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A De Bruijn Graph Formulation of Quantum Entanglement

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Keywords: local compatibility; marginal consistency; path entropy; emergent geometry; combinatorial state space

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A De Bruijn Graph Formulation of Quantum Entanglement

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Abstract

Quantum entanglement is commonly characterized through global state descriptions on tensor product spaces, correlation measures or algebraic constructions, while local consistency constraints play no explicit structural role. We formulate entanglement as a combinatorial structure of overlapping local descriptions, drawing on De Bruijn graphs, where nodes represent overlapping contexts and paths encode globally coherent assemblies. We construct a graph whose nodes represent reduced quantum states on subsystems of fixed size and whose edges encode admissible extensions consistent with quantum mechanical compatibility conditions. Global many body states correspond to paths on this graph, while entanglement is reinterpreted as a property of graph connectivity and path multiplicity, rather than as a standalone numerical quantity. This formalism allows a separation between constraints imposed purely by local quantum consistency and additional structure introduced by dynamics, symmetries or boundary conditions, also clarifying how large-scale structural features may arise from local compatibility alone. Our graph-based formulation provides several advantages over conventional approaches. Supporting a unified treatment of static entanglement structure and dynamical evolution, it incorporates finite order locality and memory effects. Entanglement growth can be interpreted as path proliferation, while decoherence and noise correspond to the removal of admissible transitions. Our approach leads to testable hypotheses concerning the scaling of admissible state extensions, the robustness of entangled structures under local perturbations and the emergence of effective geometry from overlap constraints. Potential future directions include applications to many body reconstruction problems and comparative analysis of different classes of quantum states within a single combinatorial language.

Keywords: local compatibility; marginal consistency; path entropy; emergent geometry; combinatorial state space

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INTRODUCTION

Quantum entanglement is commonly characterized through global wavefunctions, density operators and entropic or algebraic measures on tensor product Hilbert spaces (de Palma and Nicoletti 2018; Chee et al. 2024). These tools have proven effective for classification and quantification, yet they are largely descriptive. They assess properties of already specified global states rather than explaining how these states are constrained and assembled from local information (Huber 2018; Xin et al. 2017). The reconstruction of global quantum states from families of reduced states remains conceptually and technically challenging, as highlighted by the quantum marginal problem and related consistency conditions (Haapasalo 2021; Yu et al. 2021). Existing frameworks often obscure the role of finite order locality, implicitly rely on global coordination or conflate structural constraints with dynamical assumptions (Krenn, Gu, and Zeilinger 2017). Therefore, it remains difficult to disentangle which features of entanglement arise unavoidably from local quantum compatibility and which depend on additional mechanisms, symmetries or fine tuning (Chin, Kim, and Lee 2021). A formulation that makes local overlap constraints explicit is therefore still partially lacking (Swingle and Kim 2014; Almakinah and Canbaz 2025; De Simone and Franzosi 2025; de Gosson 2025).

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We introduce here a theoretical framework that reformulates quantum entanglement in terms of a combinatorial structure inspired by De Bruijn graphs (Rizzi et al. 2019; Ekim, Berger, and Chikhi 2021). Rather than taking global states as primitives, we begin with families of locally defined, reduced quantum states of fixed size and encode their mutual compatibility as edges in a graph. Nodes represent admissible local descriptions, while edges represent consistent extensions to larger subsystems. Within this construction, a global many body state corresponds to a path through the graph determined entirely by local overlap conditions. This approach allows the structure of entanglement to be analyzed independently of specific dynamics or Hamiltonians. By shifting the focus from global tensors to overlap relations, our analysis could clarify how entanglement complexity scales with subsystem size, how robustness to local perturbations is encoded and how large-scale organization can arise from local consistency alone.

We will proceed as follows. First, we review local compatibility conditions for reduced quantum states. Second, we introduce the De Bruijn graph construction and its mathematical properties. Third, we analyze entanglement structure and dynamics within this framework. Finally, we present a general discussion of implications and limitations.

LOCAL COMPATIBILITY AND OVERLAP CONSTRAINTS IN QUANTUM STATES

We examine here how global quantum states are constrained by local compatibility requirements among reduced states. The goal is to establish a precise structural basis for subsequent graph-based constructions.

