.

Note that Gómez-Ullate et al. [51] also parametrized the *t*-PT potential by means of the mentioned Jacobi indexes, except that the zero-point energy was randomly shifted by a constant dependent on the index signs. Examination of the JRef PF (4) reveals that the parameters α and β in (37) in [51] are allowed to take only positive values other than 1. In the case of $\lambda_{0;+}=1$ (or $\lambda_{0;-}=1$) the corresponding second-order pole disappears and the resultant problem with only the simple pole at +1 (or respectively at -1) requires a special attention.) It will be shown in subsection 5.1 below that one can formulate the Dirichlet problem for any positive values of $\lambda_{0;\pm}\neq 1$, with no need to exclude the intervals $0<\lambda_{0;\pm}\leq 1/2$ (in contrast with the constraint imposed on the Jacobi indexes α and β in [51]).

From a more general perspective, the absolute values of the shift parameters λ coincide with the ExpDiffs for the persistent poles of the transformed rational CSLEs in the finite plane. The latter interpretation of the shift parameters introduced in [2] allowed the cited authors to extend their approach to the rational Darboux-Crum transformations (RDCTs) of the *t*- and *h*-PT potentials [22]. We shall come back to the discussion of this issue in Section 6.

Setting

$$\lim_{\eta\to\infty} \eta^{-m} P_m^{(\lambda_+,\lambda_-)}(\eta) = 2^{-m}\langle m+\lambda_+ +\lambda_- +1\rangle_m / m! \qquad (27)$$

(λ_-, λ_+, $\lambda_- +\lambda_+ + m \neq -k$ for any positive integer $k\leq m$) (see §4.22(3) in [49]; cf. (102) in [15] or (88) in [16]), we assure that the Jacobi polynomials in question has exactly m simple zeros $\eta_l(\vec{\lambda};m)$. It is crucial that the Jacobi indexes do not depend on the polynomial degree, in contrast with the general case [54,55]. This remarkable feature of the CSLE under consideration is the direct consequence of the fact that the density function (2) has only simple poles in the finite plane [25] and as a result the ExpDiffs for the CSLE poles at ±1 become energy-independent [1].

2.3. *Prime SLE on Infinite Interval [1,∞]*

Instead of the conventional prerequisite of the spectral theory requiring the eigenfunctions to be normalizable with the weight $\rho[\eta]$ we require that the boundary conditions of our choice unambiguously select the PFS near each singular endpoint. To achieve this goal, we first convert the given CSLE to its 'prime' form (*p*-SLE) selected by the requirement that the ChExps of the two Frobenius solutions near the given end differ only by their sign and as a result the DBC pinpoints the PFS.

The rational prime form of the JRef CSLE on the interval (1,∞) is obtained by requiring the leading coefficient to be the first-degree polynomial

$$\not{p}[\eta]=\eta-1 \quad \text{for} \quad \eta\in[1,\infty), \qquad (28)$$

which brings us to the *p*-SLE [8,29]

$$\left\{-\frac{d}{d\eta}(\eta-1)\frac{d}{d\eta}+\not{q}[\eta;\vec{\lambda}_o]+\underset{\sim}{\varepsilon}\not{w}[\eta;\vec{\lambda}_o]\right\}\Psi[\eta;\vec{\lambda}_o;\underset{\sim}{\varepsilon}]=0 \quad (\eta>1) \qquad (29)$$

with the zero-energy free term

$$\not{q}[\eta;\vec{\lambda}_o]=-(\eta-1)I^o[\eta;\vec{\lambda}_o]+\frac{1}{4(\eta-1)} \quad (\eta>1) \qquad (30)$$

and the positive weight

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 18 February 2025 doi:10.20944/preprints202502.1228.v1

$$w[\eta;\vec{\lambda}_o] = (1+\eta)^{-1} < \tfrac{1}{2} \quad (\eta > 1). \tag{31}$$

Taking into account that

$$\lim_{\eta\to 1}\left[\,(\eta-1)\not{q}[\eta;\vec{\lambda}_o]\right] = \tfrac{1}{4}\lambda^2_{o;+} > 0 \tag{32}$$

we confirm that the ChExps of the Frobenius solutions for the pole at +1 have the same absolute value $\tfrac{1}{2}\lambda_{o;+}$ while differing by their sign. The ExpDiff for the pole of the JRef CSLE (1) at infinity turned out to be energy-dependent. Combining (30) with (4), one can verify that

$$\lim_{\eta\to\infty}\left[\,\eta\not{q}[\eta;\vec{\lambda}_o]\right] = -\lim_{\eta\to\infty}\left[\,\eta^2 I^o[\eta;\vec{\lambda}_o]\right] + \tfrac{1}{4} = 0\,, \tag{33}$$

confirming that the ChExps of the Frobenius solutions for the pole at ∞ are real only at negative energies and have in this case the same non-zero absolute value $\tfrac{1}{2}\sqrt{-\varepsilon}$ while differing by their sign.

Combining (33) with the similar limit

$$\lim_{\eta\to 1+}\left[\,(\eta-1)\not{q}[\eta;\vec{\lambda}_o\right] = \lambda^2_{o;\pm} > 0 \tag{34}$$

for the pole at +1, we conclude that the PFSs of the *p*-SLE (29) near both singular endpoints are unambiguously determined by the Dirichlet boundary conditions (DBCs):

$$\lim_{\eta\to r} \underset{\sim}{\Psi}_r[\eta;\vec{\lambda}_o;\underset{\sim}{\varepsilon}] = 0 \quad (r = 1,\infty) \tag{35}$$

and the sought-for eigenfunctions simply turn into the solutions of the *p*-SLE

$$\left\{-\frac{d}{d\eta}(\eta-1)\frac{d}{d\eta} + \not{q}[\eta;\vec{\lambda}_o] + \underset{\sim}{\varepsilon}_j(\vec{\lambda}_o)w[\eta;\vec{\lambda}_o]\right\}\underset{\sim}{\psi}_j[\eta;\vec{\lambda}_o] = 0 \tag{36}$$

solved under the DBCs

$$\lim_{\eta\to r} \underset{\sim}{\psi}_j[\eta;\vec{\lambda}_o] = 0 \quad (r = 1,\infty)\,. \tag{37}$$

For $\lambda_{o;+} > 1$ the DBCs (37) unambiguously specify all the possible squarely integrable solutions of the *p*-SLE (31). On the other hand, any solution of this SLE is Squarely Integrable for $0 < \lambda_{o;+} < 1$ Which Implies That the CSLE in Question and

consequently the corresponding Liouville potential (16) has the CDBESs. The OBC at η=1 simply selects the squarely integrable solution representing the PFS near this singular end. It will be demonstrated in Section 5 that the RD$\mathfrak{S}$s of the latter solution play the important role in constructing the finite EOP sequences in the LC region ($0 < \lambda_{o;+} < 1$).

2.4. R-Jacobi Polynomials

One can directly verify that the q-RSs

$$\underset{\sim}{\psi}_{-+,j}[\eta;\vec{\lambda}_o] := \frac{1}{\sqrt{\eta-1}}\phi_{-+,j}[\eta;\vec{\lambda}_o] \qquad (1 \le \eta < \infty) \tag{38}$$

$$= (1+\eta)^{\frac{1}{2}(1-\lambda_{o;-})}(1-\eta)^{\frac{1}{2}\lambda_{o;+}} P_j^{(\lambda_{o;+},-\lambda_{o;-})}(\eta)\,, \tag{39}$$

associated with the eigenvalues

$$\underset{\sim}{\varepsilon}_j(\vec{\lambda}_o) = \varepsilon_{-+,j}(\vec{\lambda}_o) := -\tfrac{1}{4}(\lambda_{o;-} - \lambda_{o;+} - 2j - 1)^2, \tag{40}$$

satisfy the DBCs (37) for

$$0 \le j \le j_{max} \equiv \lfloor \lambda_{o;-} - \lambda_{o;+} - 1 \rfloor \tag{41}$$

and therefore represent the eigenfunctions of the p-SLE (36),

$$\underset{\sim}{\psi}_j[\eta;\vec{\lambda}_o] := \underset{\sim}{\psi}_{-+,j}[\eta > 1;\vec{\lambda}_o] \text{ for } 0 \le j \le j_{max}. \tag{42}$$

which are normalizable with the weight (31):

$$\int_1^{\infty} d\eta\, \underset{\sim}{\psi}_j^2[\eta;\vec{\lambda}_o]\, w[\eta;\vec{\lambda}_o] < \infty. \tag{43}$$

Since these are the eigensolutions of the Sturm-Liouville problem solved under the DBCs they must be also mutually orthogonal [56] with the weight (31):

$$\int_1^{\infty} d\eta\, \underset{\sim}{\psi}_j[\eta;\vec{\lambda}_o]\underset{\sim}{\psi}_{j'}[\eta;\vec{\lambda}_o]\, w[\eta;\vec{\lambda}_o] = 0$$
$$\text{for } 0 \le j' < j \le j_{max}, \tag{44}$$

which brings us to the conventional orthogonality relations for the R-Jacobi polynomials

$$\int_0^{\infty} d\underset{\rightarrow}{z}\, \underset{\rightarrow}{\omega}_{\alpha,\beta}[\underset{\rightarrow}{z}] J_j^{(\alpha,\beta)}(\underset{\rightarrow}{z}) J_{j'}^{(\alpha,\beta)}(\underset{\rightarrow}{z}) = 0 \quad (j \ne j') \tag{45}$$

with the weight

$$\underset{\rightarrow}{\omega}_{\alpha,\beta}[\underset{\rightarrow}{z}] := \underset{\rightarrow}{z}^{\alpha}\ (\underset{\rightarrow}{z}+1)^{-|\beta|} \quad \text{for} \quad \underset{\rightarrow}{z} \in [1,\infty) \tag{46}$$

under the constraint:

$$\alpha := \lambda_{o;+} \equiv B - A - \tfrac{1}{2} > 0,\ \ \beta := -\lambda_{o;-} \equiv -A - B - \tfrac{1}{2} < 0 \tag{47}$$

Here we adopted Askey's [57] definition of the R-Jacobi polynomials which, as proven by Chen and Srivastava [58], is equivalent to the elementary formula

$$J_n^{(\alpha,\beta)}(\underset{\rightarrow}{z}) := P_n^{(\alpha,\beta)}(2\underset{\rightarrow}{z}+1) \text{ for } \alpha > -1,\ \beta < 0, \tag{48}$$

with

$$\underset{\rightarrow}{z} := \tfrac{1}{2}(\eta - 1). \tag{49}$$

Note that we [24,25,,29] (see also [59]) changed the symbol R for J to avoid the confusion with the Romanovski/pseudo-Jacobi [6,7] polynomials (R-Routh polynomials in our terms [24,25]) denoted in the recent publications [21, 61-64] by the same letter 'R'. A certain disarray may come from the fact that Koepf and Masjed-Jamei [65,66] used the symbol J for the R-Routh polynomials which is inconsistent with the polynomial names in our classification scheme of the Romanovsi polynomials [10]. Note also that the symbol $\mathfrak{R}$, used for the R-Jacobi polynomials in the recently published paper [67], has been reserved by us [27] for the Routh polynomials [68]:

$$\Re_m^{(\alpha_R + i\alpha_I)}(x) := (-i)^m P_m^{(\alpha_R + i\alpha_I, \alpha_R - i\alpha_I)}(ix). \qquad (50)$$

While R-Routh polynomials form finite orthogonal subsequences of the Routh DPS [24], the R-Jacobi polynomials were discovered by Romanovski [30] and have no relation to Routh' work, contrary to the statement made in [69].

Comparing (48) with (2.8) in [70] shows that the Majed-Jamei's M-polynomials are related to the R-Jacobi polynomials (50) via the elementary formula

$$M_n^{(p,q)}(\underset{\to}{z}) = (-1)^n n! J_n^{(q,-p-q)}(\underset{\to}{z}) \quad \text{for } q > -1,\ p > 0. \qquad (51)$$

(It is worth to remind the abbreviation 'OPS' in [70] stands for the orthogonal polynomial set, but not for the infinite 'orthogonal polynomial system' – the abbreviation broadly used in the modern theory of the exceptional orthogonal polynomials [15,18, 19].)

Substituting q =α, p= –α–β into (2.14) in [70] gives

$$\int_0^\infty d\underset{\to}{z}\ \underset{\to}{\omega}_{\alpha,\beta}[\underset{\to}{z}] \left| M_n^{(-\alpha-\beta,\alpha)}(\underset{\to}{z}) \right|^2 = -\frac{n!\Gamma(-\alpha-\beta-n)\Gamma(\alpha+n+1)}{(\alpha+\beta+2n+1)\Gamma(-\beta-n)}. \qquad (52)$$

As anticipated from (52), the corresponding formula for the R-Jacobi polynomials (see (5) in [71], with θ, ϑ standing for α, β here) differs by the factor $(n!)^2$. Indeed, keeping in mind that

$$(-1)^n (\beta+1)_n \Gamma(-n-\beta) = \Gamma(-\beta), \qquad (53)$$

one can directly verify that the integral (56), differs exactly by the factor $(n!)^2$ from Askey's [57] original expression for the square integrals of the R-Jacobi polynomials.

It is worth mentioning that our derivation does not cover the orthogonality region of the R-Jacobi polynomials for α varying within the nonpositive interval (-1,0]. This deficiency of our approach dealing solely with eigenfunctions of the *p*-SLEs may result in some limitations on the actual range of the indexes of the RRZ𝔍s of the R-Jacobi polynomials discussed below.

2.5. Quasi-Rational PFSs Near the Poles at +1 and Infinity

Let us specify the q-RSs (23) in the slightly way

$$\underset{\sim}{\psi}_{\vec{\sigma},m}[\eta; \vec{\lambda}_o] := \underset{\sim}{\psi}_m[\eta; \vec{\lambda}] \qquad (54)$$

with

$$\sigma_\pm := sgn\, \lambda_\pm \qquad (55)$$

Our next step is to determine all the q-RSs vanishing at one of the endpoints of the infinite interval [+1, +∞) and then select the subsets of the collected PFSs below the lowest eigenvalue. To explicitly reveal the behavior of the Jacobi-seed (JS) q-RSs (54) near the singular endpoints in question, we [1,24,59], label them as indicated in Table 1 below, with $\sigma_\pm$ and σ_∞ specifying either the decay (+) or growth (-) of the given JS at infinity, i.e., by definition

$$\underset{\sim}{\varepsilon}_{\mathbf{t}_{\vec{\sigma},m},m}(\vec{\lambda}_o) \equiv -\varepsilon_{\vec{\sigma},m}(\vec{\lambda}_o), \qquad (56)$$

Table 1. Classification of JS solutions on the infinite interval [1, ∞) based on their asymptotic behavior near the endpoints.

$\underset{\sim}{\mathbf{t}}_{\vec{\sigma},m}$	$\sigma_-\ \sigma_+\ \sigma_\infty$	m
$\underset{\sim}{\mathfrak{a}}$	+ + –	$0 \le m < \infty$

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 18 February 2025 doi:10.20944/preprints202502.1228.v1

$\underset{\sim}{\mathsf{a}}'$	− + −	$m \geq n_{\underset{\sim}{c}} = j_{max} + 1$
$\underset{\sim}{\mathsf{b}}$	− − +	$0 \leq m < \frac{1}{2}(\lambda_{o;-} + \lambda_{o;+} - 1)$
$\underset{\sim}{\mathsf{b}}'$	− + +	$0 \leq m < \frac{1}{2}(\lambda_{o;+} - \lambda_{o;-} - 1)$
$\underset{\sim}{\mathsf{c}}$	− + +	$0 \leq m \leq j_{max}$
$\underset{\sim}{\mathsf{d}}$	+ − −	$\frac{1}{2}(\lambda_{o;+} - \lambda_{o;-} - 1) \leq m < \infty$
$\underset{\sim}{\mathsf{d}}'$	− − −	$m > \frac{1}{2}(\lambda_{o;-} + \lambda_{o;+} - 1)$

where

$$\varepsilon_{\vec{\sigma},m}(\vec{\lambda}_o) := (\sigma_-\lambda_{o;-} + \sigma_+\lambda_{o;+} + 2m + 1)^2. \qquad (57)$$

In following our olden study [43] on the Darboux transforms ($D\mathfrak{T}s$) of radial potentials, we use the letters **a** and **b** to specify the PFS near the singular endpoints 1 and ∞ (cases I and II in Quesne's [72] commonly used classification scheme of q-RSs according to their behavior near the endpoints). We use the letters **c** and **d** [43] to identify the $n_{\underset{\sim}{c}}$ eigenfunctions and respectively all the q-RSs (54) not vanishing either at +1 or infinity (case III in Quesne's classification scheme). We underline the corresponding letter by tilde to indicate that the classification of the JS solutions is done on the infinite interval (1, ∞). We mark the letter by prime if the polynomial components of the given sequence of the q-RSs do not include a constant. (Note that the 'secondary' sequences of such a type do not exist for the potentials with infinitely many discrete energy levels which were the focal point of Quesne's analysis [72].)

