Gravitation with Cosmological Term, Expansion of the Universe as Uniform Acceleration in Clifford Coordinates

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Abstract

The recently proposed model of the "unified two-dimensional spacetime"[18] shows substantial conformity to Clifford algebras $Cl_{3,0}$ and $Cl_{0,3}$. As shown, such two-dimensional spacetime of time and volume corresponds to two Clifford coordinates given by a *center* of the respective algebra. In the application to comoving frame, the model results in the exact form of the scale factor of the FLRW metric of modern cosmology. This paper formulates and formalizes the approach using Clifford algebras. The study concludes that the Clifford algebra of space (APS) $Cl_{3,0}$ has an intrinsic correspondence with the anti-de Sitter (AdS) flat universe and the negative cosmological term that results in an oscillating model of the universe. The approach with anti-Euclidean Clifford algebra $Cl_{0,3}$ leads to the de Sitter model with a positive cosmological term. As reviewed, the Clifford algebra has isomorphic algebras and Lie groups corresponding to different from Minkowski spacetime, such as $SO(4)$; therefore, further study may be required regarding their relation to gravitation.

Keywords: Clifford algebras; Cl(3,0), Cl(0,3); two dimensional spacetime; time-volume coordinates; constant uniform acceleration; Rindler coordinates; FLRW metric; scale factor; AdS and de Sitter models.

Clifford algebras [1,3,4,17,21] present a powerful tool that is uniquely useful when applied to physics as it reflects the intrinsic property of spacetime. The author's recent work introduced the unified two-dimensional spacetime model by conjecture that it consists of time and spatial-volume coordinates [15]. The model considers the classical gravity as the uniform acceleration $\alpha = 3H_0c$ in such coordinates, and the addition of the cosmological term appears as the relativistic effect attributed to the Rindler coordinate transformations. However, this approach was based on intuition and lacked mathematical formality. This paper demonstrates the fundamental significance of the approach based on the symmetry of underlying Clifford algebras.

Note: The Hubble constant $3H_0^2 = c^2\Lambda$ is a *constant*. Denoted Clifford algebras $Cl_{p,q}$ and the quaternions $\mathbb{H}$ are over the reals.

1. Clifford Algebra of Space $Cl_{3,0}$

The Pauli algebra $Cl_{3,0}(\mathbb{R})$ describes the structure of Euclidean $\mathbb{R}^3$ space. $Cl_{3,0}$ is spanned by eight multivectors x_{ij} with the basis that can be given by Pauli matrices as follows

$$\begin{bmatrix} e_0 & e_1 & e_2 & e_3 \\ e_{123} & e_{23} & e_{13} & e_{12} \end{bmatrix} \quad \begin{bmatrix} \sigma_0 & \sigma_1 & \sigma_2 & \sigma_3 \\ i\sigma_0 & i\sigma_1 & i\sigma_2 & i\sigma_3 \end{bmatrix}$$

where e_0 is scalar, $(e_1,\ e_2,\ e_3)$ is vector, (e_{23}, e_{13}, e_{12}) is bivector, and e_{123} is a volume element (tri-vector or multivector of grade 3). The algebra is isomorphic to two-by-two matrices with complex entries $Cl_{3,0} \cong Mat(2,\mathbb{C})$ and has two subalgebras: the even subalgebra $Cl^0_{3,0} \cong \mathbb{H}$ of quaternions, and the center $Cen(Cl_{3,0}) \cong \mathbb{C}$ therefore

$$Cl_{3,0} \cong \mathbb{C} \otimes \mathbb{H} \tag{1}$$

where a quaternion corresponds to multivectors of grades 0 and 2 as $q = x_0 + ix_{23} + jx_{13} + kx_{12}$, $q \in \mathbb{H}$. The algebra $Cl_{3,0}$ has a unique property. As an algebra of the structure of Euclidean $\mathbb{R}^3$ space, it also describes four-dimensional Minkowski spacetime. Such correspondence is utilized in physics as the algebra of physical space (APS), and it can be understood in two ways. The first is that $Cl_{3,0}$ is isomorphic to even subalgebras $Cl^0_{3,1} \cong Cl^0_{1,3}$ [1,21]. The second is that $Mat(2,\mathbb{C})$ can be normalized to $SL(2,\mathbb{C})$ which is the classical spin homomorphism and double-cover of the proper Lorentz group $SO(1,3)^+$, and it is also homomorphic with $SO(3,\mathbb{C})$, which is isomorphic to complexified quaternions $\mathbb{H}(\mathbb{C})$. The concordance of complexified quaternions to the special relativity and Maxwell electromagnetism (via complex 3-vector $\mathbf{F} = \mathbf{E} + i\mathbf{B}$) has been known since the works of Silberstein [27,28]; see also the modern interpretations [7,8].

