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Article

A Note on Odd Perfect Numbers

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Abstract: For over two millennia, the question of whether odd perfect numbers—positive integers whose proper divisors sum to the number itself—exist has captivated mathematicians, from Euclid's elegant construction of even perfect numbers via Mersenne primes to Euler's probing of their odd counterparts. This paper resolves this ancient conjecture through a rigorous proof by contradiction, demonstrating that odd perfect numbers are impossible. We define the abundancy index, $I(n) = \frac{\sigma(n)}{n}$, where $\sigma(n)$ is the divisor sum function, and leverage its properties alongside the p-adic order and radical of a number. Assuming the existence of a smallest odd perfect number N, with $I(N) = 2$, we apply a novel lemma to express $I(N)$ as a product over its prime factors. Constraints from established results, including the requirement of at least 10 distinct prime factors and a bound on their reciprocal sum, enable us to derive a scaled inequality. By meticulously bounding the terms of this product expansion, we show that the sum falls short of the necessary threshold for $k \geq 10$ prime factors, yielding a contradiction. This proof, grounded in elementary number theory yet profound in its implications, not only settles a historic problem but also underscores the power of combining classical techniques with precise analytical bounds to unravel deep mathematical mysteries. Our findings confirm that all perfect numbers are even, closing a significant chapter in number theory.

Keywords: odd perfect numbers; divisor sum function; abundancy index function; prime numbers

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1. Introduction

For centuries, mathematicians have been captivated by the enigmatic allure of perfect numbers, defined as positive integers whose proper divisors sum precisely to the number itself [1]. This fascination traces back to ancient Greece, where Euclid devised an elegant formula for generating even perfect numbers through Mersenne primes, numbers of the form $2^p - 1$ where p is prime [1]. His discovery not only provided a systematic way to construct such numbers, like 6, 28, and 496, but also sparked a profound question that has endured through the ages: could there exist odd perfect numbers, defying the pattern of their even counterparts? This tantalizing mystery, rooted in the simplicity of natural numbers, has fueled mathematical curiosity and inspired relentless exploration.

The quest for odd perfect numbers has been marked by both ingenuity and frustration, as the absence of a definitive example or proof has kept the problem alive for millennia. Early mathematicians, guided by intuition, leaned toward the conjecture that all perfect numbers might be even, yet the lack of a rigorous disproof left room for speculation [1]. Figures like Descartes and Euler, towering giants in the history of mathematics, deepened the intrigue by investigating the potential properties of these elusive numbers [1]. Euler, in particular, highlighted the challenge, noting, "Whether ... there are any odd perfect numbers is a most difficult question". Their efforts revealed constraints—such as the necessity for an odd perfect number to have specific prime factorizations—but no concrete example emerged, leaving the question as a persistent challenge to mathematical rigor.

Today, the mystery of odd perfect numbers remains one of the oldest unsolved problems in number theory, a testament to the profound complexity hidden within simple definitions. Modern computational searches have pushed the boundaries, ruling out odd perfect numbers below staggeringly large thresholds, yet no proof confirms or denies their existence. The problem continues to captivate, not only for its historical significance but also for its ability to bridge elementary arithmetic with deep theoretical questions. As mathematicians wield advanced tools and novel approaches, the search for odd perfect numbers endures, embodying the timeless pursuit of truth in the face of uncertainty.

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This paper presents a rigorous proof by contradiction demonstrating that odd perfect numbers, positive integers whose proper divisors sum to the number itself, do not exist. Assuming the existence of a smallest odd perfect number N, we leverage its abundancy index $I(N) = 2$ and express it as a product over prime factors using a key lemma. By Proposition 4, N must have at least 10 distinct prime factors, and the sum of their reciprocals is bounded by $\ln 2$. Expanding the product and bounding its terms, we derive a scaled inequality that must hold for $I(N) = 2$. However, for $k \geq 10$ prime factors, the sum of scaled terms falls short of the required threshold, leading to a contradiction when compared to the expected value, thus proving that no such N can exist.

2. Background and Ancillary Results

Definition 1. *In number theory, the p-adic order of a positive integer n, denoted $\nu_p(n)$, is the highest exponent of a prime number p that divides n. For example, if $n = 72 = 2^3 \cdot 3^2$, then $\nu_2(72) = 3$ and $\nu_3(72) = 2$.*

The divisor sum function, denoted $\sigma(n)$, is a fundamental arithmetic function that computes the sum of all positive divisors of a positive integer n, including 1 and n itself. For instance, the divisors of 12 are $1, 2, 3, 4, 6, 12$, yielding $\sigma(12) = 1 + 2 + 3 + 4 + 6 + 12 = 28$. This function can be expressed multiplicatively over the prime factorization of n, providing a powerful tool for analyzing perfect numbers.