Note that the PFSs of the series $\underset{\sim}{\mathsf{b}}'$ may exist only if the SLE does not have the discrete energy spectrum. We thus need to consider the three sequences of the quasi-rational PFSs: two *primary* (starting from m=0) sequences $\underset{\sim}{\mathsf{a}}$ and $\underset{\sim}{\mathsf{b}}$ as well as the infinite secondary sequence $\underset{\sim}{\mathsf{a}}'$ starting from $m = n_{\underset{\sim}{c}}$.

The primary sequence $\underset{\sim}{\mathsf{a}}$ is formed by classical Jacobi polynomials and consequently may not have zeros between 1 and ∞. As expected, all the PFSs of this type lie at the energies

$$\varepsilon_{\underset{\sim}{\mathsf{a}},m}(\vec{\lambda}_o) \equiv -\varepsilon_{++,m}(\vec{\lambda}_o) \qquad (58)$$

below the lowest eigenvalue

$$\varepsilon_{\underset{\sim}{\mathsf{c}},0}(\vec{\lambda}_o) \equiv -\varepsilon_{+-,0}(\vec{\lambda}_o). \qquad (59)$$

The PFSs from the primary sequence $\underset{\sim}{\mathsf{b}}$ at the energies

$$\varepsilon_{\underset{\sim}{\mathsf{b}},m}(\vec{\lambda}_o) \equiv -\varepsilon_{--,m}(\vec{\lambda}_o) \qquad (60)$$

for

$$0 \leq m < \tfrac{1}{2}(\lambda_{o;-} + \lambda_{o;+} - 1) \qquad (61)$$

do not have real zeros larger than 1 iff

$$\varepsilon_{\underset{\sim}{\mathsf{b}},m}(\vec{\lambda}_o) - \varepsilon_{\underset{\sim}{\mathsf{c}},0}(\vec{\lambda}_o) = -4(\lambda_{o;-} - m - 1)(\lambda_{o;+} - m) < 0, \qquad (62)$$

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 18 February 2025 doi:10.20944/preprints202502.1228.v1

i.e., iff

$$0 \le \mathsf{m} < \lambda_{o;+} < \lambda_{o;-} - 1 \quad . \tag{63}$$

Similarly the PFSs from the secondary sequence $\underset{\sim}{\mathbf{a}}'$ at the energies

$$\varepsilon_{\underset{\sim}{\mathbf{a}}',\mathrm{m}}(\vec{\lambda}_o) = -\varepsilon_{+-,\mathrm{m}}(\vec{\lambda}_o) \quad \text{for } \mathrm{m} \ge \mathrm{n}_{\underset{\sim}{\mathbf{c}}} \tag{64}$$

do not have real zeros larger than 1 iff

$$\varepsilon_{\underset{\sim}{\mathbf{a}}',\mathsf{m}}(\vec{\lambda}_o) - \varepsilon_{\underset{\sim}{\mathbf{c}},0}(\vec{\lambda}_o) = -4\mathsf{m}(\lambda_{o;+} - \lambda_{o;-} + \mathsf{m} + 1) < 0 \tag{65}$$

or, in other words, iff

$$\mathsf{m} > \lambda_{o;-} - \lambda_{o;+} - 1 \quad . \tag{66}$$

Before concluding this section, let us mention that the change of variable (15) turns the quasi-rational functions (21) and (22) into the eigenfunctions of the Schrödinger equation with the h-PT potential , as prescribed by (33) and (34) in [9] with $\alpha = \lambda_{o;+} = \lambda_+$, and $\beta = \lambda_{o;-} = -\lambda_-$. Note that the second parameter represents the absolute value of the Jacobi index β defined via (51) above. Also, the h-PT potential (31) in [9] depends on the squares of both Jacobi indexes and therefore one only need to consider the positive values of α, keeping in mind that the potential is repulsive only if $\alpha > 1/2$. Moreover, all the solutions of the Schrödinger equation with the h-PT potential (31) in [9] are squarely normalizable if $1/2 < \alpha < 1$, i.e., the potential has the discrete energy spectrum only for $\alpha > 1$, i.e., iff the ExpDiff $\lambda_{o;+} \equiv \alpha$ lies within limit point (LP) range.

3. Use of RZTs for Constructing X$_m$-Jacobi DPSs

We say that the RZT is 'rational' (RRZT) if it uses the quasi-rational TF

$$\phi_m[\eta;\vec{\lambda}] = \phi_0[\eta;\vec{\lambda}]\Pi_m[\eta;\vec{\eta}^{(m)}(\vec{\lambda})] , \tag{67}$$

where $\Pi_m[\eta;\vec{\eta}]$ stands for the monomial product

$$\Pi_m[\eta;\vec{\eta}] := \prod_{\ell=1}^{m} (\eta - \eta_\ell) , \tag{68}$$

with m simple zeros η_ℓ, or, in other words, iff the TF has the rational logarithmic derivative:

$$ld\,\phi_m[\eta;\vec{\lambda}] := ld\,\phi_0[\eta;\vec{\lambda}] + \sum_{\ell=1}^{m} \frac{1}{\eta - \eta_\ell^{(m)}(\vec{\lambda})} . \tag{69}$$

By definition of the RZT (see Appendix A for details), the quasi-rational function

$$^*\phi_m[\eta;\vec{\lambda}] = \frac{1}{\sqrt{\rho[\eta]}\;\phi_m[\eta;\vec{\lambda}]} \tag{70}$$

is the solution of the transformed CSLE

$$\left\{\frac{d^2}{d\eta^2} + I^o[\eta;\vec{\lambda}\,|\,m] + \varepsilon_m(\vec{\lambda})\rho[\eta]\right\} {}^*\phi_m[\eta;\vec{\lambda}] = 0. \tag{71}$$

at the energy (25), with the density function defined via (7) for both intervals (-1,+1) and (1,∞). Each of the nodeless PFSs below the lowest eigenvalue can be used as the TF to generate the *reg*-RRZ$\mathfrak{T}$ of the CSLE (1).

Let us now take advantage of the fact that JRef CSL (1) is TFI [73], namely, that quasi-rational function

$$^*\phi_0[\eta;\vec{\lambda}] := \rho^{-1/2}[\xi] / \phi_0[\xi;\vec{\lambda}] \tag{72}$$

is the solution of the JRef CSLE (1) with $\lambda_{o;\mp}$ replaced for $|\,|\lambda_{\mp} + 1|$, namely,

 doi:10.20944/preprints202502.1228.v1

$$ {}^*\phi_0[\eta;\vec{\lambda}] = \phi_0[\eta;-\vec{\lambda}-\vec{1}] \tag{73}$$

Let us also point to another remarkable feature of the JS q-RSs (23) – the Jacobi polynomials in the given sequence are multiplied by the same quasi-rational function and moreover the Jacobi indexes are independent of polynomial degrees [1]. This is the direct consequence of the fact that this CSLE belongs to group A, which is also true for the corresponding Liouville potentials [22].

Let us re-write both RCSLE (71) and

$$\left\{\frac{d^2}{d\eta^2} + I^o[\eta;\vec{\lambda}_o] + \varepsilon_0(\vec{\lambda})\rho[\eta]\right\}\phi_0[\eta;\vec{\lambda}] = 0 \tag{74}$$

in the Riccati form:

$$I^o[\eta;\vec{\lambda}\,|\,m] = -\dot{ld}\ {}^*\phi_m[\eta;\vec{\lambda}] - ld^2\ {}^*\phi_m[\eta;\vec{\lambda}] + \varepsilon_m(\vec{\lambda})\ \rho[\eta], \tag{75}$$

and

$$I^o[\eta;\vec{\lambda}] = -\dot{ld}\ \phi_0[\eta;\vec{\lambda}] + ld^2\phi_0[\eta;\vec{\lambda}] + \varepsilon_0(\vec{\lambda})\ \rho[\eta], \tag{76}$$

respectively, where the symbolic expression $ld\, f[\eta]$ denotes the logarithmic derivative of the function $f[\eta]$. If the density function is identically equal to 1 then the derived expression turns into the standard supersymmetric representation of the quantum mechanical potential in terms of the superpotential represented by the logarithmic derivative of the TF $\phi_m[\eta;\vec{\lambda}]$. In Section 3 we will use a similar representation for the RefPFs of the RRZ𝔍s of the JRef CSLE (1) using the quasi-rational TFs (23).

Substituting the q-RS

$$ {}^*\phi_m[\eta;\vec{\lambda}] = \phi_0[\eta;-\vec{\lambda}-\vec{1}]\,/\,P_m^{(\lambda_+,\,\lambda_-)}(\eta) \tag{77}$$

into (75), coupled with (76), one finds

$$\begin{aligned} I^o[\eta;\vec{\lambda}\,|\,m] &= I^o[\eta;\vec{\lambda}+\vec{1}] + 2\hat{Q}[\eta;\vec{\eta}^{(m)}(\vec{\lambda})] + \\ &2ld\ P_m^{(\lambda_+,\,\lambda_-)}(\eta)\,ld\ \phi_0[\eta;-\vec{\lambda}-\vec{1}] + \left[\varepsilon_m(\vec{\lambda}) - \varepsilon_0(-\vec{\lambda}-\vec{1})\right]/(\eta^2-1), \end{aligned} \tag{78}$$

where [1,44]

$$\hat{Q}[\eta;\vec{\eta}] := -\tfrac{1}{2}\Pi_m[\eta;\vec{\eta}]\frac{d^2}{d\eta^2}\Pi_m^{-1}[\eta;\vec{\eta}] = \tag{79}$$

$$= \tfrac{1}{2}\frac{\ddot{\Pi}_m[\eta;\vec{\eta}]}{\Pi_m[\eta;\vec{\eta}]} - \frac{\dot{\Pi}_m^2[\eta;\vec{\eta}]}{\Pi_m^2[\eta;\vec{\eta}]} \tag{80}$$

It is worth mentioning that the derived expression (78) is valid on the both intervals (1,∞) and (-1,+1). It can be also trivially extended to two other CSLEs of group A with the Liouville potentials represented by the isotonic oscillator and by Morse potential (assuming that the Schrödinger equation in the latter case is converted to the Bessel-reference CSLE [26].

Making use of (25) and (76) , coupled with

$$ld\ \phi_0[\eta;-\vec{\lambda}-\vec{1}] = -\frac{P_1^{(\lambda_+-1,\lambda_--1)}(\eta)}{\eta^2-1}, \tag{81}$$

we can re-write (78) as

$$\begin{aligned} I^o[\eta;\vec{\lambda}\,|\,m] &= I^o[\eta;\vec{\lambda}+\vec{1}] + 2\hat{Q}[\eta;\vec{\eta}^{(m)}(\vec{\lambda})] - \\ &\frac{2P_1^{(\lambda_+-1,\lambda_--1)}(\eta)\,ld\ P_m^{(\lambda_+,\,\lambda_-)}(\eta) - m(\lambda_+ + \lambda_- + m + 1)}{\eta^2-1}. \end{aligned} \tag{82}$$

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 18 February 2025 doi:10.20944/preprints202502.1228.v1

Taking into account that

$$P_1^{(\lambda_+-1,\lambda_--1)}(\eta) = P_1^{(\lambda_+,\lambda_-)}(\eta) - \eta \,, \tag{83}$$

and

$$2[P_1^{(\lambda_+,\lambda_-)}(\eta) - \eta]\dot{P}_1^{(\lambda_+,\lambda_-)}(\eta) - (\lambda_+ + \lambda_- + 2)P_1^{(\lambda_+,\lambda_-)}(\eta)$$
$$= -\eta(\lambda_+ + \lambda_- + 2)\,, \tag{84}$$

along with

$$P_1^{(\lambda_+,\lambda_-)}(\eta) = \tfrac{1}{2}(\lambda_- + \lambda_+ + 2)[\eta - \eta_1(\vec{\lambda})] \,, \tag{85}$$

where

$$\eta_1(\vec{\lambda}) = \frac{\lambda_- - \lambda_+}{\lambda_+ + \lambda_- + 2} \,, \tag{86}$$

one can then directly verify that the derived expression turns into (27) in [29] for m=1:

$$I^o[\eta;\vec{\lambda}\,|\,1] = I^o[\eta;\vec{\lambda}+\vec{1}] - \frac{2}{[\eta-\eta_1(\vec{\lambda})]^2} + \frac{2\eta}{(\eta^2-1)[\eta-\eta_1(\vec{\lambda})]} \,. \tag{87}$$

The quartet of the Liouville potentials for the RCSLE (71) defined in the four quadrant of the vector parameter $\vec{\lambda}$ can be thus represented as follows:

$$V_{\vec{\sigma},m}[\eta;{}^*\vec{\lambda}_o] := V[\eta;\vec{\lambda}\,|\,m]$$
$$= V[\eta;{}^*\vec{\lambda}_o] + |\eta^2-1|\left\{I^o[\eta;\vec{\lambda}] - I^o[\eta;\vec{\lambda}\,|\,m]\right\} \tag{88}$$
$$= V[\eta;{}^*\vec{\lambda}_o] - 2|\eta^2-1|\,\widehat{Q}[\eta;\overline{\eta}^{(m)}(\vec{\lambda})] + \tag{89}$$
$$sgn(\eta-1)[2P_1^{(\lambda_+-1,\lambda_--1)}(\eta)\,ld\,P_m^{(\lambda_+,\lambda_-)}(\eta) - m(\lambda_+ + \lambda_- + m + 1)] \,,$$

with

$$\lambda_\mp \equiv \sigma_\mp \lambda_{o;\mp} \,, \quad {}^*\lambda_{o;\mp} := |\sigma_\mp \lambda_{o;\mp} + 1| \,. \tag{90}$$

For future references, we made the above expression to be applicable to the Liouville transformations on both intervals (1,∞) and (-1,+1), which constitutes the essence of the unified approach put forward in [26] for m=1.

Taking into account that

$$2 \lim_{\eta\to\infty} \eta^2 \widehat{Q}[\eta;\overline{\eta}] = m(m+1) \tag{91}$$

and

$$2 \lim_{\eta\to\infty} \left\{P_1^{(\lambda_+-1,\lambda_--1)}(\eta)\,ld\,P_m^{(\lambda_+,\lambda_-)}(\eta)\right\} = m(\lambda_+ + \lambda_-) \,, \tag{92}$$

0ne can verify that each of the potentials (87) vanishes at infinity, as expected.

Making use of the definition of the PF (79) for m=1:

$$\widehat{Q}[\eta;\eta_1(\vec{\lambda})] = -1/[\eta-\eta_1(\vec{\lambda})]^2 \,, \tag{93}$$

along with (83)-(85) , we can re-write the Liouville potential for m=1 as

$$V_{\vec{\sigma},1}[\eta;{}^*\vec{\lambda}_o] = V[\eta;{}^*\vec{\lambda}_o] + sgn(\eta^2-1)\left\{\frac{2(\eta^2-1)}{[\eta-\eta_1(\vec{\lambda})]^2} - \frac{2\eta}{\eta-\eta_1(\vec{\lambda})}\right\} \,. \tag{94}$$

If we set

$$2{}^*A + 1 := {}^*\lambda_{o;-} - {}^*\lambda_{o;+}, \quad 2{}^*B := {}^*\lambda_{o;-} + {}^*\lambda_{o;+} \,, \tag{95}$$

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 18 February 2025 doi:10.20944/preprints202502.1228.v1

by analogy with (20), and take into account

$$*\vec{\lambda}_o = \mp\vec{\lambda} \mp \vec{1} = \vec{\lambda}_o \mp \vec{1} \quad \text{for} \quad \vec{\sigma} = \mp\mp \quad \text{and} \qquad \lambda_{o;-}, \lambda_{o;+} > 1, \tag{96}$$

we find that

$$2*A+1 := \mp(\lambda_- - \lambda_+), \; 2*B := \mp(\lambda_- + \lambda_+ + 2) \tag{97}$$

for $\vec{\sigma} = \mp\mp$ $(\lambda_{o;-}, \lambda_{o;+} > 1$ if $\vec{\sigma} = --)$

and therefore

$$\eta_1(\vec{\lambda}) = \frac{2*A+1}{2*B} \qquad \text{for} \;\; \vec{\sigma} = \mp\mp \;\; (\lambda_{o;-}, \lambda_{o;+} > 1 \text{ if } \vec{\sigma} = --). \tag{98}$$

Substituting (98) into (94) for $\vec{\sigma} = \mp\mp$ then gives

$$V_{\mp\mp,1}[\eta; *\vec{\lambda}_o] = V_{*A,*B}[\eta \,|\, \mp\mp, 1] := V_{*A,*B}[\eta] + \tag{99}$$

$$sgn(\eta^2 - 1)\left\{\frac{2(2*A+1)}{2*B\eta - 2*A - 1} - \frac{4*B^2 - (2*A+1)^2}{(2*B\eta - 2*A - 1)^2}\right\}.$$

The change of variable (15) on the interval $(1,\infty)$ then turns (99), with $\vec{\sigma} = --$, into the potential function (9) in [20], while the change of variable $\eta = \mathit{sin}\ x$ converts (99), with $\vec{\sigma} = ++$, into (3.5) in [72], with A and B replaced in both cases for *A and *B respectively.

Let us draw reader's attention to the fact that the two generally distinct branches of the potential (87) with $\vec{\sigma} = \mp\mp$ collapse into the same Bagchi-Quesne-Roychoudhury (BQR) potential for m=1, as expected from the rigorous analysis of the latter case in [29]. The similar collapse takes place for the two other branches of this potential ($\vec{\sigma} = \mp\pm$) but the resultant potential does not have discrete energy spectrum [29] and therefore cannot be linked to any EOP sequence.

