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Article

Module Algebra Structures of Nonstandard Quantum Group $X_q(A_1)$ on $\mathbb{C}[x, y, z]$

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Abstract: In this paper, the module algebra structures of $X_q(A_1)$ on quantum polynomial algebra $\mathbb{C}_q[x, y, z]$ are investigated, and a complete classification of $X_q(A_1)$-module algebra structures on $\mathbb{C}_q[x, y, z]$ is given.

Keywords: nonstandard quantum group; quantum polynomial algebra; Hopf action; module algebra; weight

MSC: 16T20, 16S40, 17B37, 20G42

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1. Introduction

The notion of Hopf algebra actions on algebras was introduced by Beattie [6,7] in 1976. A duality theorem for Hopf module algebras was studied by Blattner and Montgomery [8] in 1985. It generalized the corresponding theorem of group actions. Moreover, the actions of Hopf algebras [21] and their generalizations (see, e.g.,[15]) play an important role in quantum group theory [19,20] and in its various applications in physics [9]. Duplij and Sinel'shchikov used a general form of the automorphism of the quantum plane to give the notion of weight for $U_q(sl_2)$-actions considered here, and completely classified quantum group $U_q(sl_2)$-module algebra structures on the quantum plane [13,14], so that the results are much richer and consist of 8 non-isomorphic cases. Moreover, In [12] the authors used the method of weights [13,14] to classify some actions in terms of action matrices, the modules algebra structures of the quantum group $U_q(sl(m+1))$ were studied on the coordinate algebra of quantum vector spaces, and the concrete actions of quantum group $U_q(sl_2)$ on $\mathbb{C}_q[x, y, z]$ were researched (also see [24]). More relevant research can be found at [10,17].

The non-standard quantum groups were studied in [16], Ge et al. obtained new solutions of Yang-Baxter equations and included the twisted extensions quantum group structures related to these new solutions explicitly. In [1] one class of non-standard quantum deformation corresponding to simple Lie algebra sl_n was given, which is denoted by $X_q(A_{n-1})$. For each vertex $i(i = 1, \cdots, n)$ of the Dynkin diagram, the parameter q_i is equal to q or $-q^{-1}$, if $q_i = q$ for all i, then $X_q(A_{n-1})$ is just $U_q(sl_n)$. However, if $q_i \neq q_{i+1}$ for some $1 \leq i \leq n$, it has the relations $E_i^2 = F_i^2 = 0$ in $X_q(A_{n-1})$. Such a $X_q(A_{n-1})$ is different to $U_q(sl_n)$. Jing et al. [18] derived a non-standard quantum group by employing the FRT constructive method, and classify all finite dimensional irreducible representations of this non-standard quantum group. Cheng and Yang [11] considered the structures and representations of weak Hopf algebras $\mathfrak{w}X_q(A_1)$, which is corresponding to non-standard quantum group $X_q(A_1)$. We [22] researched the representations of a class of small nonstandard quantum groups $\overline{X}_q(A_1)$, over which the isomorphism classes of all indecomposable modules are classified, and the decomposition formulas of the tensor product of arbitrary indecomposable modules and simple (or projective) modules are established. The projective class rings and Grothendieck rings of $\overline{X}_q(A_1)$ are also characterized. However, the research on module algebra of non-standard quantum groups has not yet yielded any results. Consequently, based on the research results of module algebra of quantum groups, we consider here the module algebra of the nonstandard quantum group $X_q(A_1)$ on the quantum polynomial

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algebra $\mathbb{C}_q[x,y,z]$. and a complete list of $X_q(A_1)$-module algebra structures on $\mathbb{C}_q[x,y,z]$ is produced and the isomorphism classes of these structures are described.

This paper is organized as follows. In Section 1, we introduce some necessary notations and the concepts. In Section 2, when $t=0$, we discuss the module algebra structures of $X_q(A_1)$ on the polynomial algebra $\mathbb{C}_q[x,y,z]$ using the method of weights [13,14]. We study the concrete actions of $X_q(A_1)$ on $\mathbb{C}_q[x,y,z]$ and characterize all module algebra structures of $X_q(A_1)$ on $\mathbb{C}_q[x,y,z]$. In Section 3, we study the module algebra structures of $X_q(A_1)$ on $\mathbb{C}_q[x,y,z]$ with $t\neq 0$. In the same way of section 2, We study the concrete actions of $X_q(A_1)$ on $\mathbb{C}_q[x,y,z]$ and characterize all module algebra structures of $X_q(A_1)$ on $\mathbb{C}_q[x,y,z]$.

2. Preliminaries

Throughout, we work over the complex field $\mathbb{C}$ unless otherwise stated. All algebras, Hopf algebras and modules are defined over $\mathbb{C}$; all maps are $\mathbb{C}$-linear.

Let $(H,m,\eta,\Delta,\varepsilon,S)$ be a Hopf algebra, here Δ and ε are the comultiplication and counit of H, respectively. Let A be a unital algebra with unit $\mathbf{1}$. We will also use the Sweedler notation $\Delta(h)=\sum_i h'_i\otimes h''_i$ [23].

Definition 2.1. *By a structure of H-module algebra on A, we mean a homomorphism $\pi: H\to \mathrm{End}_{\mathbb{C}}A$ such that:*

1. *for all $h\in H, a,b\in A$, $\pi(h)(ab)=\sum_i \pi(h'_i)(a)\cdot\pi(h''_i)(b)$;*
2. *for all $h\in H$, $\pi(h)(1)=\varepsilon(h)\mathbf{1}$.*

The structures π_1,π_2 are said to be isomorphic if there exists an automorphism Ψ of the algebra A such that $\Psi\pi_1(h)\Psi^{-1}=\pi_2(h)$ for all $h\in H$.

We assume that $q\in\mathbb{C}^*=\mathbb{C}\setminus\{0\}$ is not a root of the unit ($q^n\neq 1$ for all non-zero integers n). A class of non-standard quantum algebra $X_q(A_1)$ was studied by Jing etc. [18]. By definition the algebra $X_q(A_1)$ is a unital associative $\mathbb{C}$-algebra generated by $E,F,K_i,K_i^{-1}(i=1,2)$ subject to the relations:

$$K_iK_i^{-1}=K_i^{-1}K_i=1,\ \ K_1K_2=K_2K_1, \tag{1}$$
$$K_1E=q^{-1}EK_1, \tag{2}$$
$$K_1F=qFK_1, \tag{3}$$
$$K_2E=-q^{-1}EK_2, \tag{4}$$
$$K_2F=-qFK_2, \tag{5}$$
$$EF-FE=\frac{K_2K_1^{-1}-K_2^{-1}K_1}{q-q^{-1}}, \tag{6}$$
$$E^2=F^2=0. \tag{7}$$

The algebra $X_q(A_1)$ is also a Hopf algebra, the comultiplication Δ, counit ε and antipode S are given as the following

$$\Delta(K_i)=K_i\otimes K_i, \tag{8}$$
$$\Delta(E)=E\otimes 1+K_2K_1^{-1}\otimes E, \tag{9}$$
$$\Delta(F)=1\otimes F+F\otimes K_2^{-1}K_1, \tag{10}$$

$$\begin{aligned} \varepsilon(K_i)&=1, & \varepsilon(E)&=0, & \varepsilon(F)&=0,\\ S(K_i)&=K_i^{-1}, & S(E)&=-K_1K_2^{-1}E, & S(F)&=-FK_2K_1^{-1}. \end{aligned}$$

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We consider the quantum polynomial algebra $\mathbb{C}_q[x,y,z]$ is a unital algebra, generated by generators x,y,z, and satisfying the relations

$$yx = qxy, \tag{11}$$
$$zy = qyz, \tag{12}$$
$$zx = qxz. \tag{13}$$

Denote by $\mathbb{C}_q[x,y,z]_s$ the s-th homogeneous component of $\mathbb{C}_q[x,y,z]$, which is a linear span of the monomials $x^{m_1}y^{m_2}z^{m_3}$ with $m_1+m_2+m_3=s$. Also, given a polynomial $p\in\mathbb{C}_q[x,y,z]$, denote by p_s the s-th homogeneous component of p, that is the projection of p onto $\mathbb{C}_q[x,y,z]_s$ parallel to the direct sum of all other homogeneous components of $\mathbb{C}_q[x,y,z]$.

By [2–5], one has a description of automorphisms of the algebra $\mathbb{C}_q[x,y,z]$, as follows. Let Ψ be an automorphism of $\mathbb{C}_q[x,y,z]$, then there exist nonzero constants $\alpha,\beta,\gamma\in\mathbb{C}^*$ and $t\in\mathbb{C}$, such that

$$\Psi : x\to\alpha x,\quad y\to\beta y+txz,\quad z\to\gamma z.$$

All such automorphisms form the automorphism group of $\mathbb{C}_q[x,y,z]$ denoted by $\mathrm{Aut}(\mathbb{C}_q[x,y,z])$, one can get $\mathrm{Aut}(\mathbb{C}_q[x,y,z])\cong\mathbb{C}\rtimes(\mathbb{C}^*)^3$. In the following sections, we will explore the classificaion of $X_q(A_1)$-module algebra structures on $\mathbb{C}_q[x,y,z]$.

3. When $t=0$, Classification of $X_q(A_1)$-Module Algebra Structures on $\mathbb{C}_q[x,y,z]$

In this section, our aim is to describe the $X_q(A_1)$-module algebra structures on $\mathbb{C}_q[x,y,z]$, with $t=0$, ie. the automorphism of $\mathbb{C}_q[x,y,z]$ as follows

$$\Psi(x)=\alpha x,\quad \Psi(y)=\beta y,\quad \Psi(z)=\gamma z,\quad (\alpha,\beta,\gamma\in\mathbb{C}^*),$$

and $\mathrm{Aut}(\mathbb{C}_q[x,y,z])\cong(\mathbb{C}^*)^3$, here $K_1,K_2\in\mathrm{Aut}(\mathbb{C}_q[x,y,z])$.

