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Article

Third-Order Nonlinear Semi-Canonical Functional Differential Equations: Oscillation via New Canonical Transform

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Abstract: This paper focuses on the oscillatory properties of the semi-canonical nonlinear delay differential equation of third-order

$$(\delta_2(t)(\delta_1(t)\zeta'(t))')' + \eta(t)\zeta^\alpha(\tau(t)) = 0, \quad t \geq t_0,$$

under the conditions

$$\int_{t_0}^{\infty} \frac{1}{\delta_2(t)} dt = \infty \quad \text{and} \quad \int_{t_0}^{\infty} \frac{1}{\delta_1(t)} dt < \infty.$$

Using the new canonical transform method, we transformed the studied equation into a canonical type equation and then we find the conditions for the oscillation of the studied equation from the canonical equations. The obtained results are new and complement to the existing results mentioned in the literature. Examples are provided to illustrate the importance and novelty of the main results.

Keywords: third-order; semi canonical; functional difference equation; oscillation

MSC: 34C10, 34K11.

1. Introduction

The aim of this paper is to extend and complement the results in [20,21] for the third-order delay differential equations of the form

$$\mathfrak{L}\zeta(t) + \eta(t)\zeta^\alpha(\tau(t)) = 0, \quad t \geq t_0 > 0, \quad (E)$$

where $\mathfrak{L}$ is the differential operator defined by

$$\mathfrak{L}\zeta(t) = (\delta_2(t)(\delta_1(t)\zeta'(t))')'. \quad (1)$$

In the sequel, we assume the following hypotheses:

(H₁) $\delta_1 \in C^2([t_0, \infty))$, $\delta_2 \in C^1([t_0, \infty))$, $a_1(t) > 0$ and $a_2(t) > 0$ for all $t \geq t_0$;

(H₂) $\eta(t) \in C([t_0, \infty))$ and $\eta(t) > 0$ for all $t \geq t_0$;

(H₃) $\tau(t) \in C^1([t_0, \infty))$, $\tau(t) < t$, $\lim_{t \rightarrow \infty} \tau(t) = \infty$ and $\tau(t) > 0$ for all $t \geq t_0$;

(H₄) α is a ratio of odd positive integers;

(H₅) the operator $\mathcal{L}$ is in semi-canonical form, that is,

$$\int_{t_0}^{\infty} \frac{1}{\delta_2(t)} dt = \infty \quad \text{and} \quad \Omega_1(t_0) = \int_{t_0}^{\infty} \frac{1}{\delta_1(t)} dt < \infty.$$

Definition 1. The solution of (E) is defined to be a nontrivial function $\zeta(t) \in C([t_\zeta, \infty))$, $t_\zeta \geq t_0$ with the properties $\delta_1\zeta' \in C^1([t_\zeta, \infty))$, $\delta_2(\delta_1\zeta')' \in C^1([t_\zeta, \infty))$, that satisfies (E) on $[t_\zeta, \infty)$. We consider only those

solutions $\zeta(t)$ of (E) satisfying $\sup\{|\zeta(t)| : t \geq T\} > 0$ for all $T \geq t_\zeta$ and assume that (E) possesses such a solution.

Definition 2. A solution of (E) is called oscillatory if it has infinitely many zeros on $[t_\zeta, \infty)$; otherwise it is called nonoscillatory. Equation (E) itself is called oscillatory if all its solution are oscillatory.

Due to many practical importance of third-order functional differential equations as well as a number of mathematical problems involved [13], the area of qualitative theory of such equations received a great attention in the last several decades. Further this is very useful in predicting the similar behavior of solutions of third-order partial differential equations [9,10].

Oscillatory phenomena play a significant role in understanding the inherent vibrational patterns within dynamic systems, so investigating the oscillatory behavior of solutions to differential equations provides valuable insights into the stability and periodicity of the systems under considerations. In recent years, there has been great interest in investigating the oscillatory behavior of solutions of (E) or its particular cases or its generalizations, see, for example [1–8,11,12,15,16,19,21,22] and the references are cited therein.