Coming back to the general case $m \geq 1$, let us re-write the potential (87) for $\vec{\sigma} = \mp\mp$ as

$$V_{\mp\mp,m}[\eta; *\vec{\lambda}_o] = V_{*A,*B}[\eta \,|\, \mp\mp, m] :=$$

$$V_{*A,*B}[\eta] - \tfrac{1}{4}(\mp 2*B + m + 1)\,|\eta^2 - 1| \times \tag{100}$$

$$\left\{(\mp 2*B + m)\frac{P_{m-2}^{(\lambda_+ + 2,\, \lambda_- + 2)}(\eta)}{P_m^{(\lambda_+,\, \lambda_-)}(\eta)} - (\mp 2*B + m + 2)\left[\frac{P_{m-1}^{(\lambda_+ + 1,\, \lambda_- + 1)}(\eta)}{P_m^{(\lambda_+,\, \lambda_-)}(\eta)}\right]^2\right\} +$$

$$(\mp 2*B + m + 1)\,sgn(\eta - 1)\left[\mp[(*B \pm 1)\eta - *A - \tfrac{1}{2}]\frac{P_{m-1}^{(\lambda_+ + 1,\, \lambda_- + 1)}(\eta)}{P_m^{(\lambda_+,\, \lambda_-)}(\eta)} - m\right].$$

While our expression (100) for the BQR potential (m=1) fully agrees with (15) in [11], we detected some discrepancies between (100), with $\vec{\sigma} = --$, and the corresponding expression for this potential in [11,12].

First, both the potential (19) in [11] and the following expression (37) for the rationally-extended *t*-PT (Scarf I) potential (after being converted by the change of variable $\eta = \mathit{sin}\ x$ ($|\eta| < 1$) to its rational form (100) above) lack the term associated with the second derivative of the Jacobi polynomial in the right-hand side of (100). Since this derivative vanishes for the first-degree polynomial, one cannot detect the missed term simply by setting m=1 in the general formula. The mentioned term was also missed in (81) in [12] or in the preceding expression (64) for the rationally-extended *t*-PT (Scarf I) potential.

Disregarding the missed term, the rest of both cited expressions in [12] match (100) for $\vec{\sigma} = --$, if we set

$$\lambda_+ = -\alpha - 1, \;\; \lambda_- = \beta - 1. \tag{101}$$

or, taking into account (97),

$$\alpha = {}^*B - A - \tfrac{1}{2},\ \beta = -A - {}^*B - \tfrac{1}{2}, \tag{102}$$

in agreement with the definition of these indexes in [12]. It has been proven above that the potential (87) vanishes at infinity and therefore the function (81) in [12] does not, keeping in mind that the omitted term tends to -m(m-1) as $\eta \to \infty$. (The reason for introducing the second pair of the potentials (82) and (83) following (81) in [12] is unclear to me.)

As already pointed to in [29], the parameter swap $2B \leftrightarrow 2A+1$ does not result in the new potential. The potentials (19) in [11] and (81) in [12} differ only by notation, with the constraint $2B > 2A+1$ reversed for $2B < 2A+1$. This also true for he two potentials (12) and (15) listed in [11] for m =1.

4. Pseudo-Wronskian Representation of X_m-Jacobi DPSs

Based on Rudyak and Zakhariev's [45] generic formula for the solutions of the transformed CSLE the general solution of the RCSLE

$$\left\{\frac{d^2}{d\eta^2} + I^o[\eta;\vec{\lambda}\,|\,m] + \underset{\sim}{\varepsilon}\,\rho[\eta]\right\}\Phi[\eta;\vec{\lambda};\underset{\sim}{\varepsilon}\,|\,m] = 0 \tag{103}$$

can be represented as

$$\Phi[\eta;\vec{\lambda};\underset{\sim}{\varepsilon}\,|\,m] \equiv \Phi[\eta;\vec{\lambda}_o;\underset{\sim}{\varepsilon}\,|\,\vec{\boldsymbol{\sigma}},m] := \frac{W\{\phi_m[\eta;\vec{\lambda}],\Phi[\eta;\vec{\lambda};\underset{\sim}{\varepsilon}]\}}{\rho^{1/2}[\eta]\phi_m[\eta;\vec{\lambda}]} \tag{104}$$

$$= \rho^{-1/2}[\eta]\dot{\Phi}[\eta;\vec{\lambda};\underset{\sim}{\varepsilon}] - \rho^{-1/2}[\eta]\mathit{ld}\phi_m[\eta;\vec{\lambda}]\Phi[\eta;\vec{\lambda};\underset{\sim}{\varepsilon}]. \tag{105}$$

The gauge transformation

$$\Psi[\eta;\vec{\lambda};\underset{\sim}{\varepsilon}\,|\,m] = \rho^{1/4}[\eta]\Phi[\eta;\vec{\lambda};\underset{\sim}{\varepsilon}\,|\,m] \tag{106}$$

converts the RCSLE (103) into the algebraic Schrödinger equation

$$\left\{\frac{d}{d\eta}\sqrt{\eta^2-1}\frac{d}{d\eta} + \left(\underset{\sim}{\varepsilon} - V[\eta;\vec{\lambda}\,|\,m]\right)\sqrt{\eta^2-1}\right\}\Psi[\eta;\vec{\lambda};\underset{\sim}{\varepsilon}\,|\,m] = 0 \tag{107}$$

Substituting (106) into (104) then gives

$$\Psi[\eta;\vec{\lambda};\underset{\sim}{\varepsilon}\,|\,m] \equiv \Psi[\eta;\vec{\lambda}_o;\underset{\sim}{\varepsilon}\,|\,\vec{\boldsymbol{\sigma}},m] := \frac{W\{\psi_m[\eta;\vec{\lambda}],\Psi[\eta;\vec{\lambda};\underset{\sim}{\varepsilon}]\}}{\rho^{1/2}[\eta]\psi_m[\eta;\vec{\lambda}]} \tag{108}$$

$$= \rho^{-1/2}[\eta]\dot{\Psi}[\eta;\vec{\lambda};\underset{\sim}{\varepsilon}] + w_m[\eta;\vec{\lambda}]\Psi[\eta;\vec{\lambda};\underset{\sim}{\varepsilon}], \tag{109}$$

where

$$w_m[\eta;\vec{\lambda}] := -\rho^{-1/2}[\eta]\mathit{ld}\psi_m[\eta;\vec{\lambda}]. \tag{110}$$

Note that the latter function is holomorphic if the square-root of the density function in the JRef CSLE (1) is a PF. As a result, the RS function becomes the holomorphic 'prepotential' in Ho's terms [74-78]. In particular [76], the Rosen-Morse [79] potential represents the simplest case when $\rho^{-1/2}[\eta]$ is the first-degree polynomial [59]. The latter requirement also holds if the numerator of the PF $\rho[\eta]$ has a double zero [80], with the Manning-Rosen [81] potential ('Eckart' potential in I76[) as its limiting case).

The representation of the general solution of the algebraic Schrödinger equation in the form (109) provides the accurate mathematical basis for the conventional SUSY theory of rationally extended potentials utilized in [12,20,30,72]. It should be noticed in this connection that the renowned SUSY rules [CKS] for the changes in energy spectra due to DTs of the Schrödinger equation were originally

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 18 February 2025 doi:10.20944/preprints202502.1228.v1

formulated [38,39,82.83] for potentials with exponential tails at ±∞. The latter restriction forced the ODE under consideration to have the LP singularities at both quantization ends. The formulated rules were then extended by Sukumar [84] to radial potentials without properly treating the LC range of the centrifugal barrier. Since then, the SUSY rules for the radial Schrödinger equation were duplicated without the proper examination of this non-trivial problem, with Gangopadhyaya et al.'s paper [85] and Chapter 12 in [86] as the only known-to-us exceptions.

If the ExpDiff for the pole at the origin lies within the LC range any solution is square integrable and the *reg*-RRZT with the TF of type $\vec{\sigma} = --$ does not results in the isospectral problem if the ExpDiff for the pole of the JRef CSLE (1) lies between 1 and 2. On the other hand, as discussed in more detail in subsection 5.3, we can still construct the corresponding EOP sequence using the polynomial components of the quasi-rational eigenfunctions of the prime SLE solved under the DBCs.

4.1.⁄ . of JS Solutions

Let us consider in parallel the four RD𝔖s of the q-RS

$$\phi_j[\eta;\vec{\lambda}'] \equiv \phi_j[\eta;\vec{\sigma}' \times \vec{\lambda}_o] \quad , \tag{111}$$

using the TFs

$$\phi_m[\eta;\vec{\lambda}] \equiv \phi_m[\eta;\vec{\sigma} \times \vec{\lambda}_o], \tag{112}$$

where $\vec{\sigma}$ and $\vec{\sigma}'$ specify the quadrants of the vectors $\vec{\lambda}$ and $\vec{\lambda}'$ respectively. By definition,

$$\lambda'_{\mp} := \sigma_{\mp}\lambda_{\mp}, \tag{113}$$

where $\sigma_{\mp}$ is equal to either + or - and

$$|\lambda'_{\pm}| = |\lambda_{\pm}| = \lambda_{o;\pm}. \tag{114}$$

Let us prove that the corresponding solutions (104) of the RCSLE (103) at the common energy

$$\underset{\sim}{\varepsilon} = -\varepsilon_j(\vec{\lambda}') \tag{115}$$

have the quasi-rational form.

Theorem 1. *RCSLE (103) has four infinite sequences of q-RSs, with the polynomial components represented by polynomial Wronskians for one of these sequences and by* p-Ws *of two Jacobi polynomials for three others.*

Proof: In the simplest case $\vec{\lambda} = \vec{\lambda}'$ ($\vec{\sigma} = ++$) the Wronskian of the two JS solutions takes form

$$W\{\phi_m[\eta;\vec{\lambda}], \phi_j[\eta;\vec{\lambda}]\} = \phi_0^2[\eta;\vec{\lambda}] W^{(\lambda_+,\lambda_-)}_{m+j-1}[\eta \mid m, j] \quad , \tag{116}$$

where the polynomial of degree m+ j -1 in the right-hand side stands for the Wronskian of two Jacobi polynomials with the same pair of the indexes:

$$W^{(\lambda_+,\lambda_-)}_{m+j-1}[\eta \mid m, j] := W\{P^{(\lambda_+,\lambda_-)}_m(\eta), P^{(\lambda'_+,\lambda'_-)}_j(\eta)\}, \tag{117}$$

Taking into account that [53}

$$|1-\aleph\eta|^{1/2(\sigma_\aleph 1-1)\lambda'_\aleph} \frac{d}{d\eta}\Big[\prod_{\aleph=\mp} |1-\aleph\eta|^{1/2(\lambda'_\aleph - \sigma_\aleph \lambda'_\aleph)} P^{(\lambda'_+,\lambda'_-)}_j(\eta)\Big] =$$

$$d_{\vec{\sigma},m}(\vec{\lambda}) \prod_{\aleph=\mp} (\eta-\aleph 1)^{1/2(\sigma_\aleph 1-1)} P^{(\lambda'_+ + \sigma_+ 1, \lambda'_- + \sigma_- 1)}_{j-\sigma_+ 1/2 - \sigma_- 1/2}(\eta), \tag{118}$$

where

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 18 February 2025 doi:10.20944/preprints202502.1228.v1

$$d_{\vec{\phi},j}(\vec{\lambda}') = \begin{cases} j+\lambda'_+ & \text{if } \vec{\phi} = +-, \\ j+\lambda'_- & \text{if } \vec{\phi} = -+, \\ 2(j+1) & \text{if } \vec{\phi} = --, \\ \tfrac{1}{2}(j+\lambda'_+ +\lambda'_- +1) & \text{if } \vec{\phi} = ++, \end{cases} \tag{119}$$

(see (91) in [53] with $\alpha = \lambda'_+$, $\beta = \lambda'_-$, and n replaced for j), we introduce the p-W polynomials via the relations:

$$\mathcal{P}_{m+j-\phi_+ \frac{1}{2}-\phi_- \frac{1}{2}}[\eta;\vec{\lambda}' \,|\, \vec{\phi}, m; j] := \tag{120}$$

$$\begin{vmatrix} P_m^{(\lambda_+,\lambda_-)}(\eta) & \prod_{\aleph=\mp} (\eta-\aleph 1)^{\frac{1}{2}(1-\phi_\aleph)} P_j^{(\lambda'_+,\lambda'_-)}(\eta) \\ \dot{P}_m^{(\lambda_+,\lambda_-)}(\eta) & d_{\vec{\phi},m} j(\vec{\lambda}') P_{j-\phi_+ \frac{1}{2}-\phi_- \frac{1}{2}}^{(\lambda'_+ +\phi_+ 1,\lambda'_- +\phi_- 1)}(\eta) \end{vmatrix}.$$

and then re-write the Wronskian of the q-RSs (111) and (112),

$$W\{\phi_m[\eta;\vec{\lambda}],\phi_j[\eta;\vec{\lambda}']\} = \phi_0^2[\eta;\vec{\lambda}]\times$$
$$W\{P_m^{(\lambda_+,\lambda_-)}(\eta), |1-\aleph\eta|^{\frac{1}{2}(\lambda'_\aleph-\lambda_\aleph)} P_j^{(\lambda'_+,\lambda'_-)}(\eta)\}, \tag{121}$$

as follows:

$$W\{\phi_m[\eta;\vec{\lambda}],\phi_j[\eta;\vec{\lambda}']\} = \phi_0[\eta;\vec{\lambda}' + \vec{\phi}\times\vec{1} - \vec{1}]\phi_0[\eta;\vec{\lambda}]\times$$
$$\mathcal{P}_{m+j-\phi_+ \frac{1}{2}-\phi_- \frac{1}{2}}[\eta;\vec{\lambda}' \,|\, \vec{\phi}, m; j]. \tag{122}$$

For $\vec{\phi} = ++$ the latter expression turns into (116) with

$$\mathcal{P}_{m+j-1}[\eta;\vec{\lambda} \,|\, ++, m; j] = W_{m+j-1}^{(\lambda_+,\lambda_-)}[\eta \,|\, m, j]. \tag{123}$$

Making use of (121) and (122), the RRZ$\mathfrak{I}$ of the q-RS (111),

$$\Phi[\eta;\vec{\lambda};-\varepsilon_j(\vec{\lambda}') \,|\, m] = \sqrt{|1-\eta^2|}\, \frac{W\{\phi_m[\eta;\vec{\lambda}],\phi_j[\eta;\vec{\lambda}']\}}{\phi_0[\eta;\vec{\lambda}] P_m^{(\lambda_+,\lambda_-)}(\eta)}, \tag{124}$$

can be thus represented in the following quasi-rational form:

$$\Phi[\eta;\vec{\lambda};-\varepsilon_j(\vec{\lambda}') \,|\, m] = \frac{\phi_0[\eta;\vec{\lambda}' + \vec{\phi}\times\vec{1}]}{P_m^{(\lambda_+,\lambda_-)}(\eta)} \mathcal{P}_{m+j-\phi_+ \frac{1}{2}-\phi_- \frac{1}{2}}[\eta;\vec{\lambda}' \,|\, \vec{\phi}, m; j], \tag{125}$$

which completes the proof of the Theorem 1. □

Representing the polynomials (120) in the alternative form

$$\mathcal{P}_{m+j-\phi_+ \frac{1}{2}-\phi_- \frac{1}{2}}[\eta;\vec{\lambda}' \,|\, \vec{\phi}, m; j] := \tag{126}$$

$$\begin{vmatrix} \prod_{\aleph=\mp} (\eta-\aleph 1)^{\frac{1}{2}(1-\phi_\aleph)} P_m^{(\lambda_+,\lambda_-)}(\eta) & P_j^{(\lambda'_+,\lambda'_-)}(\eta) \\ d_{\vec{\phi},m}(\vec{\lambda}) P_{m-\phi_+ \frac{1}{2}-\phi_- \frac{1}{2}}^{(\lambda_+ +\phi_+ 1,\lambda_- +\phi_- 1)}(\eta) & \dot{P}_j^{(\lambda'_+,\lambda'_-)}(\eta) \end{vmatrix}.$$

and $\vec{\phi} = \mp\pm$, we define the Xm-Jacobi DPSs of series J1 and J2 as follows

$$\mathcal{P}_{j+m}[\eta;\vec{\lambda}'|\mp\pm,m;j]=(\eta\pm1)P_m^{(\pm\lambda'_+,\mp\lambda'_-)}(\eta)\dot{P}_j^{(\lambda'_+,\lambda'_-)}(\eta)+$$
$$(\lambda'_\mp-m)P_m^{(\pm\lambda'_+\pm1,\mp\lambda'_-\mp1)}(\eta)P_j^{(\lambda'_+,\lambda'_-)}(\eta). \quad (127)$$

The two exceptional infinite polynomial sequences are interrelated by the interchange of the two indexes accompanied by the change in the argument, namely,

$$\mathcal{P}_{j+m}[\eta;\lambda'_-,\lambda'_+|-+,m;j]=(-)^{j+m+1}\mathcal{P}_{j+m}[-\eta;\lambda'_+,\lambda'_-|+-.m;j] \quad .(128)$$

Despite the mentioned interdependence, we, in contrast with [15,16] prefer to treat these two X-OPSs separately and refer to them as being of series J1 and J2, in following the terminology suggested in [3]. Note that, contrary to the definition (2.3) of these two series in [3], with m=n,

J1: $\lambda'_-=\lambda_{o;-}\equiv h+m+\tfrac{1}{2}>0$, $\lambda'_+=\lambda_{o;+}\equiv g+m-\tfrac{3}{2}>0$,

and

J2: $\lambda'_-=\lambda_{o;-}\equiv h+m-\tfrac{3}{2}>0,\ \lambda'_+=\lambda_{o;+}\equiv g+m+\tfrac{1}{2}>0$

in our notation, the indexes of the Jacobi polynomials in the right-hand side of (128) are independent of the polynomial degrees. It will be proven below that the polynomial sequences (127) satisfy the Bochner-type ODEs but violate the Bochner theorem since they do not start from a constant. We [24] refer to them as the X_m-Jacobi DPSs of series J1 and J2, with no restrictions imposed on the Jacobi indexes other than their absolute values differ from 0 and 1.