Proposition 1. *For a positive integer $n > 1$ with prime factorization $n = \prod_{p|n} p^{\nu_p(n)}$ [2]:*

$$\sigma(n) = \prod_{p|n}\left(1 + p + p^2 + \cdots + p^{\nu_p(n)}\right) = n \cdot \prod_{p|n}\left(1 + \frac{1}{p} + \frac{1}{p^2} + \cdots + \frac{1}{p^{\nu_p(n)}}\right),$$

where $p \mid n$ indicates that p is a prime divisor of n.

The abundancy index, defined as $I(n) = \frac{\sigma(n)}{n}$, maps positive integers to rational numbers and quantifies how the divisor sum compares to the number itself. For perfect numbers, $I(n) = 2$, reflecting the defining property that the sum of proper divisors equals the number. The following proposition provides a precise formula for $I(n)$ based on the prime factorization.

Proposition 2. *Let $n = \prod_{i=1}^{j} p_i^{a_i}$ be the prime factorization of n, where $p_1 < \cdots < p_j$ are distinct primes and $a_1, \ldots, a_j$ are positive integers. Then [3]:*

$$I(n) = \prod_{i=1}^{j}\left(\sum_{k=0}^{a_i} \frac{1}{p_i^k}\right) = \prod_{i=1}^{j} \frac{p_i^{a_i+1} - 1}{p_i^{a_i}(p_i - 1)} = \left(\prod_{i=1}^{j} \frac{p_i}{p_i - 1}\right) \cdot \prod_{i=1}^{j}\left(1 - \frac{1}{p_i^{a_i+1}}\right).$$

Definition 2. *The radical of a positive integer n, denoted $\operatorname{rad}(n)$, is the largest square-free divisor of n, obtained as the product of distinct prime factors of n [4]. For example, if $n = 72 = 2^3 \cdot 3^2$, then $\operatorname{rad}(72) = 2 \cdot 3 = 6$.*

Proposition 3. *A positive integer n is a perfect number if and only if $I(n) = 2$, meaning $\sigma(n) = 2n$.*

Proposition 4. *If N is an odd perfect number, it must satisfy the following conditions [5–8]:*

- *$N = p_1^{e_1} \cdot p_2^{2e_2} \cdot p_3^{2e_3} \cdot \ldots \cdot p_k^{2e_k}$, where $p_1, p_2, \ldots, p_k$ are distinct odd primes, and $e_1, \ldots, e_k \geq 1$.*
- *N has at least 10 distinct prime factors (i.e., $k \geq 10$).*
- *The reciprocals of its prime factors satisfy $\frac{1}{p_1} + \frac{1}{p_2} + \cdots + \frac{1}{p_k} < \ln 2$, where $\ln$ denotes the natural logarithm.*

These constraints provide critical insights into the potential structure of odd perfect numbers, guiding efforts to prove their existence or non-existence.

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By establishing a contradiction in the assumed existence of odd perfect numbers, leveraging the above properties, we aim to resolve their non-existence definitively.

3. Main Result

This is a main insight.

Lemma 1. *For a positive integer $n > 1$ with prime factorization $n = \prod_{p|n} p^{\nu_p(n)}$:*

$$I(n) = \prod_{p|n}\left(1 + \frac{I(p^{\nu_p(n)-1})}{p}\right),$$

where $I(n) = \frac{\sigma(n)}{n}$ is the abundancy index, $\nu_p(n)$ is the p-adic order of n, and $\nu_p(n) - 1 \geq 0$ is a non-negative integer for all primes p dividing n.

Proof. The abundancy index is defined as $I(n) = \frac{\sigma(n)}{n}$, where $\sigma(n)$ is the sum of all positive divisors of n. Using Proposition 1, the divisor sum function for $n > 1$ is:

$$\sigma(n) = \prod_{p|n}\left(1 + p + p^2 + \cdots + p^{\nu_p(n)}\right).$$

Thus, the abundancy index becomes:

$$I(n) = \frac{\sigma(n)}{n} = \frac{1}{n} \cdot \prod_{p|n}\left(1 + p + p^2 + \cdots + p^{\nu_p(n)}\right).$$

Since $n = \prod_{p|n} p^{\nu_p(n)}$, we can rewrite the expression by factoring terms appropriately.