From the review of literature, we see that most of the oscillation results for the equation (E) under the conditions

$$\int_{t_0}^{\infty} \frac{1}{\delta_2(t)} dt = \int_{t_0}^{\infty} \frac{1}{\delta_1(t)} dt = \infty$$

or

$$\int_{t_0}^{\infty} \frac{1}{\delta_2(t)} dt < \infty \quad \text{and} \quad \int_{t_0}^{\infty} \frac{1}{\delta_1(t)} dt = \infty$$

or

$$\int_{t_0}^{\infty} \frac{1}{\delta_2(t)} dt < \infty \quad \text{and} \quad \int_{t_0}^{\infty} \frac{1}{\delta_1(t)} dt < \infty,$$

see, for example [3,4,6,9-13,17-21,24] and the references are cited therein.

In [5,20,21] the authors studied the oscillatory properties of (E) or its generalizations under the condition

$$\int_{t_0}^{\infty} \frac{1}{\delta_2(t)} dt = \infty \quad \text{and} \quad \int_{t_0}^{\infty} \frac{1}{\delta_1(t)} dt < \infty.$$

However in [20,21], the authors transformed the equation (E) into canonical form under the condition

$$\Omega_{21}(t_0) = \int_{t_0}^{\infty} \frac{\Omega_1(t)}{a_2(t)} dt = \infty, \quad (2)$$

which greatly simplified the examination of (E) by reducing the set of nonoscillatory solutions to two instead of three.

In view of the above observation, we see that if condition (2) fails to hold then (E) cannot be called as a canonical type equation. Therefore the aim of this paper is to fill this gap, that is, if $\Omega_{21}(t_0) < \infty$ then we present new transform that transforms (E) into canonical form. Thus, the results obtained here are new and complement to that of in [5,20,21]. Examples are provided to show the novelty and significance of our main results.

2. Main Results

For simplicity, we employ the following notation:

$$\Omega_1(t) = \int_t^\infty \frac{1}{\delta_1(s)} ds, \quad \Omega_{21}(t) = \int_t^\infty \frac{\Omega_1(s)}{\delta_2(s)} ds$$

$$\beta_1(t) = \frac{\delta_1(t)\Omega_1^2(t)}{\Omega_{21}(t)}, \quad \beta_2(t) = \frac{\delta_2(t)\Omega_{21}^2(t)}{\Omega_1(t)}, \quad z(t) = \frac{\zeta(t)}{\Omega_1(t)}.$$

We begin with a theorem that is adopted from [12] but presented with a different proof.

Theorem 1. *If*

$$\Omega_{21}(t_0) < \infty, \quad (3)$$

then the semi-canonical operator $\mathfrak{L}\zeta$ has the following unique canonical representation

$$\mathfrak{L}\zeta(t) = \frac{1}{\Omega_{21}(t)} \left(\frac{\delta_2(t)\Omega_{21}^2(t)}{\Omega_1(t)} \left(\frac{\delta_1(t)\Omega_1^2(t)}{\Omega_{21}(t)} \left(\frac{\zeta(t)}{\Omega_1(t)} \right)' \right)' \right)'. \quad (4)$$

Proof. By a direct calculation, we have

$$\frac{\delta_2(t)\Omega_{21}^2(t)}{\Omega_1(t)} \left(\frac{\delta_1(t)\Omega_1^2(t)}{\Omega_{21}(t)} \left(\frac{\zeta(t)}{\Omega_1(t)} \right)' \right)' = \Omega_{21}(t)\delta_2(t)(\delta_1(t)\zeta'(t))' + \Omega_1(t)\delta_1(t)\zeta'(t) + \zeta(t),$$