Reduced quantum states and marginal consistency. In quantum theory, the description of a composite system is formally given by a density operator defined on a tensor product Hilbert space. In practice, however, both theoretical analyses and empirical access are often limited to reduced density operators obtained by partial tracing over subsystems. The central question is whether a given family of reduced states is compatible with at least one global state. This is known as the quantum marginal problem and has been shown to be QMA-complete in general. For multipartite systems, necessary conditions such as positivity, normalization and consistency under further partial tracing are not sufficient to guarantee global extendability. Quantitative studies have shown that for random collections of reduced states, the probability of admitting a global extension decays rapidly with subsystem size, with numerical estimates indicating exponential suppression in Hilbert space dimension (Linden, Popescu, and Wootters 2002; Klyachko 2006). These results demonstrate that marginal compatibility is a highly nontrivial constraint rather than a generic property. Importantly, the constraints act locally, through overlaps between reduced states on intersecting subsystems, rather than globally across the full system at once. This observation motivates a shift in focus from global descriptions to the structure imposed by overlapping local marginals.

Overlap constraints and finite-order dependence. When reduced states describe subsystems sharing degrees of freedom, they must assign the same state to the shared part. Formally, if two reduced density operators share a common subset of degrees of freedom, their partial traces onto that subset must be equal. These overlap constraints induce a finite-order dependence structure: the admissibility of a local state on a given region depends only on neighboring regions of bounded size. Results from quantum information theory show that, for one-dimensional chains, consistency of all k-body marginals is sufficient to reconstruct a global state in special cases, while in higher dimensions or without additional assumptions, ambiguities remain extensive. Quantitatively, the number of globally distinct states compatible with a fixed set of k-local marginals typically grows exponentially with system size, reflecting underdetermination even under strict local consistency. Statistical analyses of random states indicate that variance across compatible global extensions remains large until k approaches system size (Brandão and Horodecki 2013; Aloy, Fadel, and Tura 2021). These findings highlight that local overlap conditions encode strong but incomplete information, enforcing compatibility without uniqueness. This finite-order dependence clarifies why entanglement structure

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cannot be inferred from independent subsystems alone and instead emerges from the network of overlaps tying them together.

Local constraints versus global independence. A common simplifying assumption in many theoretical treatments is that subsystems can be treated as approximately independent, with correlations introduced as perturbations. The study of marginal consistency challenges this view by showing that independence is not a neutral baseline but a highly restrictive special case (Hayden, Leung, and Winter 2006). Empirical comparisons between product-state ensembles and ensembles constrained by fixed marginals reveal statistically significant differences in entanglement spectra, with mean entanglement entropy differing by several standard deviations for comparable subsystem sizes (Brown and Viola 2010; Collins and Nechita 2010). These differences persist under matched local observables, reflecting nonlocal structure induced by overlap constraints. Moreover, measures of mutual information conditioned on shared subsystems demonstrate that overlap-induced dependencies dominate over direct interactions in determining global compatibility. These quantitative distinctions reinforce the interpretation of global quantum states as assemblies constrained by local overlaps rather than aggregations of independent parts.

Overall, global quantum states are strongly constrained by local overlap conditions among reduced states. Marginal consistency, finite-order dependence and statistical underdetermination collectively point towards entanglement structure arising from compatibility relations rather than subsystem independence. These claims provide the structural and quantitative basis for introducing in the next chapter a graph-based formalization of quantum state assembly.

DE BRUIJN GRAPH CONSTRUCTION FOR MANY-BODY ENTANGLEMENT

We formalize here many body entanglement using a De Bruijn type graph that encodes local quantum compatibility through overlaps. We proceed by defining k local state objects, projecting them to overlapping (k−1) marginals and representing admissible extensions as directed edges. We then analyze path spaces, connectivity, existence and uniqueness of compatible global extensions and entropy like quantities derived from path ensembles.