Consider the term inside the product for each prime $p \mid n$. We aim to express it in a form that leverages the abundancy index of prime powers. Notice that:

$$1 + p + p^2 + \cdots + p^{\nu_p(n)} = \sum_{k=0}^{\nu_p(n)} p^k.$$

We can manipulate this sum by isolating the highest term:

$$\sum_{k=0}^{\nu_p(n)} p^k = \left(\sum_{k=0}^{\nu_p(n)-1} p^k\right) + p^{\nu_p(n)}.$$

Dividing through by $p^{\nu_p(n)}$ and adjusting, we get:

$$\frac{1 + p + \cdots + p^{\nu_p(n)}}{p^{\nu_p(n)}} = \left(\frac{1}{p^{\nu_p(n)}} \sum_{k=0}^{\nu_p(n)-1} p^k\right) + 1 = \left(\sum_{k=1}^{\nu_p(n)} \frac{1}{p^k}\right) + 1 = 1 + \sum_{k=0}^{\nu_p(n)-1} \frac{1}{p^{k+1}}.$$

This suggests a connection to the abundancy index of a lower prime power. By Proposition 2, for a prime power p^m:

$$I(p^m) = \sum_{k=0}^{m} \frac{1}{p^k}.$$

Thus, for $m = \nu_p(n) - 1$:

$$I(p^{\nu_p(n)-1}) = \sum_{k=0}^{\nu_p(n)-1} \frac{1}{p^k}.$$

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We now rewrite the product term:

$$\frac{1+p+\cdots+p^{\nu_p(n)}}{p^{\nu_p(n)}} = 1 + \frac{1}{p} \cdot \sum_{k=0}^{\nu_p(n)-1} \frac{1}{p^k} = 1 + \frac{I(p^{\nu_p(n)-1})}{p}.$$

Incorporating this back into the abundancy index:

$$I(n) = \frac{1}{n} \cdot \sigma(n) = \frac{1}{\prod_{p|n} p^{\nu_p(n)}} \cdot \prod_{p|n} \left(1+p+\cdots+p^{\nu_p(n)}\right) = \prod_{p|n} \frac{1+p+\cdots+p^{\nu_p(n)}}{p^{\nu_p(n)}}.$$

Substituting the expression derived above:

$$I(n) = \prod_{p|n} \left(1 + \frac{I(p^{\nu_p(n)-1})}{p}\right).$$

Since $\nu_p(n) \geq 1$ for all $p \mid n$, the exponent $\nu_p(n) - 1 \geq 0$, ensuring that $I(p^{\nu_p(n)-1})$ is well-defined (with $I(p^0) = I(1) = 1$). This completes the proof. □

This is the main theorem.

Theorem 1. *Odd perfect numbers do not exist.*

Proof. To derive a contradiction, assume that there exists a smallest odd perfect number N. By Proposition 3, since N is perfect, its abundancy index satisfies $I(N) = \frac{\sigma(N)}{N} = 2$. From Lemma 1, we have:

$$I(N) = \prod_{p|N} \left(1 + \frac{I(p^{\nu_p(N)-1})}{p}\right),$$

where p ranges over the distinct prime factors of N, and $\nu_p(N)$ denotes the p-adic order of N. We aim to show that the equation

$$2 = \prod_{p|N} \left(1 + \frac{I(p^{\nu_p(N)-1})}{p}\right)$$

cannot hold for an odd number N. Let k be the number of distinct prime factors of N, and define:

$$a_i = \frac{I(p_i^{\nu_{p_i}(N)-1})}{p_i},$$

where $p_1, p_2, \ldots, p_k$ are the distinct prime factors of N. Thus, the abundancy index becomes:

$$I(N) = \prod_{i=1}^{k} (1 + a_i).$$

For positive terms $a_i > 0$, we expand the product as:

$$\prod_{i=1}^{k} (1 + a_i) = 1 + \sum_{i=1}^{k} a_i + \sum_{i<j} a_i a_j + \cdots + \prod_{i=1}^{k} a_i.$$

Denote the sum of terms by:

$$I(N) = 1 + A_1 + A_2 + \cdots + A_k,$$

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where:

$$\begin{aligned} A_1 &= \sum_{i=1}^{k} \frac{I(p_i^{\nu_{p_i}(N)-1})}{p_i}, \\ A_2 &= \sum_{i<j} \frac{I(p_i^{\nu_{p_i}(N)-1})}{p_i} \cdot \frac{I(p_j^{\nu_{p_j}(N)-1})}{p_j}, \\ &\vdots \\ A_k &= \prod_{i=1}^{k} \frac{I(p_i^{\nu_{p_i}(N)-1})}{p_i}. \end{aligned}$$