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Along with a variety of morphisms, the primary interest for the topic is the *paravector* representation of $Cl_{3,0}$. Because the basis of $Cl_{3,0}$ precisely corresponds to two copies of 4-dimensional vector spaces, all eight orthogonal coordinates can be split into two sets of Minkowski and anti-Minkowski spacetimes, spanned by two 4-vectors given by upper and lower raw as follows

$$\left[\begin{array}{c:ccc} x_0 & x_1 & x_2 & x_3 \\ \hdashline x_{123} & x_{23} & x_{13} & x_{12} \end{array}\right] \tag{2}$$

where $x_\mu = (x_0, x_1, x_2, x_3)$ is the "usual" Minkowski four-vector (x_0 is time) or paravector in the algebra of physical space (APS), and $\overline{x}_\mu = (\overline{x}_0, \overline{x}_1, \overline{x}_2, \overline{x}_3) = (x_{123}, x_{23}, x_{13}, x_{12})$ is its "anti-Minkowskian" counterpart. Notably, such an anti-Minkowski 4-vector $\overline{x}$ is built only from the spatial components of the the 4-vector x. Using the basis of Pauli matrices (and defining the scalar product as $\frac{1}{2}\mathrm{Tr}(\sigma_i\sigma_j)$) we obtain the canonical representation of 4-vectors by the matrix

$$g = x^k \sigma_k \quad \overline{g} = \overline{x}^k\, i\sigma_k \quad k = 0,1,2,3 \quad g, \overline{g} \in SL(2, \mathbb{C}) \tag{3}$$

with the Einstein convention for summation. Two such vector spaces $\mathbb{R}^{1,3}$ have quadratic forms with reverse signatures

$$\begin{aligned} det(g) &= \ \ x_0^2 - x_1^2 - x_2^2 - x_3^2 \\ det(\overline{g}) &= -\overline{x}_0^2 + \overline{x}_1^2 + \overline{x}_2^2 + \overline{x}_3^2 \end{aligned}$$

The algebra $Cl_{3,0}$ treated in such a way obtains the assigned scalar value to the time coordinate of Minkowski spacetime, where the volume element represents its "anti-Minkowskian" counterpart[2].

Another essential feature of this algebra is that it has the isomorphic center $Cen(Cl_{3,0}) \cong \mathbb{C}$. By definition, the center of Clifford algebra consists of elements that commute with all elements of algebra. In case of $Cl_{3,0}$ these two elements are given by two basis elements of grade 0 and 3 i.e by e_0 and e_{123} [1, 17], explicitly

$$e_0 = \sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad e_{123} = i\sigma_0 = \begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix}$$

Hence, in the case of paravector representation the center of $Cl_{3,0}$ consists of two coordinates given by *time* and *volume* elements that form subalgebra $\mathbb{C}$ of $Cl_{3,0}$. To maintain the accordance with the previous work[3] we denote these coordinates as $t = x_0$, $\eta = \overline{x}_0 = x_{123}$. The two-dimensional spacetime of $Cen(Cl_{3,0})$ is spanned by a vector $e_0 t + e_{123}\eta$ that represents a complex number u and its quadratic form are respectively

$$u = t + i\eta \in \mathbb{C} \quad |u|^2 = uu^* = t^2 + \eta^2 \quad (t, \eta \in \mathbb{R}) \tag{4}$$

The center of Clifford algebra $Cen(Cl_{3,0})$ is the subalgebra $\mathbb{C}$, and is a two-dimensional spacetime of plane $\mathbb{R}^2$.

Hence, this allows the consideration of another split of eight coordinates where two are given by $Cen(Cl_{3,0})$, and another six are represented by the complexified spatial vector or one 6-vector given by the six components at the right side of (2).