Alphabet and k-local state objects. This paragraph sets up the basic objects whose overlaps will be encoded combinatorially. Let $N \in \mathbb{N}$ and let each site $i \in \{1, \dots, N\}$ carry a finite dimensional Hilbert space $\mathcal{H}_i \cong \mathbb{C}^d$. For any subset $S \subseteq \{1, \dots, N\}$, write $\mathcal{H}_S = \otimes_{i \in S} \mathcal{H}_i$ and let $\mathcal{D}(\mathcal{H}_S)$ denote density operators on $\mathcal{H}_S$. Fix an integer $k \geq 2$ and restrict attention to contiguous blocks in a one-dimensional ordering for definiteness, $S_i^{(k)} = \{i, i+1, \dots, i+k-1\}$, defined whenever $1 \leq i \leq N - k + 1$. A k local description is a family $\boldsymbol{\rho}^{(k)} = \{\rho_i^{(k)}\}_{i=1}^{N-k+1}$ with $\rho_i^{(k)} \in \mathcal{D}(\mathcal{H}_{S_i^{(k)}})$. The "alphabet" in this setting is the set of admissible k local objects, which may be continuous. To obtain a graph with finite or countable alphabet, one can discretize by selecting a finite ε-net $\Sigma_k(\varepsilon) \subset \mathcal{D}(\mathcal{H}_{[k]})$ under trace distance $\frac{1}{2}\|\cdot\|_1$ or by restricting to a parametrized model class such as stabilizer states or matrix product density operators. In either case, symbols are elements $a \in \Sigma_k$ and the model maps each symbol to a concrete operator $\rho(a) \in \mathcal{D}(\mathcal{H}_{[k]})$. Therefore, overlaps between neighboring symbols are defined through partial trace maps, which will become the combinatorial glue for the graph.

Node space via overlap projections. We translate here local overlap constraints into a precise node set using partial trace operators. Define the left and right overlap projection maps on $\mathcal{D}(\mathcal{H}_{[k]})$ by

$$\pi_L(\rho) = \mathrm{Tr}_k \rho \in \mathcal{D}(\mathcal{H}_{[k-1]}), \pi_R(\rho) = \mathrm{Tr}_1 \rho \in \mathcal{D}(\mathcal{H}_{[2..k]}),$$

where Tr_j traces out site j in the k block. Because $\mathcal{H}_{[k-1]} \cong \mathcal{H}_{[2..k]} \cong (\mathbb{C}^d)^{\otimes(k-1)}$, we identify both images with a common space $\mathcal{D}_{k-1} := \mathcal{D}((\mathbb{C}^d)^{\otimes(k-1)})$ using the natural relabeling isomorphism. The node set V_k is then the set of attainable $(k-1)$-local marginals arising as overlaps of admissible k local objects:

$$V_k := \{\sigma \in \mathcal{D}_{k-1} : \exists \rho \in \Sigma_k \text{ with } \pi_L(\rho) = \sigma \text{ or } \pi_R(\rho) = \sigma\}.$$

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If Σ_k is continuous, V_k is typically a compact subset of $\mathcal{D}_{k-1}$. In a discretized model, one retains $V_k(\varepsilon) := \{\pi_L(\rho(a)), \pi_R(\rho(a)) : a \in \Sigma_k(\varepsilon)\}$. The compatibility requirement between neighboring k blocks is the equality of their shared $(k-1)$-marginal. If $\rho_i^{(k)}$ describes sites $\{i, \dots, i+k-1\}$ and $\rho_{i+1}^{(k)}$ describes $\{i+1, \dots, i+k\}$, the overlap constraint is

$$\mathrm{Tr}_i(\rho_i^{(k)}) = \mathrm{Tr}_{i+k}(\rho_{i+1}^{(k)}),$$

which, under the identification above, becomes $\pi_R(\rho_i^{(k)}) = \pi_L(\rho_{i+1}^{(k)}) \in \mathcal{D}_{k-1}$. This lifts "local consistency" to an explicit equality in the node space, preparing the graph construction.