3.1. Properties of $X_q(A_1)$-Module Algebras on $\mathbb{C}_q[x,y,z]$

By the definition of module algebra, it is easy to see that any action of $X_q(A_1)$ on $\mathbb{C}_q[x,y,z]$ is determined by the following 4×3 matrix with entries from $\mathbb{C}_q[x,y,z]$:

$$M \overset{definition}{=} \begin{pmatrix} K_1(x) & K_1(y) & K_1(z)\\ K_2(x) & K_2(y) & K_2(z)\\ E(x) & E(y) & E(z)\\ F(x) & F(y) & F(z)\end{pmatrix}, \tag{14}$$

which is called the full action matrix. Given a $X_q(A_1)$-module algebra structure on $\mathbb{C}_q[x,y,z]$, obviously, the action of K_1(or K_2) is determined by an automorphism of $\mathbb{C}_q[x,y,z]$, in other words, the actions of K_1 and K_2 are determined by a matrix $M_{K_1K_2}$ as follows

$$\begin{aligned} M_{K_1K_2} &\overset{definition}{=} \begin{pmatrix} K_1(x) & K_1(y) & K_1(z)\\ K_2(x) & K_2(y) & K_2(z)\end{pmatrix}\\ &= \begin{pmatrix} \alpha_1(x) & \beta_1(y) & \gamma_1(z)\\ \alpha_2(x) & \beta_2(y) & \gamma_2(z)\end{pmatrix},\end{aligned} \tag{15}$$

where $\alpha_i,\beta_i\in\mathbb{C}^*$ for $i\in\{1,2\}$. It is easy to see that every monomial $x^{m_1}y^{m_2}z^{m_3}\in\mathbb{C}_q[x,y,z]$ is an eigenvector of K_1(or K_2), and the associated eigenvalue $\alpha_1^{m_1}\beta_1^{m_2}\gamma_1^{m_3}$ (or $\alpha_2^{m_1}\beta_2^{m_2}\gamma_2^{m_3}$) is called the K_1-weight (or K_2-weight) of this monomial, which will be written as

$$wt_{K_1}(x^{m_1}y^{m_2}z^{m_3})=\alpha_1^{m_1}\beta_1^{m_2}\gamma_1^{m_3},$$

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$$wt_{K_2}(x^{m_1}y^{m_2}z^{m_3}) = \alpha_2^{m_1}\beta_2^{m_2}\gamma_2^{m_3}.$$

We will also need another matrix M_{EF} as follows

$$M_{EF} \overset{definition}{=} \begin{pmatrix} E(x) & E(y) & E(z) \\ F(x) & F(y) & F(z) \end{pmatrix}. \tag{16}$$

Obviously, all entries of M are weight vectors for K_1 and K_2, then

$$\begin{aligned} wt_{K_i}(M) \overset{definition}{=} & \begin{pmatrix} wt_{K_i}(K_1(x)) & wt_{K_i}(K_1(y)) & wt_{K_i}(K_1(z)) \\ wt_{K_i}(K_2(x)) & wt_{K_i}(K_2(y)) & wt_{K_i}(K_2(z)) \\ wt_{K_i}(E(x)) & wt_{K_i}(E(y)) & wt_{K_i}(E(z)) \\ wt_{K_i}(F(x)) & wt_{K_i}(F(y)) & wt_{K_i}(F(z)) \end{pmatrix} \\ \bowtie & \begin{pmatrix} wt_{K_i}(x) & wt_{K_i}(y) & wt_{K_i}(z) \\ wt_{K_i}(x) & wt_{K_i}(y) & wt_{K_i}(z) \\ (-1)^{i-1}q^{-1}wt_{K_i}(x) & (-1)^{i-1}q^{-1}wt_{K_i}(y) & (-1)^{i-1}q^{-1}wt_{K_i}(z) \\ (-1)^{i-1}q\,wt_{K_i}(x) & (-1)^{i-1}q\,wt_{K_i}(y) & (-1)^{i-1}q\,wt_{K_i}(z) \end{pmatrix} \\ = & \begin{pmatrix} \alpha_i & \beta_i & \gamma_i \\ \alpha_i & \beta_i & \gamma_i \\ (-1)^{i-1}q^{-1}\alpha_i & (-1)^{i-1}q^{-1}\beta_i & (-1)^{i-1}q^{-1}\gamma_i \\ (-1)^{i-1}q\alpha_i & (-1)^{i-1}q\beta_i & (-1)^{i-1}q\gamma_i \end{pmatrix}, \end{aligned} \tag{17}$$

where the relation $A = (a_{st}) \bowtie B = (b_{st})$ means that for every pair of indices s, t such that both a_{st} and b_{st} are nonzero, one has $a_{st} = b_{st}$.

We denote by $(M)_j$ the j-th homogeneous component of M, whose elements are just the j-th homogeneous components of the corresponding entries of M. Set

$$(M)_0 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ a_0 & b_0 & c_0 \\ a'_0 & b'_0 & c'_0 \end{pmatrix}_0, \tag{18}$$

where, $a_0, b_0, c_0, a'_0, b'_0, c'_0 \in \mathbb{C}$. Then, we obtain

$$wt_{K_1}((M_{EF})_0) \bowtie \begin{pmatrix} q^{-1}\alpha_1 & q^{-1}\beta_1 & q^{-1}\gamma_1 \\ q\alpha_1 & q\beta_1 & q\gamma_1 \end{pmatrix}_0 \bowtie \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}_0, \tag{19}$$

$$wt_{K_2}((M_{EF})_0) \bowtie \begin{pmatrix} -q^{-1}\alpha_2 & -q^{-1}\beta_2 & -q^{-1}\gamma_2 \\ -q\alpha_2 & -q\beta_2 & -q\gamma_2 \end{pmatrix}_0 \bowtie \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}_0. \tag{20}$$

According to q is not a root of the unit and relations (19)-(20), it means that each column of M_{EF} should contain at least one 0.

An application of E and F to the relations (11)-(13) by using equation (15), one has

$$E(y)x + \beta_1^{-1}\beta_2 yE(x) = qE(x)y + q\alpha_1^{-1}\alpha_2 xE(y), \tag{21}$$
$$E(z)y + \gamma_1^{-1}\gamma_2 zE(y) = qE(y)z + q\beta_1^{-1}\beta_2 yE(z), \tag{22}$$
$$E(z)x + \gamma_1^{-1}\gamma_2 zE(x) = qE(x)z + q\alpha_1^{-1}\alpha_2 xE(z), \tag{23}$$
$$yF(x) + \alpha_2^{-1}\alpha_1 F(y)x = qxF(y) + q\beta_2^{-1}\beta_1 F(x)y, \tag{24}$$
$$zF(y) + \beta_2^{-1}\beta_1 F(z)y = qyF(z) + q\gamma_2^{-1}\gamma_1 F(y)z, \tag{25}$$
$$zF(x) + \alpha_2^{-1}\alpha_1 F(z)x = qxF(z) + q\gamma_2^{-1}\gamma_1 F(x)z. \tag{26}$$

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After projecting equations (21)-(26) to $\mathbb{C}_q[x,y,z]_1$, we obtain

$$\begin{aligned}
b_0\left(1-q\alpha_1^{-1}\alpha_2\right)x+a_0\left(\beta_1^{-1}\beta_2-q\right)y=0,\\
c_0\left(1-q\beta_1^{-1}\beta_2\right)y+b_0\left(\gamma_1^{-1}\gamma_2-q\right)z=0,\\
c_0\left(1-q\alpha_1^{-1}\alpha_2\right)x+a_0\left(\gamma_1^{-1}\gamma_2-q\right)z=0,\\
a_0'\left(1-q\beta_1\beta_2^{-1}\right)y+b_0'\left(\alpha_1\alpha_2^{-1}-q\right)x=0,\\
b_0'\left(1-q\gamma_1\gamma_2^{-1}\right)z+c_0'\left(\beta_1\beta_2^{-1}-q\right)y=0,\\
a_0'\left(1-q\gamma_1\gamma_2^{-1}\right)z+c_0'\left(\alpha_1\alpha_2^{-1}-q\right)x=0,
\end{aligned}$$

which certainly implies

$$\begin{aligned}
b_0\left(1-q\alpha_1^{-1}\alpha_2\right)=a_0\left(\beta_1^{-1}\beta_2-q\right)=c_0\left(1-q\beta_1^{-1}\beta_2\right)=b_0\left(\gamma_1^{-1}\gamma_2-q\right)=0,\\
c_0\left(1-q\alpha_1^{-1}\alpha_2\right)=a_0\left(\gamma_1^{-1}\gamma_2-q\right)=a_0'\left(1-q\beta_1\beta_2^{-1}\right)=b_0'\left(\alpha_1\alpha_2^{-1}-q\right)=0,\\
b_0'\left(1-q\gamma_1\gamma_2^{-1}\right)=c_0'\left(\beta_1\beta_2^{-1}-q\right)=a_0'\left(1-q\gamma_1\gamma_2^{-1}\right)=c_0'\left(\alpha_1\alpha_2^{-1}-q\right)=0.
\end{aligned}$$

We will determine the weight constants α_i, β_i and $\gamma_i (i=1,2)$ as follows:

$$\begin{array}{lllll}
a_0\neq 0 & \Rightarrow & \alpha_1=q, & \alpha_2=-q, & \beta_1^{-1}\beta_2=q, & \gamma_1^{-1}\gamma_2=q;\\
b_0\neq 0 & \Rightarrow & \beta_1=q, & \beta_2=-q, & \alpha_1^{-1}\alpha_2=q^{-1}, & \gamma_1^{-1}\gamma_2=q;\\
c_0\neq 0 & \Rightarrow & \gamma_1=q, & \gamma_2=-q, & \beta_1^{-1}\beta_2=q^{-1}, & \alpha_1^{-1}\alpha_2=q^{-1};\\
a_0'\neq 0 & \Rightarrow & \alpha_1=q^{-1}, & \alpha_2=-q^{-1}, & \beta_1\beta_2^{-1}=q^{-1}, & \gamma_1\gamma_2^{-1}=q^{-1};\\
b_0'\neq 0 & \Rightarrow & \beta_1=q^{-1}, & \beta_2=-q^{-1}, & \alpha_1\alpha_2^{-1}=q, & \gamma_1\gamma_2^{-1}=q^{-1};\\
c_0'\neq 0 & \Rightarrow & \gamma_1=q^{-1}, & \gamma_2=-q^{-1}, & \beta_1\beta_2^{-1}=q, & \alpha_1\alpha_2^{-1}=q.
\end{array} \tag{27}$$

Because q is not a root of the unit, $q\neq\pm1$. Therefore at least one of a_0, b_0, c_0 and a_0', b_0', c_0' is not zero. In summary, we have obtained the following results for the 0-st homogeneous component $(M_{EF})_0$ of M_{EF}.

Lemma 3.1. *There are 7 cases for the 0-st homogeneous component $(M_{EF})_0$ of M_{EF}, as follows:*

$$\begin{pmatrix} a_0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}_0 \Rightarrow \alpha_1=q,\alpha_2=-q,\beta_1^{-1}\beta_2=q,\gamma_1^{-1}\gamma_2=q; \tag{28}$$

$$\begin{pmatrix} 0 & b_0 & 0\\ 0 & 0 & 0\end{pmatrix}_0 \Rightarrow \beta_1=q,\beta_2=-q,\alpha_1^{-1}\alpha_2=q^{-1},\gamma_1^{-1}\gamma_2=q; \tag{29}$$

$$\begin{pmatrix} 0 & 0 & c_0\\ 0 & 0 & 0\end{pmatrix}_0 \Rightarrow \gamma_1=q,\gamma_2=-q,\beta_1^{-1}\beta_2=q^{-1},\alpha_1^{-1}\alpha_2=q^{-1}; \tag{30}$$

$$\begin{pmatrix} 0 & 0 & 0\\ a_0' & 0 & 0\end{pmatrix}_0 \Rightarrow \alpha_1=q^{-1},\alpha_2=-q^{-1},\beta_1\beta_2^{-1}=q^{-1},\gamma_1\gamma_2^{-1}=q^{-1}; \tag{31}$$

$$\begin{pmatrix} 0 & 0 & 0\\ 0 & b_0' & 0\end{pmatrix}_0 \Rightarrow \beta_1=q^{-1},\beta_2=-q^{-1},\alpha_1\alpha_2^{-1}=q,\gamma_1\gamma_2^{-1}=q^{-1}; \tag{32}$$

$$\begin{pmatrix} 0 & 0 & 0\\ 0 & 0 & c_0'\end{pmatrix}_0, \Rightarrow \gamma_1=q^{-1},\gamma_2=-q^{-1},\beta_1\beta_2^{-1}=q,\alpha_1\alpha_2^{-1}=q; \tag{33}$$

$$\begin{pmatrix} 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}_0 \text{it does not determine the weight constants at all.} \tag{34}$$