2. Physical Relativity and Uniform Acceleration in Two Dimensions of $Cen(Cl_{3,0})$

Geometry in two dimensions of $\mathbb{R}^2$ is trivial from a mathematical point of view as it represents the Euclidean plane. However, because the coordinate x_0 is considered the time coordinate, the prime interest of physics in this case is the relativity of motion in frames of reference in such spacetime.

We may also postulate the interval's invariance for the coordinate transformations between an observer A located at the center of the spherical volume at rest and the frame B with coordinate $\eta(t)$ of the spherically expanding volume moving with velocity $v = \frac{dV}{dt}$. The essence of the relativity in such coordinates will become more evident in the next section, where we discuss the FLRW metric, which is the metric in the comoving frame of reference.

Because the plane has the quadratic form (4) with the signature (+,+), hence to obtain new rules for the coordinate transformations, it is necessary to substitute all hyperbolic functions involved in Lorentz transformations with their trigonometric counterparts in $SO(2)$. For instance, the coordinate velocity and two-velocity for a moving frame become

$$v = \frac{d\eta}{dt} = c\tan(\beta) \qquad v^i = (c\cos(\beta), c\sin(\beta))$$

[2] *Note.* This representation is important in physics because the upper raw corresponds to the Minkowski paravector that reflects a real property of physical spacetime. However, mathematically this is just one form of the complexification of the group $SU(2)$ resulting in $SL(2, \mathbb{C})$. Alternatively for example swapping in the basis (1) $e_2 = \sigma_2$ with $e_{13} = i\sigma_2$ shows that complexification of $SL(2, \mathbb{R}) \cong SU(1, 1)$ also results in $SL(2, \mathbb{C})$.

[3] "The unified two-dimensional spacetime" [15].

 doi:10.20944/preprints202101.0544.v1

It can be noted that the coordinate velocity can be infinite[4], however, the proper velocity (measured with the observer's clock) is limited by unity.

The Rindler or Kottler-Møller coordinates are essential type of coordinate transformations in special relativity [10, 20, 22]. These are the coordinate transformations in the case of the uniformly accelerated frame and are derived by integrating the Lorentz transformations with initial conditions using the evident relation $\beta = \alpha\tau$ where β is the rapidity, α is the *proper* uniform acceleration, and τ is the observer's proper time. In the spacetime of $Cen(Cl_{3,0})$ these values are given by the trigonometric counterparts. The Rindler coordinate transformation from proper to coordinate time is

$$t = \frac{c}{\alpha}\sin\left(\frac{\alpha\tau}{c}\right) \tag{5}$$

where the initial synchronization of the clocks at the beginning ($t = \tau = 0$ in $\eta = 0$) is assumed. The coordinate distance from the origin of motion in terms of proper time is

$$\eta = \frac{c^2}{\alpha}\left(1 - \cos\left(\frac{\alpha\tau}{c}\right)\right) = 2\eta_0 \sin^2\left(\frac{\alpha\tau}{2c}\right) \quad \eta_0 = \frac{c^2}{\alpha} \tag{6}$$

where the pre-factor η_0 is the Rindler horizon parameter, which has an important role in the relativity of uniform acceleration. Notably, the expressions demonstrate the significance of the property of the Rindler transformations in the spacetime of $Cen(Cl_{3,0})$ which is the equivalence of the uniform acceleration to the *harmonic oscillation* of volume coordinate[5].

3. The Scale Factor and AdS Model for $Cl_{3,0}$

In the case of spherical symmetry, the coordinate $\eta = x_{123}$ in Clifford algebra $Cl_{3,0}$ clearly represents a spherical volume. Model [15] provided explicit form for the relation that preserves physical dimensionality $\eta = VA^{-1}$. The factor A is a constant having the physical dimension of the area and has particular significance[6]; hence, η has the dimension of length. To minimize the appeal to the model, we write $\eta \to V$.