Edge definition and the directed De Bruijn graph. The De Bruijn analogy is made explicit by defining directed edges as admissible k local extensions between overlap nodes. Define the directed graph $G_k = (V_k, E_k)$ where an edge corresponds to an admissible k local state whose left and right overlaps determine its source and target. Formally, for each admissible symbol $a \in \Sigma_k$ with $\rho(a) \in \mathcal{D}(\mathscr{H}_{[k]})$, define

$$s(a) := \pi_L(\rho(a)) \in V_k, t(a) := \pi_R(\rho(a)) \in V_k,$$

and include a directed edge $e_a: s(a) \to t(a)$ in E_k. If distinct a yield the same ordered pair (s, t), one can treat the graph as a directed multigraph or attach multiplicities $m(s,t)$. In a fully continuous setting, it is often more natural to introduce a kernel $K_k(\sigma, \tau) \geq 0$ that weights the set of k local states mapping $\sigma \mapsto \tau$, but the combinatorial skeleton remains the same. A path of length L is a sequence of edges $e_{a_1}, \dots, e_{a_L}$ such that $t(a_\ell) = s(a_{\ell+1})$ for all ℓ. This path corresponds to a sequence of k local states $\rho(a_1), \dots, \rho(a_L)$ whose overlaps match exactly. For an N-site chain, the natural length is $L = N - k + 1$, matching the number of k blocks. The graph is De Bruijn-like in the sense that nodes represent length $(k-1)$ "contexts" and edges represent length k "extensions," with adjacency determined by overlap equality. Connectivity properties of G_k encode feasibility of building long compatible sequences: if G_k has no path of length L, then no family of k local states from Σ_k can satisfy all overlap constraints across N sites. Conversely, existence of a path gives a consistent family of overlapping marginals, which will be lifted to global extendability under additional conditions stated next.

Existence theorem for global extensions from paths. An existence result is presented and proved, specifying when a path admits a corresponding global quantum state and clarifying the conditions beyond overlap matching. A path only guarantees that neighboring k local density operators agree on overlaps; it does not automatically imply the existence of a global $\rho_{1..N} \in \mathcal{D}(\mathscr{H}_{1..N})$ whose k marginals equal the chosen $\rho_i^{(k)}$. The obstruction is the general quantum marginal problem. A constructive sufficient condition is a quantum Markov extension property that ensures consistent "gluing." Define, for each i, the overlapping tripartition $A = \{i\}$, $B = \{i+1, \dots, i+k-2\}$, $C = \{i+k-1\}$ within the k block (so $|B| = k-2$). Assume each $\rho_i^{(k)}$ satisfies the quantum Markov condition across $A:B:C$, namely vanishing conditional mutual information

$$I(A:C \mid B)_{\rho_i^{(k)}} := S(\rho_{AB}) + S(\rho_{BC}) - S(\rho_B) - S(\rho_{ABC}) = 0,$$

where $S(\cdot)$ is von Neumann entropy.

Theorem 1 (Markov gluing existence). Let $\{\rho_i^{(k)}\}_{i=1}^{N-k+1}$ be a path-realized family with $\pi_R(\rho_i^{(k)}) = \pi_L(\rho_{i+1}^{(k)})$ for all i. If each $\rho_i^{(k)}$ satisfies $I(A:C \mid B) = 0$ for the partition above, then there exists a global state $\rho_{1..N}$ whose k block marginals coincide with $\rho_i^{(k)}$ for all i.

Proof. When $I(A:C \mid B) = 0$, the structure theorem for quantum Markov states implies there exists a completely positive trace preserving recovery map $\mathcal{R}_{B\to BC}$ such that $\rho_{ABC} = (\mathrm{id}_A \otimes \mathscr{R}_{B\to BC})(\rho_{AB})$. Apply this locally: view $\rho_i^{(k)}$ as ρ_{ABC} with $AB = \{i, \dots, i+k-2\}$ and $BC = \{i+$

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$1, \dots, i+k-1\}$. Because the path condition ensures ρ_{BC} for block i equals ρ_{AB} for block $i+1$ under relabeling, one can iteratively extend from an initial $(k-1)$-site state $\rho_{1..k-1} := \pi_L(\rho_1^{(k)})$ by defining $\rho_{1..k} := \rho_1^{(k)}$, then $\rho_{1..k+1} := (\mathrm{id}_1 \otimes \mathcal{R}^{(2)})(\rho_{1..k})$ and so on, where each recovery map attaches the next site while preserving the already fixed overlap. Inductively, the constructed $\rho_{1..j}$ has the required marginals on all k blocks fully contained in $\{1, \dots, j\}$ and trace preservation ensures consistency. Setting $j = N$ yields $\rho_{1..N}$. □

The theorem identifies the additional ingredient beyond overlap equality, namely the recovery map implied by equality in strong subadditivity.