In particular, setting $\beta=\lambda'_-$, $\alpha=\lambda'_+$ in (120) and (127) we come to (72) and (71) in [16]:

$$\mathcal{P}_{m+j}[\eta;\beta,\alpha|+-,m;j]=(\alpha+j)P_j^{(\alpha-1,\beta+1)}(\eta)P_m^{(-\alpha,\beta)}(\eta) \quad (129)$$
$$-(\eta-1)P_j^{(\alpha,\beta)}(\eta)\dot{P}_m^{(-\alpha,\beta)}(\eta)$$
$$=(\alpha_+-m)P_m^{(-\alpha-1,\beta+1)}(\eta)P_j^{(\alpha,\beta)}(\eta)$$
$$+(\eta-1)P_m^{(-\alpha,\beta)}(\eta)\dot{P}_j^{(\alpha,\beta)}(\eta), \quad (130)$$

and, consequently,

$$\mathcal{P}_{j+m}[\eta;\vec{\lambda}'|+-,m;j]=(-1)^m(\lambda'_++j)\hat{P}_{m,m+j}^{(\lambda'_+-1,\lambda'_-+1)}(\eta) \quad . (131)$$

For m=1 the two X-DPSs collapse into the single X-DPS of series J [29] which contains the X_1-Jacobi OPS [31,32] often referred to simply as 'X_1-Jacobi orthogonal polynomials'. The term 'X_1-Jacobi polynomials' introduced by Yadav et al. [23] (while referring to the X_1-Jacobi DPS of series J) seems satisfactory as far as one clearly understands that that that we deal here with a larger family of polynomials which contains both the X_1-Jacobi OPS and the only existent finite EOP sequence [29].

On other hand, coming back to (101) and setting:

$$\lambda_-=\lambda'_-=\beta-1=-\lambda_{o;-},\ -\lambda_+=\lambda'_+=\alpha+1=\lambda_{o;+} \quad (132)$$

in (127) for $\vec{\sigma}=+-$ gives

$$\mathcal{P}_{j+m}[\eta;\vec{\lambda}'|+-,m;j]=(\lambda'_+-m)P_m^{(-\lambda'_+-1,\lambda'_-+1)}(\eta)P_j^{(\lambda'_+,\lambda'_-)}(\eta)$$
$$-\tfrac{1}{2}(\lambda'_++\lambda'_-+j+1)(1-\eta)P_m^{(-\lambda'_+,\lambda'_-)}(\eta)P_{j-1}^{(\lambda'_++1,\lambda'_-+1)}(\eta), \quad (133)$$

which brings us to the polynomial sequence (15) in [10] (with ν standing j here). Note that (132) is consistent the m-independent definition of the indexes α and β in [11]. With the latter correction, (15) in [10] turns into (133) here, which is nothing but the X_m-Jacobi DPS of series J2 in our terms. It does

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 18 February 2025 doi:10.20944/preprints202502.1228.v1

contain both the Xm-Jacobi OPS [15,16] and the finite EOP sequences of type $\tilde{b}$ used by Yadav et al. [10-13] to construct the analytical expression for the eigenfunctions of the rationally extended h-PT potential of the same type (and referred to in the cited papers simply as 'Xm-Jacobi polynomials'.

Similarly, the ladders of lowering and raising operators introduced in [14,89] act in the space spanned by the Xm-Jacobi DPS of series J2. The title 'Xm-Jacobi orthogonal polynomials' used for the subsection IV.A in [14] seems misleading since the ladder relations analyzed by Yadav et al. are applicable for any values of the indexes α and β.

The polynomials belong to the X-OPS iff the indexes α and β are larger than -1 and have the same sign. The lowering operators necessarily bring these indexes into region where the polynomials do not belong to the X-OPS anymore. We postpone the thorough analysis of this issue for a separate publication.

Examination of the scattering amplitude (25) in [10] reveals that it has poles on the imaginary positive *k*-axis at

$$k = (A - j)i \,. \qquad (134)$$

By choosing

$$\lambda_{\mp} = -\lambda_{o;\mp} \; (\vec{\sigma} = --), \; \lambda'_{\mp} = \mp\lambda_{o;\mp} \qquad (135)$$

and taking into account that, according to (97),

$$2 * A + 1 = 2A + 1 = -\lambda'_{-} - \lambda'_{+} \quad \text{for} \quad \vec{\sigma} = +- \,, \qquad (136)$$

we confirm that the mentioned poles correspond to the bound energies (115), as expected. We thus assert the cited scattering amplitude was obtained in [10] using the constraint 2B > 2A+1, but not the change of the notation suggested by the cited authors later in writing the potential function (19) in [11].

Let us now consider the alternative quasi-rational representation of the Wronskian (122):

$$W\{\phi_m[\eta;\vec{\lambda}], \phi_j[\eta;\vec{\lambda}']\} = \phi_0[\eta;\vec{\lambda} - \vec{1}]\phi_0[\eta;\vec{\lambda}' - \vec{1}]\mathcal{D}_{m+j+1}[\eta;\vec{\lambda}, m;\vec{\lambda}', j], \qquad (137)$$

in terms of the so-called [48] 'polynomial determinant' (PD)

$$\mathcal{D}_{m+j+1}[\eta;\vec{\lambda}, m;\vec{\lambda}', j] := \begin{vmatrix} P_m^{(\lambda_+,\lambda_-)}(\eta) & P_j^{(\lambda'_+,\lambda'_-)}(\eta) \\ S_{m+1}^{(\lambda_+,\lambda_-)}(\eta) & S_{j+1}^{(\lambda'_+,\lambda'_-)}(\eta) \end{vmatrix}, \qquad (138)$$

where

$$S_{m+1}^{(\alpha,\beta)}(\eta) := \tfrac{1}{2}[(\alpha+1)(\eta+1) + (\beta+1)(\eta-1)]P_m^{(\alpha,\beta)}(\eta) + (\eta^2 - 1)\dot{P}_m^{(\alpha,\beta)}(\eta). \qquad (139)$$

Originally [8,28] we used the PD decomposition as the universally applicable starting point for representing the RDC$\mathfrak{J}$s of JS solutions in the quasi-rational form for any admissible density function (see [88] for an example). Yet the same year Gòmez-Ullate et al. [53] developed the powerful scheme allowing one to represent the RDC$\mathfrak{J}$s of JS solutions in the quasi-rational form, with the polynomial components represented by the p-Ws. The main advantage of using the p-Ws (instead of PDs) for the RD$\mathfrak{J}$s of JS solutions discussed here is that the p-Ws, in contrast with PDs, remain finite at both singular points ∓ 1 and as a result form the Xm-Jacobi DPSs. Though we need to mention that this assertion does not generally retain for the polynomial components of the higher-order RDC$\mathfrak{J}$s in their p-W representation.

Comparing (137) with (122), one finds that

$$\mathcal{D}_{m+j+1}[\eta;\vec{\sigma}\times\vec{\lambda}', m;\vec{\lambda}', j] = (\eta-1)^{\frac{1}{2}(\sigma_- +1)}(\eta-1)^{\frac{1}{2}(\sigma_+ +1)} \times$$

$$\mathcal{P}_{m+j-\sigma_+\frac{1}{2}-\sigma_-\frac{1}{2}}[\eta;\vec{\lambda}' \,|\, \vec{\sigma}, m; j]. \qquad (140)$$

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 18 February 2025 doi:10.20944/preprints202502.1228.v1

It directly follows from (140) that the PD and p-W representations become identical for $\vec{\sigma} = --$. Evaluating the polynomials (139) at ∓ 1:

$$S_{m+1}^{(\alpha,\beta)}(\mp 1) = \tfrac{1}{2}[(\alpha+1)(\mp 1+1)+(\beta+1)(\mp 1-1)]P_m^{(\alpha,\beta)}(\mp 1) \tag{141}$$

and substituting (141) into the right-hand side of (138) gives

$$\mathcal{D}_{j+2}[\mp 1;-\vec{\lambda}',1;\vec{\lambda}',j] = \mp 2\lambda'_{\mp}P_1^{(-\lambda'_+,-\lambda'_-)}(\mp 1)P_j^{(\lambda'_+,\lambda'_-)}(\mp 1) \,. \tag{142}$$

We thus proved that the PD (138) remains finite at ∓ 1, provided that this is true for the j-th degree Jacobi polynomial on the right. Out of the four PD sequences introduced by us in [28] to generate the X_m-Jacobi DPSs, this was the only sequence which retains finite at ±1 and therefore spanned the DPS referred to by us for this reason as the X_m-Jacobi DPS of series D.

4.2. Bochner-Type ODEs for Four X_m-Jacobi DPSs

Let us draw reader's attention to the fact that the magnitudes of the exponent parameters of the basic solution $\phi_0[\eta;\vec{\sigma}\times(\vec{\lambda}+\vec{1})]$ in the right hand-side of (125):

$$|\lambda'_{\pm} + \sigma_{\pm}1| = {}^*\lambda_{o;\pm} := |\lambda_{\pm}+1| \tag{143}$$

represent the ExpDiffs for the poles of the RCSLE (15) at ±1 which implies that that each p-W polynomial remains finite at both poles at least if $\lambda_{\pm}+1$ are not negative integers with the absolute values smaller the polynomial degree. Substituting (41) into (29) and taking into account that the mentioned basis solution obeys the JRef CSLE (1) with $\lambda_{o;\pm}$ replaced for ${}^*\lambda_{o;\pm}$ we find that the p-W polynomial (38), with j standing for an arbitrary non-negative integer, satisfies the Bochner-type ODE:

$$\{\mathbf{D}_{\vec{\lambda},m}^{(\vec{\sigma})} + C_m[\eta;\vec{\lambda};\varepsilon_j(\vec{\lambda}')\,|\,\vec{\sigma}\,]\}\,\mathcal{P}_{m+j-\sigma_+\frac{1}{2}-\sigma_-\frac{1}{2}}[\eta;\vec{\lambda}\,|\,\vec{\sigma},m;j] = 0\,, \tag{144}$$

where $\mathbf{D}_{\vec{\lambda},m}^{(\vec{\sigma})}$ is the abbreviated notation for the second-order differential operator

$$\mathbf{D}_{\vec{\lambda},m}^{(\vec{\sigma})} := (1-\eta^2)P_m^{(\lambda_+,\lambda_-)}(\eta)\frac{d^2}{d\eta^2} + 2B_{m+1}[\eta;\vec{\lambda}\,|\,\vec{\sigma}]\frac{d}{d\eta} \,. \tag{145}$$

Keeping in mind that

$$ld\, P_m^{(\lambda_+,\lambda_-)}(\eta) = \sum_{l=1}^{m}\frac{1}{\eta-\eta_l(\vec{\lambda};m)} \,, \tag{146}$$

we find that the coefficient function of the first derivative is represented by the fo;;owing polynomial of degree m+1:

$$B_{m+1}[\eta;\vec{\lambda}\,|\,\vec{\sigma}] := (1-\eta^2)\,P_m^{(\lambda_+,\lambda_-)}(\eta) \times\left(\sum_{\aleph=\pm}\frac{\sigma_{\aleph}(\lambda_{\aleph}+1)+1}{2(\eta-\aleph 1)} - \sum_{l=1}^{m}\frac{1}{\eta-\eta_l(\vec{\lambda};m)}\right) \tag{147}$$

and the ε-dependent polynomial of degree m representing the free term of the ODE (142) is linear in the energy:

$$C_m[\eta;\vec{\lambda};\varepsilon\,|\,\vec{\sigma}\,] = C_m[\eta;\vec{\lambda}\,|\,\vec{\sigma}\,] - \varepsilon\,P_m^{(\lambda_+,\lambda_-)}(\eta). \tag{148}$$

Taking advantage of the definition of the second summand in (71) via (72), we assert that this term gives no contribution the polynomial (148), and therefore, bearing in mind that

$$\varepsilon_0(-\vec{\lambda}-\vec{1}) = \varepsilon_0(\vec{\lambda}), \tag{149}$$

$$\varepsilon_0(\vec{\lambda}) - \varepsilon_m(\vec{\lambda}) = -m(\lambda_- + \lambda_+ + m + 1), \tag{150}$$

and

$$P_1^{(-\lambda_+ -1, \lambda_- +1)}(\eta) - P_1^{(-\lambda_+ -1, -\lambda_- -1)}(\eta) \equiv (\lambda_- + 1)(\eta - 1), \tag{151}$$

we can represent the free-energy summand as

$$C_m[\eta;\vec{\lambda};0\,|\,\vec{\phi}] = 2(\lambda_- + 1)(\eta - 1)\times$$
$$\dot{P}_m^{(\lambda_+,\lambda_-)}(\eta) + \left[\varepsilon_0(\vec{\phi}\times\vec{\lambda}) - m(\lambda_- + \lambda_+ + m + 1)\right] P_m^{(\lambda_+,\lambda_-)}(\eta). \tag{152}$$

Setting

$$^*\lambda_\pm := \lambda'_\pm + \phi_\pm 1 \equiv \phi_\pm(\lambda_\pm + 1), \tag{153}$$

we can alternatively re-write (144) as the eigenvalue problem [15-17]

$$\left[\mathbf{T}^{(\vec{\phi})}_{^*\lambda_+, ^*\lambda_-, m} + \varepsilon_n(\vec{\lambda}')\right] \mathcal{P}_{m+n-\phi_+ 1/2 - \phi_- 1/2}[\eta;\vec{\lambda}\,|\,\vec{\phi}, m; n] = 0 \tag{154}$$

for the four exceptional differential operators

$$\mathbf{T}^{(\vec{\phi})}_{\alpha,\beta}(\eta) := \mathbf{T}_{\alpha,\beta}(\eta) - 2(1-\eta^2)\,ld\,P_m^{(\phi_+\alpha-1,\phi_-\beta-1)}(\eta)\frac{d}{d\eta}$$
$$-2\beta(1-\eta)\,ld\,P_m^{(-\alpha-1,\beta-1)}(\eta) + m(\alpha-\beta-m+1), \tag{155}$$

where

$$\mathbf{T}_{\alpha,\beta}(\eta) := (1-\eta^2)\frac{d^2}{d\eta^2} + 2P_1^{(\alpha,\beta)}(\eta)\frac{d}{d\eta}. \tag{156}$$

In particular, combining (153) with (133) for $\vec{\phi} = +-$:

$$\beta = {}^*\lambda_- = \lambda'_- + 1, \ \alpha = {}^*\lambda_+ = \lambda'_+ - 1, \tag{157}$$

one can then directly verify that the corresponding operator (155) precisely matches (2.2) in [17].

Let us remind the reader that the latter operator and the Liouville potential (100) were obtained using the same RefPF (82) as the starting point and that the Liouville potential is related to the RefPF via the elementary formula (88). This serves as the additional argument in support of the correctness of our expression for the Liouville potential of type $\tilde{b}$ compared with the ones in [11,12].

It is worth stressing that any polynomial from the X_m-Jacobi DPS of series J2 is the eigen-polynomial of this operator with the eigenvalue

$$\varepsilon_0(\vec{\lambda}') - \varepsilon_{n-m}(\vec{\lambda}') = -(n-m)(\alpha+\beta+n-m+1), \tag{158}$$

with n=m+j standing for the polynomial degree.

The X_m-Jacobi OPS of series J2 (and therefore its counter-part in the reflected argument) have been analyzed in a more rigorous way in [15-17], and for this reason we focus solely on the finite EOP sequences. In Section 5 we present the universal method for constructing the finite EOP sequences composed of RD$\mathfrak{J}$s of the R-Jacobi polynomials. In the subsections 5.1, 5.2, and 5.3 we then focus on the EOPs forming the eigenfunctions of the rationally extended *h*-PT potentials of types $\underset{\sim}{a}$, $\underset{\sim}{b}$, and $\underset{\sim}{a}'$.

5. Isospectral Triplet of RCSLEs Solved via RRZ$\mathfrak{J}$s of R-Jacobi Polynomials

Starting from this point, we discuss only the admissible RRZTs using the TFs $\phi_m[\eta;\vec{\lambda}]$ with no nodes in the interval (1,∞) for the specified ranges of the parameters λ_-, λ_+. In this Section we will consider only the TFs with the vector $\vec{\lambda}$ lying in the first three quadrants. As demonstrated in [1] using the Klein formula [49], the admissible TFs also exist for certain segments of the vector $\vec{\lambda}$ in the fourth quadrant, but this family of the finite EOP sequences lies beyond the scope of this paper.