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Next, for the 1-st homogeneous component, due to q is not a root of the unit, one has

$$wt_{K_1}(E(x)) = q^{-1}\alpha_1 = q^{-1}wt_{K_1}(x) \neq wt_{K_1}(x),$$
$$wt_{K_2}(E(x)) = -q^{-1}\alpha_2 = -q^{-1}wt_{K_2}(x) \neq wt_{K_2}(x),$$

which implies

$$(E(x))_1 = a_1 y + a_2 z,$$

for some $a_1, a_2 \in \mathbb{C}$, and in a similar way we have

$$(M_{EF})_1 = \begin{pmatrix} a_1 y + a_2 z & b_1 x + b_2 z & c_1 x + c_2 y \\ a_1' y + a_2' z & b_1' x + b_2' z & c_1' x + c_2' y \end{pmatrix}_1$$

where $b_1, b_2, c_1, c_2, a_1', a_2', b_1', b_2', c_1', c_2' \in \mathbb{C}$. Therefore

$$\begin{aligned} wt_{K_i}((M_{EF})_1) \quad & \bowtie \begin{pmatrix} (-1)^{i-1}q^{-1}\alpha_i & (-1)^{i-1}q^{-1}\beta_i & (-1)^{i-1}q^{-1}\gamma_i \\ (-1)^{i-1}q\alpha_i & (-1)^{i-1}q\beta_i & (-1)^{i-1}q\gamma_i \end{pmatrix} \\ & \bowtie \begin{pmatrix} \beta_i \text{ or } \gamma_i & \alpha_i \text{ or } \gamma_i & \alpha_i \text{ or } \beta_i \\ \beta_i \text{ or } \gamma_i & \alpha_i \text{ or } \gamma_i & \alpha_i \text{ or } \beta_i \end{pmatrix}, \end{aligned} \tag{35}$$

Now project (21)-(26) to $\mathbb{C}_q[x,y,z]_2$ to obtain

$$\begin{aligned}
&b_1\left(1 - q\alpha_1^{-1}\alpha_2\right)x^2 + b_2\left(1 - \alpha_1^{-1}\alpha_2\right)zx + a_1\left(\beta_1^{-1}\beta_2 - q\right)y^2 + a_2\left(\beta_1^{-1}\beta_2 - q^2\right)yz = 0,\\
&c_1\left(1 - q^2\beta_1^{-1}\beta_2\right)yx + c_2\left(1 - q\beta_1^{-1}\beta_2\right)y^2 + b_2\left(\gamma_1^{-1}\gamma_2 - q\right)z^2 + b_1 q\left(\gamma_1^{-1}\gamma_2 - 1\right)xz = 0,\\
&c_1\left(1 - q\alpha_1^{-1}\alpha_2\right)x^2 + qc_2\left(1 - \alpha_1^{-1}\alpha_2\right)xy + qa_1\left(\gamma_1^{-1}\gamma_2 - 1\right)yz + a_2\left(\gamma_1^{-1}\gamma_2 - q\right)z^2 = 0,\\
&a_1'\left(1 - q\beta_2^{-1}\beta_1\right)y^2 + a_2'\left(1 - q^2\beta_2^{-1}\beta_1\right)yz + b_1'\left(\alpha_2^{-1}\alpha_1 - q\right)x^2 + qb_2'\left(\alpha_2^{-1}\alpha_1 - 1\right)xz = 0,\\
&qb_1'\left(1 - \gamma_2^{-1}\gamma_1\right)xz + b_2'\left(1 - q\gamma_2^{-1}\gamma_1\right)z^2 + c_1'\left(\beta_2^{-1}\beta_1 - q^2\right)xy + c_2'\left(\beta_2^{-1}\beta_1 - q\right)y^2 = 0,\\
&qa_1'\left(1 - \gamma_2^{-1}\gamma_1\right)yz + a_2'\left(1 - q\gamma_2^{-1}\gamma_1\right)z^2 + c_1'\left(\alpha_2^{-1}\alpha_1 - q\right)x^2 + qc_2'\left(\alpha_2^{-1}\alpha_1 - 1\right)xy = 0,
\end{aligned}$$

which certainly implies

$$\begin{aligned}
&b_1\left(1 - q\alpha_1^{-1}\alpha_2\right) = b_2\left(1 - \alpha_1^{-1}\alpha_2\right) = a_1\left(\beta_1^{-1}\beta_2 - q\right) = a_2\left(\beta_1^{-1}\beta_2 - q^2\right) = 0\\
&c_1\left(1 - q^2\beta_1^{-1}\beta_2\right) = c_2\left(1 - q\beta_1^{-1}\beta_2\right) = b_2\left(\gamma_1^{-1}\gamma_2 - q\right) = b_1 q\left(\gamma_1^{-1}\gamma_2 - 1\right) = 0\\
&c_1\left(1 - q\alpha_1^{-1}\alpha_2\right) = qc_2\left(1 - \alpha_1^{-1}\alpha_2\right) = qa_1\left(\gamma_1^{-1}\gamma_2 - 1\right) = a_2 q\left(\gamma_1^{-1}\gamma_2 - q\right) = 0\\
&a_1'\left(1 - q\beta_1\beta_2^{-1}\right) = a_2'\left(1 - q^2\beta_1\beta_2^{-1}\right) = b_1'\left(\alpha_1\alpha_2^{-1} - q\right) = qb_2'\left(\alpha_1\alpha_2^{-1} - 1\right) = 0,\\
&qb_1'\left(1 - \gamma_1\gamma_2^{-1}\right) = b_2'\left(1 - q\gamma_1\gamma_2^{-1}\right) = c_1'\left(\beta_1\beta_2^{-1} - q^2\right) = c_2'\left(\beta_1\beta_2^{-1} - q\right) = 0,\\
&qa_1'\left(1 - \gamma_1\gamma_2^{-1}\right) = a_2'\left(1 - q\gamma_1\gamma_2^{-1}\right) = c_1'\left(\alpha_1\alpha_2^{-1} - q\right) = qc_2'\left(\alpha_1\alpha_2^{-1} - 1\right) = 0.
\end{aligned}$$

As a consequence, we have

$$\begin{aligned}
a_1 \neq 0 \quad &\Rightarrow \quad \beta_2\alpha_1^{-1} = q, \quad && \gamma_2\gamma_1^{-1} = 1,\\
a_2 \neq 0 \quad &\Rightarrow \quad \beta_2\alpha_1^{-1} = q^2, \quad && \gamma_2\gamma_1^{-1} = q,\\
b_1 \neq 0 \quad &\Rightarrow \quad \alpha_2\alpha_1^{-1} = q^{-1}, \quad && \gamma_2\gamma_1^{-1} = 1,\\
b_2 \neq 0 \quad &\Rightarrow \quad \alpha_2\alpha_1^{-1} = 1, \quad && \gamma_2\gamma_1^{-1} = q,\\
c_1 \neq 0 \quad &\Rightarrow \quad \beta_2\alpha_1^{-1} = q^{-2} \quad && \alpha_2\alpha_1^{-1} = q^{-1},\\
c_2 \neq 0 \quad &\Rightarrow \quad \beta_2\alpha_1^{-1} = q^{-1} \quad && \alpha_2\alpha_1^{-1} = 1,
\end{aligned} \tag{36}$$

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$$\begin{array}{llll} a_1' \neq 0 & \Rightarrow & \beta_2\alpha_1^{-1} = q, & \gamma_2\gamma_1^{-1} = 1, \\ a_2' \neq 0 & \Rightarrow & \beta_2\alpha_1^{-1} = q^2, & \gamma_2\gamma_1^{-1} = q, \\ b_1' \neq 0 & \Rightarrow & \alpha_2\alpha_1^{-1} = q^{-1}, & \gamma_2\gamma_1^{-1} = 1, \\ b_2' \neq 0 & \Rightarrow & \alpha_2\alpha_1^{-1} = 1, & \gamma_2\gamma_1^{-1} = q, \\ c_1' \neq 0 & \Rightarrow & \beta_2\alpha_1^{-1} = q^{-2} & \alpha_2\alpha_1^{-1} = q^{-1}, \\ c_2' \neq 0 & \Rightarrow & \beta_2\alpha_1^{-1} = q^{-1} & \alpha_2\alpha_1^{-1} = 1. \end{array} \tag{37}$$

From the above discussion, for the 1-st homogeneous component $(M_{EF})_1$ of M_{EF}, we have following lemma.

Lemma 3.2. *There are 13 cases for the 1-st homogeneous component $(M_{EF})_1$ of M_{EF}, as follows:*

$$\begin{pmatrix} a_1y & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}_1 \Rightarrow \beta_1 = q^{-1}\alpha_1, \beta_2 = -q^{-1}\alpha_2, \beta_1^{-1}\beta_2 = q, \gamma_1^{-1}\gamma_2 = 1; \tag{38}$$

$$\begin{pmatrix} a_2z & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}_1 \Rightarrow \gamma_1 = q^{-1}\alpha_1, \gamma_2 = -q^{-1}\alpha_2, \beta_1^{-1}\beta_2 = q^2, \gamma_1^{-1}\gamma_2 = q; \tag{39}$$

$$\begin{pmatrix} 0 & b_1x & 0 \\ 0 & 0 & 0 \end{pmatrix}_1 \Rightarrow \beta_1 = q\alpha_1, \beta_2 = -q\alpha_2, \alpha_1^{-1}\alpha_2 = q^{-1}, \gamma_1^{-1}\gamma_2 = 1; \tag{40}$$

$$\begin{pmatrix} 0 & b_2z & 0 \\ 0 & 0 & 0 \end{pmatrix}_1 \Rightarrow \gamma_1 = q^{-1}\beta_1, \gamma_2 = -q^{-1}\beta_2, \alpha_1^{-1}\alpha_2 = 1, \gamma_1^{-1}\gamma_2 = q; \tag{41}$$

$$\begin{pmatrix} 0 & 0 & c_1x \\ 0 & 0 & 0 \end{pmatrix}_1 \Rightarrow \alpha_1 = q^{-1}\gamma_1, \alpha_2 = -q^{-1}\gamma_2, \alpha_1^{-1}\alpha_2 = q^{-1}, \beta_1^{-1}\beta_2 = q^{-2}; \tag{42}$$

$$\begin{pmatrix} 0 & 0 & c_2y \\ 0 & 0 & 0 \end{pmatrix}_1 \Rightarrow \beta_1 = q^{-1}\gamma_1, \beta_2 = -q^{-1}\gamma_2, \alpha_1^{-1}\alpha_2 = 1, \beta_1^{-1}\beta_2 = q^{-1}; \tag{43}$$

$$\begin{pmatrix} 0 & 0 & 0 \\ a_1'y & 0 & 0 \end{pmatrix}_1 \Rightarrow \beta_1 = q\alpha_1, \beta_2 = -q\alpha_2, \beta_1^{-1}\beta_2 = q, \gamma_1^{-1}\gamma_2 = 1; \tag{44}$$

$$\begin{pmatrix} 0 & 0 & 0 \\ a_2'z & 0 & 0 \end{pmatrix}_1 \Rightarrow \gamma_1 = q\alpha_1, \gamma_2 = -q\alpha_2, \beta_1^{-1}\beta_2 = q^2, \gamma_1^{-1}\gamma_2 = q; \tag{45}$$

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & b_1'x & 0 \end{pmatrix}_1 \Rightarrow \beta_1 = q^{-1}\alpha_1, \beta_2 = -q^{-1}\alpha_2, \alpha_1^{-1}\alpha_2 = q^{-1}, \gamma_1^{-1}\gamma_2 = 1; \tag{46}$$

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & b_2'z & 0 \end{pmatrix}_1 \Rightarrow \gamma_1 = q\beta_1, \gamma_2 = -q\beta_2, \alpha_1^{-1}\alpha_2 = 1, \gamma_1^{-1}\gamma_2 = q; \tag{47}$$

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & c_1'x \end{pmatrix}_1 \Rightarrow \alpha_1 = q\gamma_1, \alpha_2 = -q\gamma_2, \alpha_1^{-1}\alpha_2 = q^{-1}, \beta_1^{-1}\beta_2 = q^{-2}; \tag{48}$$

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & c_2'y \end{pmatrix}_1 \Rightarrow \beta_1 = q\gamma_1, \beta_2 = -q\gamma_2, \alpha_1^{-1}\alpha_2 = 1, \beta_1^{-1}\beta_2 = q^{-1}; \tag{49}$$

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}_1 \text{ it does not determine the weight constants at all.} \tag{50}$$

3.2. The Structures of $X_q(A_1)$-Module Algebra on $\mathbb{C}_q[x,y,z]$

In this subsection, our aim is to describe the concrete $X_q(A_1)$-module algebra structures on $\mathbb{C}_q[x,y,z]$, where $K_1, K_2 \in \mathrm{Aut}(\mathbb{C}_q[x,y,z]) \cong (\mathbb{C}^*)^3$.

