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Not peer-reviewed version

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Posted Date: 5 June 2024

doi: 10.20944/preprints202406.0238.v1

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Brezis-Nirenberg Type Critical Problems on H-Type Groups

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Abstract: On the H-type group $G \subset \mathbb{R}^m \times \mathbb{R}^n$, we study the existence of solutions for the Brezis-Nirenberg problem

$$\begin{cases} -\Delta_G u = \lambda u + |u|^{Q^*-2}u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\Omega \subset G$ is a smooth bounded subset with C^2 boundary and $-\Delta_G$ is the Kohn Laplacian. Here $m \geq 2$ is an even number and $Q = m + 2n$ is the homogeneous dimension of G, $Q^* = \frac{2Q}{Q-2}$ is the critical exponent in the sense of the L^2 Folland-Stein inequality. By a classical variational method and linking argument, the Brezis-Nirenberg type results are obtained for $\lambda > 0$, a real parameter distinct from the eigenvalue of $-\Delta_G$.

Keywords: CR yamabe problem; brezis-nirenberg problem; folland-stein inequality; H-type group

1. Introduction and Main Results

In this paper, we study the Brezis-Nirenberg problem on H-type groups $G \subset \mathbb{R}^m \times \mathbb{R}^n$

$$\begin{cases} -\Delta_G u = \lambda u + |u|^{Q^*-2}u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1}$$

where $\Omega \subset G$ is a smooth bounded subset with C^2 boundary and $-\Delta_G$ is the Kohn Laplacian. Here $m \geq 2$ is an even number and $Q = m + 2n$ is the homogeneous dimension of G, $Q^* = \frac{2Q}{Q-2}$ is the critical exponent in the sense of the L^2 Folland-Stein inequality.

The classical Brezis-Nirenberg problem set on Entire space $\mathbb{R}^n$ is a model-based problems of the classical Yamabe problem (prescribed curvature problem) related to the Riemann manifolds, see Yamabe [30]. It says that for a Riemannian manifold (M, g), finding a conformal metric $\tilde{g}$ such that the scalar curvature $R_{\tilde{g}}$ is a constant. If $\tilde{g} = u^{\frac{4}{m-1}} g$ is a new metric conformal to g, the scalar curvature $R_{\tilde{g}}$ of $\tilde{g}$ is given by

$$R_{\tilde{g}} = u^{-\frac{m+2}{m-2}} \left(\frac{4(m-1)}{m-2} \Delta_g u + R_g u \right),$$

If $R_{\tilde{g}} = \mu$ is a constant, then the problem is equivalent to finding a positive C^∞ solution u to the Yamabe equation:

$$\frac{4(n-1)}{n-2} \Delta_g u + R_g u = \mu u^{\frac{m+2}{m-2}}. \tag{2}$$

This problem has the following nice variational formulation. Consider the constrained variational problem

$$\mu(M) = \inf\left\{ \int_M \left(\frac{4(m-1)}{m-2} |\nabla_g u|^2 + R_g u^2 \right) \mathrm{d}V_g : \int_M |u|^{\frac{2m}{m-2}} \mathrm{d}V_g = 1 \right\}. \tag{3}$$

Then the Yamabe problem becomes the following problems:

(a) $\mu(M)$ depends only on the conformal class of g.
(b) $\mu(M) \leq \mu(S^m)$, in which the sphere S^m has the standard metric.

(c) If $\mu(M) < \mu(S^m)$, then the infimum is attained by a positive solution. Thus the metric $\widetilde{g} = u^{\frac{4}{n-1}} g$ has constant scalar curvature $\mu(M)$.

Thanks to the contributions of Aubin [1] and Schoen [25], these issues have been completely resolved. The proof of (a) consists of the fundamental observation that problem (5) is conformally invariant. The proof of (b) begins with a thorough understanding of the special case of the sphere $\mathbb{S}^m$ in $\mathbb{R}^{m+1}$:

$$\mu(\mathbb{S}^m) = \inf\left\{ \int_{\mathbb{S}^m} \left(\frac{4(n-1)}{n-2} |\nabla_{\mathbb{S}} u|^2 + R_g u^2 \right) \mathrm{d}V_g : \int_{\mathbb{S}^m} |u|^{\frac{2m}{m-2}} \mathrm{d}V_g = 1 \right\}. \tag{4}$$

The conformal change of variables given by stereographic projection converts the variational problem on $\mathbb{S}^m$ to the more familiar problem on $\mathbb{R}^{m+1}$:

$$\mu(\mathbb{R}^{m+1}) = \inf\left\{ \int_{\mathbb{R}^{m+1}} \left(\frac{4(n-1)}{n-2} |\nabla u|^2 \right) \mathrm{d}x : \int_{\mathbb{R}^{m+1}} |u|^{\frac{2m}{m-2}} \mathrm{d}x = 1 \right\}. \tag{5}$$

In other words, by a stereographic projection, $\mu(S^m) = \mu(\mathbb{R}^m)$. While $\mu(\mathbb{R}^m)$ is the sharp constant of the Sobolev inequality on $\mathbb{R}^{m+1}$:

$$\mu(\mathbb{R}^m) \left(\int_{\mathbb{R}^m} |u|^{\frac{2m}{m-2}} \mathrm{d}x \right)^{\frac{m-2}{m}} \le \int_{\mathbb{R}^m} |\nabla u|^2 \mathrm{d}x, \tag{6}$$

and it is attained by a Talenti bubble.

$$u(x) = (a + b|x - x_0|^2)^{-\frac{m-2}{2}}. \tag{7}$$

Therefore, (c) can be easily proved by transplanting an approximate extremal function from $\mathbb{R}^{m+1}$ to a small neighborhood on M.

Actually, when further examining the apparent solution, Trudinger [27] pointed out the problem of lacking compactness. Therefore, Briezs and Nirenberg [4] proposed a detection of the Yamabe equation from an analytical perspective involves constructing model-based problems:

$$\begin{cases} -\Delta u = \lambda u + |u|^{2^*-2} u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^m$ is a bounded domain and $2^* = \frac{2m}{m-2}$ is the Sobolev critical exponent. This problem has been solved in different domains, see [6,7,13].

As we are aware, similarly to how Riemann manifolds function in Classical Mechanics, the Heisenberg group plays a significant role in Quantum Mechanics due to its noncommutative structure, see [5]. Additionally, the Heisenberg group is considered as a Cauchy-Riemann (call CR short) manifold, intersecting complex analysis, differential geometry, and the theory of partial differential equations, see [10]. On the other hand, since Wells [29] established the CR structure on complex manifold, Chern, Moser [8], Webster [28] and Tanaka[26] found a complete system of analytic invariant for two real analytic real hypersurfaces in complex manifold to be locally equivalent under biholomorphic transformation, one then can define the unique connect form θ of any CR manifold, and hence, the Webster scalar curvature R_θ. Therefore, there are far-reaching analogies between conformal and CR geometries, see [20]. The CR-Yamabe problem says that for a CR manifold (N, θ), finding a contact form $\widetilde{\theta}$ such that the Webster scalar curvature $R_{\widetilde{\theta}}$ is a constant. If $\widetilde{\theta} = u^{\frac{4}{n-1}} \theta$ is a new contact form conformal to θ, the Webster scalar curvature $R_{\widetilde{\theta}}$ of $\widetilde{\theta}$ is given by

$$R_{\widetilde{\theta}} = u^{-\frac{n+2}{n-2}} \left(\frac{4(n-1)}{n-2} \Delta_\theta u + R_\theta u \right),$$

If $R_{\widetilde{\theta}} = \lambda$ is a constant, then the problem is equivalent to finding a positive C^∞ solution u to the CR-Yamabe equation:

$$\frac{4(n-1)}{n-2}\Delta_\theta u + R_\theta u = \lambda u^{\frac{n+2}{n-2}}. \tag{8}$$

Consider the constrained variational functional

$$\lambda(N) = \inf\Big\{\int_N (\frac{4(n-1)}{n-2}|\nabla_\theta u|^2 + R_\theta u^2)\theta\wedge \mathrm{d}\theta^n : \int_N |u|^{\frac{2n}{n-2}}\theta\wedge \mathrm{d}\theta^n = 1\Big\}. \tag{9}$$

Then the CR-Yamabe problem becomes the following problems:

(d) $\lambda(N)$ depends only on the conformal class of θ.