Since $I(N) = 2$, we obtain:

$$2 = 1 + A_1 + A_2 + \cdots + A_k,$$

which implies:

$$1 = \frac{1}{2} + \frac{A_1}{2} + \frac{A_2}{2} + \cdots + \frac{A_k}{2}.$$

This can be rewritten as:

$$1 < \frac{1}{2} + A'_1 + A'_2 + \cdots + A'_k, \tag{1}$$

where:

$$\begin{aligned} \frac{A_1}{2} &= \sum_{i=1}^{k} \frac{1}{2} \cdot \frac{I(p_i^{\nu_{p_i}(N)-1})}{p_i} < \sum_{i=1}^{k} \frac{\frac{I(N)}{2}}{p_i} = \sum_{i=1}^{k} \frac{1}{p_i} = A'_1, \\ \frac{A_2}{2} &= \sum_{i<j} \frac{1}{2} \cdot \frac{I(p_i^{\nu_{p_i}(N)-1})}{p_i} \cdot \frac{I(p_j^{\nu_{p_j}(N)-1})}{p_j} < \sum_{i<j} \frac{\frac{I(N)}{2}}{p_i p_j} = \sum_{i<j} \frac{1}{p_i p_j} = A'_2, \\ &\vdots \\ \frac{A_k}{2} &= \frac{1}{2} \cdot \prod_{i=1}^{k} \frac{I(p_i^{\nu_{p_i}(N)-1})}{p_i} < \frac{I(N)}{2} \cdot \prod_{i=1}^{k} \frac{1}{p_i} = \prod_{i=1}^{k} \frac{1}{p_i} = A'_k. \end{aligned}$$

These inequalities hold due to the multiplicativity of the abundancy index and because each term

$$\frac{I(p_i^{\nu_{p_i}(N)-1}) \cdot I(p_j^{\nu_{p_j}(N)-1}) \cdots I(p_h^{\nu_{p_h}(N)-1})}{p_i \cdot p_j \cdots p_h} = \frac{I(p_i^{\nu_{p_i}(N)-1} \cdot p_j^{\nu_{p_j}(N)-1} \cdots p_h^{\nu_{p_h}(N)-1})}{p_i \cdot p_j \cdots p_h} \leq \frac{I\left(\frac{N}{\mathrm{rad}(N)}\right)}{p_i \cdot p_j \cdots p_h},$$

where:

$$I\left(\frac{N}{\mathrm{rad}(N)}\right) < I(N) = 2.$$

Multiplying both sides of inequality (1) by $\frac{1}{(\ln 2)^k}$, we obtain:

$$\frac{1}{(\ln 2)^k} < \frac{1}{2(\ln 2)^k} + \frac{A'_1}{(\ln 2)^k} + \frac{A'_2}{(\ln 2)^k} + \cdots + \frac{A'_k}{(\ln 2)^k},$$

which simplifies to:

$$\frac{1}{2(\ln 2)^k} < \frac{A'_1}{(\ln 2)^k} + \frac{A'_2}{(\ln 2)^k} + \cdots + \frac{A'_k}{(\ln 2)^k}, \tag{2}$$

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since:

$$\frac{1}{(\ln 2)^k} - \frac{1}{2(\ln 2)^k} = \frac{1}{2(\ln 2)^k}.$$

By Proposition 4, we know:

$$A'_1 = \sum_{i=1}^{k} \frac{1}{p_i} < \ln 2,$$

so:

$$\frac{A'_1}{(\ln 2)^k} < 1.$$

Next, observe that:

$$A'_2 = \sum_{i<j} \frac{1}{p_i p_j} < \sum_{i=1}^{k} \frac{A'_1}{p_i},$$

since the denominators of A'_2 are products of distinct primes (semiprimes). For instance, consider the prime p_1 among the distinct prime factors $p_1, p_2, \ldots, p_k$ of N. We can express:

$$\frac{A'_1}{p_1} = \sum_{i=1}^{k} \frac{1}{p_1 p_i} = \frac{1}{p_1^2} + \sum_{i=2}^{k} \frac{1}{p_1 p_i} > \sum_{i=2}^{k} \frac{1}{p_1 p_i},$$

where the sum $\sum_{i=2}^{k} \frac{1}{p_1 p_i}$ represents the terms in $A'_2 = \sum_{i<j} \frac{1}{p_i p_j}$ that include p_1 in the denominator. This reasoning extends to any prime p_i for $1 \leq i \leq k$, illustrating that dividing A'_1 by p_i yields terms contributing to A'_2 alongside additional positive terms like $\frac{1}{p_i^2}$. Thus:

$$\begin{aligned}
\frac{A'_1}{(\ln 2)^2} &< \frac{1}{(\ln 2)^2} \sum_{i=1}^{k} \frac{A'_1}{p_i} \\
&= \frac{1}{\ln 2} \sum_{i=1}^{k} \frac{\frac{A'_1}{\ln 2}}{p_i} \\
&< \frac{1}{\ln 2} \sum_{i=1}^{k} \frac{1}{p_i} \\
&= \frac{A'_1}{\ln 2} \\
&< 1,
\end{aligned}$$

based on $\frac{A'_1}{\ln 2} < 1$. Therefore:

$$\frac{A'_2}{(\ln 2)^k} < 1.$$

For each prime p_i among the distinct prime factors $p_1, p_2, \ldots, p_k$ of N, the term $\frac{A'_2}{p_i}$ encompasses all summands in $A'_3 = \sum_{i<j<h} \frac{1}{p_i p_j p_h}$ that include p_i in the denominator. Specifically, dividing $A'_2 = \sum_{m<n} \frac{1}{p_m p_n}$ by p_i yields terms of the form $\frac{1}{p_i p_m p_n}$, which includes the A'_3 terms with p_i. Additionally, $\frac{A'_2}{p_i}$ contains summands not present in A'_3, namely those with denominators $p_i^2 p_j$ for each $j \neq i$, arising when the pair (p_m, p_n) in A'_2 includes p_i. Similarly, we deduce:

$$\frac{A'_3}{(\ln 2)^k} < 1,$$

since:

$$\frac{A'_3}{(\ln 2)^3} < 1,$$

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given that:

$$A'_3 < \sum_{i=1}^{k} \frac{A'_2}{p_i}.$$

Indeed:

$$\begin{aligned}\frac{A'_3}{(\ln 2)^3} &< \frac{1}{(\ln 2)^3} \sum_{i=1}^{k} \frac{A'_2}{p_i} \\ &= \frac{1}{\ln 2} \sum_{i=1}^{k} \frac{\frac{A'_2}{(\ln 2)^2}}{p_i} \\ &< \frac{1}{\ln 2} \sum_{i=1}^{k} \frac{1}{p_i} \\ &= \frac{A'_1}{\ln 2} \\ &< 1,\end{aligned}$$

assuming $\frac{A'_2}{(\ln 2)^2} < 1$. We can continue this iterative process until we reach the term $\frac{A'_k}{(\ln 2)^k}$, where $A'_k = \prod_{i=1}^{k} \frac{1}{p_i}$ and $p_1, p_2, \ldots, p_k$ are the distinct prime factors of N. We show that:

$$\frac{A'_k}{(\ln 2)^k} = \prod_{i=1}^{k} \frac{1}{p_i(\ln 2)} < \prod_{i=1}^{k} \frac{2}{p_i} < 1.$$

This follows because $\frac{1}{\ln 2} \approx 1.4427 < 2$, so $\frac{1}{p_i \ln 2} < \frac{2}{p_i}$, and for any odd prime $p_i \geq 3$, we have $\frac{2}{p_i} \leq \frac{2}{3} < 1$. Thus, the product $\prod_{i=1}^{k} \frac{2}{p_i}$ is strictly less than 1, ensuring that $\frac{A'_k}{(\ln 2)^k} < 1$.

Combining these, we find:

$$\frac{1}{2(\ln 2)^k} < \frac{A'_1}{2(\ln 2)^k} + \frac{A'_2}{(\ln 2)^k} + \cdots + \frac{A'_k}{(\ln 2)^k} < 1 + 1 + \cdots + 1 = k,$$

which implies:

$$0 < k - \frac{1}{2(\ln 2)^k}.$$

Consider the function:

$$f(x) = x - \frac{1}{2(\ln 2)^x}.$$

Its derivative is:

$$f'(x) = 1 + \frac{1}{2} \ln(\ln 2) \cdot (\ln 2)^{-x},$$

which is negative in the interval $[10, \infty)$, since $\ln(\ln 2) < 0$ and $(\ln 2)^{-x} > 0$, making $f(x)$ decreasing. By Proposition 4, $k \geq 10$, and evaluating $f(10) < 0$ shows that $f(k) < 0$ for $k \geq 10$. Thus:

$$0 < k - \frac{1}{2(\ln 2)^k} < 0,$$

or equivalently:

$$0 < 0,$$

a clear contradiction. Hence, no odd perfect number N exists. □

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