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By Lemma 3.1 and 3.2, and q is not a root of the unit, it follows that if both the 0-th homogeneous component and the 1-th homogeneous component of M_{EF} are nonzero, it is easy to see that these series are empty. So, we need to consider following possibilities.

Lemma 3.3. *If the* 0*-th homogeneous component of* M_{EF} *is zero and the* 1*-st homogeneous component of* M_{EF} *is nonzero, then these series are empty.*

Proof. Now, we show that $\left[\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}_0, \begin{pmatrix} a_1 y & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}_1\right]$ -series is empty. If we suppose the contrary, then it follows from

$$EF - FE = \frac{K_2 K_1^{-1} - K_2^{-1} K_1}{q - q^{-1}}$$

that within this series, one can have

$$\frac{K_2 K_1^{-1} - K_2^{-1} K_1}{q - q^{-1}}(x) = \frac{\alpha_2 \alpha_1^{-1} - \alpha_2^{-1} \alpha_1}{q - q^{-1}} x.$$

By $a_1 \neq 0$, one can get $\beta_1 = q^{-1}\alpha_1, \beta_2 = -q^{-1}\alpha_2, \beta_2\beta_1^{-1} = q$, and $\gamma_2\gamma_1^{-1} = 1$, hence $\alpha_2\alpha_1^{-1} = -q$, and

$$\frac{K_2 K_1^{-1} - K_2^{-1} K_1}{q - q^{-1}}(x) = -x.$$

On the other hand, projecting $(EF - FE)(x)$ to $\mathbb{C}_q[x, y, z]$ we obtain

$$(EF - FE)(x) = E(F(x)) - F(E(x)) = E(0) - F(a_1 y) = 0,$$

however, $0 \neq -x$. We have obtained contradictions and proved our claims.

In a similar way, one can prove that all other series with the 0-th homogeneous component of M_{EF} is zero and the 1-th homogeneous component of M_{EF} is nonzero are empty. □

Lemma 3.4. *If the* 0*-th homogeneous component of* M_{EF} *is nonzero and the* 1*-st homogeneous component of* M_{EF} *is zero, then these series are empty.*

Proof. We only show that $\left[\begin{pmatrix} a_0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}_0, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}_1\right]$ -series is empty. in a similar way, one can prove that all other series are empty.

Consider this series, we obtain that

$$a_0 \neq 0 \quad \Rightarrow \quad \alpha_1 = q, \quad \alpha_2 = -q, \quad \beta_2\beta_1^{-1} = q, \quad \gamma_2\gamma_1^{-1} = q.$$

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and suppose that it is not empty. We set

$$\begin{aligned}
&K_1(x) = qx, && K_2(x) = -qx, \\
&K_1(y) = \beta_1 y, && K_2(y) = \beta_2 y, \\
&K_1(z) = \gamma_1 z, && K_2(z) = \gamma_2 z, \\
&E(x) = a_0 + \sum_{m_1+n_1+l_1 \geq 2} \rho^1_{m_1 n_1 l_1} x^{m_1} y^{n_1} z^{l_1} && \text{for } m_1, n_1, l_1 \in \mathbb{N}, \\
&E(y) = \sum_{m_2+n_2+l_2 \geq 2} \rho^2_{m_2 n_2 l_2} x^{m_2} y^{n_2} z^{l_2} && \text{for } m_2, n_2, l_2 \in \mathbb{N}, \\
&E(z) = \sum_{m_3+n_3+l_3 \geq 2} \rho^3_{m_3 n_3 l_3} x^{m_3} y^{n_3} z^{l_3} && \text{for } m_3, n_3, l_3 \in \mathbb{N}, \\
&F(x) = \sum_{m_4+n_4+l_4 \geq 2} \rho^4_{m_4 n_4 l_4} x^{m_4} y^{n_4} z^{l_4} && \text{for } m_4, n_4, l_4 \in \mathbb{N}, \\
&F(y) = \sum_{m_5+n_5+l_5 \geq 2} \rho^5_{m_5 n_5 l_5} x^{m_5} y^{n_5} z^{l_5} && \text{for } m_5, n_5, l_5 \in \mathbb{N}, \\
&F(z) = \sum_{m_6+n_6+l_6 \geq 2} \rho^6_{m_6 n_4 l_6} x^{m_6} y^{n_6} z^{l_6} && \text{for } m_6, n_6, l_6 \in \mathbb{N},
\end{aligned}$$

where $\beta_1, \beta_2, \gamma_1, \gamma_2 \in \mathbb{C}^*$, and $\rho^i_{m_i n_i l_i} \in \mathbb{C}, i = 1,2,3,4,5,6$. We have

$$\begin{aligned}
(K_1 E - q^{-1} E K_1)(x) =& K_1(E(x)) - q^{-1} E(K_1(x)) \\
=& K_1(a_0 + \sum_{m_1+n_1+l_1 \geq 2} \rho^1_{m_1 n_1 l_1} x^{m_1} y^{n_1} z^{l_1}) - q^{-1} q E(x) \\
=& a_0 + \sum_{m_1+n_1+l_1 \geq 2} \rho^1_{m_1 n_1 l_1} \alpha_1^{m_1} \beta_1^{n_1} \gamma_1^{l_1} x^{m_1} y^{n_1} z^{l_1} - E(x) \\
=& \sum_{m_1+n_1+l_1 \geq 2} \rho^1_{m_1 n_1 l_1} (q^{m_1} \beta_1^{n_1} \gamma_1^{l_1} - 1) x^{m_1} y^{n_1} z^{l_1} = 0,
\end{aligned}$$

then for all $m_1, n_1, l_1 \in \mathbb{N}$ with $m_1 + n_1 + l_1 \geq 2$, one has $\rho^1_{m_1 n_1 l_1} = 0$ or $q^{m_1} \beta_1^{n_1} \gamma_1^{l_1} = 1$. And

$$\begin{aligned}
(K_2 E + q^{-1} E K_2)(x) =& K_2(E(x)) + q^{-1} E(K_2(x)) \\
=& K_2(a_0 + \sum_{m_1+n_1+l_1 \geq 2} \rho^1_{m_1 n_1 l_1} x^{m_1} y^{n_1} z^{l_1}) - q^{-1} q E(x) \\
=& a_0 + \sum_{m_1+n_1+l_1 \geq 2} \rho^1_{m_1 n_1 l_1} \alpha_2^{m_1} \beta_2^{n_1} \gamma_2^{l_1} x^{m_1} y^{n_1} z^{l_1} - E(x) \\
=& \sum_{m_1+n_1+l_1 \geq 2} \rho^1_{m_1 n_1 l_1} ((-q)^{m_1} \beta_2^{n_1} \gamma_2^{l_1} - 1) x^{m_1} y^{n_1} z^{l_1} = 0,
\end{aligned}$$

then for all $m_1, n_1, l_1 \in \mathbb{N}$ with $m_1 + n_1 + l_1 \geq 2$, one has $\rho^1_{m_1 n_1 l_1} = 0$ or $(-q)^{m_1} \beta_2^{n_1} \gamma_2^{l_1} = 1$. If some $\rho^1_{m_1 n_1 l_1} \neq 0$ meet the conditions, i.e

$$\begin{cases} q^{m_1} \beta_1^{n_1} \gamma_1^{l_1} = 1, \\ (-q)^{m_1} \beta_2^{n_1} \gamma_2^{l_1} = 1, \end{cases}$$

one can get $(-1)^{m_1} q^{(n_1+l_1)} = 1$, this contradicts with q is not a unit root. Therefore, for all $m_1, n_1, l_1 \in \mathbb{N}$ with $m_1 + n_1 + l_1 \geq 2$, we have $E(x) = a_0$. By discussing $E(y)$, $E(z)$, $F(x)$, $F(y)$ and $F(z)$ using methods similar to $E(x)$, we can obtain that

$$E(y) = 0, \ \ E(z) = 0,$$

$$F(x) = \begin{cases} 0 \\ \rho^4_{200} x^2 \end{cases}, \quad F(y) = \begin{cases} 0 \\ \rho^5_{110} xy \end{cases}, \quad F(z) = \begin{cases} 0 \\ \rho^6_{101} xz \end{cases}.$$

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From $EF - FE = \frac{K_2K_1^{-1} - K_2^{-1}K_1}{q-q^{-1}}$, we have

$$\frac{K_2K_1^{-1} - K_2^{-1}K_1}{q-q^{-1}}(y) = \frac{\beta_2\beta_1^{-1} - \beta_2^{-1}\beta_1}{q-q^{-1}y} = \frac{q-q^{-1}}{q-q^{-1}y} = y,$$
$$\frac{K_2K_1^{-1} - K_2^{-1}K_1}{q-q^{-1}}(z) = \frac{\gamma_2\gamma_1^{-1} - \gamma_2^{-1}\gamma_1}{q-q^{-1}y} = \frac{q-q^{-1}}{q-q^{-1}z} = z.$$

- If $F(y) = 0$, then $(EF - FE)(y) = EF(y) - FE(y) = 0 \neq y$;
- if $F(y) = \rho^5_{110}xy$, then $(EF - FE)(y) = \rho^5_{110}E(xy) = \rho^5_{110}a_0y = y$, hence $\rho^5_{110} = a_0^{-1}$ and $F(y) = a_0^{-1}xy$.
- if $F(z) = 0$, then $(EF - FE)(z) = EF(z) - FE(z) = 0 \neq z$;
- if $F(z) = \rho^6_{101}xz$, then $(EF - FE)(z) = \rho^6_{101}E(xz) = \rho^6_{110}a_0z = z$, hence $\rho^6_{101} = a_0^{-1}$ and $F(z) = a_0^{-1}xz$.

By $E^2 = F^2 = 0$, one has $F^2(y) = F(a_0^{-1}xy) = a_0^{-1}(xF(y) + q^{-1}F(x)y)$,

- if $F(x) = 0$, then $F^2(y) = a_0^{-2}x^2y \neq 0$;
- if $F(x) = \rho^4_{200}x^2$, then $F^2(y) = a_0^{-1}(a_0^{-1}x^2y + \rho^4_{200}q^{-1}x^2y) = 0$, hence $\rho^4_{200} = -qa_0^{-1}$ and $F(x) = -qa_0^{-1}x^2$.

According to $yx = qxy$, then

$$\begin{aligned} F(yx - qxy) &= yF(x) - qxF(y) + \alpha_2^{-1}\alpha_1F(y)x - \beta_2^{-1}\beta_1qF(x)y \\ &= -a_0^{-1}(q^3 + q)x^2y \neq 0. \end{aligned}$$

In summary, this series is empty. □

Next we turn to “nonempty” series, it only has a kind of "nonempty" series.