(e) $\lambda(N) \le \lambda(S^{2n+1})$, in which the sphere S^{2n+1} has the standard CR structure.

(f) If $\lambda(N) < \lambda(S^{2n+1})$, then the infimum is attained by a positive solution. Thus the metric $\widetilde{\theta} = u^{\frac{4}{n-1}}\theta$ has constant Webster scalar curvature $\lambda(N)$.

The proof of (d-f) is quite similar to the classical case. By the Cayley transform, the variational problem on S^{2n+1}

$$\lambda(S^{2n+1}) = \inf\Big\{\int_{S^{2n+1}}\Big(\frac{4(n-1)}{n-2}|\nabla_\theta u|^2 + R_\theta u^2\Big)\theta\wedge \mathrm{d}\theta^n : \int_{S^{2n+1}} |u|^{\frac{2n}{n-2}}\theta\wedge \mathrm{d}\theta^n = 1\Big\} \tag{10}$$

can be transformed into the variational problem on $\mathbb{H}^n$

$$\lambda(\mathbb{H}^n) = \inf\Big\{\int_{\mathbb{H}^n}\Big(\frac{4(n-1)}{n-2}|\nabla_H u|^2\Big)\theta\wedge \mathrm{d}\theta^n : \int_{\mathbb{H}^n} |u|^{\frac{2n}{n-2}}\theta\wedge \mathrm{d}\theta^n = 1\Big\}. \tag{11}$$

This relates to the Folland-Stein inequality on $\mathbb{H}^n$

$$\lambda(\mathbb{H}^n)\Big(\int_{\mathbb{H}^n}|u|^{\frac{2n}{n-2}}\theta\wedge \mathrm{d}\theta^n\Big)^{\frac{n-2}{n}} \le \int_{\mathbb{H}^n}|\nabla_H u|^2\theta\wedge \mathrm{d}\theta^n. \tag{12}$$

Building on Obata's proof [24], Jerison and Lee [21] demonstrated the uniqueness of positive optimizers by deriving a Bianchi identity for the CR function that undergoes a conformal transformation. As a model-based problems, Citti [9] also consider the Brezis-Nirenber problem on $\mathbb{H}^n$

$$\begin{cases} -\Delta_{\mathbb{H}} u = \lambda u + |u|^{Q^*-2}u & \text{in } \Omega,\\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{H}^n$ is a bounded domain, $Q = 2n+2$ is the homogeneous dimension of $\mathbb{H}^n$ and $Q^* = \frac{2Q}{Q-2}$ is the critical exponent of Folland-Stein inequality. Readers can refer to [16,18] to the nonlocal case.

In the 1980s, Kaplan [22] introduced an H-type group in connection with hypoellipticity questions, showing similarities with the Heisenberg group but with more intricate properties, see Section 2. Particularly, in paper [22], Kaplan also established the similar Folland-Stein inequality on H-type group G

$$C\Big(\int_G |u|^{\frac{2Q}{Q-2}}\mathrm{d}g\Big)^{\frac{Q-2}{2Q}} \le \Big(\int_G |\nabla_G u|^2 \mathrm{d}g\Big)^{\frac{1}{2}}, \quad u \in W^{1,2}(G), \tag{13}$$

where dg is the Haar measure on G. Following this result, Han and Niu [19] established the corresponding Hardy-Sobolev type inequalities on H-type group. Interested readers can refer to [33] for the Caffarelli-Kohn-Nirenberg inequality, and to [23] for the Stein–Tomas type inequality on H-type group.

For further insights on the H-type group, Garofalo and Vassilev [14,15] presented the Yamabe-type problem concerning this group. Strangely, since Kaplan [22] determined the explicit form of the optimizers for (13), the uniqueness of the optimizers has been a longstanding question. Nearly 20 years later, Garofalo and Vassilev [14,15] provided a definitive answer when the H-type group

 doi:10.20944/preprints202406.0238.v1

is an Iwasawa-type group. Recently, Yang [31] eventually proved the uniqueness of the optimizer using a sub-critical approximate methods. However, due to the absence of Cayley transform on the general H-type, proving the the non-degeneracy of the optimizer become challenging. As a result, extending the perturbation result of the Yamabe equation from Iwasawa-type groups (see [32]) to H-type Groups remains unfeasible. This situation prompts us to examine the linear perturbation problem of the critical equation on the bounded domain, that is the Brezis-Nirenberg problem (1) on the H-type group. As we can see, the solution to the Brezis-Nirenberg problem greatly depends on the dimension and topological properties of the domain, as highlighted in [6,7]. Therefore, researching the Brezis-Nirenberg problem on an H-type group with non-homogeneous dimension holds considerable significance. By carefully analyzing and utilizing the variational method, especially the Linking argument, the primary outcome is as follows:

Theorem 1. *Let Ω be a bounded domain of G, assume that $\{\lambda_j\}$ are the sequence of eigenvalues of $-\Delta_G$ on Ω with zero Dirichlet boundary data. Then*
(i) for all $\lambda \in (0, \lambda_1)$, (1) has a nontrivial solution.
(ii) for all $\lambda \in (\lambda_j, \lambda_{j+1})$, $j \in \mathbb{N}$, (1) has a nontrivial solution.

Remark 1. *Let Ω be a star shaped domain, and $\lambda = 0$, then u is a non-negative solution of (1) implies $u \equiv 0$. This result stems from establishing a significant Pohozaev identity, as shown by Han and Niu [17, Theorem 3.6]. In the case where Ω is a half-space of H-type groups, Bonfiglioli and Uguzzoni [2] have also derived similar Nonlinear Liouville theorems.*

Turing to layout of the paper, in Section 2, we give some preliminaries knowledge on H-type group. For the sake of clarity, we establish the variational frame and make some compactness analysis in Section 3. The total proof has been divided into two cases, one is $0 < \lambda < \lambda_1$ and another is $\lambda > \lambda_1$. All of these process are completed in Section 4. Particularly, we prove the first case by a mountain-pass theorem and we prove the second case by a Linking argument.

2. Preliminaries on H-Type Groups

Let's begin with some basics of Carnot group. Carnot group (in a short CG) is a stratified, simply connected nilpotent Lie group of step r. Denote by $\mathcal{G}$ be Lie algebre of CG. It is known that $\mathcal{G} = \oplus_{i=1}^r V_i$ satisfying (see e.e. [12])
$$[V_i, V_j] = V_{j+1}, 1 < j < r-1;\ [V_1, V_r] = \{0\}.$$
As a simply connected nipotent group, CG is differential with $\mathbb{R}^N$, $N = \sum\limits_{i=1}^{r} \dim V_i$, via the exponential map $\exp : \mathcal{G} \to CG$. There is a natural family of nonisotropic dilations $\delta_\lambda : \mathcal{G} \to \mathcal{G}$ for $\lambda > 0$ and we define it as follow:
$$\delta_\lambda(X_1 + X_2 + \cdots + X_r) = \lambda X_1 + \lambda^2 X_2 + \cdots + \lambda^r X_r,\ X_j \in V_j, 1 < j < r.$$
The homogeneous dimension of CG, associated with δ_λ, is $Q = \sum\limits_{j=1}^{r} j\dim V_j$.