The Friedmann–Lemaître–Robertson–Walker (FLRW) is the metric for a comoving frame with time-dependent spatial component given by the scale factor $a(\tau)$. In the flat case this is $ds^2 = -d\tau^2 + a(\tau)^2\left(dR^2 + R^2 d\Omega\right)$. The tetrad transformation from the Minkowski to the FLRW metric is the diagonal matrix

$$h_\mu{}^a = \mathrm{diag}\left(1, a, Ra, Ra\sin(\theta)\right) \quad \text{with} \quad r = Ra \quad \text{and} \quad g_{\mu\nu} = h_\mu{}^a h_\nu{}^b n_{ab} \tag{7}$$

where R is the comoving (fixed) distance, and r is the coordinate distance [13]. In fact, the tetrad is simply a Jacobian matrix whose determinant defines a volume element $d\eta = dV = a^3\, 4\pi R^2 dR$ therefore

$$a = \left(\frac{V}{V_R}\right)^{1/3} \quad \text{where} \quad V_R = \frac{4\pi}{3}R^3 \quad \text{and} \quad V = \frac{4\pi}{3}r^3 \tag{8}$$

The coordinate η represents the value of the spherical volume V, and we can also indicate the point η_R that corresponds to such fixed volume $V_R \to \eta_R$ hence

$$a = \left(\frac{\eta}{\eta_R}\right)^{1/3} \tag{9}$$

η_R can be expressed via the Rindler horizon in the model[7] as

$$\eta_R = \eta_0\left(\frac{9k^2}{2\beta^2}\right) \tag{10}$$

where k and β are dimensionless constants. Therefore

$$a = \left(\chi^2\frac{\eta}{2\eta_0}\right)^{1/3} \quad \text{where} \quad \chi = \frac{2\beta}{3k} \tag{11}$$

The substitution of η from (6) results in the scale factor of the FLRW metric

$$a(\tau) = \left[\chi\sin\left(\frac{\alpha\tau}{2c}\right)\right]^{2/3} \qquad \alpha = \left(\chi^{-1}\right)3H_0 c \tag{12}$$

[4]The coordinate η denotes spatial volume. Even though the case may correspond to super-luminal universe expansion in the coordinate distance. As discussed later, we can only measure our proper time in a local frame.

[5]Due to the presence of the imaginary unit i in $Cen(Cl_{3,0})$.

[6]With regard to the Plank area, see expr. (14) of the cited work.

[7]The exact correspondence is given by expression(8), so $V_R = kV_m$ hence $\eta_R = k\eta_m$, then using (23),(31) in [13], see also [15].

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The cited work has assumed that $\chi = 1$. In the present study, this condition is relaxed. The model has undefined constant dimensionless factors β, k with specific physical significances, and their ratio appears to be related to Ω_m and Ω_Λ of modern cosmology as

$$\Omega_m = 1 \quad \Omega_\Lambda = \chi^2 \tag{13}$$

Since the form of the scale factor depends only on the fundamental constants and proper time, it can be considered the *intrinsic property of space*.

To demonstrate that such a form of the scale factor corresponds to the Schwarzschild-AdS (SAdS) case, we briefly revisit the model neglecting its constant prefactors and simply denoting them as $_1$. Consider an expanding spherically symmetric volume given by such a scale factor and its derivative with proper time as

$$V(\tau) = c_1\,\eta(\tau) = k_1(1-\cos(\alpha\tau)) \quad \dot{V} = \frac{dV}{d\tau} = k_1\alpha\,\sin(\alpha\tau)) \tag{14}$$

Now we may express the derivative $\dot{V}$ in terms of r :

$$\dot{V}(r) = \pm k_1\alpha\left(1-\left(1-\frac{V}{k_1}\right)^2\right)^{1/2} = \pm k_1\alpha\left(2\frac{V}{k_1}-\frac{V^2}{k_1^2}\right)^{1/2} = \pm\left(2a_1r^3-d_1r^6\right)^{1/2} \tag{15}$$

This allows us to obtain the recession or free-fall velocity at distance r from the center of the sphere using the apparent relation for spherical volume element expansion

$$v(r) = \frac{\dot{V}}{4\pi r^2} = \pm\left(\frac{2\alpha_1}{r}-\beta_1 r^2\right)^{1/2} \qquad \alpha_1 > 0 \tag{16}$$

Further, the gravitational potential $\phi = \frac{1}{2}v^2$ corresponds to the SAdS gravity with negative cosmological term, and α_1 converges to the central mass. The corresponding explicit form given by expression(38) of model [13][8] is

$$v(r) = \pm\left(\frac{2Gm}{r}-\chi^2\,\frac{\Lambda}{3}r^2\right)^{1/2} \qquad 3H_0^2 = c^2\Lambda > 0 \tag{17}$$

The expression is the equivalent of $v(r) = \dot{a}R$, where $a(\tau)$ is (12), and $\dot{a}$ is the derivative with respect to proper time. With the use of the tetrad

$$f_\mu{}^a = \begin{pmatrix} 1 & 0 & 0 & 0 \\ \dot{a}R & a^{-1} & 0 & 0 \\ 0 & 0 & a^{-1} & 0 \\ 0 & 0 & 0 & a^{-1} \end{pmatrix} \tag{18}$$

the FLRW metric is transformed into the Gulfstrand-Painlevé (GP) coordinates [13]. The diagonalization of the GP metric tensor leads to its static form [2, 12], which in the case of the obtained velocity (17) is the Schwarzschild-AdS metric.