or

$$\begin{aligned} \frac{1}{\Omega_{21}(t)} \left(\frac{\delta_2(t)\Omega_{21}^2(t)}{\Omega_1(t)} \left(\frac{\delta_1(t)\Omega_1^2(t)}{\Omega_{21}(t)} \left(\frac{\zeta(t)}{\Omega_1(t)} \right)' \right)' \right)' &= \frac{1}{\Omega_{21}(t)} \left[\Omega_{21}(t)\delta_2(t)(\delta_1(t)\zeta'(t))' \right. \\ &\quad \left. + \Omega_1(t)\delta_1(t)\zeta'(t) + \zeta(t) \right]' \\ &= \frac{1}{\Omega_{21}(t)} \left[\Omega_{21}(t) \left(\delta_2(t) \left(\delta_1(t)\zeta'(t) \right)' \right)' - \Omega_1(t) \left(\delta_1(t)\zeta'(t) \right)' \right. \\ &\quad \left. + \Omega_1(t) \left(\delta_1(t)\zeta'(t) \right)' - \zeta'(t) + \zeta'(t) \right] = \left(\delta_2(t) \left(\delta_1(t)\zeta'(t) \right)' \right)'. \end{aligned}$$

Next, we show that (4) is in canonical form, that is,

$$\int_{t_0}^\infty \frac{\Omega_1(t)}{\delta_2(t)\Omega_{21}^2(t)} dt = \infty = \int_{t_0}^\infty \frac{\Omega_{21}(t)}{\delta_1(t)\Omega_1^2(t)} dt.$$

Now,

$$\int_{t_0}^\infty \frac{\Omega_1(t)}{\delta_2(t)\Omega_{21}^2(t)} dt = \int_{t_0}^\infty d\left(\frac{1}{\Omega_{21}(t)}\right) = \lim_{t \rightarrow \infty} \frac{1}{\Omega_{21}(t)} - \frac{1}{\Omega_{21}(t_0)} = \infty.$$

Further,

$$\begin{aligned} \int_{t_0}^\infty \frac{\Omega_{21}(t)}{\delta_1(t)\Omega_1^2(t)} dt &= \int_{t_0}^\infty \Omega_{21}(t) d\left(\frac{1}{\Omega_1(t)}\right) \\ &= \frac{\Omega_{21}(t)}{\Omega_1(t)} \Big|_{t_0}^\infty + \int_{t_0}^\infty \frac{1}{\delta_2(t)} dt \\ &> \int_{t_0}^\infty \frac{1}{\delta_2(t)} dt = \infty \end{aligned}$$

by (H_5) . However, Trench proved in [24] that there exists only one canonical representation of $\mathfrak{L}$ (up to multiplicative constants with product 1) and so our canonical form is unique. The proof of the theorem

is complete.

□

Based on Theorem 2.1, one can write (E) in the canonical form as

$$\left(\beta_2(t)(\beta_1(t)z'(t))'\right)' + w(t)z^\alpha(\tau(t)) = 0, \quad t \geq t_0, \quad (E_c)$$

where $w(t) = \eta(t)\Omega_{21}(t)\Omega_1^\alpha(\tau(t))$, and the following results are immediate.

Theorem 2. Let (3) holds. Then the semi-canonical equation (E) possesses a solution $\zeta(t)$ if and only if the canonical equation (E_c) has the solution $z(t)$.

Corollary 1. Let (3) holds. The semi-canonical equation (E) has an eventually positive solution if and only if the canonical equation (E_c) has an eventually positive solution.

Corollary 2.3 simplifies significantly the investigation of (E) since for (E_c), we deal with only two classes of an eventually positive (nonoscillatory) solutions (see, Lemma 2 [14]), namely, either

$$z(t) > 0, \quad \mathcal{L}_1 z(t) < 0, \quad \mathcal{L}_2 z(t) > 0, \quad \mathcal{L}_3 z(t) < 0,$$

and in this case, we say $z \in D_0$ or

$$z(t) > 0, \quad \mathcal{L}_1 z(t) > 0, \quad \mathcal{L}_2 z(t) > 0, \quad \mathcal{L}_3 z(t) < 0,$$

and in this case we denote that $z \in D_2$, where

$$\mathcal{L}_0 z(t) = z(t), \quad \mathcal{L}_i z(t) = \beta_i(t) \left(\mathcal{L}_{i-1} z(t) \right)', \quad i = 1, 2, \quad \mathcal{L}_3 z(t) = (\mathcal{L}_2 z(t))'.$$