To formulate the SLP on the interval (1,∞) we (by analogy with the analysis) presented in subsection 2.3 for the JRef CSLE) first convert the RCSLE (103) to its prime form

$$\left\{-\frac{d}{d\eta}(\eta-1)\frac{d}{d\eta}+\mathcal{q}[\eta;\vec{\lambda}\,|\,m]+\underset{\sim}{\varepsilon}\mathcal{w}[\eta;\vec{\lambda}_o]\right\}\underset{\sim}{\Psi}[\eta;\vec{\lambda};\underset{\sim}{\varepsilon}\,|\,m]=0 \quad (\eta>1) \qquad (159)$$

and then solve it under the DBCs:

$$\lim_{\eta\to r}\underset{\sim}{\Psi}[\eta;\vec{\lambda};\underset{\sim}{\varepsilon}_j(\vec{\lambda})\,|\,m]=0 \quad (r=1,\infty) \qquad (160)$$

The zero-energy free term in the SLE (159) is related to the RefPF of the RCSLE via the same elementary formula:

$$\mathcal{q}[\eta;\vec{\lambda}\,|\,m]=-(\eta-1)I^o[\eta;\vec{\lambda}\,|\,m]+\frac{1}{4(\eta-1)} \quad (\eta>1) \qquad (161)$$

Taking into account that

$$\lim_{\eta\to 1}\left[(\eta-1)\mathcal{q}[\eta;\vec{\lambda}\,|\,m]\right]=\tfrac{1}{4}|\lambda_{+}+1|^2>0, \qquad (162)$$

we confirm that the ChExps of the Frobenius solutions for the pole at +1 have the same absolute value, while differing by their sign. The ExpDiff for the pole of the RCSLE (103) at infinity turned out to be energy-dependent. We have already proved that the second-order poles in the second and third summands nullify each other, and therefore

$$\lim_{\eta\to\infty}\left[\eta\mathcal{q}[\eta;\vec{\lambda}\,|\,m]\right]=-\lim_{\eta\to\infty}\left[\eta^2 I^o[\eta;\vec{\lambda}\,|\,m]\right]+\tfrac{1}{4}=0. \qquad (163)$$

We thus assert that the ChExps of the Frobenius solutions for the pole at ∞ are real only at negative energies and have in this case the same non-zero absolute value $\tfrac{1}{2}\sqrt{-\varepsilon}$ while differing by their sign.

Combining (163) with the similar limit

$$\lim_{\eta\to 1+}\left[(\eta-1)\mathcal{q}[\eta;\vec{\lambda}\,|\,m\right]={}^*\lambda^2_{o;+}>0 \qquad (164)$$

for the pole at +1, we conclude that the PFSs of the *p*-SLE (159) near both singular endpoints are unambiguously determined by the DBCs.

By applying the RRZT with the admissible TF $\phi_m[\eta;\vec{\lambda}]$ to the eigenfunction $\phi_j[\eta;\vec{\lambda}']$ and then converting the resultant transform (125) to its prime form:

$$\underset{\sim}{\psi}_j[\eta;\vec{\lambda}'\,|\,\vec{\sigma},m]:=\underset{\sim}{\Psi}[\eta;\vec{\sigma}\times\vec{\lambda}';-\varepsilon_j(\vec{\lambda}')\,|\,m]= \qquad (165)$$

$$=\frac{\underset{\sim}{\psi}_0[\eta;\vec{\lambda}'+\vec{\sigma}\times\vec{1}]}{P_m^{(\sigma_+\lambda'_+,\sigma_-\lambda'_-)}(\eta)}\mathcal{P}_{m,m+j-\sigma_+\frac{1}{2}-\sigma_-\frac{1}{2}}[\eta;\vec{\lambda}'\,|\,\vec{\sigma}],$$

where

$$\lambda'_{\mp}=\mp\lambda_{o;\mp}. \qquad (166)$$

Suppose that the q-RSs (166) obey the DBCs

$$\lim_{\eta\to r}\underset{\sim}{\psi}_j[\eta;\vec{\lambda}'\,|\,\vec{\sigma},m]=0 \quad (r=1,\infty) \qquad (167)$$

for any $j \le j_{max}$ and therefore represent as the eigensolutions of the Sturm–Liouville problem solved under the DBCs. (We will analyze this assertion on the case-by-case basis in subsections 2.1, 2.2, and 2.3 below for the vector $\vec{\lambda}=\vec{\sigma}\times\vec{\lambda}'$ lying in the first, second and third quadrant accordingly.) It has been proven in [56] that the eigenfunction of the generic SLE solved under the DBCs must be mutually orthogonal with the weight (31) on the infinite interval in question:

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 18 February 2025 doi:10.20944/preprints202502.1228.v1

$$\int_1^\infty d\eta\, \psi_{j'}[\eta;\vec{\lambda}'\,|\,\vec{\sigma},m]\psi_j[\eta;\vec{\lambda}'\,|\,\vec{\sigma},m]w[\eta;\vec{\lambda}_o]=0 \tag{168}$$

for $j' < j \le j_{max}$.

Consequently, the polynomial components of the quasi-rational eigenfunctions (165) must be mutually orthogonal with the m-dependent weight

$$W[\eta;\vec{\lambda}'\,|\,\vec{\sigma},m]:=\frac{\psi_0^2[\eta;\vec{\lambda}'+\vec{\sigma}\times\vec{1}]}{|P_m^{(\sigma_+\lambda'_+,\sigma_-\lambda'_-)}(\eta)|^2}w[\eta;\vec{\lambda}_o] \text{ for } \eta>1, \tag{169}$$

namely,

$$\int_1^\infty d\eta \mathcal{P}_{m,m+j'-\sigma_+\frac{1}{2}-\sigma_-\frac{1}{2}}[\eta;\vec{\lambda}'\,|\,\vec{\sigma}]\mathcal{P}_{m,m+j-\sigma_+\frac{1}{2}-\sigma_-\frac{1}{2}}[\eta;\vec{\lambda}'\,|\,\vec{\sigma}]W[\eta;\vec{\lambda}'\,|\,\vec{\sigma},m]=0$$

for $1 \le j' < j \le j_{max}$. (170)

Let us now demonstrate the power of the developed formalism by proving that the q-RSs (165) form the complete set of the eigenfunctions for the formulated Dirichlet problem.

Theorem 2: *The Dirichlet problem for the p-SLE (159) defined on the infinite positive interval [1,∞) does not have any solutions other than the eigenfunctions (176), assuming that the pole of the corresponding CSLE (103) at +1 is LP.*

Proof: Suppose that the given Dirichlet problem has a solution $\psi_n[\eta;\vec{\lambda}'\,|\,\vec{\sigma},m]$ at an energy $\varepsilon_n(\lambda'\,|\,\vec{\sigma},m) \ne -\varepsilon_j(\lambda')$ for $0 \le j \le j_{max}$.

The RRZT of the RCSLE (103) with the TF

$$*\phi_m[\eta;\vec{\lambda}]=\frac{\sqrt{1+\eta}}{\psi_m[\eta;\vec{\lambda}]} \tag{171}$$

converts the extraneous eigenfunction into the following q-RS of the *p*-SLE (29):

$$\Psi[\eta;\vec{\lambda}]:=\psi_m[\eta;\vec{\lambda}]W\left\{\sqrt{\eta+1}\psi_m^{-1}[\eta;\vec{\lambda}],\sqrt{\eta-1}\psi_n[\eta;\vec{\lambda}'\,|\,\vec{\sigma},m]\right\} \tag{172}$$

$$=\frac{\eta+1}{\psi_m[\eta;\vec{\lambda}]}\frac{d}{d\eta}\left\{\sqrt{\frac{\eta-1}{\eta+1}}\psi_m[\eta;\vec{\lambda}]\psi_n[\eta;\vec{\lambda}'\,|\,\vec{\sigma},m]\right\}. \tag{173}$$

Let now remind the reader that the extraneous eigenfunction (like any other) must decay as

$$\psi_n[\eta;\vec{\lambda}'\,|\,\vec{\sigma},m]\sim(\eta-1)^{\frac{1}{2}*\lambda_{o;+}}$$

in the limit $\eta \to 1+$. Examination of the sum:

$$\Psi[\eta;\vec{\lambda}_o]=\sqrt{\eta^2-1}\left\{\overset{\bullet}{\psi}_n[\eta;\vec{\lambda}'\,|\,\vec{\sigma},m]+ld\,\psi_m[\eta;\vec{\lambda}]\psi_n[\eta;\vec{\lambda}'\,|\,\vec{\sigma},m]\right\} \tag{174}$$

$$+\psi_n[\eta;\vec{\lambda}'\,|\,\vec{\sigma},m]/(\eta+1)$$

then shows that it vanishes at the lower end of the interval (1, ∞) as far as the pole the RCSLE (103) at +1 lies in the LP region ($*\lambda_{o;+} > 1$).

Furthermore, one can easily verify that the solution in question also vanishes in the limit $\eta \to \infty$ and therefore represents an eigenfunction of the *p*-SLE (36) with an eigenvalue differing from any of the eigenvalues (40). This result contradicts to the fact that the cited energies represent the complete discrete energy spectrum of the given Dirichlet problem. We thus confirmed that the *p*-SLE (159)

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 18 February 2025 doi:10.20944/preprints202502.1228.v1

solved under the DBCs (160) may not have eigenvalues other than (40) in the LP range of the parameter $*\lambda_{o;+}$. □

5.1. Infinitely Many EOP Sequences of Series $\underset{\sim}{\mathfrak{a}}$

Let us start from the vector $\vec{\lambda}$ lying in the first quadrant:

$$\vec{\sigma} = ++,\ \vec{\sigma} = -+. \tag{171}$$

Consequently, the seed polynomial in the denominator of the weight (169) turns into the classical Jacobi polynomial with positive indexes and therefore all the poles of the RCSLE (103) are located in the closed interval [-1,+1]. Ironically, this most obvious case has never been discussed in the literature and our study of these EOP sequences in [24] was left without a proper response.

The finite polynomial sequence under consideration belongs to the X_m-Jacobi DPS of series J1 specified by the upper sign in (127). Taking into account (166), we can represent this polynomial subset as

$$\mathcal{P}_{m,j+m}[\eta;-\lambda_{o;-},\lambda_{o;+}\,|\,-+] = (\eta+1)P_m^{(\lambda_{o;+},\lambda_{o;-})}(\eta)\,\dot{P}_j^{(\lambda_{o;+},-\lambda_{o;-})}(\eta)$$
$$-(\lambda_{o;-}+m)P_m^{(\lambda_{o;+}+1,\lambda_{o;-}-1)}(\eta)P_j^{(\lambda_{o;+},-\lambda_{o;-})}(\eta). \tag{172}$$

Setting $z=-\eta$, $\alpha=\lambda_{o;-}$, $\beta=\lambda_{o;+}$, we come to the polynomials (72) in [16]:

$$(-1)^{j+m+1}\mathcal{P}_{m,j+m}[-z;-\lambda_{o;-},\lambda_{o;+}\,|\,-+] = (\alpha+m)P_m^{(\alpha-1,\beta+1)}(z)P_j^{(-\alpha,\beta)}(z)$$
$$-(z-1)P_m^{(\alpha,\beta)}(z)\frac{d}{dz}P_j^{(-\alpha,\beta)}(z), \tag{173}$$

with the interchanged degrees m and j, i.e.,

$$(-1)^{m+1}\mathcal{P}_{m,j+m}[-z;-\alpha,\beta\,|\,-+] = (\alpha+m)\hat{P}_{j,j+m}^{(\alpha-1,\beta+1)}(z). \tag{174}$$

We thus proved that the polynomial (172) converted to its monic form coincides with the monic m-th degree polynomial from the X_j-Jacobi OPS of series J1 [24]. This implies that the polynomials (174) arranged into the infinite row (m=1,2,..,) at fixed j are mutually orthogonal with the weight

$$\hat{W}_{\lambda_{o;-},\lambda_{o;+}}[\eta;*\vec{\lambda}_o\,|\,-+] := \frac{(1+\eta)^{\lambda_{o;-}}(1-\eta)^{\lambda_{o;+}}}{|P_j^{(\lambda_{o;+}-1,-\lambda_{o;-}-1)}(\eta)|^2} \tag{175}$$

(see (87) in [16]). We thus come to the $(j_{max}+1)\times\infty$ rectilinear polynomial matrix mentioned in Introduction.

With the given choice of the quadrant for the vector $\vec{\lambda}$ and $\lambda_{o;-}>1$, the q-RSs (165) take the form:

$$\underset{\sim}{\psi}_j[\eta;-\lambda_{o;-},\lambda_{o;+}\,|\,-+,m] = \tag{176}$$
$$=\underset{\sim}{\psi}_0[\eta;-*\lambda_{o;-},*\lambda_{o;+}]\frac{\mathcal{P}_{m,m+j}[\eta;-\lambda_{o;-},\lambda_{o;+}\,|\,-+]}{P_m^{(\lambda_{o;+},\lambda_{o;-})}(\eta)},$$

making the DBC at η=1 trivially hold. (As expected, the absolute values of the power exponents $\mp\frac{1}{2}*\lambda_{o;\mp}$ of $\eta\pm1$ coincide with halves of the ExpDiffs $*\lambda_{o;\mp}=\lambda_{o;\mp}\mp1$ for the poles of the RCSLE (103) at ∓1, as far as the pole at -1 is LP.) Taking into account that

$$*\lambda_{o;-}+*\lambda_{o;+}=\lambda_{o;-}+\lambda_{o;+}, \tag{177}$$

we assert that

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 18 February 2025 doi:10.20944/preprints202502.1228.v1

$$\lim_{\eta\to\infty} \underset{\sim}{\psi}_j[\eta;-\lambda_{o;-},\lambda_{o;+}\,|\,-+,m] = \lim_{\eta\to\infty} \underset{\sim}{\psi}_j[\eta;-\lambda_{o;-},\lambda_{o;+}] = 0, \qquad (178)$$

which confirms that the q-RS (175) is the eigenfunction of the *p*-SLE (159) and therefore its polynomial components are mutually orthogonal with the weight

$$W[\eta;\vec{\lambda}'\,|\,-+,m] = \frac{\underset{\sim}{\psi}_0^2[\eta;-\lambda_{o;-},{}^*\lambda_{o;+}]}{|\,P_m^{(\lambda_{o;+},-\lambda_{o;-})}(\eta)\,|^2}. \qquad (179)$$

Since the RCSLE (103) $\vec{\sigma}=-+$. is analytically solvable only for $*\lambda_{o;+} > 1$, Theorem 2 assures the exact solvability of the Dirichlet problem in question. From the perspective of the quantum mechanical applications, the corresponding Liouville potential (100) with $1 < *\lambda_{o;+} < 2$ represents the very specific case because its SUSY partner, *h*-PT potential with the positive parameter $\lambda_{o;+}$ smaller than 1 or

$$\tfrac{1}{2} < B - A < \tfrac{3}{2}, \qquad (180)$$

has only the CDBESs. Within the specified range of the potential parameters, there is no way how our results can be reproduced using the conventional rules of the SUSY quantum mechanics.

5.2. Infinitely Many EOP Sequences of Series $\mathfrak{a}'$

Let us now consider the peculiar case when both vector $\vec{\lambda}$ and $\vec{\lambda}'$ lie in the same (second) quadrant ($\vec{\sigma}=++$). The admissible TFs of this type exist only for finite EOP sequences. According to the inequality (65), the q-RS

$$\underset{\sim}{\psi}_{\mathfrak{a}',m}(\vec{\lambda}_o) = \underset{\sim}{\psi}_{++,m}[\eta;\vec{\lambda}_o] \qquad (181)$$

is admissible for any m from the infinite sequence of the positive integers starting from $\lceil \lambda_{o;-} - \lambda_{o;+} - 1 \rceil$. Substituting (116) into (124) with $\vec{\lambda}=\vec{\lambda}'$ gives

$$\underset{\sim}{\psi}_{++,m}[\eta;\vec{\lambda}_o] = \underset{\sim}{\psi}_0[\eta;\vec{\lambda}+\vec{1}]\,\frac{W_{m+j-1}^{(\lambda_+,\lambda_-)}[\eta\,|\,m,j]}{P_m^{(\lambda_+,\lambda_-)}(\eta)}. \qquad (182)$$

As expected, the absolute values of the power exponents $\tfrac{1}{2}(\lambda_{\mp}+1)$ of $\eta\pm1$ coincide with halves of the ExpDiffs $*\lambda_{o;\mp} = |\,\lambda_{\mp} \mp 1\,|$ for the poles of the RCSLE (103) at ∓1. Since $\lambda_+ = \lambda_{o;+}$ and therefore

$$\lambda_+ + 1 = *\lambda_{o;+} = \lambda_{o;+} + 1,$$

the q-RS (182) necessarily vanishes at +1.

