Article

Not peer-reviewed version

# Is Time a Cyclic Dimension? How to Second-Quantize Fields by Means of Periodic Boundary Conditions

Donatello Dolce *

Posted Date: 25 April 2023

doi: 10.20944/preprints202304.0887.v1

Keywords: Foundations of Quantum Mechanics; Quantum Field Theory; Second Quantization; Commutation Relations; Relativistic Time; Finite Temperature Field Theory; Time Crystal

*Preprints* (www.preprints.org) | NOT PEER-REVIEWED | Posted: 25 April 2023 doi:10.20944/preprints202304.0887.v1



*Article*

# Is Time a Cyclic Dimension? How to Second-Quantize Fields by Means of Periodic Boundary Conditions

**Donatello Dolce**

University of Camerino, Piazza Cavour 19F, 62032 Camerino, Italy

Correspondence: donatello.dolce@unicam.it

**Abstract:** With simple but rigorous arguments we prove that the ordinary second quantization of bosonic and fermionc fields is formally equivalent to constraining the elementary degrees of freedom of the classical fields to have intrinsically periodic dynamics in time. This result confirms and extends the formal equivalence obtained in previous papers for general Hamiltonian systems in terms of canonical quantization, for elementary relativistic particles also in terms of the Feynman Path Integrals, as well as other remarkable correspondences of both phenomenological and theoretical fundamental interest.

**Keywords:** Foundations of Quantum Mechanics; Quantum Field Theory; Second Quantization; Commutation Relations; Relativistic Time; Finite Temperature Field Theory; Time Crystal

---

## 1. Introduction

We know from the wave-particle duality of Quantum Mechanics (QM) that each elementary particle or elementary system in general is described by a wave-function $\phi(x,t)$. That is a 'periodic phenomenon', as originally named by de Broglie [1], whose natural recurrence in time is fixed by the energy through the Planck constant (we adopt natural units $\hbar = 1$). If the system has locally varying energy $\omega(x)$ due to interactions, then the wave-function has locally varying frequency $f(x)$ and therefore the rate of the time recurrence $T(x)$ is locally modulated, such that

$$\phi(x,t) = \phi(x, t + T(x)) \,. \tag{1}$$

In a perturbative description of interactions $T(x)$ varies point by point in a causal way as the energy (undulatory mechanics and Hamilton's opto-mechanical analogy [2]).

In past papers [3–12] we have investigated the following legitimate question: what happens if the natural local recurrence in time $T(x)$ of the wave function $\phi(x,t)$ that can be associated to any generic physical system, Equation (1), is reinforced as constraint to the wave function itself by means Periodic Boundary Conditions (PBCs)? Applications of PBCs along the time dimension, all in all, has been also observed for Time Crystals [13].

The answer has been given in the form of theorem in recent [3] by using results of Geometric Quantization [14–17] and without interpretational ambiguities. Under the very general hypothesis of Hamiltonian systems, i.e., generic symplectic manifolds of even dimensions equipped with a non-degenerate, closed 2-form, and $U(1)$ symmetry, the result of such PBCs imposed as constraint is a quantization formally equivalent to the canonical quantization of the system. In particular the resulting cyclic dynamics can be described in an Hilbert space, the time evolution is given by the ordinary Schrödinger equation, the PBCs play a role formally equivalent to the canonical commutation relations and to the other Dirac's rules

$$\phi(x,t) \overset{\text{PBCs}}{\equiv} \phi(x, t + T(x)) \qquad \Longleftrightarrow \qquad [x, \hat{p}] = i \,,$$

resulting that $\hat{p} = -i\frac{\partial}{\partial x}$. Along the integral curves of the system the local temporal recurrence $T(x)$ is defined as forming a vortex tube whose circulation is invariantly equal to the Planck constant in the

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 25 April 2023 doi:10.20944/preprints202304.0887.v1



extended phase-space (Stokes' lemma) [2], in agreement with the de Broglie-Planck hypothesis. Notice that under the constraint of intrinsic periodicity a wave-function of defined local energy $\omega(x)$ becomes a vibrating string with the whole spectrum (in general anharmonic) of vibrational modes allowed by the constraint. Intuitively the idea can be regarded as a sort relativistic generalization to the time dimension of the semi-classical quantization of "a particle in a Box" where the quantization is achieved by imposing BCs in space (e.g., consider that Lorentz boosts "mix" space and time dimensions). It can be also regarded as justification of the Matsubara theory where the quantization of statistical systems at finite (constant) temperature $\mathcal{T}$ is achieved by imposing periodic BCs (or anti-periodic BCs) with period $\beta = 1/k_B\mathcal{T}$ where $k_B$ is the Boltzmann constant [18].