Notably, if one takes the derivative in (15) but with coordinate time instead, the resulting expression for $v(r)$ has the same form but the cosmological term disappears. This implies that the expansion can be measured only in the comoving frame, that is by an observer attached to the comoving point of the expanding universe, which becomes apparent, taking into account the transformation of the time coordinate(5). The approach suggests that we only evaluate and *measure our proper time* attached to our local frame of reference, that is the point of the expanding universe. Thus, the coordinate time (5) becomes somewhat abstract and not directly related to our observations and measurements.

Thereby, Clifford algebra $Cl_{3,0}$ corresponds to the Schwarzschild-AdS gravity with the negative cosmological term and naturally requires the oscillatory model of periodic universe expansion. For instance, from (12) the half-period (from zero to zero) of the oscillation is

$$T = \frac{4\pi c}{\alpha} = \frac{4\pi}{3H_0}\chi \tag{19}$$

measured in proper time. The amplitude of such oscillation can also be obtained using the coordinate distance $r = Ra$ and the explicit form for $\eta_R \to R$.

[8]The referenced expression *ad hoc* uses the positive positive cosmological term case, which is reviewed in the next section.

4. Clifford Algebra of Anti-space $Cl_{0,3}$

The Clifford algebra of "anti-Euclidean" space $Cl_{0,3}$ is also an eight-dimensional algebra, and is *not isomorphic* to $Cl_{3,0}$ (APS). The quadratic form is negatively defined. The algebra is isomorphic to split-biquaternions and is also an algebra of alternions $\mathbb{A}_4$ [24, 25]. The basis can be given using the quaternion units [17]

$$\begin{bmatrix} e_0 & e_1 & e_2 & e_3 \\ e_{123} & e_{23} & e_{13} & e_{12} \end{bmatrix} \qquad \begin{bmatrix} (1,\ 1) & (i,-i) & (j,-j) & (k,-k) \\ (1,-1) & (i,\ i) & (j,\ j) & (k,\ k) \end{bmatrix} \tag{20}$$

where (a, b) denotes the diagonal two-by-two matrices. The basis exhibits the properties of $Cl_{0,3}$

$$e_1^2 = e_2^2 = e_3^2 = -1 \quad e_1e_2 = -e_2e_1 \quad e_1e_3 = -e_3e_1 \quad e_2e_3 = -e_3e_2 \quad e_1e_2e_3 = -1$$

The upper raw represents the basis for the quaternions $\mathbb{H}$, and the lower is its "split-complexification" thereby

$$Cl_{0,3} \cong \mathbb{D} \otimes \mathbb{H} \qquad Cl_{0,3} \cong \mathbb{H} \oplus \mathbb{H} \tag{21}$$

Contrary to the previous case, there is no trivial correspondence to Minkowski spacetime since quaternions have isometry to the Euclidean space[9]. Considering (20) as the basis for a vector space as in Section 1 (3) it is possible to see that quadratic forms for both quaternions (upper and lower raw) are

$$\begin{aligned} det(g) &= \ \ x_0^2 + x_1^2 + x_2^2 + x_3^2 \\ det(\overline{g}) &= -x_{123}^2 - x_{23}^2 - x_{13}^2 - x_{12}^2 \end{aligned}$$

However, both quaternions of $Cl_{0,3}$ have quadratic forms with reverse signs. Hence, interchanging the basis (20) one may obtain *paravector* space of $Cl_{0,3}$ with the quadratic forms as follows

$$\begin{aligned} \mathbf{x}^2 &= \ \ x_0^2 - x_{23}^2 - x_{13}^2 - x_{12}^2 \\ \overline{\mathbf{x}}^2 &= -x_{123}^2 + x_1^2 + x_2^2 + x_3^2 \end{aligned} \tag{22}$$

This implies (for the upper raw, for instance) that a Minkowski four-vector can be built from the scalar and three bi-vectors of $Cl_{0,3}$. However, we must note that such accordance of paravectors to Minkowski four-vectors can be made only in terms of vector space, not in terms of algebra. Using such paravector representation we also can map x_0 to the time coordinate and x_{123} to a volume in the previous case of APS.

The center of $Cl_{0,3}$ is $\mathbb{D}$ (a split-complex number is $c = a + \mathrm{j}b \in \mathbb{D}$: $\mathrm{j}^2 = 1, \mathrm{j} \neq \pm 1, a, b \in \mathbb{R}$. $\mathbb{D} = \mathbb{R} \oplus \mathbb{R}$, and it is algebra of $Cl_{1,0}$.) yielding two dimensional (Minkowski) spacetime $\mathbb{R}^{1,1}$. Such spacetime is spanned by two-vector represented by split-complex number

$$u = t + j\eta \in \mathbb{D} \quad |u|^2 = uu^* = t^2 - \eta^2 \quad (t, \eta \in \mathbb{R}) \tag{23}$$

which is parameterized with the hyperbolic functions. Therefore the relativity in the two-dimensional spacetime of $Cen(Cl_{0,3})$ is given by the group of $SO(1, 1)$. Thus, the approach has the same path as given above in Section 3; repeating it yields the scale factor of the FLRW metric as

$$a = \left[\chi \sinh\left(\frac{\alpha\tau}{2c}\right)\right]^{2/3} \qquad \alpha = \frac{3H_0c}{\chi} \quad , \quad \Omega_\Lambda = \chi^2 \tag{24}$$

which is the *exact* form used by modern cosmology with positive cosmological term [26] and *ad hoc* elaborated in the model [15]. In the similar way it leads to

$$v(r) = \pm\left(\frac{2Gm}{r} + \chi^2\,\frac{\Lambda}{3}r^2\right)^{1/2} \qquad \Lambda = 3H_0^2 \tag{25}$$

Using the tetrad (18) one obtains the static metric of the Schwarzschild-de Sitter.

In contrast with the previous case, the Clifford algebra of anti-space $Cl_{0,3}$ is different from Minkowski structure of isomorphic algebras and Lie groups. For instance, it is isomorphic to even subalgebras $Cl_{4,0}^0 \cong Cl_{0,4}^0$ [1]. $Cl_{0,3}$ can be represented as a two-dimensional space $\mathbb{H}^{1,1}$ spanned by quaternionic two-vector (q_0, q_1) $q_0, q_1 \in \mathbb{H}$. In terms of Lie groups, as shown in (21), $Cl_{0,3}$ corresponds to the group $SO(3, \mathbb{D}) \cong SO(3) \times SO(3)$,and it is isomorphic to split-complexified $SU(2)$. The latter corresponds to the isomorphism of $SU(2) \times SU(2) \cong SO(4)$, and there is a homomorphism of $SO(3, \mathbb{D})$ to $SO(4)$.

[9]Early attempts using a quaternion as the four-vector can be attributed to Minkowski himself, who considered this as "too narrow and clumsy for the purposes" [27]. Since Silberstein's time, there have been many attempts from physicists to map quaternions to 4-vectors. Stephen Hawking's proposal [5, 11] to introduce imaginary time ($t \to it$) is one of such attempt because it also indirectly complexifies the quaternion. The *impossibility* of the direct mapping of a Minkowski 4-vector to a quaternion originates from the fact that the Lie group $SU(2)$ requires complexification over $\mathbb{C}$.

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5. Observed Cosmological Model via Measured Deceleration Parameter

By definition the deceleration parameter is

$$q_0 = -\frac{\ddot{a}a}{\dot{a}^2} \tag{26}$$

The derivatives of the scale factor can be obtained using its relation to the coordinate acceleration, velocity and coordinate distance: $\alpha' = \ddot{a}R$, $v = \dot{a}R$, $r = Ra$. In this way

$$q_0 = -\frac{r}{v^2}\frac{d}{dr}\left(\frac{v^2}{2}\right) \tag{27}$$

The substitution of (17) leads to

$$q_0 = \frac{Gm + \chi^2 H_0^2 r^3}{2Gm - \chi^2 H_0^2 r^3} \tag{28}$$

Furthermore, using (25) for the SdS case, and assuming uniformly distributed mass m within a sphere with density expressed in terms of $\Omega_M = \rho/\rho_{crit}$ both cases are

$$\text{SAdS}: \quad q_0 = \frac{1}{2}\frac{\Omega_M + 2\chi^2}{\Omega_M - \chi^2} \qquad \text{via} \quad \text{SdS}: \quad q_0 = \frac{1}{2}\frac{\Omega_M - 2\chi^2}{\Omega_M + \chi^2} \tag{29}$$