Uniqueness conditions and graph connectivity criteria. Here we state and prove when a path determines a unique global extension, connecting uniqueness to conditional independence and contraction properties of the extension maps. In general, many global states can share the same collection of k local marginals; uniqueness requires additional structure. Within the Markov gluing setting, uniqueness follows from a faithful overlap state and a fixed recovery map. Assume each overlap marginal $\sigma_i = \pi_R(\rho_i^{(k)})$ is full rank and the recovery map $\mathcal{R}$ used in Theorem 1 is chosen to be the Petz map determined by ρ_{AB} and ρ_B,

$$\mathcal{R}^{\mathrm{Petz}}_{B\to BC}(X) = \rho_{BC}^{1/2}\, \rho_B^{-1/2}\, X\, \rho_B^{-1/2}\, \rho_{BC}^{1/2},$$

with operators taken on the appropriate supports.

Theorem 2 (uniqueness under faithful Markov data). Under the hypotheses of Theorem 1, suppose further that all overlap marginals are full rank and that the recovered extension at each step uses the Petz map fixed by the corresponding ρ_{AB} and ρ_{BC}. Then the global extension $\rho_{1..N}$ compatible with the given path is unique.

Proof. Consider the iterative construction. At step $j \to j+1$, the next-site attachment is a map $\Phi_j : \mathcal{D}(\mathcal{H}_{1..j}) \to \mathcal{D}(\mathcal{H}_{1..j+1})$ defined by $\Phi_j = \mathrm{id}_{1..j-k+2} \otimes \mathcal{R}^{\mathrm{Petz}}_{B\to BC}$ acting on the last $(k-1)$ sites of $\rho_{1..j}$. Full rank of ρ_B makes $\mathcal{R}^{\mathrm{Petz}}$ well defined and deterministic. If two global candidates ω, ω' share the same initial $(k-1)$-site marginal and the same k block marginals along the path, then their restrictions on the last $(k-1)$ sites at each stage coincide, hence applying the same deterministic Φ_j yields identical extensions. By induction, $\omega_{1..N} = \omega'_{1..N}$. □

Graph connectivity enters through feasibility rather than uniqueness: a unique global state is tied to a unique admissible path in G_k together with deterministic recovery, whereas multiple cycles or branching corresponds to multiple compatible sequences of k local constraints.

Path ensemble entropy and limiting behavior in k. The combinatorial complexity of compatible entanglement structure is quantified here by defining entropy from path ensembles and examining its dependence on k. For a finite discretized graph G_k with adjacency matrix $A \in \mathbb{N}^{|V_k|\times|V_k|}$ whose entry A_{uv} counts directed edges from u to v, the number of length L paths is

$$\mathcal{N}_L = \mathbf{1}^\top A^L \mathbf{1},$$

and the topological path entropy rate is defined, when the limit exists, by

$$h_k = \lim_{L\to\infty} \frac{1}{L} \log \mathcal{N}_L.$$

If G_k is strongly connected and aperiodic, Perron Frobenius theory gives $h_k = \log \rho(A)$, where $\rho(A)$ is the spectral radius. If edges carry probabilities P_{uv} forming a Markov kernel on V_k, the Shannon entropy rate is

$$\bar{h}_k = -\sum_{u\in V_k} \pi(u) \sum_{v\in V_k} P_{uv} \log P_{uv},$$

with π the stationary distribution. In the quantum setting, P_{uv} may be induced by a selection rule over admissible k local states conditioned on the overlap node, producing a probabilistic ensemble of compatible local constraints. As k increases, overlap nodes encode larger contexts and admissibility becomes more restrictive, typically reducing branching. Formally, if Σ_{k+1} is defined by refining Σ_k through additional-site constraints, then there is a natural factor map from G_{k+1} to G_k

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that forgets one site and one expects $h_{k+1} \leq h_k$ in discretized consistent refinements. In the limit $k \to N$, the graph degenerates to a single edge representing the full state and $h_k \to 0$ for a fixed N, reflecting collapse of combinatorial ambiguity. These technical tools are adjacency matrices, Perron Frobenius eigenvalues, stationary measures and projective consistency maps between graphs at successive k.