This theorem confirms and generalizes to generic Hamiltonian systems the same formal equivalence independently obtained in previous works for elementary particles (first quantization of single degrees of freedom) in terms of both Feynman quantization and canonical quantization [4–12]. The same formal equivalence has also been applied to explain in an original way non-trivial quantum phenomena such as superconductivity and graphene physics [11,12]. The ansatz has also revealed a geometrodynamical origin of gauge interaction [5,7], and a formal derivation of the AdS/CFT correspondence [6].

In this paper we demonstrate the validity of the formal equivalence between intrinsically periodic dynamics and ordinary Quantum Field Theory (QFT) in terms of second quantization for both the bosonic and fermionic fields [19]. The result was expected as fields are anyhow Hamiltonian systems, even if with infinite degrees of freedom. In other words we establish a direct link between temporal PBCs and the field commutators of ordinary second quantization with arguments completely independent with respect to those used in the previous demonstrations.

## 2. Second Quantization of Bosons

The quantization of the Harmonic Oscillator (HO) is at the base of the second quantization and it is particularly simple to interpret in terms of intrinsic periodicity. Being an isochronous system the HO has global periodicity $T$, i.e., it does not vary along the integral curves.

### *2.1. Harmonic Oscillator*

Consider an HO of characteristic fundamental period $T = 2\pi/\omega$. This must be regarded a single degree of freedom of a Klein-Gordon (KG) field. The Hamiltonian is

$$H_{HO} = \frac{1}{2}(\hat{p}^2 + \omega^2\hat{q}^2) = \frac{1}{2}\omega(\hat{a}^\dagger\hat{a} + \hat{a}\hat{a}^\dagger)$$

where $\hat{a}$ and $\hat{a}^\dagger$ parametrize the phase-space variables as usual $\hat{q} = \sqrt{\frac{1}{2\omega}}(\hat{a} + \hat{a}^\dagger)$ and $\hat{p} = i\sqrt{\frac{\omega}{2}}(\hat{a}^\dagger - \hat{a})$.

Before starting our proof we note that the constraint of global intrinsic periodicity $\hat{q}(T) \overset{\text{PBCs}}{\equiv} \hat{q}(0)$ implies that each solution $\hat{q}(t)$ is a generic superposition of harmonic eigenmodes $q_n(t) = A_n \exp(-i\omega_n t)$ for some normalization constants $A_n$, such that $i\partial_t q(t) = \omega_n q(t)$ with harmonic spectrum $\omega_n = n\omega$ and $n \in \mathbb{Z}$. So the system can be described, at a statistical level, by an Hilbert state $|q\rangle$ defined in the infinite dimensional Hilbert space of complete orthogonal basis $|n\rangle$ associated to the harmonics $q_n(t)$. The time evolution is thus given by the Schrödinger equation $i\partial_t|q\rangle = \hat{H}_{HO}|q\rangle$, where $\hat{H}_{HO}$ is the Hilbert operator such that $\hat{H}_{HO}|n\rangle = \omega_n|n\rangle$. Similar arguments have been used by 't Hooft for continuous periodic Cellular Automata, e.g., [20–22]. Evidently, this description share some analogy with the normal ordered QHO — for a non relativistic analysis we must assume $n \in \mathbb{N}$. To establish a possible formal equivalence between the two physics we must however prove that the canonical commutation relations, and in particular the commutation relations among the ladder operators, can also be directly inferred from the condition of intrinsic periodicity.

