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On a Generic Fractional Derivative Associated with the Riemann-Liouville Fractional Integral

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Abstract: In this paper, a generic fractional derivative is defined as a set of the linear operators left-inverse to the Riemann-Liouville fractional integral. Then the theory of the left-invertible operators developed by Przeworska-Rolewicz is applied for derivation of its properties. In particular, we characterize its domain, null-space, and projector operator, establish the interrelations between its different realisations, and present a generalized fractional Taylor formula involving the generic fractional derivative. Then we consider the fractional relaxation equation containing the generic fractional derivative, derive a closed form formula for its unique solution, and study its complete monotonicity.

Keywords: Riemann-Liouville fractional integral; left-inverse operator; projector operator; generalized fractional Taylor formula; fractional differential equations; complete monotonicity

MSC: 26A33; 26A06; 26A48; 33E12; 34A08; 44A15

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1. Introduction

In the meantime, the literature devoted to the fractional integrals and derivatives and the fractional differential equations is huge and every day new contributions are published. However, some basic questions related to the nature of the fractional derivatives, the properties that they are expected to possess, and the interrelations between different kinds of the fractional derivatives are still not completely clarified even in the simplest case of the fractional derivatives of the functions of a single variable. Moreover, availability of several unequal definitions of the fractional derivatives leads to repeating of derivation of the same mathematical results formulated for different fractional derivatives and to problems in choosing an appropriate definition for the mathematical models involving fractional derivatives.

In this paper, a partial solution to the problems mentioned above is suggested. We focus on the case of the time-fractional derivatives of the functions of a single variable and refer to [1] for a discussion of the space-fractional derivatives of functions depending on several variables and to [2,3] for surveys of other types of the fractional derivatives. It is worth mentioning that in the publications [4–6], some axioms or desiderata for the one-parameter Fractional Calculus (FC) operators of the functions of a single variable were suggested. However, several questions regarding their background as well as possible realisations of these axioms systems remained open.

In [7], an abstract schema for construction of the one-parameter families of the fractional derivatives for the functions of a single variable was developed. In particular, this schema includes the Riemann-Liouville, Caputo, Hilfer, and Djrbashian-Nersessian or the nth level time-fractional derivatives that were introduced in the literature so far. In this paper, we follow and deepen the ideas suggested in [7] and embed them into the framework of the theory of the left-invertible operators suggested by Przeworska-Rolewicz in her paper [8]. It is worth mentioning that whereas many of publications by Przeworska-Rolewicz are devoted to the theory of the right-invertible operators, the case of the left-invertible ones was considered in just one of her papers, namely, in [8].

Under some very reasonable assumptions, the only family of the one-parameter fractional integrals of the functions of one variable defined on a finite interval is the family of the Riemann-Liouville fractional integrals (see [7] and [9] for details). The well-accepted definition of the one-parameter

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fractional derivatives is in form of the left-inverse operators to the corresponding fractional integrals. Thus, for the functions of a single variable, one should focus on the Riemann-Liouville fractional integral and its left-inverse operators as the fractional derivatives. The class of such operators is not empty and contains all reasonable time-fractional derivatives introduced so far including the Riemann-Liouville, Caputo, and Hilfer fractional derivatives.

According to the terminology of Przeworska-Rolewicz, the Riemann-Liouville fractional integral belongs to the class of the left-invertible operators. In this paper, for the first time, we apply the theory of the left-invertible operators developed in [8] to the Riemann-Liouville fractional integral and a generic fractional derivative associated with this integral. It is worth mentioning that many basic and advanced properties of this generic fractional derivative directly follow from its definition as a set of the linear operators left-inverse to the Riemann-Liouville fractional integral. By derivation of these properties, we do not use any explicit formulas for the fractional derivatives and thus the obtained results are generic and cover all kinds of the time-fractional derivatives introduced so far including the Riemann-Liouville, Caputo, and Hilfer fractional derivatives. The problem of providing a constructive description of all realisations of the generic fractional derivative associated with the Riemann-Liouville fractional integral is still open. However, it is known that there exist infinitely many different families of the fractional derivatives of this kind in form of the nth level fractional derivatives, see [7] for details.

The structure of the rest of the paper is as follows: In the second section, we provide a definition and the basic properties of the generic fractional derivative of the functions of a single variable including a characterization of its domain, null-space, and projector operator. In the third section, some advanced results formulated for the generic fractional derivative of the functions of a single variable are presented. In particular, we discuss the interrelations between different realisations of the generic fractional derivative and formulate a generalized fractional Taylor formula involving the Riemann-Liouville fractional integral and the generic fractional derivative associated with this integral. The fourth section is devoted to analysis of the initial-value problems for the fractional relaxation equation with the generic fractional derivative introduced in the second section. For this problem, a closed form formula for its unique solution is derived in explicit form and complete monotonicity of the solution is studied. Finally, in the last section, some conclusions and open problems for research are formulated.

2. Definition and Basic Properties of the Generic Fractional Derivative

Nowadays, several unequal definitions of the time-fractional derivatives of the functions of a single variable including the Riemann-Liouville, Caputo, Hilfer, and Djrbashian-Nersessian or the nth level fractional derivatives are actively used in the FC literature. In contrary to this situation, there exists only one family of the fractional integrals $I^\alpha,\ \alpha > 0$ that satisfies the following natural conditions on the space E of functions that can be, say, $L_p(0,1)$, $1 \le p < +\infty$ or $C[0,1]$):

A1. $(I^1\, x)(t) = \int_0^t x(\tau)\, d\tau,\ x \in E,$

A2. $(I^\alpha\, I^\beta\, x)(t) = (I^{\alpha+\beta}\, x)(t),\ \alpha, \beta > 0,\ x \in E,$

A3. $\alpha \to I^\alpha$ is a continuous map of $(0, +\infty)$ into $\mathcal{L}(E)$ for some Hausdorff topology on $\mathcal{L}(E)$, weaker than the norm topology,

A4. $x \in E$ and $x(t) \ge 0$ (a.e. for $E = L_p(0,1)$) $\Rightarrow (I^\alpha\, x)(t) \ge 0$ (a.e. for $E = L_p(0,1)$) for all $\alpha > 0$.

This family is nothing else but the well-known family of the Riemann-Liouville fractional integrals defined by the formula

$$(I^\alpha\, x)(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1}\, x(\tau)\, d\tau,\ \alpha > 0. \tag{1}$$

The actual meaning of this important result derived in the paper [9] by Cartwright and McMullen is that, in a certain sense, the only "true" one-parameter fractional integrals $I^\alpha,\ \alpha > 0$ of the functions of a single variable defined on a finite interval are the family of the Riemann-Liouville fractional integrals.

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On the other hand, in the FC literature, several different families of the operators known as the fractional derivatives of the functions of a single variable are employed. For the sake of simplicity of the formulas, in what follows, we restrict ourselves to the case of the fractional derivatives with the order $\alpha \in (0,1)$. The case of the orders $\alpha \geq 1$ can be treated in the similar manner.

For long time, the most used fractional derivative of the functions of a single variable was the Riemann-Liouville fractional derivative defined as follows:

$$(D^{\alpha}_{RL}\, x)(t) := \frac{d}{dt}\,(I^{1-\alpha}\, x)(t) = \frac{d}{dt}\,\frac{1}{\Gamma(1-\alpha)}\int_0^t (t-\tau)^{-\alpha}\, x(\tau)\, d\tau,\ 0 < \alpha < 1. \tag{2}$$

However, within the last decades, the so-called Caputo fractional derivative that was in fact introduced already by Abel in [10,11] started to be actively employed in FC in general and especially in the theory of the fractional differential equations, both ordinary and partial ones:

$$(D^{\alpha}_{C}\, x)(t) := (I^{1-\alpha}\, x')(t),\ 0 < \alpha < 1. \tag{3}$$

Recently, the generalized Riemann-Liouville or the Hilfer fractional derivative of order α, $0 < \alpha < 1$ and type β, $0 \leq \beta \leq 1$ was introduced in [12] as follows:

$$(D^{\alpha,\beta}_{H}\, x)(t) := (I^{\beta(1-\alpha)}\,\frac{d}{dt}\, I^{(1-\alpha)(1-\beta)}\, x)(t). \tag{4}$$

In this paper, we use another parametrization of the Hilfer fractional derivative that is obtained from the formula (4) by setting $\gamma_1 = \beta(1-\alpha)$:

$$(D^{\alpha,\gamma_1}_{H}\, x)(t) = (I^{\gamma_1}\,\frac{d}{dt}\, I^{1-\alpha-\gamma_1}\, x)(t),\ 0 < \alpha < 1,\ 0 \leq \gamma_1 \leq 1-\alpha. \tag{5}$$

Finally, we mention the Djrbashian-Nersessian operator from [13] in a different parametrization that was suggested in [7] in form of the nth level fractional derivative of order α, $0 < \alpha < 1$ and type $\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_n)$:

$$(D^{\alpha,(\gamma)}_{nL}\, x)(t) := \left(\prod_{k=1}^{n}(I^{\gamma_k}\,\frac{d}{dt})\right)(I^{n-\alpha-s_n}\, x)(t), \tag{6}$$

where

$$s_k := \sum_{i=1}^{k}\gamma_i,\ k = 1,2,\ldots,n \tag{7}$$

and the conditions

$$0 \leq \gamma_k \text{ and } \alpha + s_k \leq k,\ \ k = 1,2,\ldots,n \tag{8}$$

are satisfied. To avoid a reduction of an nth level fractional derivative to a derivative of a lower level, we also suppose that the conditions

$$n-1 < \alpha + s_n \text{ and } \gamma_k < 1,\ \ k = 2,\ldots,n \tag{9}$$

hold valid.

The main property of the fractional derivatives mentioned above is that, on certain spaces of functions, they are left-inverse operators to the Riemann-Liouville fractional integral (1), i.e., they satisfy the so-called first fundamental theorem of FC (see [7] for details). In particular, as shown in [7], it is the case for the linear space

$$X := \{x :\ I^{\alpha}x \in \mathrm{AC}([0,\,1]) \text{ and } (I^{\alpha}\, x)(0) = 0\} \tag{10}$$

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that can be also characterized as follows (Theorem 2.3 in [3]):

$$X = I^{1-\alpha}(L_1(0,1)). \tag{11}$$

It is worth mentioning that the space X defined by the equation (10) suits for the first fundamental theorem of FC for all fractional derivatives mentioned above (see [7] for the proofs). For some fractional derivatives, this space can be extended. Say, the Riemann-Liouville fractional derivative (2) is a left-inverse operator to the Riemann-Liouville fractional integral (1) on the space $L_1(0,1)$.

The main objective of this paper is in suggestion of a unified approach to different kinds of the time-fractional derivatives of the functions of a single variable introduced so far. For this aim, the definitions and results presented in [8] for the left-invertible operators are adjusted to the case of the fractional integrals and derivatives of the functions of a single variable.

Let X be a linear space over $\mathbb{R}$ or $\mathbb{C}$ and $\mathcal{L}(X)$ denote the set of all linear operators acting from X to X. For an operator $A \in \mathcal{L}(X)$, its domain and null-space are denoted by D_A and N_A, respectively.

Definition 1 ([8]). *An operator $R \in \mathcal{L}(X)$ with $D_R = X$ is said to be left-invertible if there exists an operator $L \in \mathcal{L}(X)$ with $D_L \subset R(X)$ such that*

$$L\,R = I \text{ on } X, \tag{12}$$

where I denotes the identity operator.

As already mentioned, the Riemann-Liouville fractional integral defined by (1) possesses the left-inverse operators (say, the Riemann-Liouville fractional derivative on the space $L_1(0,1)$) and thus it belongs to the class of the left-invertible operators. Following [7] and using Definition 1, we now introduce a natural concept of a generic fractional derivative of the functions of a single variable.

Definition 2. *Let X be a linear space over $\mathbb{R}$ or $\mathbb{C}$ and $D_{I^\alpha} = X$, where I^α, $\alpha > 0$ is the Riemann-Liouville fractional integral defined by (1).*

The set of all linear operators $\mathbb{D}^\alpha$, $\alpha > 0$ left-inverse to the Riemann-Liouville fractional integral I^α in sense of Definition 1 is called a generic fractional derivative associated with the Riemann-Liouville fractional integral.

It is worth emphasizing that the generic fractional derivative is an infinite set of linear operators that in particular includes the Riemann-Liouville, Caputo, Hilfer, and the nth level fractional derivatives (see [7] for the proofs). By $\mathbb{D}^\alpha$ we denote a realisation of the generic fractional derivative, i.e, a certain linear operator left-inverse to the Riemann-Liouville fractional integral I^α. In what follows, we refer to $\mathbb{D}^\alpha$ as to a fractional derivative associated with the Riemann-Liouville integral.

Remark 1. *For the operators $R = I^\alpha$ and $L = \mathbb{D}^\alpha$, the formula (12) from Definition 1 takes the form*

$$\mathbb{D}^\alpha\, I^\alpha = I \text{ on } X. \tag{13}$$

This formula is called the first fundamental theorem of FC for the fractional derivative $\mathbb{D}^\alpha$ and the Riemann-Liouville fractional integral I^α.

As mentioned in [7], for a concrete fractional derivative $\mathbb{D}^\alpha$, the space X, where the relation (13) is valid, can be narrower compared to the space, where the Riemann-Liouville integral is defined. Thus, in general, one has to distinguish between these two spaces. However, to keep formulations and derivations of our results clearly arranged, in what follows, we use the same notation for both spaces if its meaning is clear from the context.

Remark 2. *The properties A1-A4 of the fractional integrals along with the formula (13) can be interpreted as a system of axioms for the one-parameter families of the fractional integrals and derivatives. In contrast to the axioms suggested in [4–6], most of these axioms describe the properties of the fractional integrals and the only*

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axiom related to the fractional derivatives is the formula (13). As a consequence, the family of the fractional integrals satisfying the axioms A1-A4 is unique. As to the families of the fractional derivatives satisfying the axiom (13), there are infinitely many of such families including the Riemann-Liouville, Caputo, Hilfer, and the nth level fractional derivatives mentioned at the beginning of this section.

For investigation of the generic fractional derivative associated with the Riemann-Liouville fractional integral, a concept of its projector operator plays a very important role.

Definition 3. *Let $\mathbb{D}^\alpha$ be a fractional derivative associated with the Riemann-Liouville fractional integral I^α. The operator*

$$\mathbb{P}^\alpha = I - I^\alpha \mathbb{D}^\alpha \text{ on } D_{\mathbb{D}^\alpha}, \tag{14}$$

where I denotes the identity operator is called a projector operator of the fractional derivative $\mathbb{D}^\alpha$.

Remark 3. *The formula (14) can be rewritten in form of the so-called second fundamental theorem of FC for the fractional derivative $\mathbb{D}^\alpha$, see [7]:*

$$(I^\alpha \mathbb{D}^\alpha\, x)(t) = x(t) - (\mathbb{P}^\alpha\, x)(t),\ x \in D_{\mathbb{D}^\alpha}. \tag{15}$$

As soon as one has an explicit formula for the projector operator $\mathbb{P}^\alpha$ of a certain fractional derivative $\mathbb{D}^\alpha$, one immediately gets an explicit form of the second fundamental theorem of FC for this fractional derivative.

For the Riemann-Liouville, Caputo, and Hilfer fractional derivatives, the explicit formulas for their projector operators are well-known (see, e.g., [7]). A closed form formula for the projector operator of the nth level fractional derivative has been recently derived in [14]. For convenience, in what follows, we use the notation

$$h_\alpha(t) := \frac{t^{\alpha-1}}{\Gamma(\alpha)},\ \alpha > 0. \tag{16}$$

As already mentioned, in this paper, we restrict ourselves to the case of the fractional derivatives of order $\alpha \in (0,1)$ and then the formulas of the projector operators for the Riemann-Liouville, Caputo, Hilfer, and the nth level fractional derivatives take the following form, respectively:

$$(\mathbb{P}^\alpha_{RL}\, x)(t) := (I - I^\alpha D^\alpha_{RL}\, x)(t) = (I^{1-\alpha}\, x)(0)\, h_\alpha(t), \tag{17}$$

$$(\mathbb{P}^\alpha_{C}\, x)(t) := (I - I^\alpha D^\alpha_{C}\, x)(t) = x(0)\, h_1(t), \tag{18}$$

$$(\mathbb{P}^\alpha_{H}\, x)(t) := (I - I^\alpha D^{\alpha,\gamma_1}_{H}\, x)(t) = (I^{1-\alpha-\gamma_1}\, x)(0)\, h_{\alpha+\gamma_1}(t), \tag{19}$$

$$(P^\alpha_{nL}\, x)(t) := (I - I^\alpha D^{\alpha,(\gamma)}_{nL}\, x)(t) = \sum_{k=1}^{n} p_k\, h_{\alpha+s_k-k+1}(t), \tag{20}$$

$$s_k = \sum_{i=1}^{k} \gamma_i,\ p_k = \left(\prod_{i=k+1}^{n} \left(I^{\gamma_i} \frac{d}{dt} \right) I^{n-\alpha-s_n}\, x \right)(0),\ k = 1, 2, \ldots, n. \tag{21}$$

Remark 4. *The Riemann-Liouville, Caputo, and Hilfer fractional derivatives can be interpreted as the fractional derivatives of the 1st level because they contain just one derivative of the 1st order and thus their null-spaces are one-dimensional (of course, under the condition $\alpha \in (0,1)$). All of these derivatives are particular cases of the nth level fractional derivative $D^{\alpha,(\gamma)}_{nL}$ with $n = 1$.*

In the following theorem, some basic properties of the fractional derivatives associated with the Riemann-Liouville fractional integral I^α as well as their projector operators are given.

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Theorem 1. *Let $\mathbb{D}^\alpha$ be a fractional derivative associated with the Riemann-Liouville fractional integral I^α defined on the linear space X and $\mathbb{P}^\alpha$ be its projector operator.*

Then the following properties hold true:

P1) The operator $\mathbb{P}^\alpha$ is a projector, i.e.,

$$((\mathbb{P}^\alpha)^2\, x)(t) = (\mathbb{P}^\alpha\, x)(t),\ x \in D_{\mathbb{D}^\alpha}. \tag{22}$$

P2) The image of $\mathbb{P}^\alpha$ belongs to the null-space of $\mathbb{D}^\alpha$, i.e.,

$$(\mathbb{D}^\alpha \mathbb{P}^\alpha\, x)(t) = 0,\ x \in D_{\mathbb{D}^\alpha}. \tag{23}$$

P3) The image of I^α belongs to the null-space of $\mathbb{P}^\alpha$, i.e.,

$$(\mathbb{P}^\alpha I^\alpha\, x)(t) = 0,\ x \in X. \tag{24}$$

P4) The null-space of $\mathbb{D}^\alpha$ can be characterized in terms of its projector operator $\mathbb{P}^\alpha$ as follows:

$$N_{\mathbb{D}^\alpha} = \{x \in D_{\mathbb{D}^\alpha} :\ (\mathbb{P}^\alpha\, x)(t) = x(t)\}. \tag{25}$$

P5) The null-space of $\mathbb{P}^\alpha$ can be characterized as follows:

$$N_{\mathbb{P}^\alpha} = \{x :\ x(t) = (I^\alpha\, \phi)(t),\ \phi \in X\}. \tag{26}$$

P6) For any $n \in \mathbb{N}$, the implication

$$x(t) = (I^{\alpha\, n}\, \phi)(t),\ \phi \in X\ \Rightarrow\ (\mathbb{P}^\alpha\, (\mathbb{D}^\alpha)^k\, x)(t) = 0,\ k = 0, 1, \ldots, n-1 \tag{27}$$

holds true.

Proof. Most of the properties formulated above are valid for any left-invertible operators, their left-inverse operators and their projector operators, see [8]. Because the Riemann-Liouville fractional integral is left-invertible, we can apply these results for the generic fractional derivative in sense of Definition 2. However, for the reader's convenience, we reproduce here the proofs presented in [8] and adjust them to the case of the fractional derivatives associated with the Riemann-Liouville fractional integral.

Proof of P1: For $x \in D_{\mathbb{D}^\alpha}$, the projector operator $\mathbb{P}^\alpha$ is well-defined. Employing the definitions of the fractional derivative $\mathbb{D}^\alpha$ and its projector operator, we arrive at the following chain of equations:

$$((\mathbb{P}^\alpha)^2\, x)(t) = ((I - I^\alpha\mathbb{D}^\alpha)(I - I^\alpha\mathbb{D}^\alpha)\, x)(t) = x(t) - (I^\alpha\mathbb{D}^\alpha\, x)(t) - (I^\alpha\mathbb{D}^\alpha\, x)(t) +$$
$$(I^\alpha\mathbb{D}^\alpha I^\alpha\mathbb{D}^\alpha\, x)(t) = x(t) - (I^\alpha\mathbb{D}^\alpha\, x)(t) - (I^\alpha\mathbb{D}^\alpha\, x)(t) + (I^\alpha\mathbb{D}^\alpha\, x)(t) =$$
$$x(t) - (I^\alpha\mathbb{D}^\alpha\, x)(t) = (\mathbb{P}^\alpha\, x)(t).$$

Proof of P2: For $x \in D_{\mathbb{D}^\alpha}$, we get

$$(\mathbb{D}^\alpha\mathbb{P}^\alpha\, x)(t) = (\mathbb{D}^\alpha(I - I^\alpha\mathbb{D}^\alpha)\, x)(t) = (\mathbb{D}^\alpha\, x)(t) - (\mathbb{D}^\alpha I^\alpha\mathbb{D}^\alpha\, x)(t) =$$
$$(\mathbb{D}^\alpha\, x)(t) - (\mathbb{D}^\alpha\, x)(t) = 0.$$

Proof of P3: For $x \in X$, the definitions of the fractional derivative $\mathbb{D}^\alpha$ and its projector operator lead to the formula

$$(\mathbb{P}^\alpha I^\alpha\, x)(t) = ((I - I^\alpha\mathbb{D}^\alpha)I^\alpha\, x)(t) = (I^\alpha\, x)(t) - (I^\alpha\mathbb{D}^\alpha I^\alpha\, x)(t) = (I^\alpha\, x)(t) - (I^\alpha\, x)(t) = 0.$$

Proof of P4: Let $x \in D_{\mathbb{D}^\alpha}$ and $(\mathbb{P}^\alpha\, x)(t) = x(t)$. Then

$$(\mathbb{D}^\alpha\, x)(t) = (\mathbb{D}^\alpha \mathbb{P}^\alpha\, x)(t) = 0$$

according to the property P2 and we have the inclusion $x \in N_{\mathbb{D}^\alpha}$.

Now let $x \in N_{\mathbb{D}^\alpha}$, i.e., $(\mathbb{D}^\alpha\, x)(t) = 0$. Then

$$(\mathbb{P}^\alpha\, x)(t) = ((I - I^\alpha \mathbb{D}^\alpha)\, x)(t) = x(t) - (I^\alpha \mathbb{D}^\alpha\, x)(t) = x(t) - (I^\alpha\, 0)(t) = x(t)$$

and the proof of P4 is completed.

Proof of P5: Let $x(t) = (I^\alpha\, \phi)(t),\ \phi \in X$. Then the property P3 implicates the formula

$$(\mathbb{P}^\alpha\, x)(t) = (\mathbb{P}^\alpha I^\alpha\, \phi)(t) = 0$$

and $x \in N_{\mathbb{P}^\alpha}$.

Now let $x \in N_{\mathbb{P}^\alpha}$, i.e., $(\mathbb{P}^\alpha\, x)(t) = 0$. Then

$$((I - I^\alpha \mathbb{D}^\alpha)\, x)(t) = 0 \ \Rightarrow\ x(t) = (I^\alpha \mathbb{D}^\alpha\, x)(t) \ \Rightarrow\ x(t) = (I^\alpha \phi)(t),\ \phi(t) = (\mathbb{D}^\alpha\, x)(t).$$

Proof of P6: Let $n \in \mathbb{N}$ and $x(t) = (I^{\alpha\, n}\, \phi)(t),\ \phi \in X$. Then, for $k = 0, 1, \ldots, n-1$, we get

$$(\mathbb{P}^\alpha\, (\mathbb{D}^\alpha)^k\, x)(t) = ((I - I^\alpha \mathbb{D}^\alpha)\, (\mathbb{D}^\alpha)^k x)(t) = ((I - I^\alpha \mathbb{D}^\alpha)\, (\mathbb{D}^\alpha)^k (I^{\alpha\, n}\, \phi))(t) =$$

$$((I - I^\alpha \mathbb{D}^\alpha)\, (\mathbb{D}^\alpha)^k (I^\alpha)^n\, \phi)(t) = ((I - I^\alpha \mathbb{D}^\alpha)\, (I^\alpha)^{n-k}\, \phi)(t) =$$

$$((I^\alpha)^{n-k}\, \phi)(t) - (I^\alpha \mathbb{D}^\alpha (I^\alpha)^{n-k}\, \phi)(t) = ((I^\alpha)^{n-k}\, \phi)(t) - (I^\alpha (I^\alpha)^{n-k-1}\, \phi)(t) =$$

$$((I^\alpha)^{n-k}\, \phi)(t) - ((I^\alpha)^{n-k}\, \phi)(t) = 0.$$

The proof of the theorem is completed. □

Remark 5. *It is worth mentioning that in the proof of Theorem 1, we did not employ any explicit formulas for the fractional derivatives or for their projector operators. The main tool used in the proof was the 1st fundamental theorem of FC valid for all realisations of the generic fractional derivative by definition. Thus, the properties P1-P6 are valid for all realisations of the generic fractional derivative, both for the known and for not yet introduced ones.*

The results formulated in Theorem 1 can be used for a concrete fractional derivative that leads to some known or new formulas. Say, the property P4 applied to the Riemann-Liouville, Caputo, Hilfer, and the nth level fractional derivatives in combination with the formulas (17)-(20) for their projector operators lead to the following characterisation of their null-spaces (see [7], [14]):

$$N_{D^\alpha_{RL}} = \{c_1 h_\alpha(t),\ c_1 \in \mathbb{R}\}, \tag{28}$$

$$N_{D^\alpha_{C}} = \{c_1 h_1(t),\ c_1 \in \mathbb{R}\}, \tag{29}$$

$$N_{D^{\alpha,\gamma_1}_{H}} = \{c_1 h_{\alpha+\gamma_1}(t),\ c_1 \in \mathbb{R}\}, \tag{30}$$

$$N_{D^{\alpha,(\gamma)}_{nL}} = \{\sum_{k=1}^{n} c_k\, h_{\alpha+s_k-k+1}(t),\ s_k = \sum_{i=1}^{k} \gamma_i,\ c_k \in \mathbb{R},\ k = 1, 2, \ldots, n\}. \tag{31}$$

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3. Advanced Properties of the Generic Fractional Derivative

In this section, we derive some advanced properties of the generic fractional derivative introduced in the previous section including a characterization of its domain, a formula connecting its different realisations, and the generalized fractional Taylor formula involving the generic fractional derivative.

We start with a simple but important result regarding the domain of the generic fractional derivative.

Theorem 2. *Let $\mathbb{D}^\alpha$, $\alpha \in (0,1)$ be a fractional derivative associated with the Riemann-Liouville fractional integral I^α defined on the linear space X.*

Then for any $x \in D_{\mathbb{D}^\alpha}$, there exist the functions $x_1 \in X$ and $x_2 \in N_{\mathbb{D}^\alpha}$, such that

$$x(t) = (I^\alpha\, x_1)(t) + x_2(t),\ x_1 \in X,\ x_2 \in N_{\mathbb{D}^\alpha}. \tag{32}$$

Proof. Let $x \in D_{\mathbb{D}^\alpha}$. By definition, $\mathbb{D}^\alpha \in \mathcal{L}(X)$ and then

$$(\mathbb{D}^\alpha\, x)(t) = x_1(t),\ x_1 \in X. \tag{33}$$

Now we apply the Riemann-Liouville fractional integral I^α to both sides of the formula (33) and get the relation

$$(I^\alpha \mathbb{D}^\alpha\, x)(t) = (I^\alpha x_1)(t),\ x_1 \in X \tag{34}$$

that can be rewritten in terms of the projector operator of the fractional derivative $\mathbb{D}^\alpha$ as follows:

$$(I^\alpha \mathbb{D}^\alpha\, x)(t) = x(t) - (\mathbb{P}^\alpha\, x)(t) = (I^\alpha x_1)(t),\ x_1 \in X. \tag{35}$$

According to the property P2 of Theorem 1, the image of the projector operator belongs to the null-space of $\mathbb{D}^\alpha$ and thus we arrive at the representation

$$x(t) = (I^\alpha x_1)(t) + (\mathbb{P}^\alpha\, x)(t) = (I^\alpha x_1)(t) + x_2(t),\ x_1 \in X,\ x_2 = \mathbb{P}^\alpha\, x \in N_{\mathbb{D}^\alpha} \tag{36}$$

that completes the proof of the theorem. □

The representation (32) and the formula (13) (the 1st fundamental theorem of FC) clarify the reason for discrepancy in the mapping properties of different realisations of the generic fractional derivative that is caused by their unequal domains and null-spaces.

Now we derive a formula connecting different realisations of the generic fractional derivative under the assumption that their null-spaces are finite-dimensional as it is the case for the Riemann-Liouville, Caputo, Hilfer, and the nth level fractional derivatives, see the formulas (28), (29), (30), (31).

Theorem 3. *Let $\mathbb{D}_1^\alpha$ and $\mathbb{D}_2^\alpha$ be two fractional derivatives associated with the Riemann-Liouville fractional integral, the null-space of $\mathbb{D}_1^\alpha$ be finite-dimensional, i.e.,*

$$N_{\mathbb{D}_1^\alpha} = \{c_1\phi_1(t) + \cdots + c_n\phi_n(t),\ c_1, \ldots, c_n \in \mathbb{R}\}, \tag{37}$$

where the functions $\phi_1, \ldots, \phi_n$ are linear independent, and the inclusion

$$N_{\mathbb{D}_1^\alpha} \subset D_{\mathbb{D}_2^\alpha} \tag{38}$$

hold true.

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Then the fractional derivatives $\mathbb{D}_1^\alpha$ and $\mathbb{D}_2^\alpha$ are connected by the relation

$$(\mathbb{D}_2^\alpha\, x)(t) = (\mathbb{D}_1^\alpha\, x)(t) + \sum_{j=1}^{n} a_j (\mathbb{D}_2^\alpha\, \phi_j)(t),\ x \in D_{\mathbb{D}_1^\alpha}, \tag{39}$$

where the coefficients $a_j,\ j = 1, \ldots, n$ are as in the representation

$$(\mathbb{P}_1^\alpha\, x)(t) = \sum_{j=1}^{n} a_j \phi_j(t) \tag{40}$$

of the element $\mathbb{P}_1^\alpha x$ from the null-space $N_{\mathbb{D}_1^\alpha}$, where $\mathbb{P}_1^\alpha$ is the projector of the fractional derivative $\mathbb{D}_1^\alpha$.

Proof. According to Theorem 2, any $x \in D_{\mathbb{D}_1^\alpha}$ can be represented as follows:

$$x(t) = (I^\alpha\, x_1)(t) + x_2(t),\ x_1 \in X,\ x_2 \in N_{\mathbb{D}_1^\alpha}.$$

The null-space of $\mathbb{D}_1^\alpha$ is finite-dimensional and is given by (37). Thus, we get the relation

$$x(t) = (I^\alpha\, x_1)(t) + \sum_{j=1}^{n} c_j \phi_j(t). \tag{41}$$

Because of the inclusion (38) and the formula (13) (the 1st fundamental theorem of FC), we can apply the operator $\mathbb{D}_2^\alpha$ to the right-hand side of the formula (41) and thus also to its left-hand side and arrive at the equation

$$(\mathbb{D}_2^\alpha\, x)(t) = x_1(t) + \sum_{j=1}^{n} c_j (\mathbb{D}_2^\alpha\, \phi_j)(t). \tag{42}$$

Acting on the representation (41) with the fractional derivative $\mathbb{D}_1^\alpha$ immediately leads to the relation

$$(\mathbb{D}_1^\alpha\, x)(t) = x_1(t) \tag{43}$$

and thus

$$(\mathbb{D}_2^\alpha\, x)(t) = (\mathbb{D}_1^\alpha\, x)(t) + \sum_{j=1}^{n} c_j (\mathbb{D}_2^\alpha\, \phi_j)(t). \tag{44}$$

Now let us determine the coefficients c_j in the last representation. According to the property P2 from Theorem 1, the inclusion $\mathbb{P}_1^\alpha x \in N_{\mathbb{D}_1^\alpha}$ holds true that leads to the representation (40) with certain coefficients $a_j,\ j = 1, \ldots, n$ that depend on the function x.

On the other hand, applying the projector operator $\mathbb{P}_1^\alpha$ to the representation (41), we obtain the equation

$$(\mathbb{P}_1^\alpha x)(t) = (\mathbb{P}_1^\alpha I^\alpha\, x_1)(t) + \sum_{j=1}^{n} c_j (\mathbb{P}_1^\alpha \phi_j)(t). \tag{45}$$

The property P3 of Theorem 1 implicates

$$(\mathbb{P}_1^\alpha I^\alpha\, x_1)(t) = 0, \tag{46}$$

whereas the property P4 leads to the relations

$$(\mathbb{P}_1^\alpha \phi_j)(t) = \phi_j(t),\ j = 1, \ldots, n. \tag{47}$$

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Taking into account the formulas (46) and (47), we rewrite the equation (45) as follows

$$(\mathbb{P}_1^{\alpha} x)(t) = \sum_{j=1}^{n} c_j \phi_j(t). \tag{48}$$

Comparing this equation with the relation (40), we see that $c_j = a_j,\ j = 1, \ldots, n$, where the coefficients a_j are defined as in the representation (40). The proof of the theorem is completed. □

Remark 6. *To employ the formula (39) for the fractional derivatives whose null-spaces are linear combinations of some power law functions, we need the formula*

$$(\mathbb{D}^{\alpha}\, h_{\beta})(t) = h_{\beta-\alpha}(t),\ \alpha > 0,\ h_{\beta} \in D_{\mathbb{D}^{\alpha}} \setminus N_{\mathbb{D}^{\alpha}}. \tag{49}$$

This formula easily follows from the 1st fundamental theorem of FC valid for any realisation of the generic fractional derivative by definition and the well-known formula

$$(I^{\alpha}\, h_{\beta})(t) = (h_{\alpha} * h_{\beta})(t) = h_{\alpha+\beta}(t),\ \alpha > 0,\ \beta > 0.$$

Example 1. *We start with derivation of the known relation between the Riemann-Liouville and Caputo fractional derivatives.*

Setting $\mathbb{D}_1^{\alpha} = D_C^{\alpha}$ and $\mathbb{D}_2^{\alpha} = D_{RL}^{\alpha}$ in Theorem 3, we immediately see that the inclusion $N_{D_C^{\alpha}} = \{c_1 h_1(t),\ c_1 \in \mathbb{R}\} \subset D_{D_{RL}^{\alpha}}$ holds true and thus this theorem is applicable.

The relation (39) from Theorem 3 adjusted to the case of the Riemann-Liouville and Caputo fractional derivatives by means of the formula (18) for the projector operator of the Caputo fractional derivative takes then the well-known form:

$$(D_{RL}^{\alpha}\, x)(t) = (D_C^{\alpha}\, x)(t) + x(0)(D_{RL}^{\alpha}\, h_1)(t) = (D_C^{\alpha}\, x)(t) + x(0)h_{1-\alpha}(t),\ x \in D_{D_C^{\alpha}}. \tag{50}$$

It is worth mentioning that we cannot apply Theorem 3 in the case $\mathbb{D}_1^{\alpha} = D_{RL}^{\alpha}$ and $\mathbb{D}_2^{\alpha} = D_C^{\alpha}$ because the null-space $N_{D_{RL}^{\alpha}} = \{c_1 h_{\alpha}(t),\ c_1 \in \mathbb{R}\}$ does not belong to the domain $D_{D_C^{\alpha}}$ of the Caputo fractional derivative.

Example 2. *Now we derive a new relation between the Riemann-Liouville and the nth level fractional derivatives.*

In Theorem 3, we set $\mathbb{D}_1^{\alpha} = D_{nL}^{\alpha,(\gamma)}$ and $\mathbb{D}_2^{\alpha} = D_{RL}^{\alpha}$. The null-space of the nth level fractional derivative is given by the formula (31). It is n-dimensional and the inclusion $N_{D_{nL}^{\alpha,(\gamma)}} = \{\sum_{k=1}^{n} c_k\, h_{\alpha+s_k-k+1}(t),\ c_1, \ldots, c_n \in \mathbb{R}\} \subset D_{D_{RL}^{\alpha}}$ holds true because all of the exponents $\sigma_k = \alpha + s_k - k,\ k = 1, \ldots, n$ of the power law functions $h_{\alpha+s_k-k+1}$ satisfy the inequality $-1 < \sigma_k$ that is ensured by the conditions (8) and (9) posed on the parameters of the nth level fractional derivative D_{nL}^{α}.

Taking into consideration the formula (20) for the projector operator of the nth level fractional derivative as well as the formula (49), the relation (39) from Theorem 3 leads to the formula

$$(D_{RL}^{\alpha}\, x)(t) = (D_{nL}^{\alpha,(\gamma)}\, x)(t) + \sum_{k=1}^{n} p_k\, h_{s_k-k+1}(t),\ x \in D_{D_{nL}^{\alpha,(\gamma)}}, \tag{51}$$

$$s_k = \sum_{i=1}^{k} \gamma_i,\ p_k = \left(\prod_{i=k+1}^{n} \left(I^{\gamma_i} \frac{d}{dt} \right) I^{n-\alpha-s_n}\, x \right)(0),\ k = 1, 2, \ldots, n.$$

In the rest of this section, we present a generalized Taylor formula for the generic fractional derivative associated with the Riemann-Liouville fractional integral. To derive this formula, we do not use any explicit expressions for the fractional derivatives and just employ their definition as the left-inverse operators to the Riemann-Liouville fractional integral.

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Theorem 4. *Let $\mathbb{D}^\alpha$ be any fractional derivative associated with the Riemann-Liouville fractional integral I^α and $x \in D_{(\mathbb{D}^\alpha)^n}$, $n \in \mathbb{N}$.*

Then the generalized Taylor formula

$$x(t) = \sum_{k=0}^{n-1} (I^{\alpha k}\, (\mathbb{P}^\alpha((\mathbb{D}^\alpha)^k\, x)))(t) + r_n(t),\ n \in \mathbb{N} \tag{52}$$

holds valid, where the remainder r_n is given by the formula

$$r_n(t) = (I^{\alpha n}((\mathbb{D}^\alpha)^n\, x))(t) = ((\mathbb{D}^\alpha)^n\, x)(c)\, h_{\alpha n+1}(t), \tag{53}$$

where c is a suitably chosen value from the interval $(0, t)$.

Proof. The basic element of the proof is the operator identity

$$I = \sum_{k=0}^{n-1} I^{\alpha k}\mathbb{P}^\alpha(\mathbb{D}^\alpha)^k + I^{\alpha n}(\mathbb{D}^\alpha)^n,\ n \in \mathbb{N} \tag{54}$$

on the space $D_{(\mathbb{D}^\alpha)^n}$. A formula of type (54) holds true for any left-invertible operator, see [8]. For the reader's convenience, we provide a proof of (54) by the method of mathematical induction.

For $n = 1$, the formula (54) takes the form

$$I = \mathbb{P}^\alpha + I^\alpha\mathbb{D}^\alpha \tag{55}$$

that is nothing else but the definition of the projector operator.

Let the identity (54) be true for $n = N$, i.e.,

$$I = \sum_{k=0}^{N-1} I^{\alpha k}\mathbb{P}^\alpha(\mathbb{D}^\alpha)^k + I^{\alpha N}(\mathbb{D}^\alpha)^N. \tag{56}$$

Using the formula (56) and the definition of the projector operator, for $n = N + 1$, we then get the following chain of equations:

$$\sum_{k=0}^{N} I^{\alpha k}\mathbb{P}^\alpha(\mathbb{D}^\alpha)^k + I^{\alpha(N+1)}(\mathbb{D}^\alpha)^{N+1} = \sum_{k=0}^{N-1} I^{\alpha k}\mathbb{P}^\alpha(\mathbb{D}^\alpha)^k + I^{\alpha N}\mathbb{P}^\alpha(\mathbb{D}^\alpha)^N + I^{\alpha(N+1)}(\mathbb{D}^\alpha)^{N+1} =$$

$$I - I^{\alpha N}(\mathbb{D}^\alpha)^N + I^{\alpha N}\mathbb{P}^\alpha(\mathbb{D}^\alpha)^N + I^{\alpha(N+1)}(\mathbb{D}^\alpha)^{N+1} = I - I^{\alpha N}(\mathbb{D}^\alpha)^N + I^{\alpha N}(I - I^\alpha\mathbb{D}^\alpha)(\mathbb{D}^\alpha)^N +$$
$$I^{\alpha(N+1)}(\mathbb{D}^\alpha)^{N+1} = I - I^{\alpha N}(\mathbb{D}^\alpha)^N + I^{\alpha N}(\mathbb{D}^\alpha)^N - I^{\alpha N}I^\alpha\mathbb{D}^\alpha(\mathbb{D}^\alpha)^N + I^{\alpha(N+1)}(\mathbb{D}^\alpha)^{N+1} = I$$

that completes the proof of the identity (54). This identity can be immediately interpreted as the generalized fractional Taylor formula (52) with the remainder in form

$$r_n(t) = (I^{\alpha n}((\mathbb{D}^\alpha)^n\, x))(t).$$

By means of the mean value theorem applied to the fractional integral $I^{\alpha n}$, we can represent the remainder as follows:

$$r_n(t) = ((\mathbb{D}^\alpha)^n\, x)(c)\, (I^{\alpha n}h_1)(t) = ((\mathbb{D}^\alpha)^n\, x)(c)\, h_{\alpha n+1}(t)$$

where c is a suitably chosen value from the interval $(0, t)$. This completes the proof of the theorem. □

It is worth mentioning that the remainder r_n defined as in (53) is provided in the Lagrange form. Indeed, for $\alpha = 1$ (the case of the Taylor formula with the integer order derivatives), we have the formula

$$r_n(t) = (I^n\, x^{(n)})(t) = x^{(n)}(c)\, h_{n+1}(t) = x^{(n)}(c)\frac{t^n}{\Gamma(n+1)} = \frac{x^{(n)}(c)}{n!}t^n, \tag{57}$$

where c is a suitable chosen value from the interval $(0, t)$.

Let us now consider some particular cases of the generalized Taylor formula (52), both the known and the new ones.

Example 3. *For the Riemann-Liouville fractional derivative with the projector operator given by the relation (17), the generalized Taylor formula (52) takes the known form ([15], [16]):*

$$x(t) = \sum_{k=0}^{n-1} (I^{1-\alpha}\, (D^{\alpha}_{RL})^k\, x)(0) h_{\alpha k+\alpha}(t) + r_n(t),\ n \in \mathbb{N} \tag{58}$$

with the remainder r_n given by the formula

$$r_n(t) = (I^{\alpha n}((D^{\alpha}_{RL})^n\, x))(t) = ((D^{\alpha}_{RL})^n\, x)(c)\, h_{\alpha n+1}(t), c \in (0, t). \tag{59}$$

Example 4. *The projector operator for the Caputo fractional derivative is given by the formula (18). Employing this formula, we represent the generalized Taylor formula for the Caputo fractional derivative as follows ([16]):*

$$x(t) = \sum_{k=0}^{n-1} ((D^{\alpha}_{C})^k\, x)(0) h_{\alpha k+1}(t) + r_n(t),\ n \in \mathbb{N} \tag{60}$$

with the remainder r_n given by the formula

$$r_n(t) = (I^{\alpha n}((D^{\alpha}_{C})^n\, x))(t) = ((\mathbb{D}^{\alpha}_{C})^n\, x)(c)\, h_{\alpha n+1}(t), c \in (0, t). \tag{61}$$

Example 5. *In this example, for the first time, we present a generalized Taylor formula for the Hilfer fractional derivative. Its projector operator is given by the relation (19). Then the generalized Taylor formula (52) takes the following form:*

$$x(t) = \sum_{k=0}^{n-1} (I^{1-\alpha-\gamma_1}\, (D^{\alpha,\gamma_1}_{H})^k\, x)(0) h_{\alpha k+\alpha+\gamma_1}(t) + r_n(t),\ n \in \mathbb{N} \tag{62}$$

with the remainder r_n given by the formula

$$r_n(t) = (I^{\alpha n}((D^{\alpha,\gamma_1}_{H})^n\, x))(t) = ((\mathbb{D}^{\alpha,\gamma_1}_{H})^n\, x)(c)\, h_{\alpha n+1}(t), c \in (0, t). \tag{63}$$

As expected, for $\gamma_1 = 0$ and $\gamma_1 = 1 - \alpha$, the generalized Taylor formula (62) for the Hilfer fractional derivative is reduced to the generalized Taylor formula (58) for the Riemann-Liouville fractional derivative and the generalized Taylor formula (60) for the Caputo fractional derivative, respectively.

4. Fractional Relaxation Equation with the Generic Fractional Derivative

In this section, we deal with an initial-value problem for the fractional relaxation equation with the generic fractional derivative associated with the Riemann-Liouville fractional integral. Its particular cases with the Riemann-Liouville, Caputo, and Hilfer fractional derivatives are well-known. Recently, the fractional relaxation equation with the nth level fractional derivative has been considered in [14]. In the particular cases mentioned above, the fractional relaxation equations were solved by employing the individual form of the corresponding fractional derivatives. In this paper, we derive an explicit formula for solution of the fractional relaxation equation with the generic fractional derivative just by using its definition as a set of the linear operators left-inverse to the Riemann-Liouville fractional integral.

Theorem 5. *The initial-value problem for the fractional relaxation equation*

$$\begin{cases} (\mathbb{D}^\alpha\, y)(t) = -\lambda y(t), & 0 < \alpha < 1,\ \lambda > 0,\ t > 0, \\ (\mathbb{P}^\alpha\, y)(t) = y_0(t), & y_0 \in N_{\mathbb{D}^\alpha}, \end{cases} \tag{64}$$

where $\mathbb{D}^\alpha$ is any fractional derivative associated with the Riemann-Liouville fractional integral I^α and $\mathbb{P}^\alpha$ is its projector operator, possesses a unique solution given by the formula

$$y(t) = y_0(t) - \lambda(\tau^{\alpha-1}E_{\alpha,\alpha}(-\lambda\tau^\alpha) * y_0(\tau))(t), \tag{65}$$

where $E_{\alpha,\beta}$ is the two-parameters Mittag-Leffler function defined by the convergent series

$$E_{\alpha,\beta}(z) := \sum_{k=0}^{+\infty} \frac{z^k}{\Gamma(\alpha k + \beta)},\ z, \beta \in \mathbb{C},\ \alpha > 0. \tag{66}$$

Proof. To derive the solution formula (65), we first act with the Riemann-Liouville fractional integral on the fractional relaxation equation (first line in (64)) and use the initial condition (second line in (64)) as well as the definition of the projector operator:

$$(\mathbb{D}^\alpha\, y)(t) = -\lambda y(t) \Leftrightarrow (I^\alpha\, \mathbb{D}^\alpha\, y)(t) = -\lambda(I^\alpha\, y)(t) \Leftrightarrow y(t) - (\mathbb{P}^\alpha\, y)(t) = -\lambda(I^\alpha\, y)(t) \Leftrightarrow$$

$$y(t) - y_0(t) = -\lambda(I^\alpha\, y)(t) \Leftrightarrow y(t) + \lambda(I^\alpha\, y)(t) = y_0(t).$$

The last equation in this chain of equations is the conventional Abel integral equation of the second kind whose solution in explicit form is given by the formula (65) (see, e.g., [3], [17], [18]). This completes the proof of the theorem. □

Remark 7. *It is worth mentioning that the solution formula (65) holds valid for any fractional derivative associated with the Riemann-Liouville fractional integral I^α including the Riemann-Liouville, Caputo, Hilfer, and the nth level fractional derivatives. However, the form of the projector operator and thus the form of the initial condition (second line of (64)) is not the same for different realisations of the generic fractional derivative. This leads to essentially different forms of the solution formula (65) for the particular fractional derivatives introduced so far.*

Remark 8. *The initial condition (second line in (64)) for the fractional relaxation equation is provided in terms of the projector operator $\mathbb{P}^\alpha$. Using an individual form of the projector operator (say, given by the formulas (17), (18), (19), or (20)), this initial condition can be represented in the conventional form, i.e., in terms of the values of certain FC operators applied to the unknown function y evaluated at the point zero.*

The projector operators for the Riemann-Liouville, Caputo, Hilfer, and the nth level fractional derivatives given by the formulas (17), (18), (19), (20), respectively, are expressed in terms of the linear combinations of the power law functions. To specify the solution formula (65) for these fractional derivatives, we start with the relation

$$(\tau^{\alpha-1}E_{\alpha,\alpha}(-\lambda\tau^\alpha) * h_\beta(\tau))(t) = t^{\alpha+\beta-1}E_{\alpha,\alpha+\beta}(-\lambda t^\alpha),\ \alpha > 0,\ \beta > 0 \tag{67}$$

that easily follows from the well-known formula

$$(h_\alpha(\tau)\ *\ h_\beta(\tau))(t) = h_{\alpha+\beta}(t),\ \alpha > 0,\ \beta > 0 \tag{68}$$

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and the term by term integration of the series for the Mittag-Leffler function from the left-hand side of the formula (67).

Using the relation (67), the solution formula (65) for the initial-value problem (64) with the for the initial condition $y_0(t) = h_\beta(t)$ takes the following form:

$$y(t) = h_\beta(t) - \lambda(\tau^{\alpha-1}E_{\alpha,\alpha}(-\lambda\tau^\alpha) * h_\beta(\tau))(t) = h_\beta(t) - \lambda t^{\alpha+\beta-1}E_{\alpha,\alpha+\beta}(-\lambda t^\alpha) =$$

$$t^{\beta-1}E_{\alpha,\beta}(-\lambda t^\alpha),\ \alpha > 0,\ \beta > 0. \tag{69}$$

The representation (69) and the formulas (17), (18), (19), (20) for the projector operators of the Riemann-Liouville, Caputo, Hilfer, and the nth level fractional derivatives immediately lead to the known solution formulas for the fractional relaxation equations with these fractional derivatives.

Example 6. *For the nth level fractional derivative, the formula* (20) *for its projector operator leads to the following formulation of the initial-value problem* (64) *for the nth level fractional derivative ([14]):*

$$\begin{cases} (D_{nL}^{\alpha,(\gamma)}\, y)(t) = -\lambda\, y(t),\ 0 < \alpha < 1,\ \lambda > 0,\ t > 0, \\ \left(\prod_{i=k+1}^{n}\left(I^{\gamma_i}\,\frac{d}{dt}\right)\, I^{n-\alpha-s_n}\, y\right)(0)\ =\ y_k,\ k = 1,\ldots,n, \end{cases} \tag{70}$$

where $s_k,\ k = 1,\ldots,n$ *are given by the formula* (7).

Using the formula (20) *for the projector operator and the representation* (69)*, the solution formula* (65) *for the problem* (70) *can be represented as follows:*

$$y(t) = \sum_{k=1}^{n} y_k\, t^{\alpha+s_k-k}\, E_{\alpha,\alpha+s_k-k+1}(-\lambda t^\alpha), \tag{71}$$

where $s_k,\ k = 1,\ldots,n$ *are given by the formula* (7)*. The solution formula* (71) *was derived for the first time in* [14] *using the Laplace transform method.*

The probably most important particular cases of the initial-value problem (70) for the fractional relaxation equation are those with the first-level fractional derivatives ($n = 1$ in (70)), i.e., with the Riemann-Liouville, Caputo, and Hilfer fractional derivatives:

1) The problem (70) with the Hilfer fractional derivative ($n = 1, 0 \le \gamma_1 \le 1 - \alpha$) takes the form

$$\begin{cases} (D_H^{\alpha,\gamma_1}\, y)(t) = -\lambda\, y(t),\ 0 < \alpha < 1, \lambda > 0,\ t > 0, \\ (I^{1-\alpha-\gamma_1}\, y)(0)\ =\ y_1. \end{cases} \tag{72}$$

Its solution is given by the formula

$$y(t) = y_1\, x^{\alpha+\gamma_1-1}\, E_{\alpha,\alpha+\gamma_1}(-\lambda t^\alpha). \tag{73}$$

2) The problem (70) with the Riemann-Liouville fractional derivative ($n = 1,\ \gamma_1 = 0$) takes the form

$$\begin{cases} (D_{RL}^{\alpha}\, y)(t) = -\lambda\, y(t),\ 0 < \alpha < 1,\ \lambda > 0,\ t > 0, \\ (I^{1-\alpha}\, y)(0)\ =\ y_1. \end{cases} \tag{74}$$

Its solution is given by the formula

$$y(t) = y_1\, x^{\alpha-1}\, E_{\alpha,\alpha}(-\lambda t^\alpha), \tag{75}$$

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3) The problem (70) with the Caputo fractional derivative ($n = 1, \gamma_1 = 1 - \alpha$) takes the form

$$\begin{cases} (D_C^\alpha y)(t) = -\lambda\, y(t),\ 0 < \alpha < 1,\ \lambda > 0,\ t > 0, \\ y(0) \ = \ y_1. \end{cases} \tag{76}$$

Its solution is given by the formula

$$y(t) = y_1\, E_{\alpha,1}(-\lambda t^\alpha). \tag{77}$$

Finally, let us discuss complete monotonicity of the solution (65) to the initial-value problem (64) for the fractional relaxation equation in the case the null-space of the fractional derivative $\mathbb{D}^\alpha$ is built by the linear combinations of some power law functions, i.e.,

$$N_{\mathbb{D}^\alpha} = \{c_1 h_{\sigma_1}(t) + \cdots + c_m h_{\sigma_m}(t),\ c_1, \ldots, c_m \in \mathbb{R}\}. \tag{78}$$

Theorem 6. *Let a fractional derivative $\mathbb{D}^\alpha$ associated with the Riemann-Liouville fractional integral satisfy the condition (78), the parameters σ_k fulfill the restrictions $\alpha \le \sigma_k \le 1,\ k = 1, \ldots, m$, and $y_k \ge 0,\ k = 1, \ldots, m$.*
Then the unique solution to the initial-value problem

$$\begin{cases} (\mathbb{D}^\alpha\, y)(t) = -\lambda y(t),\ 0 < \alpha < 1,\ \lambda > 0,\ t > 0, \\ (\mathbb{P}^\alpha\, y)(t) = y_1 h_{\sigma_1}(t) + \cdots + y_m h_{\sigma_m}(t) \end{cases} \tag{79}$$

is complete monotonic.

Proof. According to Theorem 5, the unique solution of the initial-value problem (79) is given by the formula (65). For the fractional derivatives with the null-space given by (78), we employ the formula (69) and represent the solution as follows:

$$y(t) = \sum_{k=1}^{m} y_k\, t^{\sigma_k - 1} E_{\alpha,\sigma_k}(-\lambda t^\alpha). \tag{80}$$

According to a result derived in [14], the function

$$h_{\alpha,\beta,\gamma,\lambda}(t) := t^{\gamma-1}\, E_{\alpha,\beta}(-\lambda t^\alpha) \tag{81}$$

is completely monotonic under the conditions

$$0 < \alpha \le 1,\ \alpha \le \beta,\ 0 < \gamma \le 1,\ \lambda > 0. \tag{82}$$

In our case, the parameters of the problem (79) satisfy the conditions $\alpha \le \sigma_k \le 1,\ k = 1, \ldots, m$, $\lambda > 0$ and thus all of the functions $t^{\sigma_k-1} E_{\alpha,\sigma_k}(-\lambda t^\alpha),\ k = 1, \ldots, m$ from the sum at the right-hand side of the solution formula (80) are completely monotonic. Because $y_k \ge 0,\ k = 1, \ldots, m$ and any linear combination of completely monotonic functions with non-negative coefficients is completely monotonic, the proof of the theorem is completed. □

Example 7. *Let us consider the initial-value problem (70) for the fractional relaxation equation with the nth level fractional derivative. The null-space of the nth level fractional derivative has the form (78), see the formula (31). Thus we can apply Theorem (6) that implicates that the unique solution to the initial-value problem (70) is completely monotonic under the conditions*

$$k - 1 \le s_k,\ k = 1, \ldots, n, \tag{83}$$

where $s_k,\ k = 1, \ldots, n$ are given by the formula (7).

For $n = 1$ and $\gamma_1 \geq 0$, the condition (83) is satisfied because $s_1 = \gamma_1 \geq 0$ and thus we arrive at the well-known result that the solutions (73), (75), and (77) to the initial-value problems (72), (74), and (76) with the initial condition $y_1 \geq 0$ and the Hilfer, Riemann-Liouville, and Caputo fractional derivatives, respectively, are complete monotonic.

5. Conclusions and Open Problems

The main objective of this paper is in suggestion of a unified approach to different kinds of the time-fractional derivatives of the functions of a single variable introduced so far. In the framework of this approach, we introduced a generic fractional derivative that was defined as a set of the linear operators left-inverse to the Riemann-Liouville fractional integral. In particular, the Riemann-Liouville, Caputo, Hilfer, and the nth level fractional derivatives are different realisations of the generic fractional derivative associated with the Riemann-Liouville fractional integral.

Another contribution of the paper is in derivation of some basic and advanced properties of the generic fractional derivative that are valid for all of its realisations. In particular, we provided a characterization of its domain, null-space, and the projector operator. In the case, the null-spaces of two different realisations of the generic fractional derivative are finite-dimensional, we derived a formula connecting these realizations. Its particular case is a formula for the nth level fractional derivative in terms of the Riemann-Liouville fractional derivative that was derived in this paper for the first time.

One of the most important results derived in this paper is a generalized fractional Taylor formula valid for any fractional derivative associated with the Riemann-Liouville fractional integral. For the proof of this formula, we did not use any concrete form of a particular fractional derivative. However, the generic fractional Taylor formula can be specified for a given fractional derivative. In particular, a generalized Taylor formula for the Hilfer fractional derivative was presented in this paper for the first time.

In the last part of the paper, we dealt with an initial-value problem for the fractional relaxation equation with the generic fractional derivative and the initial condition formulated in terms of its projector operator. First, we derived a closed form formula for its unique solution that is independent on the concrete form of the fractional derivative involved in this problem. Then some results regarding complete monotonicity of this solution were presented.

As to the open problems related to the generic fractional derivative, it is not clear at the moment if there exist its realisations with a finite-dimensional null-space that are different from the nth level fractional derivatives. The question if there are some fractional derivatives associated with the Riemann-Liouville integrals whose null-space is infinite-dimensional is also still open. Another direction of research would be derivation of other advanced properties of the generic fractional derivative that are independent on its realisations and are based just on its definition as the linear operators left-inverse to the Riemann-Liouville fractional integral.

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