Theorem 3. Let (3) holds. Assume that there exists a function $\sigma \in C^1([t_0, \infty))$ such that

$$\sigma'(t) > 0, \quad \sigma(t) > t, \quad g(t) = \tau(\sigma(\sigma(t))) < t. \quad (5)$$

If both the first-order delay differential equations

$$u'(t) + Q_1(t)u^\alpha(\tau(t)) = 0 \quad (6)$$

and

$$u'(t) + Q_2(t)u^\alpha(g(t)) = 0, \quad (7)$$

where

$$\begin{aligned} Q_1(t) &= w(t) \left(\int_{t_1}^{\tau(t)} \frac{1}{\beta_1(s)} \int_{t_1}^s \frac{1}{\beta_2(s_1)} ds_1 ds \right)^\alpha \\ Q_2(t) &= \frac{1}{\beta_1(t)} \int_t^{\sigma(t)} \frac{1}{\beta_2(s)} \int_s^{\sigma(s)} w(s_1) ds_1 ds \end{aligned}$$

for all $t_1 \geq t_0$, are oscillatory, then equation (E) is oscillatory.

Proof. Let $\zeta(t)$ be an eventually positive solution of (E). Then by Corollary 2.3, $z(t)$ is also a positive solution of (E_c) and either $z(t) \in D_0$ or $z(t) \in D_2$ for all $t \geq t_1 \geq t_0$.

First assume that $z(t) \in D_2$. Then using the fact that $\mathcal{L}_2 z(t) > 0$ and decreasing, we have

$$\mathcal{L}_1 z(t) \geq \int_{t_1}^t \frac{\mathcal{L}_2 z(s)}{\beta_2(s)} ds \geq \mathcal{L}_2 z(t) \int_{t_1}^t \frac{1}{\beta_2(s)} ds,$$

or

$$z'(t) \geq \mathcal{L}_2 z(t) \left(\frac{1}{\beta_1(t)} \int_{t_1}^t \frac{1}{\beta_2(s)} ds \right).$$

Integrating again from t_1 to t , we get

$$z(t) \geq \mathcal{L}_2 z(t) \int_{t_1}^t \frac{1}{\beta_1(s)} \left(\int_{t_1}^s \frac{1}{\beta_2(s_1)} ds_1 \right) ds. \quad (8)$$

Let $u(t) = \mathcal{L}_2 z(t)$. Then combining (8) with (E_c), we see that

$$u'(t) + w(t) \left(\int_{t_1}^{\tau(t)} \frac{1}{\beta_1(s)} \int_{t_1}^s \frac{1}{\beta_2(s_1)} ds_1 ds \right)^\alpha u^\alpha(\tau(t)) \leq 0$$

for $t \geq t_1$. Integrating the latter inequality from t to ∞ , we have

$$u(t) \geq \int_t^\infty Q_1(s) u^\alpha(\tau(s)) ds$$

for $t \geq t_1$. The function $u(t)$ is clearly decreasing on $[t_1, \infty)$ and hence by Theorem 1 in [18], we conclude that there exists a positive solution $u(t)$ of (6) with $\lim_{t \rightarrow \infty} u(t) = 0$, which contradicts the fact that (6) is oscillatory.