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 25 April 2023 doi:10.20944/preprints202304.0887.v1



Let us thus impose intrinsic periodicity $q(0) \overset{\text{PBCs}}{\equiv} q(T)$ to the classical solutions $q(t)$ and generically describe such an intrinsically periodic HO in an Hilbert space with temporal evolution given by the Schrödinger equation. We prove that the canonical quantization of the HO is formally equivalent to the constrain of intrinsic periodicity. In particular we want to prove that

$$\left.\begin{array}{rcl} \hat{q}(T) & \overset{\text{PBCs}}{\equiv} & \hat{q}(0) \\ T & = & 2\pi/\omega \end{array}\right\} \Leftrightarrow \quad [\hat{q},\hat{p}]_- = i\,. \tag{2}$$

The demonstration starts by assuming, for the parameters $\hat{a}$ and $\hat{a}^\dagger$ promoted to generic Hilbert operators and keeping Weyl ordering, the most generic commutation relation

$$[\hat{a},\hat{a}^\dagger]_\pm = \hat{a}\hat{a}^\dagger \pm \hat{a}^\dagger\hat{a} = \delta\,,$$

for some constant $\delta \in \mathbb{R}$ ($\delta$ cannot be a function of time).

For any Hilbert operator $\hat{F}$ associated to the intrinsically periodic HO the time evolution resulting from the Schrödinger equation is (Heisenberg representation)

$$\hat{F}(t) = e^{i\hat{H}t}\hat{F}(0)e^{-i\hat{H}t}\,. \tag{3}$$

This can be evaluated by using the Lie series (Baker-Campbell-Hausdorff lemma), see [23],

$$e^{\hat{X}}\hat{Y}e^{-\hat{X}} = \hat{Y} + [\hat{X},\hat{Y}]_- + \frac{[\hat{X},[\hat{X},\hat{Y}]_-]_-}{2!} + \dots \tag{4}$$

$$= e^{2X}\left(e^{-\hat{X}}Ye^{-\hat{X}}\right)$$

$$= e^{2X}\left(\hat{Y} - [\hat{X},\hat{Y}]_+ + \frac{[\hat{X},[\hat{X},\hat{Y}]_+]_+}{2!} + \dots\right), \tag{5}$$

or simply by using the general identities $e^{\hat{A}} = \sum_n \frac{\hat{A}^n}{n!}$ and $(\hat{A}+\hat{B})^n = \sum_k \binom{n}{k}\hat{A}^{n-k}\hat{B}^k$.

We write $\hat{a} = a(0)$ and $a^\dagger = a^\dagger(0)$ and study their time evolution. Since $\hat{a}(\hat{a}^\dagger\hat{a})^n = (\hat{a}\hat{a}^\dagger)^n\hat{a}$ and $\hat{a}(\hat{a}\hat{a}^\dagger)^n = (\delta \mp \hat{a}\hat{a}^\dagger)\hat{a}$, we have

$$\begin{array}{rcl} \hat{a}(t) & = & e^{i\hat{H}t}\hat{a}(0)e^{-i\hat{H}t} \\ & = & e^{i\hat{H}t}e^{-\frac{i}{2}\omega t(\hat{a}\hat{a}^\dagger \mp \hat{a}\hat{a}^\dagger + \delta)}\hat{a}(0) \\ & = & e^{-\frac{i}{2}\omega t(\delta \mp \delta)}\hat{a}(0)\,. \end{array} \tag{6}$$

and an equivalent result for its conjugate $\hat{a}^\dagger(t)$.

We must conclude that

$$\left\{\begin{array}{rcl} \hat{a}(T) & \overset{\text{PBCs}}{\equiv} & \hat{a}(0) \\ T & = & 2\pi/\omega \end{array}\right. \Leftrightarrow \quad [\hat{a},\hat{a}^\dagger]_- = 1\,. \tag{7}$$

In fact the anti-commutation case $[\hat{a},\hat{a}^\dagger]_+ = \delta$ as well as the commutation $[\hat{a},\hat{a}^\dagger]_- = 0$ must be excluded since in both cases we get a trivial evolution $a(t) = const$. We can also exclude $\delta \in \mathbb{C}$ since its imaginary part would imply a damping in the solutions.

The remaining possibilities compatible with the constraint of intrinsic periodicity are $[\hat{a},\hat{a}^\dagger]_- = n$ with $n \in \mathbb{N}$. The negative integers $\delta = -n$ imply an inversion of role between the two ladder operators with respect to the corresponding positive integers $\delta = n$.