An H-type group G is a Carnot group with the following properties (see Kaplan [22]): the Lie algebra $\mathcal{G}$ of G is endowed with an inner product $\langle \cdot, \cdot \rangle$ such that, if $\mathcal{Z}$ is the center of $\mathcal{G}$, then $[\mathcal{Z}^\perp, \mathcal{Z}^\perp] = \mathcal{Z}$ and moreover, for every fixed $z \in \mathcal{Z}$, the map $J_z := \mathcal{Z}^\perp \to \mathcal{Z}^\perp$ defined by
$$\langle J_z(v), \omega \rangle = \langle z, [v, \omega] \rangle,\ \forall \omega \in \mathcal{Z}^\perp$$

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is an orthogonal map whenever $\langle z,z\rangle = 1$. Set $m = \dim \mathcal{Z}^\perp$ and $n = \dim \mathcal{Z}$. Since G has step two, we can fix on G a system of coordinates (x,t) such that the group law on G has the form (see [2])

$$(x,t)\circ(x',t') = \begin{bmatrix} x_i + x_i',\ i = 1,2,\cdots,m \\ t_j + t_j' + \frac{1}{2}\langle x, U^{(j)}x'\rangle,\ j = 1,2,\cdots,n \end{bmatrix} \tag{14}$$

for suitable skew-symmetric matrices $U^{(j)}$'s. Note that $U^{(j)}$ is also a orthogonal matrix, hence m has to be an even number, see [2, Remark 7.3]. The vector field in the Lie algebra $\mathcal{G}$ that agrees at the origin with $\frac{\partial}{\partial x_j}(j = 1,\cdots,m)$ is given by

$$X_j = \frac{\partial}{\partial x_j} + \frac{1}{2}\sum_{k=1}^{n}\left(\sum_{i=1}^{m} U_{i,j}^{(k)} x_i\right)\frac{\partial}{\partial t_k}$$

and $\mathcal{G}$ is spanned by the left-invariant vector fields $X_1,\cdots,X_m$, $T_1 = \frac{\partial}{\partial t_1},\cdots,T_n = \frac{\partial}{\partial t_n}$. Furthermore (see [2], Page 200, (A.4)),

$$[X_i, X_j] = \sum_{r=1}^{n} U_{i,j}^{(r)} T_r,\ i,j \in \{1,2,\cdots,n\}.$$

The horizontal gradient on G is $\nabla_G = (X_1,\cdots,X_m)$. The sub-Laplacian on G is given by (see [2], Remark A.6.)

$$\begin{aligned}\Delta_G = \sum_{j=1}^{m} X_j^2 &= \sum_{j=1}^{m}\left(\frac{\partial}{\partial x_j} + \frac{1}{2}\sum_{k=1}^{n}\left(\sum_{i=1}^{m} U_{i,j}^{(k)} x_i\right)\frac{\partial}{\partial t_k}\right)^2 \\ &= \Delta_x + \frac{1}{4}|x|^2\Delta_t + \sum_{k=1}^{n}\langle x, U^k\nabla_x\rangle\frac{\partial}{\partial t_k},\end{aligned}$$

where

$$\Delta_x = \sum_{j=1}^{m}\left(\frac{\partial}{\partial x_j}\right)^2,\ \ \Delta_t = \sum_{k=1}^{n}\left(\frac{\partial}{\partial t_k}\right)^2.$$

The Sobolev space $W_0^{1,2}(G)$ is the closure of $C_0^\infty(G)$ with respect to the norm

$$||u||_{W_0^{1,2}(G)} = \left(\int_G |\nabla_G u|^2 \mathrm{d}g\right)^{\frac{1}{2}},$$

where $\mathrm{d}g$ is the Haar measure on G. We remark that the Haar measure on G, induced by the exponential mapping from the Lebesgue measure on $\mathcal{G} = \mathbb{R}^N$, coincides the Lebesgue measure on $\mathbb{R}^N$. Correspondingly, in a bounded domain $\Omega \subset G$, we define the Sobolev space as $W_{0,G}^{1,2}(\Omega)$.

The Folland-Stein inequality on G reads that there exist some constant C such that for each $u \in W_0^{1,p}(G)$ (see [11,20,21])

$$C\left(\int_G |u|^{\frac{pQ}{Q-2}}\mathrm{d}g\right)^{\frac{Q-2}{2Q}} \le \left(\int_G |\nabla_G u|^2\mathrm{d}g\right)^{\frac{1}{2}},\ 1 < p < Q. \tag{15}$$

More precisely, Yang [31] obtained the sharp constant $S_{m,n} = C$ and obtained the uniqueness of the extremal unction $U(\xi)$. Namely, set

$$U(\xi) = \left[\left(1 + \frac{|x|^2}{4}\right)^2 + |t|^2\right]^{-\frac{Q-2}{4}},\ \xi = (x,t) \in G,$$

$$U_{\lambda,\eta}(\xi) = \lambda^{\frac{Q-2}{2}} U(\delta_\lambda(\eta^{-1}\circ\xi)),\ \eta \in G,$$

it has been shown that $[\frac{m(Q-2)}{4}]^{\frac{Q-2}{4}}U_{\lambda,\eta}(\xi)$ satisfies the Yamabe-type equation (see [14,15])

$$\Delta_G[\frac{m(Q-2)}{4}]^{\frac{Q-2}{4}}U_{\lambda,\eta}+\{[\frac{m(Q-2)}{4}]^{\frac{Q-2}{4}}U_{\lambda,\eta}\}^{\frac{Q+2}{Q-2}}=0,$$

or equivalently,

$$\Delta_G U_{\lambda,\eta}+\frac{m(Q-2)}{4}U_{\lambda,\eta}^{\frac{Q+2}{Q-2}}=0. \tag{16}$$

Based on the group law (14), the nonisotropic dilations δ_λ on G is

$$\delta_\lambda(x,t)=(\lambda x,\lambda^2 x).$$

For $(x,t)\in G$, the homogeneous norm of (x,t) is

$$\rho(x,t)=\left(\frac{|x|^4}{16}+|t|^2\right)^{\frac{1}{4}}.$$

With this norm ρ, we can define the ball centered at origin with radius R

$$B_R(0)=\{(x,t)\in G:\rho(x,t)<R\}.$$

3. Variational Framework

In order to study the problem by variational methods, we introduce the energy functional associated to (1) by

$$J_\lambda(u)=\frac{1}{2}\int_\Omega|\nabla_G u|^2\mathrm{d}g-\frac{\lambda}{2}\int_\Omega|u|^2\mathrm{d}g-\frac{1}{Q^*}\int_\Omega|u|^{Q^*}\mathrm{d}g,\quad u\in W^{1,2}_{0,G}(\Omega).$$

Firstly, we check that the energy functional $J_\lambda(u)$ has a mountain structure as $\lambda\in(0,\lambda_1)$.

Lemma 1. *If $\lambda\in(0,\lambda_1)$, then J_λ satisfies the following properties:*
(i) There exists $\alpha,\rho>0$ such that $J_\lambda(u)\geq\alpha$ for $||u||=\rho$.
(ii) There exists $e\in W^{1,2}_{0,G}(\Omega)$ with $||e||\geq\rho$ such that $J_\lambda(e)<0$.