Next, assume that $z(t) \in D_0$. Integrating (E_c) from t to $\sigma(t)$ gives

$$\mathcal{L}_2 z(t) \geq \int_t^{\sigma(t)} w(s) z^\alpha(\tau(s)) ds \geq z^\alpha(\tau(\sigma(t))) \int_t^{\sigma(t)} w(s) ds.$$

Then

$$(\beta_1(t) z'(t))' \geq \frac{z^\alpha(\tau(\sigma(t)))}{\beta_2(t)} \int_t^{\sigma(t)} w(s) ds.$$

Integrating last inequality from t to $\sigma(t)$, we get

$$-z'(t) \geq \frac{z^\alpha(g(t))}{\beta_1(t)} \int_t^{\sigma(t)} \frac{1}{\beta_2(s)} \int_s^{\sigma(s)} w(s_1) ds_1 ds.$$

Finally, integrating from t to ∞ yields

$$z(t) \geq \int_t^\infty \frac{z^\alpha(g(s))}{\beta_1(s)} \int_s^{\sigma(s)} \frac{1}{\beta_2(s_1)} \int_{s_1}^{\sigma(s_1)} w(s_2) ds_2 ds_1 ds.$$

Set the right hand side of the last inequality by $u(t)$, we have $z(t) \geq u(t) > 0$. Then, it is easy to see that

$$0 = u'(t) + Q_2(t) z^\alpha(g(t))$$

$$0 \geq u'(t) + Q_2(t) u^\alpha(g(t)).$$

Since $u(t)$ is a positive bounded solution of the last inequality, then by Corollary 1 of [18], we see that the corresponding differential equation (7) has also a positive solution. This is a contradiction to our assumption, and we conclude that (E) oscillates. The proof of the theorem is complete. $\square$

Employing criteria for oscillation of (6) and (7), we immediately obtain explicit criteria for oscillation of (E) for different value of α .

Corollary 2. Let (3) holds. Assume that there exists a function $\sigma \in C^1([t_0, \infty))$ such that (5) holds. If $\alpha = 1$,

$$\liminf_{t \rightarrow \infty} \int_{\tau(t)}^t Q_1(s) ds > \frac{1}{e} \quad (9)$$

and

$$\liminf_{t \rightarrow \infty} \int_{g(t)}^t Q_2(s) ds > \frac{1}{e}, \quad (10)$$

then (E) is oscillatory.

Proof. Application of Theorem 2.1.1 of [9] with (9) and (10) imply that the equations (6) and (7) are oscillatory. Now, the proof follows from Theorem 2.4. This ends the proof. $\square$

Corollary 3. Let (3) holds. Assume that there exists a function $\sigma \in C^1([t_0, \infty))$ such that (5) holds. If $0 < \alpha < 1$

$$\int_{t_*}^{\infty} Q_1(t) dt = \infty \quad (11)$$

and

$$\int_{t_*}^{\infty} Q_2(t) dt = \infty \quad (12)$$

for all $t_* \geq t_0$, then equation (E) is oscillatory.

Proof. Application of Theorem 3.9.3 of [9] with (11) and (12) imply that the equations (6) and (7) are oscillatory. Now, the proof follows from Theorem 2.4. This ends the proof. $\square$

Corollary 4. Let (3) holds. Assume that there exists a function $\sigma \in C^1([t_0, \infty))$ such that (5) holds. Further assume that $\alpha > 1$, $\tau(t) = \theta_1 t$, $\sigma(t) = \theta_2(t)$, $\theta_1 \in (0, 1)$, $\theta_2 > 1$ and $g(t) = \theta_3(t)$ with $\theta_3 = \theta_1 \theta_2^2 < 1$. If there exist

$$\lambda_1 > -\ln \alpha / \ln \theta_1, \quad \lambda_2 > -\ln \alpha / \ln \theta_3 \quad (13)$$

such that

$$\liminf_{t \rightarrow \infty} [Q_1(t) \exp(-t^{\lambda_1})] > 0 \quad (14)$$

and

$$\liminf_{t \rightarrow \infty} [Q_2(t) \exp(-t^{\lambda_2})] > 0 \quad (15)$$

hold, then (E) is oscillatory.