Finally it is easy to realize that $[\hat{a},\hat{a}^\dagger]_- = 1$ is the most stringent condition allowed. The quantization obtained with commutator $[\hat{a},\hat{a}^\dagger]_- = n$ and $n \in \mathbb{N}/\{0,1\}$ is implicit in the most general quantization $[\hat{a},\hat{a}^\dagger]_- = 1$. For instance, acting on the vacuum with such a creation operator satisfying

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 25 April 2023 doi:10.20944/preprints202304.0887.v1



commutation equal to generic $n \in \mathbb{N}/\{0,1\}$ is equivalent to acting $n$ times on the vacuum with the creation operator satisfying $[\hat{a}, \hat{a}^\dagger]_- = 1$. In the former case only states $|n\rangle$, $|2n\rangle$, $|3n\rangle \dots$, can be created. This would also mean that the fundamental period of the harmonic system is $T/n$, in contradiction with the hypothesis that the HO has fundamental period $T$.

Our thesis Equation (2) follows directly from Equation (7). The dynamics of an ordinary Quantum HO (QHO) obtained by imposing canonical commutation relations among the phase-space operators are therefore formally equivalent, at a statistical level, to the dynamics of a corresponding classical HO constrained to have fundamental periodicity $T = 2\pi/\omega$.

The same conclusion holds for the normal ordered Hamiltonian : $H := \omega \hat{a}\hat{a}^\dagger$. This shows that the vacuum energy $\frac{1}{2}\omega$ depends on the ordering and not only on the BCs. By taking track of the operators ordering, the energy eigenstates and eigenvalues of the QHO can be therefore inferred directly from the condition of intrinsic periodicity, as discussed in the premise of our proof.

### 2.2. Second Quantization of Bosonic Fields

Consider now a classical KG field, for the sake of simplicity we will always assume one spatial dimension,

$$\Phi(q,t) = \int dp \phi_p(q,t) = \int dp (f_p(q,t)\hat{a}_p + f_p^*(q,t)\hat{a}_p^\dagger)\,,$$

where $f_p(q,t)$, $\forall p$, clearly form a complete set of orthogonal solutions due to the PBCs. The Hamiltonian is

$$H_{KG} = \frac{1}{2}\int dp\omega(p)(\hat{a}_p\hat{a}_p^\dagger + \hat{a}_p^\dagger\hat{a}_p)\,.$$

We have infinite degrees of freedom but each field component $\phi_p(q,t)$ can be regarded as an elementary relativistic HO of fundamental periodicity $T_p = 2\pi/\omega(p)$ so that the formal equivalence obtained for the HO can be generalized. The ordinary second quantization condition of the KG field is thus formally equivalent to the constraint of intrinsic periodicity applied for each elementary HO constituting the field. From Equation (7), $\forall p, p'$, we have

$$\begin{cases} \phi_p(q,T_p) & \overset{\text{PBCs}}{\equiv} & \phi_p(q,0) \\ T_p & = & 2\pi/\omega(p) \end{cases} \Leftrightarrow \; [\hat{a}_p, \hat{a}_{p'}^\dagger]_- = \delta(p-p')\,.$$

The KG field $\Phi(q,t)$ has not a well defined periodicity being the integral of all the momentum components $\phi_p(q,t)$. Each component has however a well defined periodicity $T_p = 2\pi/\omega(p)$. We introduce a *modular* time evolution in which each field component $\phi_p(q,t)$ has a different time evolution rate $t_p$:

$$e^{i\hat{\theta}[t]} = e^{\frac{i}{2}\int dp\theta_p[t](\hat{a}_p\hat{a}_p^\dagger + \hat{a}_p^\dagger\hat{a}_p)} \tag{8}$$

with $\theta_p[t] = t_p\omega_p$, $\forall p$.

We denote by $\Phi(q,[T])$ the KG field in which each component $\phi_p(q,t)$ has evolved in time exactly of an integer number of corresponding periods $T_p$ with respect to the initial configuration $\Phi(q,0)$. That is,

$$\Phi(q,[T]) = e^{i\hat{\theta}[T]}\Phi(q,0)e^{-i\hat{\theta}[T]}$$

where $\theta_p[T] = T_p\omega(p) = 2\pi$, $\forall p$.