Proof. (i) By the Poincare inequality and Sobolev embedding inequality, we have

$$\begin{aligned}J_\lambda(u)&\geq\frac{1}{2}\int_\Omega|\nabla_G u|^2\mathrm{d}g-\frac{\lambda}{2\lambda_1}\int_\Omega|\nabla_G u|^2\mathrm{d}g-\frac{1}{Q^*}|u|^{Q^*}_{Q^*}\\&\geq\frac{1}{2}(1-\frac{\lambda}{\lambda_1})||u||^2-\frac{1}{Q^*}C_1||u||^2.\end{aligned}$$

Since $\lambda\in(0,\lambda_1)$, we can chose some $\alpha,\rho>0$ such that $J_\lambda(u)\geq\alpha$ for $||u||=\rho$.

(ii) For some $u_1\in W^{1,2}_{0,G}(\Omega)\setminus\{0\}$, we have

$$J_{\lambda_1}(tu_1)=\frac{t^2}{2}\int_\Omega|\nabla_H u_1|^2\mathrm{d}g-\frac{\lambda t^2}{2}\int_\Omega|u_1|^2\mathrm{d}g-\frac{t^{Q^*}}{Q^*}\int_\Omega|u_1|^{Q^*}\mathrm{d}g<0$$

for $t>0$ large enough. Hence we can take an $e:=t_1u_1$ for some $t_1>0$ and (ii) follows. □

Proposition 1. *If $\lambda\in(0,\lambda_1)$, there exits a (PS) sequence $\{u_n\}$ such that $J_\lambda(u_n)\to c^*$ and $J_\lambda(u_n)\to 0$ in $W^{1,2}_{0,G}(\Omega)$ at the the minimax level*

$$c^*=\inf_{\gamma\in\Gamma}\inf_{t\in[0,1]}J_\lambda(\gamma(t))>0,$$

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where

$$\Gamma := \{\gamma \in C([0,1], W_{0,G}^{1,2}(\Omega)) : \gamma(0) = 0, J_\lambda(\gamma(1)) < 0\}.$$

Proof. The proposition is a direct conclusion of Lemma 1 and the mountain pass theorem without (PS) condition. □

As $\lambda \in [\lambda_j, \lambda_{j+1})$ for some $j \in \mathbb{N}$, we then easily check that J_λ satisfies a Link structure. Set $\{e_j\}_{j\in\mathbb{N}} \subset L^\infty(\Omega)$ be the sequence of eigenfunctions corresponding to λ_j. We know that this sequence is an orthonormal basis of $L^2(\Omega)$ and the orthogonal basis of $W_{0,G}^{1,2}(\Omega)$. We denote

$$\mathbb{E}_{j+1} := \{u \in W_{0,G}^{1,2}(\Omega) : \langle u, e_i\rangle_{W_0^{1,2}} = 0,\ \forall i = 1,2,\cdots,j\}, \tag{17}$$

and denote $\mathrm{Y}_j := \mathrm{span}\{e_1,\cdots,e_j\}$ by the linear subspace generated by the first j eigenfunctions of $-\Delta_G$ for any $j \in \mathbb{N}$. Note that $\mathbb{Y}_j$ is finite dimensional and $\mathbb{Y}_j \oplus \mathbb{E}_{j+1} = W_{0,G}^{1,2}(\Omega)$, then J_λ satisfies the following properties:

Lemma 2. *If $\lambda \in [\lambda_j, \lambda_{j+1})$ for some $j \in \mathbb{N}$, then*
(i) There exists $\alpha, \rho > 0$ such that for any $u \in \mathbb{E}_{j+1}$ with $||u|| = \rho$ such that $J_\lambda(u) \geq \alpha$.
(ii) $J_\lambda(u) < 0$ for any $u \in \mathbb{Y}_j$.
(iii) Let $\mathbb{F}$ be a finite dimensional subspace of $W_{0,G}^{1,2}(\Omega)$. There exists $R > \rho$ such that for any $u \in \mathbb{F}$ with $||u|| \geq R$ such that $J_\lambda(u) \leq 0$.

Proof. (i) By the Poincare inequality and Sobolev embedding inequality, for any $u \in \mathbb{E}_{j+1} \setminus \{0\}$, we have

$$\begin{aligned} J_\lambda(u) &\geq \frac{1}{2}\int_\Omega |\nabla_G u|^2 dg - \frac{\lambda}{2\lambda_{j+1}}\int_\Omega |\nabla_G u|^2 dg - \frac{1}{Q^*}|u|_{Q^*}^{Q^*} \\ &\geq \frac{1}{2}(1 - \frac{\lambda}{\lambda_{j+1}})||u||^2 - \frac{1}{Q^*}C_1||u||^2. \end{aligned}$$

Since $\lambda \in (\lambda_j, \lambda_{j+1})$, we can chose some $\alpha, \rho > 0$ such that $J_\lambda(u) \geq \alpha$ for all $u \in \mathbb{E}_{j+1} \setminus \{0\}$ with $||u|| = \rho$.

(ii) Let $u \in \mathbb{Y}_j$, i.e., $u = \sum_{i=1}^{j} l_i e_i$, where $l_i \in \mathbb{R}$, $i = 1,\cdots,j$. Since $\{e_i\}_{i\in\mathbb{N}}$ is an orthonormal basis of $L^2(\Omega)$ and $W_{0,G}^{1,2}(\Omega)$, we have

$$\int_\Omega u^2 dg = \sum_{i=1}^{j} l_i^2 \quad \text{and} \quad \int_\Omega |\nabla_G u|^2 dg = \sum_{i=1}^{j} |\nabla e_i|_2^2.$$

Then, we get

$$\begin{aligned} J_\lambda(u) &= \frac{1}{2}\sum_{i=1}^{j} l_i^2(|\nabla e_i|_2^2 - \lambda) - \frac{1}{Q^*}\int_\Omega |u|^{Q^*} dg \\ &< \frac{1}{2}\sum_{i=1}^{j} l_i^2(\lambda_i - \lambda) \leq 0. \end{aligned}$$

(iii) Since all norms on finite dimensional space are equivalent, the non-negativity of λ gives

$$J_\lambda(u) = \frac{1}{2}||u||^2 - \frac{\lambda}{2}|u|_2^2 - \frac{1}{Q^*}|u|_{Q^*}^{Q^*} \leq \frac{1}{2}||u||^2 - \frac{1}{Q^*}|u|_{Q^*}^{Q^*} \leq \frac{1}{2}||u||^2 - \frac{C_1}{Q^*}||u||^{Q^*}$$

for some positive constant C_1. So, $J_\lambda \to \infty$ as $||u|| \to +\infty$. Hence, there exists $R > \rho$ such that for any $u \in \mathbb{F}$ with $||u|| \geq R$ it results that $J_\lambda(u) \leq 0$. □

Similar to Proposition 1, we have the next proposition for $\lambda \in [\lambda_j, \lambda_{j+1}]$.

 doi:10.20944/preprints202406.0238.v1

Proposition 2. *If $\lambda \in [\lambda_j, \lambda_{j+1}]$, there exists a (PS) sequence $\{u_n\}$ such that $J_\lambda(u_n) \to c^\star$ and $J'_\lambda(u_n) \to 0$ in $W^{1,2}_{0,G}(\Omega)^{-1}$ at the minimax level*

$$c^\star = \inf_{\gamma\in\Gamma} \max_{u\in V} J_\lambda(\gamma(u)) > 0, \tag{18}$$

where

$$\Gamma := \{\gamma \in C(\overline{V}, W^{1,2}_{0,G}(\Omega)) : \gamma = \text{id} \;\text{ on } \partial V\}$$

and

$$V = (\overline{B_R} \cap \mathbb{Y}_j) \oplus \{ru : r \in (0,R), u \in \mathbb{E}_j\}.$$

Now, we are discuss the boundedness of these (PS) sequence.