To confirm the DBCs (167) at the upper end of the quantization interval, first note that the PF in the right-hand side of (182) grows at large η as η^{j-1}. Taking into account that the q-RSs $\underset{\sim}{\psi}_0[\eta;\vec{\lambda}]$ and $\underset{\sim}{\psi}_0[\eta;\vec{\lambda}+\vec{1}]$ have the same asymptotics at infinity while the eigenfunction $\underset{\sim}{\psi}_j[\eta;\vec{\lambda}]$ vanishes at this endpoint, we conclude that this should be the more so true for the q-RS (182).

Since $*\lambda_{o;+} > 1$, Theorem 2 again assures the exact solvability of the Dirichlet problem in question. Once more, the corresponding Liouville potential (100) with $1 < *\lambda_{o;+} < 2$ and $\vec{\sigma}=++$ is not covered by the conventional rules of the SUSY quantum mechanics. In Grandati's notation [9]: $\lambda_{o;+}=\alpha$, $\lambda_{o;-}=\beta$ so his analysis is applicable solely to the parameter range $\beta > \alpha + 1 > 2$. On the contrary, the technique developed here made it possible to extend the discussion of the rationally-extended *h*-PT potentials to the parameter range $0 < \alpha < 1$. The border case $\alpha = 1$ requires a more cautious analysis.

I doubt that the extension of the domain definition for the parameter α to negative values (α > -½ in (31) in [9]) makes any sense in the quantum-mechanical studies of the *h*-PT potential and its rational extensions. Though it is possible that formulating the appropriate PSLPs would allow one to construct finite EOP sequences of types $\underset{\sim}{\mathbf{a}}$ and $\underset{\sim}{\mathbf{a}}'$ for -1< α < 0 the q-RSs composed of these polynomials could be hardly applied in physics.

5.3. Finitely Many EOP Sequences of Series $\underset{\sim}{\mathbf{b}}$

By placing the vector $\vec{\lambda}$ into the third quadrant, we finally come to the case which so far has attracted much more attention from the physicists [10-13] – the rationally extended *h*-PT potential of type $\underset{\sim}{\mathbf{b}}$ constructed using the finite number of the nodeless TFs $\phi_m[\eta;-\vec{\lambda}_o]$ under the constraint (63). In the sharp contrast with the two cases discussed above, the RRZTs in question decreases by 1 the ExpDiff for the pole at +1. As a result, the *p*-SLE (159) can be analytically solved within the LC range of the ExpDiff $\lambda_{o;+}$. However, we were unable to prove that the SLP is exactly solvable in this case.

With $\vec{\lambda}=-\vec{\lambda}_o$, $\lambda_{o;+}>1$ and $\vec{\sigma}=+-$, the q-RSs (165) take the form:

$$\underset{\sim}{\psi}_j[\eta;-\vec{\lambda}_o\,|+-,m]=\underset{\sim}{\psi}_0[\eta;-{}^*\lambda_{o;-},{}^*\lambda_{o;+}]\frac{\mathcal{P}_{m,m+j}[\eta;-\lambda_{o;-},\lambda_{o;+}\,|+-]}{P_m^{(\lambda_{o;+},\lambda_{o;-})}(\eta)} \quad (183)$$

As expected, the absolute values of the power exponents $\mp\frac{1}{2}{}^*\lambda_{o;\mp}$ of $\eta\pm1$ coincide with halves of the ExpDiffs ${}^*\lambda_{o;\mp}=\lambda_{o;\mp}\pm1$ for the poles of the RCSLE (103) at ∓1, as far as the pole at +1 is LP. Though the q-RSs automatically vanish at +1, it should be noticed that we excluded out of consideration the LC region for the pole of the JRef CSLE (1) at +1, when ${}^*\lambda_{o;+}=1-\lambda_{o;+}$. Since $\underset{\sim}{\psi}_j[\eta;-{}^*\lambda_{o;-},{}^*\lambda_{o;+}]$ is the eigenfunction of the *p*-SLE (29) solved under the DBCs (35), the q-RS (183) must also vanish at infinity and therefore represents the eigenfunction of the SLP under discussion.

Note that, compared with the two other SLPs discussed above, the *p*-SLE (159) with $\vec{\sigma}=+-$ is analytically solvable in the LC region for its pole at +1. While Theorem 2 does not guarantee the exact solvability of this SLP in this case, we can point to another remarkable development by Yadav et al. [10]. Namely, they calculated the scattering amplitude for the rationally extended *h*-PT potential of the given type and the inspection of the derived expression (25) in [10] reveals that it has exactly the same poles on the imaginary positive *k*-axis as the scattering amplitude for the *h*-PT potential. This confirms that the DT in question does not create new bound energy states, in agreement with Theorem 2. It should be however stressed that the expression for the scattering amplitude (25) in [10] makes no distinction between the LP and LC singularities. It seems interesting to re-examine the derivation of the cited expression to make sure that the specifics of the LC region was correctly accounted for but we postpone this analysis for future studies.

It is also interesting to mention the basic PFS of the given type (m=0) have been used in [85] to study the DTs between the LP and LC regions of the h-PT potential. Keeping in mind that

$$A:=\lambda_{o;-}={}^*\lambda_{o;-}-1,\ B=\lambda_{o;+}-1={}^*\lambda_{o;+}<1$$

in our terms, we find the close connection between the analysis presented in [85] and the current discussion. Namely, the cited authors pick up the $D\mathfrak{I}$ of each eigenfunction (well-defined in the LP region) as the special representative of the CDBESs for the eigenvalue. As more recently proven in [48], the RRZ$\mathfrak{I}$ of the eigenfunction of the p-SLE with the LP singularities at the ends necessarily obeys

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 18 February 2025 doi:10.20944/preprints202502.1228.v1

the DBCs under consideration. In other words, Gangopadhyaya et al select the bound energy states described by the PFSs at the origin, with a clear resemblance to our prescriptions.

6. Discussion

The paper is based on the three core notions advanced by the author in the aforementioned publications. One of them is the scrupulous analysis of the RCSLEs obtained by the RRZTs of the JRef CSLE (1). As pointed to in Introduction, any RRZT of the RCSLE is directly related to a DT of the corresponding Liouville potential and as a result this part of our studies lays the rigorous foundation for the SUSY theory of the quantum-mechanical potentials solvable by polynomials.

Another important element of our approach is the concept of the 'prime' SLE chosen in such a way that the two ChExps for the poles at the endpoints differ only by sign. As a result, the energy spectrum of the given Sturm-Liouville problem can be obtained by solving the prime SLE under the DBCs. This in turn allows one to take advantage of the rigorous theorems proven in [56] for eigenfunctions of the generic SLE solved under the DBCs.

Finally, we put forward the concept of the X-Jacobi DPSs formed by polynomial solutions of the Bochner-type ODEs. Since the X-DPSs either do not start from a constant or lack a first-degree polynomial, they do not satisfy the Bochner theorem [36], as originally noticed by Gòmez-Ullate et al. [33] in the context of the discovered by them X_1-Jacobi OPS. In general, each X-Jacobi OPS belongs to one of the X-Jacobi DPSs. If the given X-Jacobi DPS does contain a X-Jacobi OPS, then we say that they belong to the same series. This is why we [8] started referring to the X_m-Jacobi DPS of series D [24] as being of series J3, after becoming aware of Grandati and Bérard's [50] discovery of the X_m-Jacobi OPS of series J3 for even m. (We still continue to refer the X_m-Jacobi DPS of series D as being of series J3 for odd m even though the latter do not contain any OPSs.) (As proven in [28], the X_1-Jacobi DPS of series D does not contain any finite EOP sequences either.)

In addition to the two finite EOP sequences identified in [9] and in [10,11] accordingly, our original research [24] discovered the (undeniably new to our knowledge) finite EOP sequences of type $\underset{\sim}{\mathfrak{a}}$, which were constructed using the q-RTFs composed of the classical Jacobi polynomials. Since all the zeros of the latter polynomials lie between -1 and +1, the q-RTFs of this kind (representing the PFSs at the origin) do not have nodes within the quantization interval (+1,∞). The very remarkable feature of the finite EOP sequences constructed in such a way [24] is that they can be arranged into the rectilinear polynomial matrix with a finite number of rows and an infinite number of columns representing the X-Jacobi OPSs of series J1 and RD$\mathfrak{I}$s of the R-Jacobi polynomials accordingly.

Compared with the cited preprints [1,8,24], the brand new element of the current analysis, is the proof that the X_m-Jacobi DPSs of series J1 and J2 are composed of the $\mathfrak{p}$-Ws of two Jacobi polynomials [51] , and therefore this is also true for their infinite and finite X-orthogonal subsets. In particular the mentioned rectilinear polynomial matrix is formed by the $\mathfrak{p}$-Ws of the classical Jacobi and R-Jacobi polynomials, respectively.

This result for the X_m-Jacobi DPS of series J1 has a very interesting far-reaching implications, making it possible to obtain analytical expressions for both infinite and finite EOP sequences using the PFSs of the same type $\underset{\sim}{\mathfrak{a}}$ as seed functions. In [59] we have discussed the RCSLEs obtained from the JRef CSLE (1) using the seed Jacobi polynomials with the same indexes. Since the RDCTs using the seed functions of types $\underset{\sim}{\mathfrak{a}}$ and $\underset{\sim}{\mathfrak{b}}$ are specified by same series of the Maya diagrams, any RCSLE using an arbitrary combination of these seed functions can be alternatively obtained by considering only infinitely many combinations $\bar{M}_p := \{m_1, m_2, ..., m_p\}$ of the PFSs of types $\underset{\sim}{\mathfrak{a}}$ [51,73]. In particular, the Liouville potentials of type $\underset{\sim}{\mathfrak{b}}$ discussed above can be alternatively obtained by means of the $\mathfrak{m}^{th}$-order RDCT with the seed functions $\underset{\sim}{\mathfrak{a}}$,m=1,...,$\mathfrak{m}$ [22,73].

One can easily verify that the quasi-rational functions defined via the relations

$$\mathcal{P}_{\mathfrak{N}(\bar{M}_p)+j}[\eta;\vec{\lambda}'|-+,\bar{M}_p;j] := \prod_{\aleph=\mp} (\eta-\aleph 1)^{-\frac{1}{2}p(\lambda_\aleph+1)-\frac{1}{2}(\lambda'_\aleph+1)}$$

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 18 February 2025 doi:10.20944/preprints202502.1228.v1

$$\times(\eta+1)^{-p}\, W\{\phi_{m_1}[\eta;\vec{\lambda}],\ldots,\phi_{m_p}[\eta;\vec{\lambda}],\phi_j[\eta;\vec{\lambda}']\}, \tag{184}$$

where $\lambda'_{\mp} = \mp\lambda_{\mp}$, constitute polynomials of degree $\mathfrak{N}(\bar{M}_p) + j$, where

$$\mathfrak{N}(\bar{M}_p) = \sum_{k=1}^{p} m_k - \tfrac{1}{2}\, p(p-1). \tag{185}$$

The explicit expression for the 'simple' p-W polynomials (184) can be easily obtained using (92) in [51] and will be examined in a separate paper. As expected, the 'simple' p-W polynomials (184) turn into the Xm-Jacobi DPSs of series J1 for p=1.

It has been proven in [25] that the ExpDiffs for the poles of the RCSLE

$$\left\{\frac{d^2}{d\eta^2} + I^o[\eta;\vec{\lambda}\,|\,\bar{M}_p] + \underset{\sim}{\varepsilon}\,\rho[\eta]\right\}\Phi[\eta;\vec{\lambda};\underset{\sim}{\varepsilon}\,|\,\bar{M}_p] = 0 \tag{186}$$

at ∓ 1 are given by the simple formula

$$*\lambda^{(p)}_{o;\mp} = |\,\lambda_{\mp} + p\,|\,. \tag{187}$$

The corresponding eigenfunctions has the form [90]:

$$\Phi[\eta;\vec{\lambda};\varepsilon_j(\vec{\lambda}')\,|\,++\vdots\bar{M}_p;-+,j] = \frac{W\{\phi_{m_1}[\eta;\vec{\lambda}],\ldots,\phi_{m_p}[\eta;\vec{\lambda}],\phi_j[\eta;\vec{\lambda}']\}}{\rho^{p/2}[\eta]\, W\{\phi_{m_1}[\eta;\vec{\lambda}],\ldots,\phi_{m_p}[\eta;\vec{\lambda}]\}} \tag{188}$$

$$= \left(\frac{\eta-1}{\eta+1}\right)^{\frac{1}{2}p} \phi_j[\eta;\vec{\lambda}']\, \frac{\mathcal{P}_{\mathfrak{N}(\bar{M}_p)+j}[\eta;\vec{\lambda}\,|\,-+\vdots\bar{M}_p;j]}{W^{(\lambda_+,\lambda_-)}_{\mathfrak{N}(\bar{M}_p)}[\eta\,|\,\bar{M}_p]}, \tag{189}$$

where

$$W^{(\lambda_+,\lambda_-)}_{\mathfrak{N}(\bar{M}_p)}[\eta\,|\,\bar{M}_p] := W\{P^{(\lambda_+,\lambda_-)}_{m_1}(\eta),\ldots,P^{(\lambda_+,\lambda_-)}_{m_p}(\eta)\}\,. \tag{190}$$

Representing (189) as

$$\Phi[\eta;\vec{\lambda};\varepsilon_j(\vec{\lambda}')\,|\,++\vdots\bar{M}_p;-+,j]$$
$$= \phi_j[\eta;-\lambda_{o;-}-p;\lambda_{o;+}+p]\,\frac{\mathcal{P}_{\mathfrak{N}(\bar{M}_p)+j}[\eta;\vec{\lambda}'\,|\,-+\vdots\bar{M}_p;j]}{W^{(\lambda_+,\lambda_-)}_{\mathfrak{N}(\bar{M}_p)}[\eta\,|\,\bar{M}_p]}, \tag{191}$$

we find that the absolute values of the power exponents of $\eta \pm 1$ coincide with halves of the ExpDiffs (187) and therefore the simple p-W polynomials (184) form a X-Jacobi DPS, which will be referred to by us as being of series $-+\vdots\bar{M}_p$. A similar X-Jacobi DPS of series $+-\vdots\bar{M}_{p'}$ simply constitutes another representation of one of the X-DPSs mentioned above and might be dropped though it would come with a catch: the corresponding net of the X-Jacobi OPSs ($\vec{\sigma} = +-, \lambda'_{\mp} = \lambda'_{o;\mp}$) starts from the infinite manifold of the Xm-Jacobi OPSs in the conventional sense [15-17].

If we choose $\lambda_{\mp} = \lambda_{o;\mp}$ and restrict j from above by the constraint $j \le j_{max}$, then the resultant DC$\mathfrak{I}$s of the R-Jacobi polynomials form the finite EOP sequences of series $-+\vdots\bar{M}_p$. We thus come to the very broad brand families of both infinite and finite EOP sequences, which will be examined in detail in a separate publication.

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 18 February 2025 doi:10.20944/preprints202502.1228.v1

One can also combine the nodeless PFSs of type $\underset{\sim}{\mathbf{a}}'$ with the juxtaposed pairs of the eigenfunctions (type $\underset{\sim}{\mathbf{c}}$) to construct the RCSLEs quantized by the finite EOP sequences composed of the Wronskian transforms of the R-Jacobi polynomials, similar to the finite EOP sequences formed by the Wronskian transforms of the R-Bessel and R-Jacobi polynomials [26,28].

The general case using the seed functions of all the four types, $\underset{\sim}{\mathbf{a}}$, $\underset{\sim}{\mathbf{b}}$, $\underset{\sim}{\mathbf{a}}'$, and $\underset{\sim}{\mathbf{c}}$ represent a much more challenging problem.