In terms of such *modular* time evolution we have

$$\begin{cases} \Phi(q,[T]) & \overset{\text{PBCs}}{\equiv} & \Phi(q,0) \\ \theta_p[T] = T_p\omega_p & = & 2\pi \end{cases} \Leftrightarrow [\hat{a}_p, \hat{a}_{p'}^\dagger]_- = \delta(p-p')\,.$$

This also means that the second quantization expressed in terms of field commutators holds if and only if each field component is constrained to assume the same configuration after an integer number of its cycles

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 25 April 2023 doi:10.20944/preprints202304.0887.v1



$$\begin{cases} \Phi(q,[T]) & \overset{\text{PBCs}}{\equiv} & \Phi(q,0) \\ \theta_p[T] = T_p\omega_p & = & 2\pi, \ \forall p \end{cases}$$
$$\Updownarrow$$
$$[\Phi(x,t), \dot{\Phi}(y,t)]_- = i\delta(x-y)\,.$$

Summarizing, a classical KG field $\Phi(q,t)$ is formally equivalent to a second quantized KG field if each component $\phi_p(q,t)$ is constrained to be intrinsically periodic with respect to its natural recurrence. In a similar way we can infer the formal equivalence between second quantization and the constraint of intrinsic periodicity for vectorial fields, with all the due prescriptions of the ordinary second quantization of gauge theories, see also [3,5].

## 3. Second Quantization of Fermions

The analysis of the Fermi oscillator will allow us to easily generalize the above formal equivalence to Dirac fields.

### *3.1. Fermi and Dirac Oscillator*

We consider a Fermi oscillator of fundamental periodicity $T = 2\pi/\omega$, as described by the Hamiltonian

$$\hat{H}_F = \omega \hat{f}^\dagger \hat{f}\,.$$

We want to prove that the condition of fundamental periodicity imposed as constraint $\hat{f}(T) \overset{\text{PBCs}}{\equiv} \hat{f}(0)$ necessarily implies the canonical anti-commutators of fermionic degrees of freedom.

Again the condition of intrinsic periodicity guaranties that the system can be described, at a statistical level, in a Hilbert space and that its temporal evolution is given by the Schrödinger equation. Thus we promote $\hat{f}$ and $f^\dagger$ to Hilbert operators and we assume generic anti-commutation

$$[\hat{f}, \hat{f}^\dagger]_+ = \delta\,. \tag{9}$$

with $\delta \in \mathbb{R}$ .

We want to obtain the Fermi-Dirac statistics so we directly exclude the possibility of commutation relations $[\hat{f}, \hat{f}^\dagger]_- = \delta$ and we assume $\hat{f}^2 = 0$ for obvious reasons. Without loss of generality we temporarily assume $\omega = 1$. Since $\hat{H}_f^2 = \delta \hat{H}_f$ the eigenvalues of $\hat{H}_f$ must be $\lambda^2 = \delta\lambda$ so that $\lambda = 0$ and $\delta$. let us investigate the quantization resulting from the generic anti-commutation relation Equation (9). We define $|0\rangle$ such that $\hat{H}_f|0\rangle = 0$ with $\langle 0|0\rangle = 1$. Thus $|a\rangle = \hat{f}|0\rangle = 0$ as $||a\rangle|^2 = \langle 0|\hat{f}^\dagger f|0\rangle = 0$. Also we define $|\delta\rangle = \hat{f}^\dagger|0\rangle$ such that $||\delta\rangle|^2 = \langle 0|(\delta - \hat{f}^\dagger \hat{f})|0\rangle = \delta$.

In analogy with the bosonic case, from Equations (3) and (4) we find the following time evolution

$$\hat{f}(t) = e^{-i\omega t(\delta - 2\hat{f}^\dagger \hat{f})}\hat{f}(0) = e^{i2\hat{H}_f t} e^{-i\omega t\delta}\hat{f}(0)\,.$$

Notice the *Zitterbewegung* term in this single-particle description, e.g., [24].