Corollary 5. Let (3) holds. Assume that there exists a function $\sigma \in C^1([t_0, \infty))$ such that (5) holds. Further assume that $\alpha > 1$, $\tau(t) = t^{\theta_1}$, $\sigma(t) = t^{\theta_2}$, $\theta_1 \in (0, 1)$, $\theta_2 > 1$ and $g(t) = t^{\theta_3}$ with $\theta_3 = \theta_1 \theta_2^2 < 1$. If there exist

$$\lambda_1 > -\ln \alpha / \ln \theta_1, \quad \lambda_2 > -\ln \alpha / \ln \theta_3 \quad (16)$$

such that

$$\liminf_{t \rightarrow \infty} [Q_1(t) \exp(-(\ln t)^{\lambda_1})] > 0 \quad (17)$$

and

$$\liminf_{t \rightarrow \infty} [Q_2(t) \exp(-(\ln t)^{\lambda_2})] > 0 \quad (18)$$

hold, then (E) is oscillatory.

The proofs of Corollary 2.7 and Corollary 2.8 follow by using Theorem 4 and Theorem 5 of [23] with Theorem 2.4, respectively.

Next, we present another criteria for the oscillation of (E) when $\alpha = 1$ and $\alpha < 1$.

Theorem 4. Let (3) and $\alpha = 1$ hold. If

$$\limsup_{t \rightarrow \infty} \int_{\tau(t)}^t \left(\frac{1}{\beta_1(s)} \int_s^t \frac{1}{\beta_2(s_1)} \int_{s_1}^t w(s_2) ds_2 ds_1 \right) ds > 1 \quad (19)$$

and

$$\liminf_{t \rightarrow \infty} \int_{\tau(t)}^t Q_1(s) ds > \frac{1}{e}, \quad (20)$$

then equation (E) is oscillatory.

Proof. Let $\zeta(t)$ be an eventually positive solution of (E). Then proceeding as in the proof of Theorem 2.4, we see that $z(t)$ is a positive solution of (E_c) and belongs to either D_0 or D_2 for all $t \geq t_1 \geq t_0$.

If $z(t) \in D_2$, then as in the Proof of Corollary 2.5, we obtain a contradiction with (20) and so $z(t) \in D_0$

Integrating (E_c) from s to t gives

$$\begin{aligned} (\beta_1(s)z'(s))' &\geq \frac{1}{\beta_2(s)} \int_s^t w(s_1)z^\alpha(\tau(s_1))ds_1 \\ &\geq \frac{z^\alpha(\tau(t))}{\beta_2(s)} \int_s^t w(s_1)ds_1. \end{aligned}$$

Again integrating the inequality twice from s to t , we have

$$z(s) \geq z^\alpha(\tau(t)) \int_s^t \left(\frac{1}{\beta_1(s_1)} \int_{s_1}^t \frac{1}{\beta_2(s_2)} \int_{s_2}^t w(s_3) ds_3 ds_2 \right) ds_1.$$

Letting $s = \tau(t)$ and $\alpha = 1$, we obtain a contradiction with (19). Hence we conclude that (E) is oscillatory. This completes the proof. $\square$

Theorem 5. Let (3) and $0 < \alpha < 1$ hold. If

$$\limsup_{t \rightarrow \infty} \int_{\tau(t)}^t \left(\frac{1}{\beta_1(s)} \int_s^t \frac{1}{\beta_2(s_1)} \int_{s_1}^t w(s_2) ds_2 ds_1 \right) ds = \infty \quad (21)$$

and

$$\lim_{t \rightarrow \infty} \int_{t_1}^\infty Q_1(s) ds = \infty \quad (22)$$

for all $t_1 \geq t_0$, then equation (E) is oscillatory.

Proof. Let $\zeta(t)$ be an eventually positive solution of (E). Then proceeding as in the proof of Theorem 2.4, we see that $z(t)$ is a positive solution of (E_c) that belongs to either D_0 or D_2 for all $t \geq t_1 \geq t_0$.