Lemma 3. *For any $\lambda > 0$ but $\lambda \neq \lambda_k$, if $\{u_n\}$ is a $(PS)_{c^*}$ or $(PS)_{c^\star}$ sequence of J_λ, then $\{u_n\}$ is bounded in $W^{1,2}_{0,G}(\Omega)$. Let $u_0 \in W^{1,2}_{0,G}(\Omega)$ be the weak limit of $\{u_n\}$, then u_0 is a weak solution of (1)*

Proof. It is easy to see that there exists $C_1 > 0$ such that

$$||J_\lambda(u_n)|| \leq C_1, \;\; |\langle J'_\lambda(u_n), \frac{u_n}{||u_n||}\rangle| \leq C_1. \tag{19}$$

We firstly prove the case $\lambda \in (0, \lambda_1)$. For n large enough, then by (19) we have

$$C_1(1+||u_n||) \geq J_\lambda(u_n) - \frac{1}{Q^*}\langle J'_\lambda(u_n, u_n)\rangle = (\frac{1}{2} - \frac{1}{Q^*})(||u_n||^2 - \lambda|u_n|_2^2) \geq (\frac{1}{2} - \frac{1}{Q^*})\delta_1||u_n||^2$$

for some $\delta_1 > 0$. Thus $\{u_n\}$ is bounded in $W^{1,2}_{0,G}(\Omega)$.
We Secondly prove the case $\lambda \in (\lambda_j, \lambda_{j+1})$. Let $\beta \in (\frac{1}{Q^*}, \frac{1}{2})$, then for n large enough we have

$$\begin{aligned}
C_1(1+||u_n||) &\geq J_\lambda(u_n) - \beta\langle J'(\lambda)(u_n), u_n\rangle \\
&= (\frac{1}{2} - \beta)(||u_n||^2 - \lambda|u_n|_2^2) + (\beta - \frac{1}{Q^*})|u|_{Q^*}^{Q^*} \\
&= (\frac{1}{2} - \beta)(||z_n||^2 + ||y_n||^2 - \lambda|z_n|_2^2 - \lambda|y_n|_2^2) + (\beta - \frac{1}{Q^*})|u_n|_{Q^*}^{Q^*} \\
&\geq (\frac{1}{2} - \beta)(\delta_2||z_n||^2 + (\lambda_1 - \lambda)|y_n|_2^2) + (\beta - \frac{1}{Q^*})|u_n|_{Q^*}^{Q^*}.
\end{aligned}$$

for some $\delta_2 > 0$, where $u_n = z_n + y_n$, $z_n \in \mathbb{E}_{j+1}$, $y_n \in \mathbb{Y}_j$, where $\mathbb{E}_{j+1}$ is defined in (17). It is then easy to verify that $\{u_n\}$ is bounded in $W^{1,2}_{0,G}(\Omega)$ using the fact that $\mathbb{Y}_j$ is finite dimension.

Next, we are going to prove that u_0 is a weak solution. Since $W^{1,2}_{0,G}(\Omega)$ is reflexive, up to a subsequence, still denoted by u_n, there exists $u_0 in W^{1,2}_{0,G}(\Omega)$ such that $u_n \rightharpoonup u_0$ in $W^{1,2}_{0,G}(\Omega)$ as $u_n \rightharpoonup u_0$ in $L^{Q^*}_G(\Omega)$ as $n \to +\infty$. Then

$$|u_n|^{Q^*-2}u_{u_n} \rightharpoonup |u_0|^{Q^*-2}u_0 \;\text{ in } L^{\frac{2Q}{Q+2}}$$

as $n \to +\infty$. Since $\varphi \in W^{1,2}_{0,G}(\Omega)$, we have

$$\langle J'_\lambda(u_n), \varphi\rangle = \int_\Omega \nabla u_n \nabla\varphi \mathrm{d}g - \lambda\int_\Omega u_n\varphi \mathrm{d}g - \int_\Omega |u|^{Q^*-2}u_n\varphi \mathrm{d}g \to 0,$$

passing to the limit as $n \to +\infty$, we obtain

$$\int_\Omega \nabla u_0 \nabla\varphi \mathrm{d}g - \lambda\int_\Omega u_0\varphi \mathrm{d}g - \int_\Omega |u_0|^{Q^*-2}u_0\varphi \mathrm{d}g = 0$$

 doi:10.20944/preprints202406.0238.v1

for any $\varphi \in W^{1,2}_{0,G}(\Omega)$, this means that u_0 is a weak solution of (1). $\square$

To make sure the weak solution $u_0 \in W^{1,2}_{0,G}(\Omega)$, we shall prove the (PS) sequence has a strong convergence subsequence by controlling the energy of the functionals. Before this, we need a Brezis-Lieb lemma to describe the properties of the (PS) sequence.

Lemma 4. *Let $u_m \in L^p_G(\Omega)$, $1 \leq p < +\infty$ be such that $|u_m|_p \leq \text{constant}$, and $u_m \to u$ a.e. in Ω. Then*

$$\lim_{m\to+\infty} |u_m - u|^p_p = \lim_{m\to+\infty} |u_n|^p_p - |u|^p_p. \tag{20}$$

If replace $L^p_G(\Omega)$ with any Hilbert space $W^{1,2}_{0,G}(\Omega)$, on has that

$$\lim_{m\to+\infty} ||u_m - u||^2 = \lim_{m\to+\infty} ||u_n||^2 - ||u||^2 + 2(u_m|u). \tag{21}$$

Hence if u_m converges weakly to u then (20) holds.

Proof. The proof is similar to the classical case, we refer readers to Brezis-Lieb [3]. $\square$

Based on this Lemma, we have the following claim.

Lemma 5. *For any $\lambda > 0$ and $\lambda \neq \lambda_k$, if $\{u_n\}$ is a $(PS)_c$ sequence of J_λ with*

$$c < \frac{1}{Q} S^{\frac{Q}{2}},$$

then $\{u_n\}$ has a convergent subsequence.

Proof. Let u_0 be the weak limit of $\{u_n\}$ obtained in Lemma 3 and define $v_n := u_n - u_0$. Then we know $v_n \rightharpoonup 0$ in $W^{1,2}_{0,G}(\Omega)$ and $v_n \longrightarrow 0$ a.e. in Ω. By the Brezis-Lieb lemma 4, we know

$$\int_\Omega |\nabla u_n|^2 \mathrm{d}g = \int_\Omega |\nabla v_n|^2 \mathrm{d}g + \int_\Omega |\nabla u_0|^2 \mathrm{d}g + o_n(1),$$

and

$$\int_\Omega |u_n|^{Q^*} \mathrm{d}g = \int_\Omega |v_n|^{Q^*} \mathrm{d}g + \int_\Omega |u_0|^{Q^*} \mathrm{d}g + 0_n(1).$$

Then, since $J_\lambda(u_0) \geq 0$ and $\int_\Omega |u_n|^2 \mathrm{d}g \to \int_\Omega |u_0|^2 \mathrm{d}g$, we have

$$\begin{aligned} c \leftarrow J_\lambda(u_n) &= \frac{1}{2}\int_\Omega |\nabla u_n|^2 \mathrm{d}g - \frac{\lambda}{2}\int_\Omega |u_n|^2 \mathrm{d}g - \frac{1}{Q^*}\int_\Omega |u_n|^{Q^*} \mathrm{d}g \\ &= \frac{1}{2}\int_\Omega |\nabla v_n|^2 \mathrm{d}g + \frac{1}{2}\int_\Omega |\nabla v_0|^2 \mathrm{d}g - \frac{\lambda}{2}\int_\Omega |u_0|^2 \mathrm{d}g - \frac{1}{Q^*}\int_\Omega |v_n|^{Q^*} \mathrm{d}g - \frac{1}{Q^*}\int_\Omega |v_0|^{Q^*} \mathrm{d}g + o_n(1) \\ &= J_\lambda(u_0) + \frac{1}{2}\int_\Omega |\nabla v_n|^2 \mathrm{d}g - \frac{1}{Q^*}\int_\Omega |v_0|^{Q^*} \mathrm{d}g + o_n(1) \\ &\geq \frac{1}{2}\int_\Omega |\nabla v_n|^2 \mathrm{d}g - \frac{1}{Q^*}\int_\Omega |v_0|^{Q^*} \mathrm{d}g + o_n(1). \end{aligned} \tag{22}$$

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Similarly, since $\langle J'_\lambda(u_0), u_0\rangle = 0$, we have

$$\begin{aligned}\langle J'_\lambda(u_n), u_n\rangle &= \int_\Omega |\nabla u_n|^2 \mathrm{d}g - \lambda\int_\Omega |u_n|^2 \mathrm{d}g - \int_\Omega |u|^{Q^*}\mathrm{d}g \\ &= \int_\Omega |\nabla v_n|^2 \mathrm{d}g + \int_\Omega |\nabla v_0|^2 \mathrm{d}g - \lambda\int_\Omega |u_0|^2 \mathrm{d}g - \int_\Omega |v_n|^{Q^*}\mathrm{d}g - \int_\Omega |v_0|^{Q^*}\mathrm{d}g + o_n(1) \\ &= \langle J'_\lambda(u_0), u_0\rangle + \int_\Omega |\nabla v_n|^2 \mathrm{d}g - \int_\Omega |v_0|^{Q^*}\mathrm{d}g + o_n(1) \\ &\geq \int_\Omega |\nabla v_n|^2 \mathrm{d}g - \int_\Omega |v_0|^{Q^*}\mathrm{d}g + o_n(1).\end{aligned} \tag{23}$$

From (23), we know there exits a non-negative constant b such that

$$\int_\Omega |\nabla v_n|^2 \mathrm{d}g \to b,$$

and

$$\int_\Omega |u|^{Q^*}\mathrm{d}g \to b,$$

as $n \to +\infty$. From (22) and (23), we obtain

$$c \geq (\frac{1}{2} - \frac{1}{Q^*})b. \tag{24}$$

On the other hand, By the definition of the best constant $S_{m,n}$ in (15), we have

$$S_{m,n}\Big(\int_\Omega |u|^{Q^*}\mathrm{d}g\Big)^{\frac{2}{Q^*}} \leq \Big(\int_\Omega |\nabla v_n|^2 \mathrm{d}g\Big),$$

which deduce that $b \geq S_{m,n} b^{\frac{2}{Q^*}}$.

Thus we have either $b = 0$ or $b \geq S_{m,n}^{\frac{Q}{2}}$. If $b = 0$, the proof is completed. Otherwise $b \geq S_{m,n}^{\frac{Q}{2}}$, then we obtain from (24) that

$$c \geq (\frac{1}{2} - \frac{1}{Q^*})b \geq (\frac{1}{2} - \frac{1}{Q^*})S_{m,n}^{\frac{Q}{2}},$$

which contradicts with the fact that $c < \frac{1}{Q}S_{m,n}^{\frac{Q}{2}}$. Therefore, $b = 0$ and $||u_n - u_0|| \to 0$ as $n \to 0$.
$\square$

To control $c^* < \frac{1}{Q}S_{m,n}^{\frac{Q}{2}}$ and $c^\star < \frac{1}{Q}S_{m,n}^{\frac{Q}{2}}$ by some λ, we recall some invariant properties of extremal function for the sharp constant $S_{m,n}$. Since the functional associated the Sobolev inequality is invariant under dilation, define

$$U_\varepsilon(\xi) := \varepsilon^{\frac{Q-2}{2}} U(\frac{\xi}{\varepsilon}) = \frac{\varepsilon^{\frac{Q-2}{2}}}{\Big((1+\frac{\varepsilon^2|x|^2}{4})^2 + \varepsilon^4|t|^2\Big)^{\frac{Q-2}{4}}},$$

we then have

$$||\nabla U_\varepsilon||_2^2 = |U_\varepsilon|_{Q^*}^{Q^*} = S_{m,n}^{\frac{Q}{2}}.$$

Observe that the critical point of c^* or $c^\star$ is vary closed to the extermal function, we take a cut-off method to estimate every term of $J_\lambda(u)$. Without loss of generality, we may assume that $0 \in \Omega$ and $B_R \subset \Omega \subset B_{2R}$ for some positive R. Let $\psi \in C_0$ such that

$$\psi(x) = \begin{cases} 1, & \text{if } x \in B_0(R), \\ 0, & \text{if } x \in G \setminus B_0(2R), \end{cases}$$

and $0 \le \psi(x) \le 1$, $|\nabla_G \psi(x)| \le C$ for all $x \in G$. Define

$$u_\varepsilon(x) := \psi(x) U_\varepsilon(x),$$

we then have the following estimate:

Lemma 6.

$$\int_\Omega |u_\varepsilon|^{Q^*} \mathrm{d}g = S_{m,n}^{\frac{Q}{2}} + O(\varepsilon^{-Q}), \tag{25}$$

$$\int_\Omega |\nabla_G u_\varepsilon|^2 \mathrm{d}g = S_{m,n}^{\frac{Q}{2}} + O(\varepsilon^{-Q+2}), \tag{26}$$

$$\int_\Omega |u_\varepsilon|^2 \mathrm{d}g \ge \begin{cases} c_1 \varepsilon^{-2} |\log \varepsilon| + O(\varepsilon^{-2}), & Q = 4, \\ c_1 \varepsilon^{-2} + O(\varepsilon^{-Q+2}), & Q \ge 6, \end{cases} \tag{27}$$

where c_1 is a positive constant.

Proof. (i) By the define of u_ε, we have

$$\begin{aligned} \int_\Omega |u_\varepsilon|^{Q^*} \mathrm{d}g &= \int_G |U_\varepsilon|^{Q^*} \mathrm{d}g - \int_{G \setminus B_0(R)} |U_\varepsilon|^{Q^*} \mathrm{d}g + \int_{B_0(2R) \setminus B_0(R)} |\psi U_\varepsilon|^{Q^*} \mathrm{d}g \\ &= S_{m,n}^{\frac{Q}{2}} - \int_{G \setminus B_0(R)} |U_\varepsilon|^{Q^*} \mathrm{d}g + \int_{B_0(2R) \setminus B_0(R)} |\psi U_\varepsilon|^{Q^*} \mathrm{d}g. \end{aligned}$$

For the second term, we have

$$\begin{aligned} \int_{G \setminus B_0(R)} |U_\varepsilon|^{Q^*} \mathrm{d}g &= \int_{G \setminus B_0(R)} \frac{\varepsilon^Q}{\left((1 + \frac{\varepsilon^2 |x|^2}{4})^2 + \varepsilon^4 t^2 \right)^{\frac{Q}{2}}} \mathrm{d}g \\ &\le C \int_{G \setminus B_0(R)} \frac{\varepsilon^{-Q}}{\left(\frac{|x|^2}{4} + t^2 \right)^{\frac{Q}{2}}} \mathrm{d}g \\ &= C \int_{G \setminus B_0(R)} \frac{\varepsilon^{-Q}}{|\xi|^{2Q}} \mathrm{d}g = O(\varepsilon^{-Q}) \end{aligned}$$

For the third term, we have

$$\int_{B_0(2R) \setminus B_0(R)} |\psi U_\varepsilon|^{Q^*} \mathrm{d}g \le C \int_{B_0(2R) \setminus B_0(R)} \varepsilon^{-Q} \mathrm{d}g \le O(\varepsilon^{-Q}).$$

Hence, (25) holds.

(ii) Since U_ϵ satisfies the Yamabe equation (16), we have

$$\begin{aligned} \int_\Omega |\nabla_G u_\varepsilon|^2 \mathrm{d}g &\le \int_G |\nabla_G u_\varepsilon|^2 \mathrm{d}g \\ &= \int_G |\nabla_G U_\epsilon|^2 \psi^2 \mathrm{d}g + \int_G |\nabla_G \psi|^2 U_\varepsilon^2 \mathrm{d}g + 2 \int_G \nabla_G \psi \cdot \nabla_G U_\varepsilon U_\varepsilon \psi \mathrm{d}g \\ &= \int_G \nabla_G U_\varepsilon \cdot \nabla_G (U_\varepsilon \psi^2) \mathrm{d}g + \int_G |\nabla_G \psi|^2 U_\varepsilon^2 \mathrm{d}g \\ &= \int_G |U_\varepsilon|^{Q^*} \psi^2 \mathrm{d}g + \int_G |\nabla_G \psi|^2 U_\varepsilon^2 \mathrm{d}g \\ &= \int_G |U_\varepsilon|^{Q^*} \mathrm{d}g - \int_G |U_\varepsilon|^{Q^*} (1 - \psi^2) \mathrm{d}g + \int_G |\nabla_G \psi|^2 U_\varepsilon^2 \mathrm{d}g \\ &\le S_{m,n}^{\frac{Q}{2}} + \int_G |U_\varepsilon|^{Q^*} (1 - \psi^2) \mathrm{d}g + \int_G |\nabla_G \psi|^2 U_\varepsilon^2 \mathrm{d}g. \end{aligned}$$

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For the second term in the last inequality, we have

$$\int_G |U_\varepsilon|^{Q^*}(1-\psi^2)\mathrm{d}g \le C\int_{G\setminus B_0(R)} |U_\varepsilon|^{Q^*}\mathrm{d}g \le O(\varepsilon^{-Q}).$$

For the third term in the last inequality, we have

$$\int_G |\nabla_G\psi|^2 U_\varepsilon^2\mathrm{d}g \le C\int_{B_0(2R)\setminus B_0(R)} |U_\varepsilon|^2\mathrm{d}g \le O(\varepsilon^{-Q+2}).$$

Hence, (26) holds.

(iii) A direct calculation shows that

$$\begin{aligned}
\int_\Omega |u_\varepsilon|^2\mathrm{d}g &= C\int_{B_0(2R)} \frac{\varepsilon^{Q-2}|\psi|^2}{\left((1+\frac{\varepsilon^2|x|^2}{4})^2+\varepsilon^4|t|^2\right)^{\frac{Q-2}{2}}}\mathrm{d}g \\
&\ge C\int_{B_0(R)} \frac{\varepsilon^{Q-2}}{\left((1+\frac{\varepsilon^2|x|^2}{4})^2+\varepsilon^4|t|^2\right)^{\frac{Q-2}{2}}}\mathrm{d}g \\
&= C\int_{B_0(\varepsilon R)} \frac{\varepsilon^{-2}}{\left((1+\frac{|x|^2}{4})^2+|t|^2\right)^{\frac{Q-2}{2}}}\mathrm{d}g \\
&\ge C\varepsilon^{-2}\left(\int_{B_0(1)} 1\mathrm{d}g + \int_{B_\varepsilon(R)\setminus B_0(1)} \frac{1}{|\xi|^{2(Q-2)}}\mathrm{d}g\right) \\
&\ge C\varepsilon^{-2}\left(C_1+\int_1^{\varepsilon R}\rho^{-Q+3}\mathrm{d}\rho\right).
\end{aligned}$$

If $Q>4$, then

$$\int_\Omega |u_\varepsilon|^2\mathrm{d}g \ge C\varepsilon^{-2}\left(C_1+\int_1^{\varepsilon R}\rho^{-Q+3}\mathrm{d}\rho\right) = C\varepsilon^{-2}+O(\varepsilon^{-Q+2}),$$

if $Q=4$, then

$$\int_\Omega |u_\varepsilon|^2\mathrm{d}g \ge C\varepsilon^{-2}\left(C_1+\int_1^{\varepsilon R}\rho^{-1}\mathrm{d}\rho\right) = C\varepsilon^{-2}+C\varepsilon^{-2}|\log\varepsilon|.$$

This implies that (27) holds. □

Based on the above estimate, we have the following inequality holds.

Lemma 7. *For any $\lambda>0$, there exists $u_\varepsilon \in W^{1,2}_{0,G}(\Omega)\setminus\{0\}$ such that*

$$\frac{||u_\varepsilon||^2-\lambda|u_\varepsilon|_2^2}{|u|_{Q^*}^2} < S_{m,n}.$$

Proof. For the case $Q>4$ and ε large enough, we have

$$\frac{||u_\varepsilon||^2-\lambda|u_\varepsilon|_2^2}{|u|_{Q^*}^2} \le \frac{S_{m,n}^{\frac{Q}{2}}+O(\varepsilon^{-Q+2})-\lambda(\varepsilon^{-2}+O(\varepsilon^{-Q+2}))}{\left(S_{m,n}^{\frac{Q}{2}}+O(\varepsilon^{-Q})\right)^{\frac{Q-2}{Q}}} < S_{m,n}.$$

For the case $Q=4$ and ε large enough, we have

$$\frac{||u_\varepsilon||^2-\lambda|u_\varepsilon|_2^2}{|u|_{Q^*}^2} \le \frac{S_{m,n}^2+O(\varepsilon^{-2})-\lambda(\varepsilon^{-2}|\log\varepsilon|+O(\varepsilon^{-2}))}{(S_{m,n}^2+O(\varepsilon^{-4}))^{\frac{1}{2}}} < S_{m,n}.$$

 doi:10.20944/preprints202406.0238.v1

□

4. Proof of the Main Theorem

4.1. Case $\lambda \in (0, \lambda_1)$

Proof of Theorem 1 (i). Follow the Mountain-Pass Theorem, it suffices to proof that the c^* defined in Proposition 1 satisfies that $c^* < \frac{1}{Q} S_{m,n}^{\frac{Q}{2}}$ for any $\lambda \in (0, \lambda_1)$, since we has checked the mountain pass structure and $(PS)_{c^*}$ condition in Lemma 1 and Lemma 5 respectively. Indeed, taking the half-line $\{tu_\varepsilon : t \geq 0\}$ as a test curve for the MP level, we have that

$$c_* \leq \max_{t \geq 0} J_\lambda(tu_\varepsilon). \tag{28}$$

On the other hand, since $\max\limits_{t \geq 0} J_\lambda(tu_\varepsilon)$ is achieved at

$$\tau = \left(\frac{||u_\varepsilon||_2^2 - \lambda |u_\varepsilon|_2^2}{|u_\varepsilon|_{Q^*}^{Q^*}} \right)^{\frac{1}{Q^*-2}}, \tag{29}$$

then

$$J_\lambda(\tau u_\varepsilon) = \frac{1}{Q} \left(\frac{||u_\varepsilon||_2^2 - \lambda |u_\varepsilon|_2^2}{|u_\varepsilon|_{Q^*}^2} \right)^{\frac{Q}{2}}. \tag{30}$$