From the condition of intrinsic periodicity $\hat{f}(T) \overset{\text{PBCs}}{\equiv} \hat{f}(0)$ we immediately have $\delta \in \mathbb{N}$. In fact, $\hat{H}_f|\delta\rangle = \hat{f}^\dagger \hat{f}\hat{f}^\dagger|0\rangle = \hat{f}^\dagger(\delta - \hat{f}^\dagger f)|0\rangle = f^\dagger \delta|0\rangle = \delta|\delta\rangle$. Thus $(\delta - 2\delta) \in \mathbb{Z}$ and finally $\delta \in \mathbb{Z}$. Similarly, $\hat{H}_f|0\rangle$ =0 and again $\delta \in \mathbb{Z}$. The choice $-\delta \in \mathbb{N}$ is physically equivalent to $\delta \in \mathbb{N}$ for the same reasons of the bosonic case.

The case $\delta = 0$ is trivial and must be excluded because it yields $f(t) = const$. The case $\delta = 1$ works fine. In particular $\hat{f}(t)|0\rangle = \exp(-i\omega t)\hat{f}(0)|0\rangle$ and $\hat{f}(t)|1\rangle = \exp(+i\omega t)\hat{f}(0)|1\rangle$. The case $\delta = n$ with $n \in \mathbb{N}/\{0,1\}$ must be excluded as well. It implies that $\hat{f}(t)|0\rangle = \exp(-in\omega t)\hat{f}(0)|0\rangle$ and $\hat{f}(t)|n\rangle = \exp(+in\omega t)\hat{f}(0)|n\rangle$. However $|n\rangle$ cannot be interpreted as $n$ fermions in the same state due to the underlying Dirac-Fermi statistics. It must be rather interpreted as a fermion of fundamental energy $n\omega$ in its state $|1\rangle$ since no lower energetic states can be created. In other words, similarly to the bosonic case, it means that the period $T$ of the Hamiltonian is not the fundamental period of the system. The fundamental period in this case would be $T_n = T/n$ in contradiction with our hypothesis.

The only possible choice in Equation (9) is $\delta = 1$. Thus,

$$\begin{cases} \hat{f}(T) & \stackrel{\text{PBCs}}{\equiv} & \hat{f}(0) \\ T & = & 2\pi/\omega \end{cases} \quad \Leftrightarrow \quad [\hat{f}, \hat{f}^\dagger]_+ \;=\; 1\,.$$

The same conclusions also apply to

$$\hat{H}'_F = \frac{1}{2}\omega(\hat{f}^\dagger \hat{f} - \hat{f}\hat{f}^\dagger) \tag{10}$$

where the time evolution is now a pure *Zitterbewegung*

$$\hat{f}(t) = e^{-i\omega t(\delta - 2\hat{f}^\dagger \hat{f})}\hat{f}(0) = e^{i2\hat{H}'_f t}\hat{f}(0)\,, \tag{11}$$

with the same Fock space of the previous case.

In analogy to the Fermi oscillator we may consider the Dirac oscillator defined by the Hamiltonian

$$\hat{H}_D = \omega(\hat{b}^\dagger \hat{b} - \hat{d}\hat{d}^\dagger)\,.$$

Similarly to Equation (10), the generic anti-commutation relations Equations (9) assumed separately for both $\hat{b}$ and $\hat{d}$ operators lead to the evolution

$$\hat{b}(t) = e^{-i\omega t(\delta - 2\hat{b}^\dagger \hat{b})}\hat{b}(0)\,,\;\; \hat{d}(t) = e^{-i\omega t(\delta - 2\hat{d}^\dagger \hat{d})}\hat{d}(0)\,.$$

The *Zitterbewegung* is disappeared in the Dirac oscillator and thus it will disappear in the Dirac field description, as noticed in literature, e.g., [24,25].