First assume that $z(t) \in D_2$. In view of condition (22) and by Corollary 2.6, we get that D_2 is empty and so $z(t) \in D_0$. Thus proceeding as in the proof of Theorem 2.9, we obtain

$$z^{1-\alpha}(\tau(t)) \geq \int_{\tau(t)}^t \left(\frac{1}{\beta_1(s)} \int_s^t \frac{1}{\beta_2(s_1)} \int_{s_1}^t w(s_2) ds_2 ds_1 \right) ds. \quad (23)$$

Since $z(t)$ is decreasing and $\alpha < 1$, we see that $z^{1-\alpha}(\tau(t)) \leq M$ for all $t \geq t_1 \geq t_0$, and using this in (23), one gets a contradiction with (21). The proof of the theorem is complete. $\square$

Remark 1. Here, we use the canonical equation (E_c) to get oscillation criteria of (E). Therefore, it is clear that one may use the results in [1–3,6,11,15,16] to get several oscillatory and asymptotic behaviors of (E_c) which in turn imply that of (E). The details are left to the reader.

3. Examples

In this section, we present some examples to show the importance of the main results.

Example 1. Consider the semi-canonical third-order linear delay differential equation

$$(t(t^2\zeta'(t)))' + \eta_0\zeta(\lambda t) = 0, \quad t \geq 1, \quad (24)$$

where $\eta_0 > 0$ and $\lambda \in (0, 1)$.

Here $\delta_1(t) = t^2$, $\delta_2(t) = t$, $\eta(t) = \eta_0$, $\tau(t) = \lambda t$ and $\alpha = 1$. By a simple calculation, we have $\Omega_1(t) = \frac{1}{t}$, $\Omega_{21}(t) = \frac{1}{t}$, $\beta_1(t) = t$, $\beta_2(t) = 1$ and $w(t) = \frac{\eta_0}{\lambda t^2}$. The transformed equation is

$$(tz'(t))'' + \frac{\eta_0}{\lambda^2 t} z(\lambda t) = 0, \quad t \geq 1,$$

which is in canonical form. The condition (3) is clearly satisfied. Choose $\sigma(t) = \mu t$ with $1 < \mu < \frac{1}{\sqrt{\lambda}}$ and $g(t) = \lambda\mu^2 t < t$, we see that

$$Q_1(t) = \eta_0 \left(\frac{1}{t} - \frac{1}{\lambda t^2} - \frac{\ln \lambda}{\lambda t^2} - \frac{\ln t}{\lambda t^2} \right),$$

$$Q_2(t) = \frac{\eta_0}{\lambda t} \left(1 - \frac{\lambda}{\mu} \right) \ln \mu.$$

The condition (9) becomes

$$\eta_0 \ln \left(\frac{1}{\lambda} \right) > \frac{1}{e}$$

and the condition (10) becomes

$$\frac{\eta_0}{\lambda} \left(1 - \frac{1}{\mu} \right) \ln \mu \ln \left(\frac{1}{\lambda\mu^2} \right) > \frac{1}{e}.$$

Therefore, by Corollary 2.5, the equation (24) is oscillatory if

$$\eta_0 > \max \left\{ \frac{1}{e \ln \left(\frac{1}{\lambda} \right)}, \frac{\lambda\mu}{(\mu - 1) \ln \mu \ln \left(\frac{1}{\lambda\mu^2} \right)} \right\}.$$

Example 2. Consider the third-order semi-canonical sub-linear delay differential equation

$$(t^3\zeta'(t))'' + \eta_0 t \zeta^{\frac{1}{3}}(\lambda t) = 0, \quad t \geq 1, \quad (25)$$

where $\eta_0 > 1$ and $\lambda \in (0, 1)$.