Theorem 3.5. *The* $\left[\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}_0, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}_1\right]$ *-series has* $X_q(A_1)-$*module algebra structures on* $\mathbb{C}_q[x, y, z]$ *given by*

$$K_1(x) = \lambda_1x, \ \ K_2(x) = \pm\lambda_1x, \tag{51}$$
$$K_1(y) = \lambda_2y, \ \ K_2(y) = \pm\lambda_2y, \tag{52}$$
$$K_1(z) = \lambda_3z, \ \ K_2(z) = \pm\lambda_3z, \tag{53}$$
$$E(x) = F(x) = E(y) = F(y) = E(z) = F(z) = 0, \tag{54}$$

where $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{C}^*$*, they are pairwise nonisomorphic.*

Proof. It is easy to check that (51)-(54) determine a well-defined $X_q(A_1)$-action consistent with the multiplication in $X_q(A_1)$ and in $\mathbb{C}_q[x, y, z]$, as well as with comultiplication in $X_q(A_1)$. Prove that there are no other $X_q(A_1)$-actions here. Note that an application of (6) to x, y or z has zero projection to $\mathbb{C}_q[x, y, z]_1$, ie. $(EF - FE)(x) = (EF - FE)(y) = (EF - FE)(z) = 0$, because in this series E and F send any monomial to a sum of the monomials of higher degree. Therefore,

$$\frac{K_2K_1^{-1} - K_2^{-1}K_1}{q-q^{-1}}(x) = \frac{\alpha_2\alpha_1^{-1} - \alpha_1\alpha_2^{-1}}{q-q^{-1}}x = 0,$$
$$\frac{K_2K_1^{-1} - K_2^{-1}K_1}{q-q^{-1}}(y) = \frac{\beta_2\beta_1^{-1} - \beta_1\beta_2^{-1}}{q-q^{-1}}y = 0,$$
$$\frac{K_2K_1^{-1} - K_2^{-1}K_1}{q-q^{-1}}(z) = \frac{\gamma_2\gamma_1^{-1} - \gamma_2^{-1}\gamma_1}{q-q^{-1}}z = 0,$$

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and hence

$$\alpha_2\alpha_1^{-1} - \alpha_2^{-1}\alpha_2 = \beta_2\beta_1^{-1} - \beta_2^{-1}\beta_1 = \gamma_2\gamma_1^{-1} - \gamma_2^{-1}\gamma_2 = 0,$$

which leads to $\alpha_1^2 = \alpha_2^2$, $\beta_1^2 = \beta_2^2$ and $\gamma_1^2 = \gamma_2^2$, let $\alpha_1 = \lambda_1$, $\beta_1 = \lambda_2$ and $\gamma_1 = \lambda_3$, we have $\alpha_2 = \pm\lambda_1$, $\beta_2 = \pm\lambda_2$ and $\gamma_2 = \pm\lambda_3$. To prove (54), note that if $E(x) \neq 0$ or $F(y) \neq 0$, then they are a sum of the monomials that their degrees are greater than 1. It is similar to the proof of Lemma 3.4, we get that this is impossible, because they can not satisfy conditions of $X_q(A_1)$-module algebra on $\mathbb{C}_q[x,y,z]$.

To see that the $X_q(A_1)$-module algebra structures are pairwise nonisomorphic, observe that all the automorphisms of $\mathbb{C}_q[x,y,z]$ commute with the actions of K_1 and K_2. □

4. When $t \neq 0$, Classification of $X_q(A_1)$-Module Algebra Structures on $\mathbb{C}_q[x,y,z]$

In this section, we suppose the automorphism Ψ of $\mathbb{C}_q[x,y,z]$ as follows:

$$\Psi(x) = \alpha x, \quad \Psi(y) = \beta y + txz, \quad \Psi(z) = \gamma z, \quad (\alpha, \beta, \gamma, t \in \mathbb{C}^*),$$

and $\mathrm{Aut}(\mathbb{C}_q[x,y,z]) \cong \mathbb{C} \rtimes (\mathbb{C}^*)^3$. One can have

$$\Psi^{-1}(x) = \alpha^{-1}x, \quad \Psi^{-1}(y) = \beta^{-1}y - t\alpha^{-1}\beta^{-1}\gamma^{-1}xz, \quad \Psi^{-1}(z) = \gamma^{-1}z.$$

In the following, we will begin to discuss the $X_q(A_1)$-module algebra structures on $\mathbb{C}_q[x,y,z]$ with $t \neq 0$, ie. here $K_1, K_2 \in \mathbb{C} \rtimes (\mathbb{C}^*)^3$. In this Section, our research method is similar to Section 3.

4.1. Properties of $X_q(A_1)$-Module Algebras on $\mathbb{C}_q[x,y,z]$

It is easy to see that any action of $X_q(A_1)$ on $\mathbb{C}_q[x,y,z]$ is determined by the following 4×3 matrix with entries from $\mathbb{C}_q[x,y,z]$:

$$M = \begin{pmatrix} K_1(x) & K_1(y) & K_1(z) \\ K_2(x) & K_2(y) & K_2(z) \\ E(x) & E(y) & E(z) \\ F(x) & F(y) & F(z) \end{pmatrix}. \tag{55}$$

Given a $X_q(A_1)$-module algebra structure on $\mathbb{C}_q[x,y,z]$, obviously, the action of K_1(or K_2) is determined by an automorphism of $\mathbb{C}_q[x,y,z]$, in other words, the actions of K_1 and K_2 are determined by a matrix $M_{K_1K_2}$ as follows

$$M_{K_1K_2} \overset{definition}{=} \begin{pmatrix} K_1(x) & K_1(y) & K_1(z) \\ K_2(x) & K_2(y) & K_2(z) \end{pmatrix} = \begin{pmatrix} \alpha_1(x) & \beta_1(y) + t_1xz & \gamma_1(z) \\ \alpha_2(x) & \beta_2(y) + t_2xz & \gamma_2(z) \end{pmatrix}, \tag{56}$$

where $\alpha_i, \beta_i, \gamma_i, t_i \in \mathbb{C}^*$ for $i \in \{1,2\}$.

Lemma 4.1. *For all $\alpha_i, \beta_i, \gamma_i, t_i \in \mathbb{C}^*$, $i \in \{1,2\}$, either $\beta_i = \alpha_i\gamma_i$ or $t_i = (\beta_i - \alpha_i\gamma_i)t$, where $t \in \mathbb{C}^*$.*

Proof. For all $\alpha_i, \beta_i, \gamma_i, t_i \in \mathbb{C}^*$, $i \in \{1,2\}$, we have

$$K_i(y) = \beta_i y + t_i xz \;\text{ and }\; K_i^{-1}(y) = \beta_i^{-1}y - t_i\alpha_i^{-1}\beta_i^{-1}\gamma_i^{-1}xz$$

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by (56). It is to easy check $K_iK_i^{-1}(y) = y$, and

$$\begin{aligned} K_1K_2(y) &= K_1(\beta_2 y + t_2xz) \\ &= \beta_1\beta_2 y + \beta_2 t_1 xz + t_2\alpha_1\gamma_1 xz, \\ K_2K_1(y) &= K_2(\beta_1 y + t_1xz) \\ &= \beta_1\beta_2 y + \beta_1 t_2 xz + t_1\alpha_2\gamma_2 xz. \end{aligned}$$

By the definition of module algebra and (1), we have $t_1(\beta_2 - \alpha_2\gamma_2) = t_2(\beta_1 - \alpha_1\gamma_1)$, for $t_i \in \mathbb{C}^*, i = 1, 2$, hence, either $\beta_i = \alpha_i\gamma_i$ or $\frac{t_1}{t_2} = \frac{\beta_1 - \alpha_1\gamma_1}{\beta_2 - \alpha_2\gamma_2}$, we can write the latter as $t_i = (\beta_i - \alpha_i\gamma_i)t$, where $t \in \mathbb{C}^*$. $\square$

It is easy to see that every monomial $x^{m_1}z^{m_3} \in \mathbb{C}_q[x, y, z]$ is an eigenvector of K_1(or K_2), and the associated eigenvalue $\alpha_1^{m_1}\gamma_1^{m_3}$ (or $\alpha_2^{m_1}\gamma_2^{m_3}$) is called the K_1-weight (or K_2-weight) of this monomial, which will be written as

$$wt_{K_1}(x^{m_1}z^{m_3}) = \alpha_1^{m_1}\gamma_1^{m_3},$$
$$wt_{K_2}(x^{m_1}z^{m_3}) = \alpha_2^{m_1}\gamma_2^{m_3}.$$

We will also need another matrix M_{EF} as follows

$$M_{EF} \overset{definition}{=} \begin{pmatrix} E(x) & E(y) & E(z) \\ F(x) & F(y) & F(z) \end{pmatrix}. \tag{57}$$

Obviously, $K_1(x), K_2(x), E(x), F(x)$ and $K_1(z), K_2(z), E(z), F(z)$ are weight vectors for K_1 and K_2, then

$$\begin{aligned} wt_{K_i}(M) &\overset{definition}{=} \begin{pmatrix} wt_{K_i}(K_1(x)) & wt_{K_i}(K_1(z)) \\ wt_{K_i}(K_2(x)) & wt_{K_i}(K_2(z)) \\ wt_{K_i}(E(x)) & wt_{K_i}(E(z)) \\ wt_{K_i}(F(x)) & wt_{K_i}(F(z)) \end{pmatrix} \\ &\bowtie \begin{pmatrix} wt_{K_i}(x) & wt_{K_i}(z) \\ wt_{K_i}(x) & wt_{K_i}(z) \\ (-1)^{i-1}q^{-1}wt_{K_i}(x) & (-1)^{i-1}q^{-1}wt_{K_i}(z) \\ (-1)^{i-1}q\,wt_{K_i}(x) & (-1)^{i-1}q\,wt_{K_i}(z) \end{pmatrix} \\ &= \begin{pmatrix} \alpha_i & \gamma_i \\ \alpha_i & \gamma_i \\ (-1)^{i-1}q^{-1}\alpha_i & (-1)^{i-1}q^{-1}\gamma_i \\ (-1)^{i-1}q\alpha_i & (-1)^{i-1}q\gamma_i \end{pmatrix}. \end{aligned} \tag{58}$$

Same as Section 3, we denote by $(M)_j$ the j-th homogeneous component of M. Obviously, if $(M)_j$ is nonzero, one can calculate the associated eigenvalues.

Set $a_0, b_0, c_0, a_0', b_0', c_0' \in \mathbb{C}$, we obtain the 0-th homogeneous component of M as follow:

$$(M)_0 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ a_0 & b_0 & c_0 \\ a_0' & b_0' & c_0' \end{pmatrix}_0. \tag{59}$$

Then, we have

$$wt_{K_i}((M_{EF})_0) = \begin{pmatrix} (-1)^{i-1}q^{-1}\alpha_i & (-1)^{i-1}q^{-1}\gamma_i \\ (-1)^{i-1}q\alpha_i & (-1)^{i-1}q\gamma_i \end{pmatrix}_0 \bowtie \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}_0, \tag{60}$$

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According to q is not a root of the unit and relation (60), it means that a_0 and a_0' (c_0 and c_0') should contain at least one 0.