By introducing the ordinary concept of Dirac vacuum and in analogy with the Fermi oscillator we obtain

$$\begin{cases} \hat{b}(T) & \stackrel{\text{PBCs}}{\equiv} & \hat{b}(0) \\ \hat{d}(T) & \stackrel{\text{PBCs}}{\equiv} & \hat{d}(0) \\ T & = & 2\pi/\omega \end{cases} \quad \Leftrightarrow \quad \begin{cases} [\hat{b}, \hat{b}^\dagger]_+ & = & 1 \\ [\hat{d}, \hat{d}^\dagger]_+ & = & 1\,, \end{cases}$$

and all the other possible anti-commutators must be equal to zero.

### 3.2. Second Quantization for Dirac Fields

We want to establish the relationship between PBCs and the second quantization of Dirac fields in analogy to the previous analysis of bosonic fields.

The Dirac field can be written in the following form

$$\Psi(q,t) = \int dp \psi_p(q,t) = \int dp \left( u_p(q,t)\hat{b}_p + v_p(q,t)\hat{d}_p^\dagger \right)\,,$$

where $u_p(q,t)$ and $v_p(q,t)$ are spinors (we have suppressed the spin index). The Hamiltonian is

$$H_\Psi = \int dp \omega(p)(b_p^\dagger b_p - d_p d_p^\dagger)\,.$$

By investigating the time evolutions of these ladder operators, in complete analogy with the Dirac HO , $\forall p, p'$ we obtain the following formal equivalence

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 25 April 2023 doi:10.20944/preprints202304.0887.v1



$$\begin{cases} [\hat{b}_p, \hat{b}^\dagger_{p'}]_+ = \delta(p - p') \\ [\hat{d}_p, \hat{d}^\dagger_{p'}]_+ = \delta(p - p') \end{cases}$$
$$\Updownarrow$$
$$\begin{cases} \psi_p(q, T_p) & \stackrel{\text{PBCs}}{\equiv} & \psi(q, 0)\,, \\ T_p & = & 2\pi/\omega(p)\,. \end{cases}$$

and all the other anti-commutators equal to zero.

In terms of the *modular* time evolution analogous to Equation (8) the above result can be written as

$$\begin{cases} [\hat{b}_p, \hat{b}^\dagger_{p'}]_+ = \delta(p - p') \\ [\hat{d}_p, \hat{d}^\dagger_{p'}]_+ = \delta(p - p') \end{cases} \Leftrightarrow \begin{cases} \Psi(q, [T]) \stackrel{\text{PBCs}}{\equiv} \Psi(q, 0), \\ \theta_p[T] = T_p\omega_p = 2\pi\,. \end{cases}$$

and finally

$$\begin{cases} \Psi(q, [T]) & \stackrel{\text{PBCs}}{\equiv} & \Psi(q, 0) \\ \theta_p[T] = T_p\omega_p & = & 2\pi, \ \forall p\,. \end{cases}$$
$$\Updownarrow$$
$$[i\Psi^\dagger(x,t), \Psi(y,t)]_+ = i\delta(x - y) \quad . \tag{12}$$

The condition of intrinsic periodicity is formally equivalent to the second quantization condition also for Dirac fields.

In finite temperature field theory we are used to quantize bosonic degree of freedom by imposing PBCs and femionic degrees of freedom by imposing anti-PBCs along the Euclidean time dimension, see e.g., [18]. These are common BCs for all the thermal fields mode as they all have the same temperature, contrarily to the BCs considered here which must be imposed by means of the *modular* time operator. Also, the BCs must always be interpreted up to a twist factor $e^{-i\theta}$ since both bosonic and ferminic degrees of freedom typically have $U(1)$ symmetry. For instance the PBCs of the fermionic degrees of freedom can be replaced with anti-PBCs (twist $e^{-i\pi}$) and the effect on the energy spectrum is a shift of the vacuum energy similarly to that resulting from a different ordering of the operators [3,5,7–12], see also the *zitterbewegung* in Equation (11).

## 4. Interactions

Field theory is based on the physics of the HO and interactions are typically described in a perturbative way, by means of a scattering matrix with in and out states of constant energies, and therefore of in and out constant recurrences in time. Hence, we don't have to worry about the complication resulting from the local modulations of the periodicity rate $T(x)$ associated to the local variations of energy $E(x)$ of a perturbative description of interactions. Anyhow, in the non-perturbative formalization, the local modulations of periodicity occurring during interaction has been formalized in [3,5–7]. They imply local space-time geometrodynamics which has revealed a novel formulation of gauge interaction analogous to that of gravitational interaction.