Using the fact showed in Lemma 7, we have

$$c_* \leq \max_{t \geq 0} J_\lambda(tu_\varepsilon) = J_\lambda(\tau u_\varepsilon) = \frac{1}{Q} \left(\frac{||u_\varepsilon||_2^2 - \lambda |u_\varepsilon|_2^2}{|u_\varepsilon|_{Q^*}^2} \right)^{\frac{Q}{2}} = \frac{1}{Q} S_{m,n}^{\frac{Q}{2}}, \tag{31}$$

this completes the proof. □

4.2. Case $\lambda \in (\lambda_j, \lambda_{j+1})$

In this case, $J_\lambda(u)$ perform a strong indefiniteness structure. To prove the existence of the critical for $J_\lambda(u)$, we need to use the Link argument. Define

$$\mathrm{G}_{j,\varepsilon} := \mathrm{span}\{e_1.e_2, \cdots, u_\varepsilon\} \tag{32}$$

and set

$$m_{j,\varepsilon} := \max_{u \in \mathrm{G}_{j,\varepsilon}, |u|_{Q^*}=1} \left(\int_\Omega |\nabla_G u|^2 \mathrm{d}g - \lambda \int_\Omega |u|^2 \mathrm{d}g \right). \tag{33}$$

Lemma 8. *Let $\lambda \in [\lambda_j, \lambda_{j+1})$ for some $j \in \mathbb{N}$, then*

(i) $m_{j,\varepsilon}$ is achieved at some $u_m \in \mathbb{G}_{j,\varepsilon}$ and u_m can be written as $u_m = v + tu_\varepsilon$ with $v \in \mathbb{Y}_j$ and $t \geq 0$.

(ii) The following estimate holds true:

$$m_{j,\varepsilon} \leq \begin{cases} (\lambda_j - \lambda)|v|_2^2 & \text{if } t = 0, \\ (\lambda_j - \lambda)|v|_2^2 + A_\varepsilon(1 + |z|_\infty O(\varepsilon^{-\frac{Q-2}{2}})) + O(\varepsilon^{-\frac{Q-2}{2}})|z|_\infty & \text{if } t > 0, \end{cases} \tag{34}$$

as $\varepsilon \longrightarrow +\infty$, where $A_\varepsilon = \frac{||u_\varepsilon||^2 - \lambda |u_\varepsilon|_2^2}{|u_\varepsilon|_{Q^*}^2}$.

(iii) $m_{j,\varepsilon} < S$.

Proof. (1) ince $\mathbb{G}_{\daleth,\nsubseteq}$ is a finite dimensional space, there exists $0 \neq u_m \in \mathbb{G}_{j,\varepsilon}$ such that

$$m_{j,\varepsilon} = ||u_m||^2 - \lambda |u_m|_2^2, \text{ and } |u|_{Q^*} = 1. \tag{35}$$

From the definition of $G_{j,\varepsilon}$ it is clear that $u_m = z + tu_\varepsilon$ where $z \in \mathbb{Y}_j$ and $t \in \mathbb{R}$. We can suppose that $t \geq 0$ otherwise we can replace u_m with $-u_m$.

(ii) If $t = 0$, then $u_m = v \in \mathbb{Y}_j$ and

$$m_{j,\varepsilon} = ||u_m||^2 - \lambda |u_m|_2^2 = |\nabla v|_2^2 - \lambda |v|_2^2 \leq (\lambda_j - \lambda)|z|_2^2. \tag{36}$$

If $t > 0$, then for any $u_m = z + tu_\varepsilon$, we have

$$\begin{aligned} m_{j,\varepsilon} &= ||z + tu_\varepsilon||^2 - \lambda |z + tu_\varepsilon|_2^2 \\ &\leq (\lambda_j - \lambda)|z|_2^2 + A_\varepsilon \left(\int_\Omega |tu_\varepsilon|^{Q^*} \mathrm{d}g \right)^2 + C|z|_\infty |u_\varepsilon|_2. \end{aligned} \tag{37}$$

On the other hand, since $|u_m|_{Q^*} = 1$, we have

$$\begin{aligned} 1 &= \int_\Omega |u_m|^{Q^*} \mathrm{d}g \\ &= \int_\Omega |z + tu_\varepsilon|^{Q^*} \mathrm{d}g \\ &\geq \int_\Omega |z|^{Q^*} \mathrm{d}g + C \int_\Omega |tu_\varepsilon|^{Q^*-1} z \mathrm{d}g + \int_\Omega |tu_\varepsilon|^{Q^*} \mathrm{d}g \\ &\geq \int_\Omega |tu_\varepsilon|^{Q^*} \mathrm{d}g - \int_\Omega |tu_\varepsilon|^{Q^*-1} z \mathrm{d}g. \end{aligned} \tag{38}$$

That is

$$\int_\Omega |tu_\varepsilon|^{Q^*} \mathrm{d}g \leq 1 + |z|_\infty \int_\Omega |tu_\varepsilon|^{Q^*-1} \mathrm{d}g \leq 1 + t^{Q^*-1} O(\varepsilon^{-\frac{Q-2}{2}}). \tag{39}$$

Connected (37), we have

$$m_{j,\varepsilon} \leq (\lambda_j - \lambda)|z|_2^2 + A_\varepsilon (1 + |z|_\infty O(\varepsilon^{-\frac{Q-2}{2}}) + C|z|_\infty O(\varepsilon^{-\frac{Q-2}{2}}). \tag{40}$$

(iii) If $t = 0$, we have $m_{j,\varepsilon} = (\lambda_j - \lambda)|v|_2^2 < S_{m,n}$. If $t > 0$, then by the estimate of A_ε in Lemma 8, we also have $m_{j,\varepsilon} < S_{m,n}$. □

Proof of Theorem 1 (ii). By the Link argument, we also just to show that $c^\star$ defined in Proposition 2 satisfies that $c^\star < \frac{1}{Q} S_{m,n}^{\frac{Q}{2}}$ for every $\lambda \in [\lambda_j, \lambda_{j+1})$, since we have verified that $J_\lambda(u)$ satisfying the the link structure and $(PS)_{c^\star}$ condition in Lemma 2 and Lemma 5. Actually, by the definition of $c^\star$, see (18), for any $\gamma \in \Gamma$ we have $c^\star \leq \max\limits_{u \in V} J_\lambda(\gamma(u))$. In particular, if we take $\gamma = id$ on $\overline{V}$ then $c^\star \leq \max\limits_{u \in V} J_\lambda(u) \max\limits_{u \in G_{j,\varepsilon}} J_\lambda(u)$. Since $G_{j,\varepsilon}$ is a linear space, we have

$$\max_{u \in \mathbb{G}_{j,\varepsilon}} J_\lambda(u) = \max_{u \in \mathbb{G}_{j,\varepsilon}, t \neq 0} J_\lambda(\frac{|t|u}{|t|}) = \max_{u \in \mathbb{G}_{j,\varepsilon}, t > 0} J_\lambda(tu) \leq \max_{u \in \mathbb{G}_{j,\varepsilon}, t \geq 0} J_\lambda(u). \tag{41}$$

On the other hand, by a direct calculation, we have

$$\max_{u \in \mathbb{G}_{j,\varepsilon}, t \geq 0} J_\lambda(u) = \frac{1}{Q} \left(\frac{||u||^2 - \lambda |u|_2^2}{|u|^{Q^*}} \right)^{\frac{Q}{2}} \tag{42}$$

Then by Lemma 8 we know that $c^\star \leq \max\limits_{u \in \mathbb{G}_{j,\varepsilon}, t \geq 0} J_\lambda(u) < \frac{1}{Q} S_{m,n}^{\frac{Q}{2}}$. This completes the proof. □

Author Contributions: The authors declare that they contribute to the paper equally, they all joined in the work of analysis, calculation and organizing the paper.

Funding: The research is supported by the National Natural Science Foundation of China (No. 12061010), Science and Technology Planning Project of Jiangxi Province (No. GJJ2201204).

Data Availability Statement: All of the data for the research is included in the manuscript.

Conflicts of Interest: The authors declare that they have no conflict of interest between each other.

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