An application of E and F to the relations (11)-(13) by using equation (56), one has

$$E(y)x - qE(x)y + K_2K_1^{-1}(y)E(x) - qK_2K_1^{-1}(x)E(y) = 0, \tag{61}$$
$$E(z)y - qE(y)z + K_2K_1^{-1}(z)E(y) - qK_2K_1^{-1}(y)E(z) = 0, \tag{62}$$
$$E(z)x - qE(x)z + K_2K_1^{-1}(z)E(x) - qK_2K_1^{-1}(x)E(z) = 0, \tag{63}$$
$$yF(x) - qxF(y) + F(y)K_2^{-1}K_1(x) - qF(x)K_2^{-1}K_1(y) = 0, \tag{64}$$
$$zF(y) - qyF(z) + F(z)K_2^{-1}K_1(y) - qF(y)K_2^{-1}K_1(z) = 0, \tag{65}$$
$$zF(x) - qxF(z) + F(z)K_2^{-1}K_1(x) - qF(x)K_2^{-1}K_1(z) = 0. \tag{66}$$

Which certainly implies

$$b_0\left(1 - q\alpha_1^{-1}\alpha_2\right) = a_0\left(\beta_1^{-1}\beta_2 - q\right) = c_0\left(1 - q\beta_1^{-1}\beta_2\right) = b_0\left(\gamma_1^{-1}\gamma_2 - q\right) = 0,$$
$$c_0\left(1 - q\alpha_1^{-1}\alpha_2\right) = a_0\left(\gamma_1^{-1}\gamma_2 - q\right) = a_0'\left(1 - q\beta_1\beta_2^{-1}\right) = b_0'\left(\alpha_1\alpha_2^{-1} - q\right) = 0,$$
$$b_0'\left(1 - q\gamma_1\gamma_2^{-1}\right) = c_0'\left(\beta_1\beta_2^{-1} - q\right) = a_0'\left(1 - q\gamma_1\gamma_2^{-1}\right) = c_0'\left(\alpha_1\alpha_2^{-1} - q\right) = 0.$$

Because q is not a root of the unit, $q \neq \pm 1$, and from the above discussion, for the 0-st homogeneous component $(M_{EF})_0$ of M_{EF}, we have following lemma.

Lemma 4.2. *There are 8 cases for the 0-st homogeneous component $(M_{EF})_0$ of M_{EF}, as follows:*

$$\begin{pmatrix} a_0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}_1 \Rightarrow \alpha_1 = q, \alpha_2 = -q, \beta_1^{-1}\beta_2 = q, \gamma_1^{-1}\gamma_2 = q; \tag{67}$$
$$\begin{pmatrix} 0 & b_0 & 0 \\ 0 & 0 & 0 \end{pmatrix}_1 \Rightarrow \alpha_1^{-1}\alpha_2 = q^{-1}, \gamma_1^{-1}\gamma_2 = q; \tag{68}$$
$$\begin{pmatrix} 0 & 0 & c_0 \\ 0 & 0 & 0 \end{pmatrix}_1 \Rightarrow \gamma_1 = q, \gamma_2 = -q, \beta_1^{-1}\beta_2 = q^{-1}, \alpha_1^{-1}\alpha_2 = q^{-1}; \tag{69}$$
$$\begin{pmatrix} 0 & 0 & 0 \\ a_0' & 0 & 0 \end{pmatrix}_1 \Rightarrow \alpha_1 = q^{-1}, \alpha_2 = -q^{-1}, \beta_1\beta_2^{-1} = q^{-1}, \gamma_1\gamma_2^{-1} = q^{-1}; \tag{70}$$
$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & b_0' & 0 \end{pmatrix}_1 \Rightarrow \alpha_1\alpha_2^{-1} = q, \gamma_1\gamma_2^{-1} = q^{-1}; \tag{71}$$
$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & c_0' \end{pmatrix}_1 \Rightarrow \gamma_1 = q^{-1}, \gamma_2 = -q^{-1}\beta_1\beta_2^{-1} = q, \alpha_1\alpha_2^{-1} = q; \tag{72}$$

$$\begin{pmatrix} 0 & b_0 & 0 \\ 0 & b_0' & 0 \end{pmatrix}_1 \Rightarrow \alpha_1^{-1}\alpha_2 = q^{-1}, \gamma_1^{-1}\gamma_2 = q; \tag{73}$$
$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}_1 \Rightarrow \text{it does not determine the weight constants at all.} \tag{74}$$

Next, for the 1-st homogeneous component, due to q is not a root of the unit, one has

$$wt_{K_1}(E(x)) = q^{-1}\alpha_1 = q^{-1}wt_{K_1}(x) \neq wt_{K_1}(x),$$
$$wt_{K_2}(E(x)) = -q^{-1}\alpha_2 = -q^{-1}wt_{K_2}(x) \neq wt_{K_2}(x),$$

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which implies

$$(E(x))_1 = a_1 z,$$

for some $a_1 \in \mathbb{C}$, and in a similar way we have

$$(M_{EF})_1 = \begin{pmatrix} a_1 z & b_1 x + b_2 y + b_3 z & c_1 x \\ a_1' z & b_1' x + b_2' y + b_3' z & c_1' x \end{pmatrix}_1$$

where $b_1, b_2, b_3, c_1, a_1', b_1', b_2', b_3', c_1' \in \mathbb{C}$. In fact,

$$wt_{K_i}((M_{EF})_1) \quad \bowtie \begin{pmatrix} (-1)^{i-1}q^{-1}\alpha_i & (-1)^{i-1}q^{-1}\gamma_i \\ (-1)^{i-1}q\alpha_i & (-1)^{i-1}q\gamma_i \end{pmatrix} \bowtie \begin{pmatrix} \gamma_i & \alpha_i \\ \gamma_i & \alpha_i \end{pmatrix}. \tag{75}$$

Now, we project (61)-(66) to $\mathbb{C}_q[x, y, z]_2$, one can obtain the following conclusion.
If $a_0 \neq 0$, then $\alpha_1 = q, \alpha_2 = -q, \beta_1^{-1}\beta_2 = q, \gamma_1^{-1}\gamma_2 = q$, and we have

$$(E(y))_1 x - q(E(x))_1 y + qy(E(x))_1 + (t_2\beta_1^{-1} + qt_1\beta_1^{-1})xz(E(x))_0 + qx(E(y))_1 = 0,$$

$$b_1(1+q)x^2 + 2qb_2 xy + a_1 q(1-q)yz + (2qb_3 + a_0 t_2\beta_1^{-1} + qa_0 t_1\beta_1^{-1})xz = 0,$$

$$\begin{cases} a_1 = 0, \\ b_1 = 0, \\ b_2 = 0, \\ b_3 = \frac{-a_0(t_2+qt_1)}{2q\beta_1}. \end{cases} \tag{76}$$

If $a_0 = 0$, then

$$(E(y))_1 x - q(E(x))_1 y + \beta_1^{-1}\beta_2 y(E(x))_1 - q\alpha_1^{-1}\alpha_2 x(E(y))_1 = 0,$$

$$b_1\left(1 - q\alpha_1^{-1}\alpha_2\right)x^2 + b_2\left(1 - \alpha_1^{-1}\alpha_2\right)yx + b_3\left(1 - \alpha_1^{-1}\alpha_2\right)zx + a_1\left(\beta_1^{-1}\beta_2 - q^2\right)yz = 0,$$

$$\begin{cases} b_1 \neq 0 \Rightarrow \alpha_1^{-1}\alpha_2 = q^{-1}, \\ b_2 \neq 0 \Rightarrow \alpha_1^{-1}\alpha_2 = 1, \\ b_3 \neq 0 \Rightarrow \alpha_1^{-1}\alpha_2 = 1, \\ a_1 \neq 0 \Rightarrow \beta_1^{-1}\beta_2 = q^2. \end{cases} \tag{77}$$

If $c_0 \neq 0$, then $\gamma_1 = q, \gamma_2 = -q, \beta_1^{-1}\beta_2 = q^{-1}, \alpha_1^{-1}\alpha_2 = q^{-1}$, and we have

$$(E(z))_1 y - q(E(y))_1 z - z(E(y))_1 - y(E(z))_1 - q(t_2\beta_1^{-1} + q^{-1}t_1\beta_1^{-1})xz(E(z))_0 = 0,$$

$$c_1(1-q)xy - 2qb_2 yz - b_3(1+q)z^2 - (2qb_1 + qc_0 t_2\beta_1^{-1} + c_0 t_1\beta_1^{-1})xz = 0,$$

$$\begin{cases} b_2 = 0, \\ b_3 = 0, \\ c_1 = 0, \\ b_1 = \frac{-c_0(t_1+qt_2)}{2q\beta_1}. \end{cases} \tag{78}$$

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If $c_0 = 0$, then

$$(E(z))_1 y - q(E(y))_1 z + \gamma_1^{-1}\gamma_2 z(E(y))_1 - q\beta_1^{-1}\beta_2 y(E(z))_1 = 0,$$

$$c_1\left(1 - q^2\beta_1^{-1}\beta_2\right)xy + qb_1\left(\gamma_1^{-1}\gamma_2 - 1\right)xz + qb_2\left(\gamma_1^{-1}\gamma_2 - 1\right)yz + b_3\left(\gamma_1^{-1}\gamma_2 - q\right)z^2 = 0,$$

$$\begin{cases} c_1 \neq 0 \Rightarrow \beta_1^{-1}\beta_2 = q^{-2}, \\ b_1 \neq 0 \Rightarrow \gamma_1^{-1}\gamma_2 = 1, \\ b_2 \neq 0 \Rightarrow \gamma_1^{-1}\gamma_2 = 1, \\ b_3 \neq 0 \Rightarrow \gamma_1^{-1}\gamma_2 = q. \end{cases} \tag{79}$$

$$(E(z))_1 x - q(E(x))_1 z + \gamma_1^{-1}\gamma_2 z(E(x))_1 - q\alpha_1^{-1}\alpha_2 x(E(z))_1 = 0,$$
$$c_1(1 - q\alpha_1^{-1}\alpha_2)x^2 + a_1(\gamma_1^{-1}\gamma_2 - q)z^2 = 0,$$

$$\begin{cases} c_1 \neq 0 \Rightarrow \alpha_1^{-1}\alpha_2 = q^{-1}, \\ a_1 \neq 0 \Rightarrow \gamma_1^{-1}\gamma_2 = q. \end{cases} \tag{80}$$

If $a_0' \neq 0$, then $\alpha_1 = q^{-1}, \alpha_2 = -q^{-1}, \beta_1\beta_2^{-1} = q^{-1}, \gamma_1\gamma_2^{-1} = q^{-1}$, and we have

$$y(F(x))_1 - qx(F(y))_1 - (F(y))_1 x - (F(x))_1 y - q(F(x))_0(t_1\alpha_2^{-1}\gamma_2^{-1} - q^{-1}t_2\alpha_2^{-1}\gamma_2^{-1})xz = 0,$$

$$a_1'(1-q)yz - b_1'(1+q)x^2 - 2qb_2'xy + (-2qb_3' + a_0't_2\alpha_2^{-1}\gamma_2^{-1} - qa_0't_1\alpha_2^{-1}\gamma_2^{-1})xz = 0,$$

$$\begin{cases} a_1' = 0, \\ b_1' = 0, \\ b_2' = 0, \\ b_3' = \frac{a_0'(t_2 - qt_1)}{2q\alpha_2\gamma_2}. \end{cases} \tag{81}$$

If $a_0' = 0$, then

$$y(F(x))_1 - qx(F(y))_1 + \alpha_1\alpha_2^{-1}(F(y))_1 x - q\beta_1\beta_2^{-1}(F(x))_1 y = 0,$$

$$b_1'\left(\alpha_1\alpha_2^{-1} - q\right)x^2 + qb_2'\left(\alpha_1\alpha_2^{-1} - 1\right)xy + qb_3\left(\alpha_1\alpha_2^{-1} - 1\right)xz + a_1'\left(1 - q^2\beta_1\beta_2^{-1}\right)yz = 0,$$

$$\begin{cases} a_1' \neq 0 \Rightarrow \beta_1\beta_2^{-1} = q^{-2}, \\ b_1' \neq 0 \Rightarrow \alpha_1\alpha_2^{-1} = q, \\ b_2' \neq 0 \Rightarrow \alpha_1\alpha_2^{-1} = 1, \\ b_3' \neq 0 \Rightarrow \alpha_1\alpha_2^{-1} = 1. \end{cases} \tag{82}$$