Even though the constraint of intrinsic periodicity can be safely used (even in the exception of constant recurrences) as a simple mathematical trick to quantize fields and calculate quantum observables, we strongly believe that its origin has strong physical motivations in a renewed formulation of the concept of relativistic time.

## 5. Conclusions

The constraint of intrinsic time periodicity can be formally used in place of the canonical commutators in the second quantization for both bosonic and fermionic fields. This study lays

 doi:10.20944/preprints202304.0887.v1



the ground of a new mathematical method potentially applicable in calculating quantum observables in QFT, such as Feynman diagrams, and essentially based on the physics of classical-relativistic harmonic systems.

## References


1. Broglie, L.d. *Phil. Mag.* **1924**, *47*, 446.
2. Arnol'd, V. Mathematical Methods of Classical Mechanics; Springer: New York, NY, USA, 2013. Available online: https://books.google.it/books?id=5OQlBQAAQBAJ.
3. Dolce, D. Is time a cyclic dimension? canonical quantization implicit in classical cyclic dynamics. *Ann. Phys.* **2023**, *448*, 169182. Available online: https://www.sciencedirect.com/science/article/pii/\S0003491622002925. [CrossRef].
4. Dolce, D. New Stringy Physics beyond Quantum Mechanics from the Feynman Path Integral. *Int. J. Quantum Found.* **2022**, *8*, 125. [CrossRef].
5. Dolce, D. Gauge Interaction as Periodicity Modulation, *Ann. Phys.* **2012**, *327*, 1562–1592. [CrossRef].
6. Dolce, D. Classical geometry to quantum behavior correspondence in a Virtual Extra Dimension. *Ann. Phys.* **2012**, *327*, 2354–2387. [CrossRef].
7. Dolce, D. Elementary spacetime cycles. *EPL* **2013**, *102*, 31002. [CrossRef].
8. Dolce, D. Compact Time and Determinism for Bosons: foundations. *Found. Phys.* **2011**, *41*, 178–203. [CrossRef].
9. Dolce, D. Introduction to the quantum theory of elementary cycles. In *Beyond Peaceful Coexistence*; Imperial College Press: 2016; pp. 93–135.
10. Dolce, D. Unification of Relativistic and Quantum Mechanics from Elementary Cycles Theory. *Electron. J. Theor. Phys.* **2016**, *12*, 29–86. [CrossRef].
11. Dolce, D.; Perali, A. On the Compton clock and the undulatory nature of particle mass in graphene systems. *Eur. Phys. J. Plus* **2015**, *130*, 41. [CrossRef].
12. Dolce, D.; Perali, A. The role of quantum recurrence in superconductivity, carbon nanotubes and related gauge symmetry breaking. *Found.Phys.* **2014**, *44*, 905–922. [CrossRef].
13. Wilczek, F. Quantum Time Crystals. *Phys. Rev. Lett.* **2012**, *109*, 160401. [CrossRef].
14. Todorov, I. Quantization Is a Mystery. *Bulg. J. Phys.* **2012**, *39*, 107–149.
15. Nair, V.P. Elements of Geometric Quantization and Applications to Fields and Fluids. *arXiv* **2016**, arXiv:1606.06407.
16. Moshayedi, N. Notes on Geometric Quantization. *arXiv* **2020**, arXiv:2010.15419.
17. Carosso, A. Quantization: History and problems. *Stud. Hist. Philos. Sci.* **2022**, *96*, 35–50. Available online: https://www.sciencedirect.com/science/article/pii/\S0039368122001339. [CrossRef].
18. Kapusta, J.I.; Gale, C. Finite-Temperature Field Theory: Principles and Applications; Cambridge Monographs on Mathematical Physics; Cambridge University Press: 2011. [CrossRef].
19. Greiner, W.; Reinhardt, J. Field quantization. 1996.
20. 't Hooft, G. Deterministic Quantum Mechanics: The Mathematical Equations. *Front. Phys.* **2020**, *8*, 253. [CrossRef].
21. 't Hooft, G. Classical cellular automata and quantum field theory. *Int. J. Mod. