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Chains of Dense G_δ Sets in Perfect Polish Spaces

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Abstract

We prove that in every nonempty perfect Polish space, every dense G_δ subset contains strictly decreasing and strictly increasing chains of dense G_δ subsets of length $\mathfrak{c}$, the cardinality of the continuum. As a corollary, this holds in $\mathbb{R}^n$ for each $n \geq 1$. This provides an easy answer to a question of Erdős since the set of Liouville numbers admits a descending chain of cardinality $\mathfrak{c}$, each member of which has the Erdős property. We also present counterexamples demonstrating that the result fails if either the perfection or the Polishness assumption is omitted. Finally, we show that the set $\mathcal{T}$ of real Mahler T-numbers is a dense Borel set and contains a strictly descending chain of length $\mathfrak{c}$ of proper dense Borel subsets.

Keywords: Baire category; dense G_δ-set; perfect Polish spaces; Borel set; Liouville numbers; Erdős property; Mahler T-numbers; descriptive set theory

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1. Introduction

Dense G_δ-subsets of Polish spaces play a central role in descriptive set theory and in Baire category methods. The Baire category theorem asserts that such sets are generic, and they are often used to witness "largeness" in a topological sense.

In this paper we study the internal structure of dense G_δ-sets. Our main result shows that in any nonempty perfect Polish space, each dense G_δ set admits chains of proper dense G_δ-subsets of continuum length, ordered either ascendingly or descendingly by inclusion. The proof proceeds by constructing a Cantor set inside the given dense G_δ, then using partitions of this Cantor set to build a continuum-scale chain of meager F_σ sets and their dense G_δ complements.

The result illustrates the ubiquity of continuum-sized order-theoretic structures inside dense G_δ-sets. As an application, we derive the corresponding statement for $\mathbb{R}^n$. Finally, we show that the assumptions of perfection and Polishness are both essential by giving counterexamples.

This work builds on classical constructions of Cantor sets in perfect Polish spaces (see e.g. Oxtoby [6], Kechris [3]) and connects with the study of special subsets of the real line (see Miller [7]).

2. Main Result on Descending Chains

Theorem 1. *Let X be a nonempty perfect Polish space and let $G \subseteq X$ be a dense G_δ-set. Then there exists a strictly decreasing chain*

$$\{H_t : t \in (0,1)\}$$

of proper dense G_δ-subsets of X, all contained in G, indexed by $(0,1)$ and ordered by inclusion: for $s < t$,

$$H_t \subsetneq H_s \subsetneq G.$$

In particular, the chain has length $\mathfrak{c}$.

Proof. We divide the proof into four steps.

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Step 1: A Cantor set inside G. Write $G = \bigcap_{k\in\mathbb{N}} U_k$ with each U_k open dense. Since X is perfect and Polish, we may construct a Cantor scheme of compact sets $\{F_\sigma : \sigma \in 2^{<\omega}\}$ such that:

(i) $F_\varnothing \subseteq U_0$ is nonempty compact with empty interior in X;
(ii) $F_{\sigma 0}, F_{\sigma 1} \subseteq F_\sigma$ are disjoint compact subsets with $\operatorname{diam}(F_{\sigma i}) \le 2^{-|\sigma|-2}$;
(iii) $F_\sigma \subseteq \bigcap_{k<|\sigma|} U_k$.

Define

$$P := \bigcap_{n=0}^{\infty} \bigcup_{\sigma\in 2^n} F_\sigma.$$

Then P is homeomorphic to the middle-thirds Cantor set, nowhere dense in X, and contained in G.

Step 2: Partition into disjoint compact nowhere dense sets. Partition P into a sequence $(P_n)_{n\in\mathbb{N}}$ of pairwise disjoint, nonempty clopen (in P) Cantor subsets. Each P_n is compact and nowhere dense in X.

Step 3: A continuum chain of meager F_σ sets. Enumerate $\mathbb{Q} \cap (0,1)$ as $\{q_n : n \in \mathbb{N}\}$. For $t \in (0,1)$ set

$$A_t = \{n : q_n < t\}, \quad K_t = \bigcup_{n\in A_t} P_n.$$

Then K_t is a meager F_σ. Moreover, if $s < t$ then there exists q_n with $s < q_n < t$, hence $K_s \subsetneq K_t$.

Step 4: Dense G_δ complements. Let $D_t := X \setminus K_t$, a dense G_δ. Define

$$H_t = G \cap D_t.$$

Each H_t is a dense G_δ-subset of X, properly contained in G. For $s < t$ we have $H_t \subsetneq H_s$. This yields the desired chain. □

3. Ascending Chains

By dualizing the above argument we obtain:

Corollary 1. *In the setting of Theorem 1, there exists a strictly increasing chain $\{K_t : t \in (0,1)\}$ of proper dense G_δ-subsets of X, all contained in G, of length $\mathfrak{c}$.*

Corollary 2. *Let X be a nonempty locally compact perfect Polish space. Then every dense G_δ subset $G \subseteq X$ admits both a strictly increasing and a strictly decreasing chain of proper dense G_δ-subsets of length $\mathfrak{c}$.*

Proof. By hypothesis, X is perfect and Polish. Therefore Theorem 1 applies directly to X and to the given dense G_δ set $G \subseteq X$, yielding a strictly decreasing chain

$$\{H_t : t \in (0,1)\}$$

of proper dense G_δ subsets of X contained in G, with $H_t \subsetneq H_s$ for $s < t$.

For the strictly increasing chain, apply the "ascending" construction (see the corollary following Theorem 1) in the same setting to obtain a chain

$$\{K_t : t \in (0,1)\}$$

of proper dense G_δ-subsets of X contained in G with $K_s \subsetneq K_t$ for $s < t$. In both cases, the index set $(0,1)$ has cardinality $\mathfrak{c}$, so the chains have length $\mathfrak{c}$. □

Remark 1. *Typical examples include any nonempty open subset of $\mathbb{R}^n$ and, more generally, any separable manifold without boundary and without isolated points; these are locally compact, perfect, and Polish, hence fall under Corollary 2.*

Definition 1. *A subset $X \subseteq \mathbb{R}$ has the* Erdős property *if $X + X = \mathbb{R}$; that is, for every $r \in \mathbb{R}$ there exist $x, y \in X$ with $r = x + y$.*

Erdős proved that the set of all Liouville numbers, despite being thin (Lebesgue measure zero and of Hausdorff dimension zero), has the Erdős property. In fact, he showed that every dense G_δ subset of $\mathbb{R}$ has this property. Erdős asked whether other proper subsets of the set of Liouville numbers have the property. This was answered in [2], where it was shown (with significant effort) that the set of Liouville numbers has $2^{\mathfrak{c}}$ subsets with the Erdős property. Our main theorem now gives an immediate family: since the Liouville numbers form a dense G_δ-subset of $\mathbb{R}$, they admit a descending chain of length $\mathfrak{c}$ of proper dense G_δ-subsets, and each of these $\mathfrak{c}$ sets has the Erdős property. For the record, there are only $\mathfrak{c}$ many G_δ (hence Borel) subsets of $\mathbb{R}$; the additional Erdős sets found in [2] are not Borel.

4. Applications to Euclidean Spaces

Corollary 3. *For each $n \geq 1$, every dense G_δ-subset of $\mathbb{R}^n$ admits both a strictly increasing and a strictly decreasing chain of proper dense G_δ subsets of length $\mathfrak{c}$.*

Proof. The Euclidean space $\mathbb{R}^n$ is a perfect Polish space. Apply Theorem 1. □

5. Necessity of Hypotheses

We illustrate that both hypotheses in Theorem 1 are essential.

Example 1 (Failure without perfection). *Let $X = \mathbb{N}$ with the discrete topology. Then X is Polish but not perfect. Every dense G_δ is X itself, so no proper dense G_δ subset exists.*

Example 2 (Failure without Polishness). *Let $X = \mathbb{Q}$ with the usual topology. Then X is not Polish (not complete). Every dense G_δ in X is co-meager, but since X itself is meager, no chain of proper dense G_δ subsets of continuum length can be constructed.*

Open Question Let X be a nonempty perfect Polish space. Does there exist a dense G_δ-set $U \subseteq X$ such that the poset

$$(\mathcal{D}(U), \subseteq),$$

where $\mathcal{D}(U)$ denotes the family of dense G_δ subsets of U ordered by inclusion, is Tukey universal [3,10] among all such posets arising from dense G_δ subsets of perfect Polish spaces?

6. Connections with Mahler's T-Numbers

Mahler's classification of transcendental numbers divides them into classes $\mathcal{S}, \mathcal{T}$, and $\mathcal{U}$, according to the growth of the approximation exponents $w_d(\xi)$. A real number ξ is a T-number precisely when

$$0 < \sup_d w_d(\xi) = \infty \quad \text{while every } w_d(\xi) < \infty.$$

The set $\mathcal{T}$ of all T-numbers is known to be nonempty and uncountable by Schmidt's seminal work [8,9].

From the perspective of descriptive set theory, Ki established that $\mathcal{T}$ has precise Borel complexity:

$$\mathcal{T} \in \Pi^0_4 \setminus \Sigma^0_3,$$

so $\mathcal{T}$ is a Borel set, in fact at a relatively low level of the Borel hierarchy [4].

A striking structural property of $\mathcal{T}$ is its density. By Mahler's invariance under algebraic dependence, if $\xi \in \mathcal{T}$ and $q \in \mathbb{Q}$, then $\xi + q \in \mathcal{T}$. Since $\xi + \mathbb{Q}$ is dense in $\mathbb{R}$, it follows that $\mathcal{T}$ itself is dense. Consequently, results such as Theorem 1, which produce continuum-length chains of descending dense analytic sets, apply to $\mathcal{T}$ as well. In particular, $\mathcal{T}$ contains strictly descending chains of proper dense Borel subsets of length $\mathfrak{c}$.

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These parallels emphasize that both in number theory (via Mahler's classification) and in descriptive set theory (via dense G_δ-sets in Polish spaces), large sets naturally accommodate long descending chains of dense subsets.

7. Mahler's Classification and the Set $\mathcal{T}$

For a real (or complex) number ξ, let $w_d(\xi)$ be the supremum of exponents w such that

$$0 < |P(\xi)| \leq H(P)^{-w}$$

for infinitely many integer polynomials P of degree at most d, where $H(P)$ denotes the height. Mahler's classes A, S, T, U are defined by the growth of the sequence $d \mapsto w_d(\xi)$. A number ξ is a *T-number* iff

$$0 < \sup_d w_d(\xi) = \infty \quad \text{while each } w_d(\xi) < \infty.$$

Mahler proved the following fundamental property.

Theorem 2 (Invariance under algebraic dependence)**.** *If ξ, η are algebraically dependent over $\mathbb{Q}$, then they belong to the same Mahler class.*

In particular, for any rational q, ξ and $\xi + q$ are algebraically dependent; hence $\xi \in \mathcal{T} \Rightarrow \xi + q \in \mathcal{T}$ [1, Thm. 2.3].

The set $\mathcal{T}$ of real T-numbers is known to be nonempty (indeed uncountable) by Schmidt's work [8,9]. Moreover, its descriptive set-theoretic complexity is sharp: $\mathcal{T} \in \Pi^0_4$ but $\mathcal{T} \notin \Sigma^0_3$ [4].

8. Density of $\mathcal{T}$

Theorem 3. *The set $\mathcal{T} \subseteq \mathbb{R}$ is dense.*

Proof. By Schmidt's seminal work on T-numbers [8,9], there exists $\tau \in \mathcal{T}$. By Theorem 2, $\tau + q \in \mathcal{T}$ for all $q \in \mathbb{Q}$ [1, Thm. 2.3]. Since $\tau + \mathbb{Q}$ is dense in $\mathbb{R}$, and $\tau + \mathbb{Q} \subseteq \mathcal{T}$, it follows that $\mathcal{T}$ is dense, because any superset of a dense set is dense. □

9. Complexity Remark

From Ki's work [4] we have $\mathcal{T} \in \Pi^0_4 \setminus \Sigma^0_3$. In particular, $\mathcal{T}$ is Borel and hence automatically analytic ($\mathbf{\Sigma}^1_1$).

10. Perfect-Set Input for Analytic Sets

We use the standard perfect-set theorem: every uncountable analytic subset of a Polish space contains a nonempty perfect subset (hence a homeomorphic copy of the Cantor set) [3, Perfect Set Theorem].

Proposition 1. *Let $A \subseteq \mathbb{R}$ be a dense analytic set. Then there exists a strictly descending chain $\{A_\alpha : \alpha < \mathfrak{c}\}$ of* proper *dense analytic subsets of $\mathbb{R}$ with $A_\beta \subsetneq A_\alpha \subseteq A$ for $\alpha < \beta < \mathfrak{c}$. Moreover, if A is Borel then each A_α can be chosen Borel.*

Construction. Since A is dense it is uncountable. By the perfect-set theorem, fix a Cantor set $P \subseteq A$ [3]. Let $h : 2^\omega \to P$ be a homeomorphism (so P is closed, perfect, and nowhere dense in $\mathbb{R}$).

Equip 2^ω with the lexicographic order $\leq_{\text{lex}}$. Choose a well-order $(s_\alpha)_{\alpha<\mathfrak{c}}$ of 2^ω that is cofinal in $(2^\omega, \leq_{\text{lex}})$. For each $\alpha < \mathfrak{c}$, set

$$K_\alpha := h\big(\{x \in 2^\omega : x \leq_{\text{lex}} s_\alpha\}\big).$$

Each K_α is closed in P, hence closed in $\mathbb{R}$, and $K_\alpha \subseteq P \subseteq A$. Moreover $\alpha < \beta \Rightarrow K_\alpha \subsetneq K_\beta$, and no K_α has interior in $\mathbb{R}$ (since $K_\alpha \subseteq P$ and P has empty interior), so each K_α is nowhere dense.

Now define

$$A_\alpha := A \setminus K_\alpha.$$

Then:

- A_α is analytic (analytic minus closed is analytic); if A is Borel then A_α is Borel.
- A_α is dense: removing a closed nowhere dense set from a dense set preserves density.
- The chain is strict and descending since $K_\alpha \subsetneq K_\beta \subseteq A$ implies $A_\beta = A \setminus K_\beta \subsetneq A \setminus K_\alpha = A_\alpha$.

This gives the required chain of length $\mathfrak{c}$. □

11. Descending Chain of Borel Sets in $\mathcal{T}$

Theorem 4. *Let $\mathcal{T} \subseteq \mathbb{R}$ be the set of Mahler T-numbers. Then $\mathcal{T}$ is dense and Borel (Π^0_4), and $\mathcal{T}$ contains a strictly descending chain of length $\mathfrak{c}$ of proper dense Borel (hence analytic) subsets.*

Proof. Density follows from Theorem 3, and the Borel complexity from [4]. Applying Proposition 1 with $A = \mathcal{T}$, we obtain the chain of dense Borel subsets. □

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