Appendix A. Rudjak-Zakhariev Transformation of Generic CSLE

Let $\phi_\tau[\eta;\vec{\lambda}_o]$ be a nodeless solution of a CSLE

$$\left\{\frac{d^2}{d\eta^2} + I^o[\eta;\vec{\lambda}_o] + \varepsilon\,\rho[\eta]\right\}\Phi[\eta;\vec{\lambda}_o;\varepsilon] = 0 \tag{A1}$$

at the energy

$$\varepsilon = \varepsilon_\tau(\vec{\lambda}_o), \tag{A2}$$

i.e.,

$$\left\{\frac{d^2}{d\eta^2} + I^o[\eta;\vec{\lambda}_o] + \varepsilon_\tau(\vec{\lambda}_o)\,\rho[\eta]\right\}\phi_\tau[\eta;\vec{\lambda}_o] = 0 \,. \tag{A3}$$

We define the Rudyak-Zahariev transformation of the given CSLE via the requirement that the function

$${}^*\phi_\tau[\eta;\lambda_o] = \frac{\rho^{-1/2}[\eta]}{\phi_\tau[\eta;\lambda_o]} \tag{A4}$$

is the solution of the transformed CSLE:

$$\left\{\frac{d^2}{d\eta^2} + I^o[\eta;\vec{\lambda}_o\,|\,\tau] + \varepsilon\,\rho[\eta]\right\}\Phi[\eta;\vec{\lambda}_o;\varepsilon] = 0 \tag{A5}$$

at the same energy (A2), i.e.,

$$\left\{\frac{d^2}{d\eta^2} + I^o[\eta;\vec{\lambda}_o\,|\,\tau] + \varepsilon_\tau(\vec{\lambda}_o)\rho[\eta]\right\}{}^*\phi_\tau[\eta;\vec{\lambda}_o] = 0. \tag{A6}$$

Representing both CSLEs (A3) and (A6) in the Riccati form:

$$I^o[\eta;\vec{\lambda}_o] = -ld^{\,2}\phi_\tau[\eta;\vec{\lambda}_o] - \dot{ld}\,\phi_\tau[\eta;\vec{\lambda}_o] - \varepsilon_\tau(\vec{\lambda}_o)\,\rho[\eta] \tag{A7}$$

and

$$I^o[\eta;\vec{\lambda}_o\,|\,\tau] := -ld^{\,2}\,{}^*\phi_\tau[\eta;\vec{\lambda}_o] - \dot{ld}\,{}^*\phi_\tau[\eta;\vec{\lambda}_o] - \varepsilon_\tau(\vec{\lambda}_o)\rho[\eta], \tag{A8}$$

subtracting one from another, and also taking into account that the logarithmic derivatives of the TF $\phi_\tau[\eta;\vec{\lambda}_o]$ and its reciprocal (A4) are related in the elementary fashion:

$$ld\,{}^*\phi_\tau[\eta;\lambda_o] = -ld\,\phi_\tau[\eta;\lambda_o] - \tfrac{1}{2}\,ld\,\rho[\eta] \tag{A9}$$

one finds [8]

$$I^o[\eta;\vec{\lambda}_o\,|\,\tau] = I^o[\eta;\vec{\lambda}_o] + 2\sqrt{\rho[\eta]}\,\frac{d}{d\eta}\frac{ld\,\phi_\tau[\eta;\vec{\lambda}_o]}{\sqrt{\rho[\eta]}} + \mathfrak{I}\{\rho[\eta]\}, \tag{A10}$$

where the last summand represents the so-called [48] 'universal correction' defined via the generic formula

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 18 February 2025 doi:10.20944/preprints202502.1228.v1

$$\mathcal{G}\{f[z]\} := \frac{1}{2}\sqrt{f[z]}\frac{d}{dz}\frac{ld\, f[z]}{\sqrt{f[z]}}. \quad \text{(A11)}$$

Alternatively, we can represent (A10) as

$$I^{o}[\eta;\vec{\lambda}_o \mid \tau] = I^{o}[\eta;\vec{\lambda}_o] + 2\sqrt{\rho[\eta]}\frac{d}{d\eta}\frac{ld\,\psi_\tau[\eta;\vec{\lambda}_o]}{\sqrt{\rho[\eta]}}, \quad \text{(A12)}$$

where

$$\psi_\tau[\eta;\vec{\lambda}_o] := \rho^{1/4}[\eta]\phi_\tau[\eta;\vec{\lambda}_o]. \quad \text{(A13)}$$

By defining the 'Riccati-Schrödinger' (RS) function [9] as

$$w_\tau[\eta;\vec{\lambda}_o] := -\rho^{-1/2}[\eta]ld\,\psi_\tau[\eta;\vec{\lambda}_o] \quad \text{(A14)}$$

we come to the following formula relating the Liouville potentials of the CSLEs (A5) and (A6):

$$V[\eta;\vec{\lambda}_o \mid \tau] = V[\eta;\vec{\lambda}_o] + \frac{2}{\sqrt{\rho[\eta]}}\frac{d}{d\eta}\frac{ld\, w_\tau[\eta;\vec{\lambda}_o]}{\sqrt{\rho[\eta]}}. \quad \text{(A15)}$$

The change of variable $\eta(x)$ determined by the first-order ODE

$$\eta'(x) = \rho^{-1/2}[\eta(x)] \quad \text{(A16)}$$

then converts (A15) to the potential (5) in [9].

Appendix B. Nodelessness of PFSs Below the Lowest Eigenvalue

Examination of the asymptotic behavior of the PFSs $\underset{\sim}{\Psi}_{r}[\eta;\vec{\lambda}_o;\underset{\sim}{\varepsilon}]$ near the corresponding endpoints reveals that

$$\lim_{\eta\to r}\left((\eta-1)\dot{\underset{\sim}{\Psi}}_{r}[\eta;\vec{\lambda}_o;\underset{\sim}{\varepsilon}]\right) = 0 \quad (r = 1,\infty) \quad \text{(A17)}$$

and therefore the so-called [90] 'generalized' Wronskian (g-W) of two PFSs near the endpoint 0 or 1 accordingly,

$$\underset{\sim}{\mathbf{W}}_{r}[\eta;\vec{\lambda}_o;\underset{\sim}{\varepsilon}] := \not{p}[\eta]\,W\{\underset{\sim}{\Psi}_{r}[\eta;\vec{\lambda}_o;\underset{\sim}{\varepsilon}],\underset{\sim}{\psi}_0[\eta;\vec{\lambda}_o]\} \quad \text{for } \eta \ge 1 \quad \text{(A18)}$$

(the symplectic form in Everitt's terms [91]) vanishes at the endpoint in question [92]:

$$\lim_{\eta\to 1+}\underset{\sim}{\mathbf{W}}_{1}[\eta;\vec{\lambda}_o;\underset{\sim}{\varepsilon}] = \lim_{\eta\to\infty}\underset{\sim}{\mathbf{W}}_{\infty}[\eta;\vec{\lambda}_o;\underset{\sim}{\varepsilon}] = 0 \quad . \quad \text{(A19)}$$

Another important relation pointed to in [92]:

$$\dot{\underset{\sim}{\mathbf{W}}}_{r}[\eta;\vec{\lambda}_o;\underset{\sim}{\varepsilon}] = [\underset{\sim}{\varepsilon}_0(\vec{\lambda}_o) - \underset{\sim}{\varepsilon}]\,\not{w}[\eta;\vec{\lambda}_o]\underset{\sim}{\psi}_0[\eta;\vec{\lambda}_o]\underset{\sim}{\Psi}_{r}[\eta;\vec{\lambda}_o;\underset{\sim}{\varepsilon}] \quad \text{(A20)}$$

reveals that the g-W is a monotonically decreasing function of η as far as both solutions remain positive and $\underset{\sim}{\varepsilon} < \underset{\sim}{\varepsilon}_0(\vec{\lambda}_o)$.

Selecting the PFS by the DBC allows one to prove that any PFS below the lowest energy level keeps its sign inside the quantization interval $(1,\infty)$:

Theorem B. *No PFS below the lowest eigenvalue may have nodes inside the quantization interval.*

Proof of Theorem B. Suppose that that the g-W (A18) evaluated at the energy for $\underset{\sim}{\varepsilon} < \underset{\sim}{\varepsilon}_0(\vec{\lambda}_o)$ vanishes at $\eta = \eta_r \in (1,\infty)$, i.e.

$$\underset{\sim}{\Psi}_{r}[\eta_r;\vec{\lambda}_o;\underset{\sim}{\varepsilon}] = 0. \quad \text{(A21)}$$

Without loss of generality we can assume that lowest-energy eigenfunction and the PFS near the singular endpoint $\eta = r$ at an energy $\underset{\sim}{\varepsilon} < \underset{\sim}{\varepsilon}_0(\vec{\lambda}_o)$ are both positive for $1 < \eta < \eta_1$ or $\eta > \eta_\infty$

 doi:10.20944/preprints202502.1228.v1

accordingly (flopping signs if necessary). Under the latter constraint the g-W must be either positive or respectively negative at this point:

$$\underset{\sim}{\mathbf{W}}_1[\eta_1;\vec{\lambda}_o;\underset{\sim}{\varepsilon}] > 0 \qquad \text{(A22)}$$

and

$$\underset{\sim}{\mathbf{W}}_\infty[\eta_\infty;\vec{\lambda}_o;\underset{\sim}{\varepsilon}] < 0, \qquad \text{(A23)}$$

which contradicts to the relations

$$\underset{\sim}{\mathbf{W}}_1[\eta_1;\vec{\lambda}_o;\underset{\sim}{\varepsilon}] = (\eta_1 - 1)\,\psi_0[\eta_1;\vec{\lambda}_o]\,\dot{\Psi}_1[\eta_1;\vec{\lambda}_o;\underset{\sim}{\varepsilon}] < 0 \qquad \text{(A24)}$$

and accordingly

$$\underset{\sim}{\mathbf{W}}_\infty[\eta_\infty;\vec{\lambda}_o;\underset{\sim}{\varepsilon}] = (\eta_\infty - 1)\,\psi_0[\eta_\infty;\vec{\lambda}_o]\,\dot{\Psi}_\infty[\eta_\infty;\vec{\lambda}_o;\underset{\sim}{\varepsilon}] < 0, \qquad \text{(A25)}$$

keeping in mind that

$$\dot{\Psi}_1[\eta_1;\vec{\lambda}_o;\underset{\sim}{\varepsilon}] < 0 \qquad \text{(A26)}$$

and

$$\dot{\Psi}_\infty[\eta_\infty;\vec{\lambda}_o;\underset{\sim}{\varepsilon}] > 0. \qquad \text{(A27)}$$

We thus confirmed that the PFS $\underset{\sim}{\Psi}_r[\eta_r;\vec{\lambda}_o;\underset{\sim}{\varepsilon}]$ is necessarily nodeless at any energy $\underset{\sim}{\varepsilon} < \underset{\sim}{\varepsilon}_0(\vec{\lambda}_o)$. □

Funding: This research received no funding

Data Availability Statement: No new data were created.

Conflicts of Interest: The author declares no conflict of interest.

Abbreviations

BQR	Bagchi-Quesne-Roychoudhury
CDBESs	continuously degenerate bound energy states
ChExp	characteristic exponent
CSLE	canonical Sturm-Liouville equation
DBC	Dirichlet boundary condition
DPS	differential polynomial system
DCT	Darboux-Crum transformation
DC$\mathfrak{J}$	Darboux-Crum transform
DT	Darboux deformation
DT	Darboux transformation
D$\mathfrak{J}$	Darboux transform
EOP	exceptional orthogonal polynomial
ExpDiff	exponent difference
GDT	generalized Darboux transformation
h-PT	hyperbolic Pöschl-Teller
JRef	Jacobi-reference
JS	Jacobi-seed
LC	limit circle
LDT	Liouville-Darboux transformation
LP	limit point
ODE	ordinary differential equation
OPS	orthogonal polynomial system
PD	polynomial determinant
PF	polynomial fraction
PFS	principal Frobenius solution

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 18 February 2025 doi:10.20944/preprints202502.1228.v1

p-SLE	prime Sturm-Liouville equation
p-W	pseudo-Wronskian
q-RS	quasi-rational solution
q-RTF	quasi-rational transformation function
RCSLE	rational canonical Sturm-Liouville equation
RDCT	rational Darboux-Crum transformation
RDC$\mathfrak{T}$	rational Darboux-Crum transform
RDT	rational Darboux transformation
RD$\mathfrak{T}$	rational Darboux transform
restr-HRef	restrictive Heun-reference
R-Jacobi	Romanovski-Jacobi
R-Routh	Romanovski-Routh
RRZ$\mathfrak{T}$	rational Rudjak-Zakharov transform
RRZT	rational Rudjak-Zakharov transformation
RSLP	rational Sturm-Liouville problem
RS	Riccati-Schrödinger
RTSI	rational translationally shape-invariant
RZ$\mathfrak{T}$	Rudjak-Zakharov transform
RZT	Rudjak-Zakharov transformation
SLE	Sturm-Liouville equation
SLP	Sturm-Liouville problem
t-PT	trigonometric Pöschl-Teller
TF	transformation function
TFI	translationally form-invariant
TSI	translationally shape-invariant

References

1. Natanson G. Gauss-seed nets of Sturm-Liouville problems with energy-independent characteristic exponents and related sequences of exceptional orthogonal polynomials I. Canonical Darboux transformations using almost-everywhere holomorphic factorization functions. *arXiv* **2013**. Available online: arXiv:1305.7453v1 (accessed on May 31 2013). https://doi.org/10.48550/arXiv.1305.7453.
2. Odake, S.; Sasaki, R. Infinitely many shape invariant potentials and new orthogonal polynomials. *Phys. Lett. B* **2009**, *679*, 414–417. https://doi.org/10.1016/j.physletb.2009.08.004.
3. Ho, C.-L.; Odake, S.; Sasaki, R. Properties of the Exceptional (X_l) Laguerre and Jacobi Polynomials. *SIGMA* **2011**, *7*, 107. https://doi.org/10.3842/SIGMA.2011.107.
4. Pöschl G.; Teller F. Bemerkungen zur Quantenmechanik des anharmonischen Oszillators. *Zs. Phys.* **1933**, *83*, 143 -151 https://doi.org/10.1007/BF01331132.
5. Romanovski, V.I. Sur quelques classes nouvelles de polynomes orthogonaux. *C.R. Acad. Sci.* **1929**, *188*, 1023–1025.
6. Lesky, P.A. Vervollstandigungderklassischen Orthogonalpolynomedurch Erganzungen zum Askey—Schema der hypergeometrischen orthogonalen Polynome. *Ost. Ak. Wiss.* **1995**, *204*, 151–166.
7. Lesky, P.A. Endliche und unendliche Systeme von kontinuierlichen klassichen Othogonalpolynomen. *Z. Angew. Math. Mech.* **1996**, *76*, 181–184. https://doi.org/10.1002/zamm.19960760317.
8. Natanson, G. Breakup of SUSY Quantum Mechanics in the Limit-Circle Region of the Darboux-Pöschl-Teller Potential. 2019. Available online: www.researchgate.net/publication/334960618 (accessed on 1 October 2019). https://doi.org/10.13140/RG.2.2.35328.17926/4.
9. Grandati Y. New rational extensions of solvable potentials with finite bound state spectrum. *Phys. Lett. A* **2012**, *376*, 2866-2872. https://doi.org/10.1016/j.physleta.2012.09.037.

10. Yadav, R.K.; Khare, A.A.; Mandal, B.P. The scattering amplitude for one parameter family of shape invariant potentials related to X_m Jacobi polynomials. *Phys. Lett. B* **2013**, *723*, 433–435. https://doi.org/10.1016/j.physletb.2013.05.03.
11. Yadav, R. K.; Khare A.; Kumari N.; Bagchi B.; Mandal B. P, Parametric symmetries in exactly solvable real and P T symmetric complex potentials. *J. Math. Phys*. **2016**, *57*, 062106. https:/z/doi.org/10.1063/1.4954330.
12. Yadav, R.K.; Banerjee S., Kumari N.; Khare, A.A.; Mandal, B.P. One parameter family of rationally extended isospectral potentials, *Ann. Phys.* **2022**, *436*, 168679, 27 pages. https://doi.org/10.1016/j.aop.2021.168679.
13. Banerjee S.; Yadav, R.K.; Khare, A.A.; Kumari N.; Mandal, B.P. Solutions of one-dimensional Dirac equation associated with exceptional orthogonal polynomials and the parametric symmetry *Int. J. Modern. Phys A* **2023**, *38*, 2350069, 23 pages. https://doi.org/10.1142/S0217751X23500690.
14. Yadav R. K.; Khare A.; Mandal, B.P. Supersymmetry and shape invariance of exceptional orthogonal polynomials. *Ann. Phys.* **2022**, *444*, 169064. https://doi.org/10.1016/j.aop.2022.169064.
15. Gȯmez-Ullate D.; Kamran N.; Milson R. On orthogonal polynomials spanning a non-standard flag. *Contemp. Math* **2012**, *563*, 51-71. https://doi.org/10.1090/conm/563/11164.
16. Gȯmez-Ullate D.; Marcellan, F.; Milson R., Asymptotic and interlacing properties of zeros of exceptional Jacobi and Laguerre polynomials," *J. Math. Anal. Appl.* **2013**, *399*, 480-495. https://doi.org/10.1016/j.jmaa.2012.10.032.
17. Liaw, C.; Littlejohn, L.L.; Stewart, J. Spectral analysis for the exceptional Xm-Jacobi equation. *Electr. J. Diff. Equ.* **2015**, *194*, 1–10.
18. Garcia-Ferrero M.; Gomez-Ullate D.; Milson R. A Bochner type classification theorem for exceptional orthogonal polynomials. *J. Math. Anal. & Appl.* **2019**, *472*, 584-626. https://doi.org/10.1016/j.jmaa.2018.11.042.
19. Garcia-Ferrero M.; Gomez-Ullate D.; Milson R. Classification of exceptional Jacobi polynomials, 2024. Available online: arXiv:2409.02656v1 (accessed on Sept 4 2024). https://doi.org/10.48550/arXiv.2409.026.
20. Bagchi B.; Quesne C.; Roychoudhury R. Isospectrality of conventional and new extended potentials, second-order supersymmetry and role of PT symmetry. *Pramana J. Phys.* **2009**, *73*, 337-347. https://doi.org/10.1007/s12043-009-0126-4.
21. Quesne C, Extending Romanovski polynomials in quantum mechanics. *J. Math. Phys.* **2013,** *54,* 122103, 15 pages. https://doi.org/10.1063/1.4835555.
22. Odake S. and Sasaki R. Krein-Adler transformations for shape-invariant potentials and pseudo virtual states. *J Phys A* **2013**, *46*, 245201, 24pp. https://doi.org/10.1088/1751-8113/46/24/245201.
23. Yadav R. K.; Khare A.; Mandal, B.P. The scattering amplitude for newly found exactly solvable potential. *Ann. Phys.* **2013**, *331*, 313-316. https://doi.org/10.1016/j.physletb.2013.05.036.
24. Natanson G. Biorthogonal differential polynomial system composed of X-Jacobi polynomials from different sequences. **2018**. Available online: https://www.researchgate.net/publication/322851312 (accessed on Jan. 22, 2018). https://doi.org/10.13140/RG.2.2.12116.50569.
25. Natanson G. Use of Wronskians of Jacobi polynomials with common complex indexes for constructing X-DPSs and their infinite and finite orthogonal subsets, **2019**. Available from www.researchgate.net/publication/331638063 (accessed on March 10 2019). https://doi.org/10.13140/RG.2.2.10090.36809/1.
26. Natanson G. Quantization of rationally deformed Morse potentials by Wronskian transforms of Romanovski-Bessel polynomials. *Acta* Polytech. **2022**, *62*, 100-117. https://doi.org/10.14311/AP.2022.62.0100.
27. Natanson, G. Rediscovery of Routh polynomials after hundred years in obscurity. In *Recent Research in Polynomials*; Özger, F., Ed.; IntechOpen: London, UK**, 2023**; 27p. Available online:

https://www.intechopen.com/chapters/1118656 (accessed on 26 January 2023). https://doi.org/10.5772/intechopen.1000855.

28. Natanson G. Uniqueness of Finite Exceptional Orthogonal Polynomial Sequences Composed of Wronskian Transforms of Romanovski-Routh Polynomials. *Symmetry* **2024**, *16*, 282, 42 pages. https:// doi.org/10.3390/sym16030282. Corrected online: www.researchgate.net/publication/387949136 (accessed on 15 Jan 2025).
29. Natanson G. X1-Jacobi differential polynomials systems and related double-step shape-invariant Liouville potentials solvable by exceptional orthogonal polynomials. *Symmetry* **2025**, *17*, 109, 51 pages. Corrected online: www.researchgate.net/publication/387949136 (accessed on 15 Jan 2025). https://doi.org/10.3390/sym17010109.
30. Soltész, T.; Pethő L.; Lévai G. Unified supersymmetric description of shape-Invariant potentials within and beyond the Natanzon class. *Symmetry* **2024**, *16*, 174; 25 pp. https://doi.org/10.3390/sym16020174.
31. Gòmez-Ullate, D.; Grandati, Y.; Milson, R. Extended Krein-Adler theorem for the translationally shape invariant potentials. *J. Math. Phys.* **2014**, *55*, 043510. https://doi.org/10.1063/1.4871443.
32. Gòmez-Ullate D.; Kamran N.; Milson R. Exceptional orthogonal polynomials and the Darboux transformation. *J. Physics A* **2010**, *43*, 434016, 16 pages. https://doi.org/10.1088/1751-8113/43/43/434016.
33. Gòmez-Ullate, D.; Kamran, N.; Milson, R. An extension of Bochner's problem: Exceptional invariant subspaces. *J. Approx. Theory* **2010**, *162*, 987–1006. https://doi.org/10.1016/j.jat.2009.11.002.
34. Everitt, W.N.; Littlejohn, L.L. Orthogonal polynomials and spectral theory: A survey, In *Orthogonal Polynomials and Their Applications. IMACS Annals on Computing and Applied Mathematics*; Brezinski, C., Gori, L., Ronveaux, A., Eds.; J.C. Baltzer: Basel, Switzerland, 1991; Volume 9, pp. 21–55, ISSN 1012-2435.
35. Everitt, W.N.; Kwon, K.H.; Littlejohn, L.L.; Wellman, R. Orthogonal polynomial solutions of linear ordinary differential equations. *J. Comp. Appl. Math.* **2001**, *133*, 85–109. https://doi.org/10.1016/S0377-0427(00)00636-1.
36. Bochner, S. Über Sturm-Liouvillesche Polynomsysteme. *Math. Z.* **1929**, 29, 730–736. https://doi.org/ 10.1007/BF01180560.
37. Kwon, K.H.; Littlejohn, L.L. Classification of classical orthogonal polynomials. *J. Korean Math. Soc.* **1997**, 34, 973–1008.
38. Andrianov, A.N.; Ioffe, M.V. The factorization method and quantum systems with equivalent energy spectra. *Phys. Lett. A* **1984**, *105*, 19–22. https://doi.org/10.1016/0375-9601(84)90553-X.
39. Andrianov A. A ; Borisov N. V.; Ioffe M. V. Factorization method and Darboux transformation for multidimensional Hamiltonians," *Sov. Phys. JETP Lett.* **61** (1984), 1078-1088. https://doi.org/10.1007/BF01029109.
40. Darboux G. Leçons sur la théorie générale des surfaces et les applications géométriques du calcul infinitésimal. Vol 2. Paris : Gauthier-Villars, 1915: 210-5.
41. Spiridonov V.; Zhedanov A. Discrete Darboux transformations, the discrete-time Toda lattice, and the Askey-Wilson polynomials," *Methods and Appl. Anal.* **2** (1995), 369–398. https://doi.org/10.4310/MAA.1995.v2.n4.a1.
42. Gòmez-Ullate, D.; Kamran, N.; Milson, R. Supersymmetry and algebraic deformations. *J. Phys. A* **2004**, *37*, 10065, 16 pages. https://doi.org/10.1088/0305-4470/37/43/004.
43. Natanzon G. A. Use of the Darboux theorem for constructing the general solution of the Schrödinger equation with the Pöschl-Teller potential. *Vestn. Leningr. Univ.* (in Russian), No 16, (1977), 33-39. Available online: https://www.researchgate.net/publication/316150022.
44. Natanson G. Single-source nets of algebraically-quantized reflective Liouville potentials on the line I. Almost-everywhere holomorphic solutions of rational canonical Sturm-Liouville equations with second-order poles. *arXiv* **2015**. Available online: arXiv:1503.04798v2 (accessed on Dec 9 2015).

45. Rudyak B. V.; Zakhariev B. N. New exactly solvable models for Schrödinger equation. *Inverse Problems.* **1987**, *3*, 125-133. https://doi.org/10.1088/0266-5611/3/1/014.
46. Schnizer W.A. Leeb H. Exactly solvable models for the Schrödinger equation from generalized Darboux transformations. *J Phys A* **1993**, *26*. 5145-5156. https://doi.org/10.1088/0305-4470/26/19/041.
47. Schnizer W.A.; Leeb H. Generalizes Darboux transformations: classification of inverse scattering methods for the
radial Schrodinger equation. *J Phys A* **1994**, *27*, 2605-2614. https://doi.org/10.1088/0305-4470/27/7/035
48. Natanson, G. Darboux-Crum Nets of Sturm-Liouville Problems Solvable by Quasi-Rational Functions I. General Theory. **2018**. pp. 1–99. Available online: https://www.researchgate.net/publication/323831953.
49. Szego G. *Orthogonal Polynomials*. (New York: Amer. Math. Soc., 1959) p.150.
50. Grandati Y.; Bérard A. Rational solutions for the Riccati–Schrödinger equations associated to translationally shape invariant potentials. *Ann. Phys*. **2010**, *325*, 1235-1259.
51. Gómez-Ullate D.; Grandati, Y.; Milson R. Shape invariance and equivalence relations for pseudo-Wronskians of Laguerre and Jacobi polynomials. *J. Phys.* A **2018,** *51*, 345201, 28 pages. https://doi.org/10.1088/1751-8121/aace4b.
52. Erdelyi, A.; Bateman, H. *Transcendental Functions*; McGraw Hill: New York, NY, USA, 1953; Volume 1.
53. Cooper, F.; Khare, A.; Sukhatme, U.P. Supersymmetry and quantum mechanics. *Phys. Rep*. **1995**, *251*, 267–385. https://doi.org/10.1016/0370-1573(94)00080-M.
54. Natanzon, G.A. Study of the one-dimensional Schrödinger equation generated from the hypergeometric equation. Vestn. Len ingr. Univ. 1971, 10, 22–28 (in Russian). [English translation is available online: https://doi.org/10.48550/arXiv.phys ics/9907032.]
55. Natanson G. Survey of nodeless regular almost-everywhere holomorphic solutions for Exactly solvable Gauss-reference Liouville potentials on the line I. Subsets of nodeless Jacobi-seed solutions co-existent with discrete energy spectrum. *arXiv* **2016**. Available online: arXiv:1606.08758. (accessed on 29 Oct 2016). https://doi.org/10.48550/arXiv.1606.08758.
56. Gesztesy, F.; Simon, B.; Teschl, G. Zeros of the Wronskian and renormalized oscillation theory. *Am. J. Math.* **1996**, *118*, 571–594. https://doi.org/10.1353/ajm.1996.0024.
57. Askey R. An integral of Ramanujan and orthogonal polynomials," *J. Indian Math. Soc.* **51** (1987), 27-36
58. Chen, M.P.; Srivastava, H.M. Orthogonality relations and generating functions for Jacobi polynomials and related hypergeometric functions, *Appl. Math. Comput.* **1995**, *68*, 153–188. https://doi.org/10.1016/0096-3003(94)00092-I.
59. Natanson G. Use of Wronskians of Jacobi polynomials with common complex indexes for constructing X-DPSs and their infinite and finite orthogonal subsets, **2019**. Available from https://www.researchgate.net/publication/331638063 (accessed on March 10 2019).
60. Alhaidari A,D,; AssiI.A. Finite-Series Approximation of the Bound States for Two Novel Potentials. *Physics* **2022**, *4*, 1067-1080; https://doi.org/10.3390/physics4030070.
61. Raposo A.P.; Weber H. J.; Alvarez-Castillo D.E; Kirchbach M. Romanovski polynomials in selected physics prob-lems," Centr. Eur. J. Phys. **2007**, 5, 253-284. https://doi.org/10.2478/s11534-007-0018-5
62. Weber H. J. Connections between Romanovski and other polynomials," C. Eur. J. Math. 5 (2007), 581-595
63. Avarez-Castillo D. E.; Kirchbach M., Exact spectrum and wave functions of the hyperbolic Scarf potential in terms of finite Romanovski polynomials," *Rev. Mex. Fis.E* **2007**, ***53***, 143-154.
64. Martınez-Finkelshtein A., Silva Ribeiro L.L.; Sri Ranga A.; Tyaglov M. Complementary Romanovski-Routh polynomials: From orthogonal polynomials on the unit circle to Coulomb wave functions. Proc. Amer. Math. Soc. **2019**, *147*, 2625–2640. https://doi.org/10.1090/proc/14423.
65. Masjed-Jamei M. *Special Functions and Generalized Sturm-Liouville Problems* (Springer Nature Switzerland AG 2020), pp. 21-23. springer.com/book/10.1007/978-3-030-32820-7.

66. Koepf W.; Masjed-Jamei M. A generic polynomial solution for the differential equation of hypergeometric type and six sequences of orthogonal polynomials related to it," *Integral Transforms and Special Functions* **2006**, *17*, 559-576. https://doi.org/10.1080/10652460600725234.
67. Youssri Y.H.; Zaky M.A.; Hafezd R.M. Romanovski-Jacobi spectral schemes for high-order differential equations. *Appl. Numer. Math.* **2023**, *198*, 148 – 159. https://doi.org/10.1016/j.apnum.2023.12.015.
68. Routh E.J. On some properties of certain solutions of a differential equation of second order. *Proc. London Math. Soc.* **1884**, 16, 245-261. https://doi.org/10.1112/plms/s1-16.1.245.
69. Das S.; Swaminathan A. Higher order derivatives of R-Jacobi polynomials," *AIP Conference Proceed.* **2016**, *1739*, 020058, 8 pages. https://doi.org/10.1063/1.4952538.
70. Masjedjamei M. Three finite classes of hypergeometric orthogonal polynomials and their application in functions approximation," *Integral Transforms and Special Functions* **2002**, *13*, 169-190. https://doi.org/10.1080/10652460212898.
71. Quesne, C. Solvable Rational Potentials and Exceptional Orthogonal Polynomials in Supersymmetric Quantum Mechanics. *SIGMA* **2009**, *5*, 084, 24 pages. https://doi.org/10.3842/SIGMA.2009.084.
72. Natanson G. Equivalence relations for Darboux-Crum transforms of translationally form-invariant Sturm-Liouville equations. **2021**. Available from www.researchgate.net/publication/353131294 (accessed on 9 Aug 2021). DOI: 10.13140/RG.2.2.36086.93768/1.
73. Ho C.-L. Prepotential approach to exact and quasi-exact solvabilities. *Ann. Phys.* **2008**, *323*, 2241-2252. https://doi.org/10.1016/j.aop.2008.04.010.
74. Ho C-L, Prepotential approach to exact and quasi-exact solvabilities of Hermitian and non-Hermitian Hamiltonians. **2008**. Available online: arXiv:0801.0944 [hep-th]. (accessed on Feb 7 2008). https://doi.org/10.48550/arXiv.0801.0944
75. Ho C.-L. Simple unified derivation and solution of Coulomb, Eckart and Rosen-Morse potentials in prepotential approach. *Ann. Phys.* **2009**, *324*, 1095-1104. https://doi.org/10.1016/j.aop.2008.10.004
76. Ho C-L Prepotential approach to solvable rational potentials and exceptional orthogonal polynomials. *Prog. Theor. Phys.* **2011**, *126*, 185-201. https://doi.org/10.1143/PTP.126.185.
77. Ho C-L Prepotential approach to solvable rational extensions of Harmonic Oscillator and Morse potentials *J. Math. Phys.* **2011**, *52*, 122107, 8 pages. https://doi.org/10.1063/1.3671966.
78. Rosen N.; Morse P.M. On the vibrations of polyatomic molecules," *Phys. Rev,* **1932**, *42*, 210-217. https://doi.org/10.1103/PhysRev.42.210.
79. Natanson, G. Dutt-Khare-Varshni potential and its quantized-by-Heun-polynomials SUSY partners as nontrivial examples of solvable potentials explicitly expressible in terms of elementary functions. *arXiv* **2015**, pp. 1-51 Available online: arXiv:1508.04738v1 (accessed on 11 Aug 2015). https://doi.org/10.48550/arXiv.1508.04738.
80. Manning M. F.; Rosen N. A potential function for the vibrations of diatomic molecules. *Phys. Rev.* **1933**, *44*, 953.
81. Sukumar C.V. Supersymmetry, factorization of the Schrodinger equation and a Hamiltonian hierarchy. *J. Phys. A* **1985**, *18*, L57-L61. https://doi.org/10.1088/0305-4470/18/2/001.
82. Sukumar C. V. Supersymmetric quantum mechanics of one-dimensional system. *J. Phys. A* **1985**, *18*, 2917–2936. https://doi.org/10.1088/0305-4470/18/15/020.
83. Sukumar, C. V. Supersymmetric quantum mechanics and the inverse scattering method. *J. Phys.* A **1985**, *18*, 2937–2955. https://doi.org/10.1088/0305-4470/18/15/021.
84. Gangopadhyaya A.; Panigrahi P. K. ; Sukhatme U. P. Analysis of inverse-square potentials using supersymmetric quantum mechanics. *J. Phys. A.* **1994**, *27*, 4295-4300. https://doi.org/10.1088/0305-4470/27/12/032.
85. Gangopadhyaya, A.; Mallow J. V.; Rasinariu C. *Supersymmetric Quantum Mechanics. An Introduction.* (London: World Scientific Publishing, 2011).

86. Weisstein, E.W. Jacobi Polynomials. *MathWorld.* 2018. Available online: https://en.wikipedia.org/wiki/Jacobi_Recurrence_relations (accessed on 15 July 2024).
87. Natanson G. Use of normalizable seed solutions for constructing exactly-solvable rational Darboux-Crum transforms of Cooper-Ginocchio-Khare potential and its siblings. **2023**. europepmc.org/article/PPR/PPR707751 (Accessed Aug 18, 2023). https://doi.org/10.20944/preprints202308.1263.v1.
88. Yadav, R. K.; Kumari N.; Khare; A.; Mandal B. P. Group theoretic approach to rationally extended shape invariant potentials *Ann. Phys*. **2015**, *359*. 46-54. https://doi.org/10.1016/j.aop.2015.04.002.
89. Schulze-Halberg, A. Higher-order Darboux transformations with foreign auxiliary equations and equivalence with generalized Darboux transformations. *Appl. Math. Lett*. **2012**, *25*, 1520-1527. https://doi.org/10.1016/j.aml.2012.01.008.
90. Weidmann J. Spectral Theory of Sturm-Liouville Operators Approximation by Regular Problems, in *Sturm-Liouville Theory, Past and Present*, ed. W.O. Amrein, A. M. Hinz and D.B. Pearson (Birkhäuser Verlag, Basel), pp 75-98 (2005). http://dx.doi.org/10.1007/3-7643-7359-8_4.
91. Everitt, W.N. Note on the X_1-Jacobi orthogonal polynomials. *arXiv* **2008**, arXiv:0812.0728v1.
92. Everitt, W.N. *A Catalogue of Sturm-Liouville Differential Equations.* In *Sturm-Liouville Theory, Past and Present;* Amrein, W.O., Hinz, A.M., Pearson, D.B., Eds.; Birkhäuser Verlag: Basel, Switzerland, 2005; pp. 271–331. https://doi.org/10.1007/3-7643-7359-8_12.