If $c_0' \neq 0$, then $\gamma_1 = q^{-1}, \gamma_2 = -q^{-1}, \beta_1\beta_2^{-1} = q, \alpha_1\alpha_2^{-1} = q$, and we have

$$z(F(y))_1 - qy(F(z))_1 + q(F(z))_1 y + (F(z))_0(t_1\alpha_2^{-1}\gamma_2^{-1} - qt_2\alpha_2^{-1}\gamma_2^{-1})xz + q(F(y))_1 z = 0,$$

$$2qb_2'yz + b_3'(1+q)z^2 + qc_1'(1-q)xy + (2qb_1' - qc_0't_2\alpha_2^{-1}\gamma_2^{-1} + c_0't_1\alpha_2^{-1}\gamma_2^{-1})xz = 0,$$

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$$\begin{cases} b'_2 = 0, \\ b'_3 = 0, \\ c'_1 = 0, \\ b'_1 = \frac{c'_0(qt_2 - t_1)}{2q\alpha_2\gamma_2}. \end{cases} \tag{83}$$

If $c'_0 = 0$, then

$$z(F(y))_1 - qy(F(z))_1 + \beta_1\beta_2^{-1}(F(z))_1 y - q\gamma_1\gamma_2^{-1}(F(y))_1 z = 0,$$

$$qb'_1\left(1 - \gamma_1\gamma_2^{-1}\right)xz + qb'_2\left(1 - \gamma_1\gamma_2^{-1}\right)yz + b'_3\left(1 - q\gamma_1\gamma_2^{-1}\right)z^2 + c'_1\left(\beta_1\beta_2^{-1} - q^2\right)xy = 0,$$

$$\begin{cases} b'_1 \neq 0 \Rightarrow \gamma_1\gamma_2^{-1} = 1, \\ b'_2 \neq 0 \Rightarrow \gamma_1\gamma_2^{-1} = 1, \\ b'_3 \neq 0 \Rightarrow \gamma_1\gamma_2^{-1} = q^{-1}, \\ c'_1 \neq 0 \Rightarrow \beta_1\beta_2^{-1} = q^2. \end{cases} \tag{84}$$

$$z(F(x))_1 x - qx(F(z))_1 + \alpha_1\alpha_2^{-1}(F(z))_1 x - q\gamma_1\gamma_2^{-1}(F(x))_1 z = 0,$$
$$c'_1(\alpha_1\alpha_2^{-1} - q)x^2 + a'_1(1 - q\gamma_1\gamma_2^{-1})z^2 = 0,$$

$$\begin{cases} a'_1 \neq 0 \Rightarrow \gamma_1\gamma_2^{-1} = q^{-1}, \\ c'_1 \neq 0 \Rightarrow \alpha_1\alpha_2^{-1} = q. \end{cases} \tag{85}$$

From the above discussion, and due to q is not a root of the unit, we can obtain the following lemma.

Lemma 4.3. *There are 18 cases for the 1-st homogeneous component $(M_{EF})_1$ of M_{EF}, as follows:*

$$\begin{pmatrix} a_1 y & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}_1 \Rightarrow \beta_1 = q^{-1}\alpha_1, \beta_2 = -q^{-1}\alpha_2, \beta_1^{-1}\beta_2 = q, \gamma_1^{-1}\gamma_2 = 1; \tag{86}$$

$$\begin{pmatrix} a_2 z & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}_1 \Rightarrow \gamma_1 = q^{-1}\alpha_1, \gamma_2 = -q^{-1}\alpha_2, \beta_1^{-1}\beta_2 = q^2, \gamma_1^{-1}\gamma_2 = q; \tag{87}$$

$$\begin{pmatrix} 0 & b_1 x & 0 \\ 0 & 0 & 0 \end{pmatrix}_1 \Rightarrow \beta_1 = q\alpha_1, \beta_2 = -q\alpha_2, \alpha_1^{-1}\alpha_2 = q^{-1}, \gamma_1^{-1}\gamma_2 = 1; \tag{88}$$

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$$\begin{pmatrix} 0 & b_2z & 0 \\ 0 & 0 & 0 \end{pmatrix}_1 \Rightarrow \gamma_1 = q^{-1}\beta_1, \gamma_2 = -q^{-1}\beta_2, \alpha_1^{-1}\alpha_2 = 1, \gamma_1^{-1}\gamma_2 = q; \tag{89}$$

$$\begin{pmatrix} 0 & 0 & c_1x \\ 0 & 0 & 0 \end{pmatrix}_1 \Rightarrow \alpha_1 = q^{-1}\gamma_1, \alpha_2 = -q^{-1}\gamma_2, \alpha_1^{-1}\alpha_2 = q^{-1}, \beta_1^{-1}\beta_2 = q^{-2}; \tag{90}$$

$$\begin{pmatrix} 0 & 0 & c_2y \\ 0 & 0 & 0 \end{pmatrix}_1 \Rightarrow \beta_1 = q^{-1}\gamma_1, \beta_2 = -q^{-1}\gamma_2, \alpha_1^{-1}\alpha_2 = 1, \beta_1^{-1}\beta_2 = q^{-1}; \tag{91}$$

$$\begin{pmatrix} 0 & 0 & 0 \\ a_1'y & 0 & 0 \end{pmatrix}_1 \Rightarrow \beta_1 = q\alpha_1, \beta_2 = -q\alpha_2, \beta_1^{-1}\beta_2 = q, \gamma_1^{-1}\gamma_2 = 1; \tag{92}$$

$$\begin{pmatrix} 0 & 0 & 0 \\ a_2'z & 0 & 0 \end{pmatrix}_1 \Rightarrow \gamma_1 = q\alpha_1, \gamma_2 = -q\alpha_2, \beta_1^{-1}\beta_2 = q^2, \gamma_1^{-1}\gamma_2 = q; \tag{93}$$

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & b_1'x & 0 \end{pmatrix}_1 \Rightarrow \beta_1 = q^{-1}\alpha_1, \beta_2 = -q^{-1}\alpha_2, \alpha_1^{-1}\alpha_2 = q^{-1}, \gamma_1^{-1}\gamma_2 = 1; \tag{94}$$

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & b_2'z & 0 \end{pmatrix}_1 \Rightarrow \gamma_1 = q\beta_1, \gamma_2 = -q\beta_2, \alpha_1^{-1}\alpha_2 = 1, \gamma_1^{-1}\gamma_2 = q; \tag{95}$$

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & c_1'x \end{pmatrix}_1 \Rightarrow \alpha_1 = q\gamma_1, \alpha_2 = -q\gamma_2, \alpha_1^{-1}\alpha_2 = q^{-1}, \beta_1^{-1}\beta_2 = q^{-2}; \tag{96}$$

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & c_2'y \end{pmatrix}_1 \Rightarrow \beta_1 = q\gamma_1, \beta_2 = -q\gamma_2, \alpha_1^{-1}\alpha_2 = 1, \beta_1^{-1}\beta_2 = q^{-1}; \tag{97}$$

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}_1 \textit{it does not determine the weight constants at all;} \tag{98}$$

If $a_0 \neq 0$, *then* $\alpha_1 = q, \alpha_2 = -q, \beta_1^{-1}\beta_2 = q, \gamma_1^{-1}\gamma_2 = q$, *and we have*

$$\begin{cases} a_1 = 0, \\ b_1 = 0, \\ b_2 = 0, \\ b_3 = \frac{-a_0(t_2+qt_1)}{2q\beta_1}. \end{cases} \tag{99}$$

If $c_0 \neq 0$, *then* $\gamma_1 = q, \gamma_2 = -q, \beta_1^{-1}\beta_2 = q^{-1}, \alpha_1^{-1}\alpha_2 = q^{-1}$, *and we have*

$$\begin{cases} b_2 = 0, \\ b_3 = 0, \\ c_1 = 0, \\ b_1 = \frac{-c_0(t_1+qt_2)}{2q\beta_1}. \end{cases} \tag{100}$$

If $a_0' \neq 0$, *then* $\alpha_1 = q^{-1}, \alpha_2 = -q^{-1}, \beta_1\beta_2^{-1} = q^{-1}, \gamma_1\gamma_2^{-1} = q^{-1}$, *and we have*

$$\begin{cases} a_1' = 0, \\ b_1' = 0, \\ b_2' = 0, \\ b_3' = \frac{a_0'(t_2-qt_1)}{2q\alpha_2\gamma_2}. \end{cases} \tag{101}$$

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If $c'_0 \neq 0$, then $\gamma_1 = q^{-1}, \gamma_2 = -q^{-1}, \beta_1\beta_2^{-1} = q, \alpha_1\alpha_2^{-1} = q$, and we have

$$\begin{cases} b'_2 = 0, \\ b'_3 = 0, \\ c'_1 = 0, \\ b'_1 = \frac{c'_0(qt_2 - t_1)}{2q\alpha_2\gamma_2}. \end{cases} \tag{102}$$

4.2. The structures of $X_q(A_1)$-module algebra on $\mathbb{C}_q[x,y,z]$

Through the previous discussion, we found that both the 0-st homogeneous component $(M_{EF})_0$ and the 1-st homogeneous component $(M_{EF})_1$ determine the eigenvalues of x and z. By lemma 4.2 and lemma 4.3, and q is not a root of the unit, it follows that there are 91 kinds of $[(M_{EF})_0, (M_{EF})_1]$ are empty. Hence, we only discuss the following cases.

Lemma 4.4. *If the 0-th homogeneous component of M_{EF} is zero and the 1-st homogeneous component of M_{EF} is nonzero, then these series are empty.*

Proof. The proof is similar to the proof of Lemma 3.3. □

Lemma 4.5. *If the 0-th homogeneous component of M_{EF} is nonzero and the 1-st homogeneous component of M_{EF} is zero, then these series are empty.*

Proof. The proof is similar to the proof of Lemma 3.4. □

Lemma 4.6. *The* $\left[\begin{pmatrix} a_0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}_0, \begin{pmatrix} 0 & \frac{-a_0(t_2+qt_1)}{2q\beta_1}z & 0 \\ 0 & 0 & 0 \end{pmatrix}_1\right]$ *-series is empty.*

Proof. By (67), one has $\alpha_1 = q, \alpha_2 = -q, \beta_1^{-1}\beta_2 = q, \gamma_1^{-1}\gamma_2 = q$, and

$$\alpha_1^{-1}\alpha_1\gamma_1^{-1}\gamma_2 = -q \neq q = \beta_1^{-1}\beta_2.$$

From Lemma 4.1, it can be concluded that $t_i = (\beta_i - \alpha_i\gamma_i)t$ $(i = 1,2)$, where $t \in \mathbb{C}^*$, and $\frac{-a_0(t_2+qt_1)}{2q\beta_1}z = -a_0tz$.