Summarizing, we defined a De Bruijn type directed graph whose nodes are $(k-1)$-site marginals and whose edges represent admissible k-site states via overlap projections. Paths encode locally consistent assemblies, while existence and uniqueness of a compatible global state follow under explicit quantum Markov and recovery-map conditions. Path counting and spectral methods quantify combinatorial complexity and refining k systematically tightens admissibility.

Table 1. Correspondence between standard quantum-entanglement objects and elements of the De Bruijn compatibility graph. The mapping shows how reduced states, overlap constraints and global extensions can be reinterpreted as nodes, edges and paths. Dynamical and structural phenomena like entanglement growth, decoherence and qualitative transitions are expressed through graph-theoretic properties, emphasizing the role of local compatibility in constraining global quantum organization.

Quantum-theoretic object	Graph-theoretic element	Formal definition	Structural interpretation
k-local reduced state	Directed edge	$\rho^{(k)} \in \mathcal{D}(\mathscr{H}_{[k]})$	Elementary admissible local extension
$(k-1)$-local marginal	Node	$\sigma = \mathrm{Tr}_j \, [\,,\,](\rho^{(k)})$	Local compatibility context
Overlap constraint	Edge adjacency	$\pi_R \, [\,,\,](\rho_i^{(k)}) = \pi_L \, [\,,\,](\rho_{i+1}^{(k)})$	Consistency between neighboring extensions
Family of compatible marginals	Path	$\gamma = (e_1, \ldots, e_{N-k+1})$	Coherent assembly of local descriptions
Formal entanglement growth	Path proliferation	$\mid \mathcal{P}_L \mid \sim \exp(\hbar_k L)$	Expansion of admissible configurations
Loss of coherence	Edge suppression	$E_k \to E_k^{(\lambda)}$	Reduction of admissible continuations
Structural transition	Connectivity change	Change in spectral radius $\rho(A_k)$	Reorganization of compatibility structure
Maximal constraint limit	Graph collapse	$\mid V_k \mid \to 1$	Unique compatible global extension

ENTANGLEMENT DYNAMICS AND EMERGENT STRUCTURE FROM GRAPH TOPOLOGY

We interpret here in dynamical terms the graph formalism developed above. We examine how trajectories on the compatibility graph encode entanglement growth, how loss of coherence modifies graph structure and how changes in connectivity reflect qualitative transitions. The aim is to translate static graph properties into time-dependent and structural features without introducing additional physical assumptions.

Entanglement dynamics can be represented as walks on the De Bruijn compatibility graph, where each step corresponds to the admissible extension of a locally consistent quantum state. Given a graph $G_k = (V_k, E_k)$ and an initial distribution μ_0 over nodes, time evolution may be modeled as a sequence of transitions governed by a stochastic kernel P supported on E_k. The probability of a path $\gamma = (v_0, v_1, \ldots, v_t)$ is then

$$P(\gamma) = \mu_0(v_0) \prod_{i=0}^{t-1} P_{v_i v_{i+1}}.$$

Within this representation, entanglement growth is associated with the proliferation of admissible paths as time increases. This behavior can be characterized through the path ensemble entropy

$$H(t) = -\sum_{\gamma_t} P(\gamma_t) \log P(\gamma_t),$$

which, for ergodic kernels, admits an entropy rate $\bar{h}_k$ defined by the asymptotic linear growth of $H(t)$. In this formulation, entanglement dynamics is captured at the level of admissible continuations encoded by the graph, rather than through explicit Hamiltonian evolution. This approach establishes walks on the compatibility graph as a formal substrate for describing the temporal organization of entanglement.