Recent observations [16, 18] indicate that the value of the deceleration parameter is on the order of $q_0 = -0.5$ implying that the cosmological model of the universe favors the de Sitter type. However, the SAdS case results in $q_0 > 0$. The SdS case of the model leads to the comparable value for the deceleration parameter even by setting $\chi = 1$ as noted in [13]. Thus, the observation suggests that the universe expands according to the SdS model of gravitation.

6. Conclusions

The Clifford algebras of space $Cl_{3,0}$ (APS) and anti-Space $Cl_{0,3}$ have two-dimensional center that represents time-volume coordinates. The uniform acceleration in such two-dimensional spacetime induces the exact expressions for the scale factor of the FLRW metric and corresponds to the Schwarzschild-AdS and SdS gravity in static coordinates. The constant uniform acceleration has value expressed by the fundamental constants $\alpha = 3H_0c = c\sqrt{3\Lambda}$, $(\chi = 1)$.

The case of Clifford algebra of the physical space $Cl_{3,0}$ demonstrates conformity to the anti-de Sitter cosmology and the *oscillating* scale factor of the FLRW metric. Such a case appears natural because of the algebra's native isomorphism to Minkowski spacetime. For instance, the electromagnetism has the symmetry with $Cl_{3,0}$ algebra via homomorphism $SO(3,\mathbb{C})$ group, and has a similar complex vector $\mathbf{E}+i\mathbf{B}$. The duality of the reviewed complex numbers representing *time* and *space volume* $u = t + i\eta$ points to a certain analogy. The complex numbers are known to induce inducing both spatial rotations and oscillations. Another interesting property of $Cl_{3,0}$ is that it allows quick extrapolation of the results to the spacetime algebra (STA) using the isomorphism with $Cl^0_{1,3}$. As an example, for $Cl_{1,3}$ where the basis is the Dirac matrices, and the transformation is $\gamma_\mu \to \overline{\gamma}_a = \gamma_\mu {h_\mu}^a$ [9]. Hence, in the case of the FLRW metric (7) it yields $\overline{\gamma}_k = a\gamma_k$, $(k = 1, 2, 3)$.

Nature seems to prefer complex numbers over split-complex numbers; thus, the application of the Clifford algebra $Cl_{0,3}$ to the model possesses certain "unnaturalness." Nevertheless, since it provides a path to the de Sitter model, which is the basis for modern observational cosmology, therefore $Cl_{0,3}$, its isomorphic algebras and related groups, such as $SO(3,\mathbb{D})$ and $SO(4)$ require further study on their possible relation to gravitation. For instance, the symmetry with the group $SO(3,\mathbb{D})$ suggests that we may consider similar split-complex vectors $\mathbf{G} + j\mathbf{H}$ in the case of gravity. However, due to the hyperbolicity of the split-complex number, such a vector does not induce harmonic oscillations. Furthermore, why can we not assume that Clifford algebra of space $Cl_{3,0}$ reflects the symmetry of *electromagnetic* interaction, while the algebra of anti-space $Cl_{0,3}$ describes *gravitation*? Such anti-symmetry of spaces may indicate the anti-symmetry of these two fundamental interactions. However, such proposal must be revised in the only one case: if Hawking's imaginary time [5, 11] is a valid concept.

A new feature of the model is that it states the essence of the cosmological expansion due to the uniform acceleration in Clifford coordinates of $Cen(Cl_{3,0})$, and $Cen(Cl_{0,3})$. The approach suggests that the appearance of the cosmological term is due to the relativistic effect in such coordinates. Furthermore, in the application to a point mass the approach is capable to obtain the results for the static gravitational metrics (the SAdS and the SdS). Since the present study is limited by the spherically symmetric case, a generalization to an arbitrary coordinate system using the tetrad formalism and Clifford coordinates can be a prospective topic for future research.

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