If we suppose this series is not empty, we have $K_1(x) = qx, K_2(x) = -qx$, and $wt_{K_i}(E(x)) = (-1)^{i-1}q^{-1}\alpha_i = 1$, hence $E(x) = a_0$. Set

$$\begin{aligned}
&K_1(y) = \beta_1 y + (\beta_1 - q\gamma_1)txz, && K_2(y) = \beta_2 y + (\beta_2 + q\gamma_2)txz, \\
&K_1(z) = \gamma_1 z, && K_2(z) = \gamma_2 z, \\
&E(y) = -a_0tz + \sum_{m_2+n_2+l_2\geq 2} \psi^2_{m_2n_2l_2} x^{m_2}y^{n_2}z^{l_2} && \text{for } m_2, n_2, l_2 \in \mathbb{N}, \\
&E(z) = \sum_{m_3+l_3\geq 2} \psi^3_{m_3l_3} x^{m_3}z^{l_3} && \text{for } m_3, l_3 \in \mathbb{N}, \\
&F(x) = \sum_{m_4+l_4\geq 2} \psi^4_{m_4l_4} x^{m_4}z^{l_4} && \text{for } m_4, l_4 \in \mathbb{N}, \\
&F(y) = \sum_{m_5+n_5+l_5\geq 2} \psi^5_{m_5n_5l_5} x^{m_5}y^{n_5}z^{l_5} && \text{for } m_5, n_5, l_5 \in \mathbb{N}, \\
&F(z) = \sum_{m_6+l_6\geq 2} \psi^6_{m_6l_6} x^{m_6}z^{l_6} && \text{for } m_6, l_6 \in \mathbb{N},
\end{aligned}$$

where $\beta_1, \beta_2, \gamma_1, \gamma_2, t \in \mathbb{C}^*$, and $\psi^2_{m_2n_2l_2}, \psi^3_{m_3l_3}, \psi^4_{m_4l_4}, \psi^5_{m_5n_5l_5}, \psi^6_{m_6l_6} \in \mathbb{C}$. According to (61), it can be obtained

$$\sum_{m_2+n_2+l_2\geq 2} \psi^2_{m_2n_2l_2}(q^{n_2+l_2} + q)x^{m_2+1}y^{n_2}z^{l_2} = 0,$$

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then for all $m_2, n_2, l_2 \in \mathbb{N}$ with $m_2 + n_2 + l_2 \geq 2$, one has $\psi^2_{m_2 n_2 l_2} = 0$ and $E(y) = -a_0 tz$. Similarly, we can get $E(z) = 0$. By (3) and (5), we have

$$\begin{aligned}(K_1F - qFK_1)(x) =& K_1(F(x)) - qF(K_1(x))\\ =& K_1(\sum_{m_4+l_4\geq 2} \psi^4_{m_4 l_4} x^{m_4} z^{l_4}) - q^2 F(x)\\ =& \sum_{m_4+l_4\geq 2} \psi^4_{m_4 l_4}(q^{m_4}\gamma_1^{l_4} - q^2) x^{m_4} z^{l_4} = 0,\\ (K_2F + qFK_2)(x) =& K_2(F(x)) + qF(K_2(x))\\ =& \sum_{m_4+l_4\geq 2} \psi^4_{m_4 l_4}[(-q)^{m_4}\gamma_2^{l_4} - q^2] x^{m_4} z^{l_4} = 0,\end{aligned}$$

then for all $m_4, l_4 \in \mathbb{N}$ with $m_4 + l_4 \geq 2$, one has

$$\psi^4_{m_4 l_4} = 0 \ \text{ or } \ (-q)^{m_4}\gamma_2^{l_4} = q^2,$$

and $F(x) = 0$ or $F(x) = \psi^4_{20} x^2$.

- If $F(x) = 0$, it is easy to get $F(y) = F(z) = 0$, then

$$(EF - FE)(z) = 0 \neq \frac{K_2K_1^{-1} - K_2^{-1}K_1}{q - q^{-1}}(z) = z,$$

  this contradicts our hypothesis.
- If $F(x) = \psi^4_{20} x^2$, by (66), one can get $F(z) = \frac{\psi^4_{20}(q^2-1)}{2q} xz$, and

$$(EF - FE)(z) = \frac{K_2K_1^{-1} - K_2^{-1}K_1}{q - q^{-1}}(z) = z,$$

  after calculation, we can conclude that $F(x) = \frac{2qa_0}{q^2-1} x^2$ and $F(z) = a_0 xz$. However

$$F^2(z) = F(a_0 xz) = a_0^2 \frac{q^2+1}{q^2-1} x^2 z \neq 0,$$

  this contradicts our hypothesis.

  In summary, this series is empty. □

Similar to Lemma 3.6, we can obtain $\left[\begin{pmatrix} 0 & 0 & c_0 \\ 0 & 0 & 0 \end{pmatrix}_0, \begin{pmatrix} 0 & \frac{-c_0(t_1+qt_2)}{2q\beta_1}x & 0 \\ 0 & 0 & 0 \end{pmatrix}_1\right]$, $\left[\begin{pmatrix} 0 & 0 & 0 \\ a_0' & 0 & 0 \end{pmatrix}_0, \begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{a_0'(t_2-qt_1)}{2q\alpha_2\gamma_2}z & 0 \end{pmatrix}_1\right]$, $\left[\begin{pmatrix} 0 & 0 & 0 \\ C_0' & 0 & 0 \end{pmatrix}_0, \begin{pmatrix} 0 & 0z & 0 \\ 0 & \frac{c_0'(qt_2-t_1)}{2q\alpha_2\gamma_2}z & 0 \end{pmatrix}_1\right]$ are empty series.
Next we turn to "nonempty" series, it only has one "nonempty" series.

Theorem 4.7. *The* $\left[\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}_0, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}_1\right]$ *-series has two types of $X_q(A_1)$-module algebra structures on the $\mathbb{C}_q[x, y, z]$ given by*

1. *for all $\lambda, \mu, t \in \mathbb{C}^*$, we have*

$$\begin{aligned}&K(x) = \lambda x, && K_2(x) = \pm\lambda x\\ &K(y) = \lambda\mu y + txz, && K_2(x) = \pm(\lambda\mu y + txz)\\ &K(z) = \mu z, && K_2(z) = \pm\mu z\\ &E(x) = E(y) = E(z) = 0, && F(x) = F(y) = F(z) = 0,\end{aligned} \tag{103}$$

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they are pairwise nonisomorphic.

2. *for all $\lambda, \sigma, \mu, \widetilde{t} \in \mathbb{C}^*$, we have*

$$\begin{array}{ll} K(x) = \lambda x, & K_2(x) = \pm\lambda x \\ K(y) = \sigma y + \widetilde{t}xz, & K_2(x) = \pm(\sigma y + \widetilde{t}xz) \\ K(z) = \mu z, & K_2(z) = \pm\mu z \\ E(x) = E(y) = E(z) = 0, & F(x) = F(y) = F(z) = 0, \end{array} \quad (104)$$

where $\widetilde{t} = (\lambda\mu - \sigma)t \in \mathbb{C}^*$, they are pairwise nonisomorphic.

Proof. The proof is similar to the proof of Theorem 3.5. □

5. Conclusions

We investigate the module algebra structures of $X_q(A_1)$ on quantum polynomial algebra $\mathbb{C}_q[x, y, z]$. Our main contributions are as follows.

1. When $t = 0$, the classification of $X_q(A_1)$-module algebra structures are given on $\mathbb{C}_q[x, y, z]$. We obtain the following:
   - there are 7 cases for the 0-st homogeneous component $(M_{EF})_0$ of M_{EF}, see Lemma 3.1;
   - there are 13 cases for the 1-st homogeneous component $(M_{EF})_1$ of M_{EF}, see Lemma 3.2;
   - there are 90 kinds of $[(M_{EF})_0], (M_{EF})_1]$-series are empty;
   - $X_q(A_1)$-module algebra structures are given and classified on $\mathbb{C}_q[x, y, z]$, see Theorem 3.5.
2. When $t \neq 0$, the classification of $X_q(A_1)$-module algebra structures were given on $\mathbb{C}_q[x, y, z]$.
   - there are 8 cases for the 0-st homogeneous component $(M_{EF})_0$ of M_{EF}, see Lemma 4.2;
   - there are 18 cases for the 1-st homogeneous component $(M_{EF})_1$ of M_{EF}, see Lemma 4.3;
   - $X_q(A_1)$-module algebra structures are given and classified on $\mathbb{C}_q[x, y, z]$, see Theorem 4.7.

These researches make some preparations on the classification of module algebra structures of $X_q(A_n)$ on $\mathbb{C}_q[x, y, z]$ for $n \geq 2$.

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References

1. A. Aghamohammadi, V. Karimipour, S. Rouhani, The multiparametric non-standard deformation of A_{n-1}. *J. Phys. A: Math. Gen. 1993, 26*, L75-L82.
2. J. Alev, M. Chamarie, Dérivations et automorphismes de quelques algèbres quantiques. *Comm. Algebra 1992, 20*, 1787-1802.
3. V. A. Artamonov, Quantum polynomial algebras. *J. Math. Sci. 1997, 87*, 3441-3462.
4. V. A. Artamonov, Actions of pointlike Hopf algebras on quantum polynomials. *Russian Math. Surveys 2000, 55*, 1137-1138.
5. V. A. Artamonov, R. Wisbauer, Homological properties of quantum polynomials. *Algebr. Represent. Theory 2001, 4*, 219-247.
6. M. Beattie, A direct sum for the Brauer group of H-module algebras. *J. Algebra 1976, 43*, 686-693.
7. M. Beattie, Brauer groups of H-module and H-dimodule algebras. Ph.D. thesis, Queen's University (Canada), ProQuest LLC, Ann Arbor, MI, 1977.
8. R. J. Blattner, S. Montgomery, A duality theorem for Hopf module algebras. *J. Algebra 1985, 95*, 153-172.
9. L. Castellani, J. Wess, *Quantum groups and their applications in physics*; IOS Press, Amsterdam, Netherlands, 1996; PP. 56-89.
10. K. Chan, C. Walton Y. Wang, J. Zhang, Hopf actions on filtered regular algebras. *J. Algebra 2014, 397*, 68-90.

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 30 April 2025 doi:10.20944/preprints202504.2576.v1

11. C. Cheng, S. Yang, Weak Hopf algebras corresponding to non-standard quantum groups.*Bull. Korean Math. Soc. 2017*, *54*, 463-484.
12. S. Duplij, Y. Hong, F. Li, $U_q(sl(m+1))$-module algebra structures on the coordinate algebra of a quantum vector space. *J. Lie Theory 2015 25*, 327-361.
13. S. Duplij, S. Sinel'shchikov, Classification of $U_q(sl_2)$-module algebra structures on the quantum plane. *J. Math. Phys. Anal. Geom. 2010, 6*, 406-430, 436, 439.
14. S. Duplij, S. Sinel'shchikov, On $U_q(sl_2)$-actions on the quantum plane. *Acta Polytechnica 2010, 50*, 25-29.
15. B. Drabant, A. Van Daele, Y. Zhang, Actions of multiplier Hopf algebras. *Comm. Algebra 1999, 27*, 4117-4172.
16. M. Ge, G. Liu, K. Xue, New solutions of Yang-Baxter equations: Birman-Wenzl algebra and quantum group structures. *J. Phys. A: Math. Gen. 1991, 24*, 2679-2690.
17. N. Hu, Quantum divided power algebra, q-derivatives, and some new quantum groups. *J. Algebra 2000, 232*, 507-540.
18. N. Jing, M. Ge, Y. Wu, A New Quantum Group Associated with a Non-standard Braid Group Representation. *Lett. Math. Phys. 1991, 21*, 193-203.
19. C. Kassel, *Quantum groups*; Springer-Verlag, New York, USA, 1995; pp. 146-157.
20. A. Klimyk, K. Schmüdgen, *Quantum groups and their representations*; Springer-Verlag, Berlin, Germany, 1997; pp.68-87.
21. S. Montgomery, *Hopf Algebras and their actions on rings*; the Conference Board of the Mathematical Sciences, Washington, USA, 1993; 74-93.
22. D. Su, S. Yang, Representations of the small nonstandard quantum groups $\overline{X}_q(A_1)$. *Comm. Algebra 2019, 47*, 5039-5062.
23. M. E. Sweedler, *Hopf Algebras*; W. A. Benjamin, Inc., New York, USA, 1969, 56-73.
24. M. Zhu, Classification of quantum group A_n-module algebra structures on quantum polynomial algebras. Ph.D. thesis, Yangzhou University, Yangzhou University, 2012.

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