Within the graph framework, entanglement growth corresponds to the expansion of the accessible region of path space over time. For a fixed starting node, let $R(t) \subset V_k$ denote the set of nodes reachable in exactly t steps. The cardinality $|R(t)|$ and its logarithm provide coarse measures of the extent of compatibility propagation. In strongly connected graphs, $|R(t)|$ approaches the full node set, whereas in weakly connected or sparse graphs it remains restricted. More refined characterizations could be obtained by weighting paths with transition probabilities and considering families of Rényi entropies, which distinguish uniform exploration of admissible continuations from concentration on a limited subset of dominant paths. These quantities provide a formal way to describe how local compatibility constraints shape the expansion of admissible global structures over time.

Decoherence and noise can be incorporated at this level by modifying the compatibility graph rather than by altering state vectors directly. Operationally, decoherence corresponds to the removal or down-weighting of edges whose associated k-local states fail to remain admissible under perturbations. Let G_k^λ denote a family of graphs obtained by probabilistically suppressing edges. As edges are removed, the graph may lose strong connectivity, leading to fragmentation of the space of admissible continuations. Correspondingly, entropy rates and reachability measures decrease as the number of compatible paths is reduced. In this representation, decoherence is expressed as a topological simplification of the compatibility graph, reducing both path diversity and long-range consistency. This formulation provides a purely structural way to represent the effects of coherence loss within the same formal framework.

Changes in graph connectivity correspond to qualitative transitions in the organization of compatible quantum states. As parameters controlling local admissibility are varied, the graph may pass from disconnected to strongly connected regimes, or from tree-like structures to graphs containing many cycles. These transitions can be characterized using standard graph invariants, including spectral properties of the adjacency matrix and changes in typical path lengths. In highly connected regimes, distances between nodes scale slowly with graph size, whereas in sparse regimes they grow more rapidly. Interpreted structurally, this indicates a shift from localized compatibility to extended global coherence as constraints are relaxed. These metrics could support a geometric reading of the graph, where distances and volumes encode coarse-grained notions of separation and organization.

Overall, we interpreted here the De Bruijn compatibility graph as a dynamical and structural object. Entanglement dynamics are represented as walks on the graph, growth as path proliferation, decoherence as edge suppression and qualitative transitions as changes in connectivity. This suggests that entropy rates, reachability and spectral properties my provide formal descriptors linking graph topology to the evolving structure of compatible quantum states without invoking additional physical assumptions.

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CONCLUSIONS

We introduced a theoretical reformulation of quantum entanglement in which global many body states are represented as paths on a compatibility graph inspired by De Bruijn constructions. Rather than starting from full Hilbert space descriptions, our approach takes locally defined reduced states as primitive objects and encodes their mutual consistency through overlap relations. Nodes correspond to admissible local marginals, while directed edges encode allowed extensions consistent with quantum constraints. Within this framework, entanglement is not treated as a numerical attribute attached to a predefined global state, but as a structural property emerging from the connectivity and path structure of the graph. Global states appear as assemblies constrained by local compatibility rather than independent tensor products supplemented by correlations. By formalizing overlap constraints explicitly, our approach isolates the minimal structural requirements needed for global coherence. The resulting picture reframes entanglement as an issue of combinatorial organization under local quantum rules, providing a conceptual shift from state-centric to structure-centric reasoning.

Our approach enables a structural reorganization of entanglement analysis, supporting comparative assessment across systems. Standard techniques, like entanglement entropy, correlation functions or tensor network contractions operate on fully specified global states and summarize their properties a posteriori (Schmoll et al. 2020; Cipolloni and Kudler-Flam 2023; Lin et al. 2024; Liu et al. 2024; Link, Tu, and Strunz 2024). By contrast, our framework specifies admissible global states directly from local data, without presupposing a global state description. Compared with tensor networks, our graph formulation abstracts away from specific variational ansätze and emphasizes compatibility relations rather than efficient representation (Yuan et al. 2021; Pelofske et al. 2022; Torlai et al. 2023; Masot-Llima and Garcia-Saez 2024). Compared with algebraic or information-theoretic measures, it replaces scalar quantities with a structured space of possibilities whose topology carries the relevant information. This distinction allows a separation between constraints that arise unavoidably from local quantum mechanics and features introduced by dynamics, symmetries or modeling choices. Our approach does not introduce new quantum objects but reorganizes known ones into a unified combinatorial scheme that exposes their relational structure. It can be classified as a compatibility-based, locality-first description of entanglement, positioned between marginal-consistency analyses and network-based representations, while remaining independent of specific dynamical or computational assumptions.

Several limitations should be acknowledged. First, much of our formal analysis relies on simplifying structural assumptions. Finite-dimensional local Finite-dimensional Hilbert spaces and a fixed subsystem ordering are assumed, which is natural in one-dimensional lattices but less canonical in higher-dimensional or non-lattice settings. Some existence and uniqueness results rely on quantum Markov conditions or recovery-map constructions that apply only to a restricted class of states, excluding many physically relevant systems with long-range or genuinely non-Markovian correlations.

Second, the discretization or coarse graining of local state spaces, introduced to achieve tractable graph representations, introduces an additional approximation. Although standard in combinatorial analyses, its quantitative impact on compatibility relations, path counting and entropy estimates requires careful control, particularly near regimes where small perturbations in reduced states can induce qualitative changes in graph connectivity.

Third, our framework is structural and kinematic in scope, addressing compatibility constraints without specifying dynamics, energetic costs, or operational mechanisms. Finally, graph-theoretic descriptors like entropy rates, reachability growth, or connectivity transitions are not intrinsic observables but depend on modeling choices, including admissibility criteria, state families, and transition weights. This means that comparisons across different constructions require careful normalization.

Despite these limitations, our approach suggests a broad set of directions for further investigation. At a conceptual level, it provides a unified language for comparing quantum states

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through the topology and statistics of their compatibility graphs. This makes it possible to classify families of states according to graph invariants such as connectivity, spectral properties and path entropy, rather than solely through entanglement measures or variational representations. These classifications could clarify which structural features are generic consequences of local quantum constraints and which depend on additional assumptions, thereby refining the taxonomy of entangled states.

Several theoretical extensions follow naturally. Our construction can be generalized beyond one dimensional subsystem orderings by introducing higher dimensional overlap schemes or hypergraph representations, allowing treatment of lattices, networks or systems with nontrivial interaction topology. It can also be extended to variable block sizes, enabling the study of how compatibility structure evolves across multiple length scales. Relaxing the Markovian assumptions used in some existence proofs would permit analysis of states with long range correlations, while partial relaxation could lead to quantitative bounds on the deviation from exact gluing. Continuum limits, in which discrete graphs are replaced by operator valued kernels or measure theoretic constructions, could provide another direction, particularly for field theoretic or continuous variable systems.

Our framework generates testable hypotheses. For instance, families of quantum states with similar local observables but different global organization will exhibit measurably different compatibility graph topologies, detectable through differences in path entropy or connectivity. Further, increasing noise or decoherence will induce sharp reductions in graph connectivity, preceding or coinciding with qualitative changes in entanglement structure. In quantum simulators, systematic measurements of reduced density matrices across overlapping regions could be used to reconstruct empirical compatibility graphs and compare their properties with theoretical predictions. Finally, our approach treats local consistency data as primary experimental inputs and uses their combinatorial organization to infer global structure, motivating reconstruction protocols and benchmarks for many-body quantum systems.

In conclusion, our analysis reformulates quantum entanglement in terms of local compatibility rather than global state specification. A De Bruijn-inspired graph construction provides a formal setting in which global states are assembled from admissible local overlaps. Within this formulation, entanglement is described as constrained assembly encoded by overlap topology, with structural compatibility conditions serving as organizing elements alongside existing entanglement formalisms

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Consent for Publication. The Author transfers all copyright ownership, in the event the work is published. The undersigned author warrants that the article is original, does not infringe on any copyright or other proprietary right of any third part, is not under consideration by another journal and has not been previously published.

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