Here $\delta_2(t) = 1$, $\delta_1(t) = t^3$, $\eta(t) = \eta_0 t$, $\tau(t) = \lambda t$ and $\alpha = \frac{1}{3}$. By a simple calculation, we have $\Omega_1(t) = \frac{1}{2t^2}$, $\Omega_{21}(t) = \frac{1}{2t}$, $\beta_1(t) = \beta_2(t) = \frac{1}{2}$, $w(t) = \frac{\eta_1}{t^{\frac{2}{3}}}$ where $\eta_1 = \eta_0 \left(\frac{2}{\lambda}\right)^{\frac{2}{3}}$. The condition (3) clearly holds and the transformed equation is

$$z'''(t) + \frac{\eta}{t^{\frac{2}{3}}} z^{\frac{1}{3}}(\lambda t) = 0, \quad t \geq 1,$$

which is in canonical form. With a further calculation, we see that

$$Q_1(t) = 2^{\frac{1}{3}} \eta_0.$$

Choose $\sigma(t) = \mu t$ with $1 < \mu < \frac{1}{\lambda}$, we see that $g(t) = \lambda \mu^2 t < t$ and so condition (5) holds. Also

$$Q_2(t) = \frac{9}{4} \eta_1 (\mu - 1)^{\frac{5}{3}} t^{\frac{4}{3}}.$$

The conditions (11) and (12) are clearly hold. Therefore, by Corollary 2.6, the equation (25) is oscillatory for all $\eta_0 > 0$.

Example 3. Consider the third-order semi-canonical super-linear delay differential equation

$$(t^4 \zeta'(t))'' + \eta_0 t^{12} e^{t^2} \zeta^3\left(\frac{t}{4}\right) = 0, \quad t \geq 1, \quad (26)$$

where $\eta_0 > 0$.

Here $\delta_1(t) = t^4$, $\delta_2(t) = 1$, $\eta(t) = \eta_0 t^{12}$, $\tau(t) = \frac{t}{4}$ and $\alpha = 3$. By a simple computation, we see that

$$\Omega_1(t) = \frac{1}{3t^3} \quad \text{and} \quad \Omega_{21}(t) = \frac{1}{6t^2}$$

and hence condition (3) is satisfied. The transformed equation is

$$\left(\frac{1}{t} z''(t)\right)' + \eta_1 t e^{t^2} z^3\left(\frac{t}{4}\right) = 0, \quad t \geq 1,$$

where $\eta_1 = \frac{\eta_0(4)^9}{9}$. Choose $\sigma(t) = \frac{3}{2}t$, we have $g(t) = \frac{9}{16}t$ and so condition (5) hold. With a further calculation, we see that

$$Q_1(t) = \frac{\eta_0}{9(6^3)} t^{10} e^{t^2}, \quad Q_2(t) = \frac{\eta_1}{36} \left(4e^{\frac{81}{16}t^2} - 13e^{\frac{9}{4}t^2} + 9e^{t^2}\right).$$

Let $\lambda_1 = 1$, then $1 > \frac{\ln 3}{\ln 4}$ and let $\lambda_2 = 2$, then $2 > \frac{\ln 3}{\ln\left(\frac{16}{9}\right)}$, so conditions in (13) holds. The condition (14) becomes

$$\liminf_{t \rightarrow \infty} \left[\frac{\eta_0}{9(6^3)} t^{10} \left(e^{t^2-t} \right) \right] > 0,$$

that is, condition (14) holds. The condition (15) becomes

$$\liminf_{t \rightarrow \infty} \left[\frac{\eta_1}{36} \left(4e^{\frac{81}{16}t^2} - 13e^{\frac{9}{4}t^2} + 9e^{t^2} \right) e^{-t^2} \right] > \lim_{t \rightarrow \infty} \frac{\eta_1}{36} \left(3e^{\frac{5}{4}t^2} + 9 - 16e^{-t^2} \right) > 0,$$

that is, condition (15) hold. Therefore, by Corollary 2.7, the equation (26) is oscillatory.

4. Conclusions

In this paper, by using a new canonical transform, we changed the shape of the equation (E) into a canonical type equation. This significantly simplifies the examination of the equation (E). Therefore, the results obtained here are new and complement to the existing results.

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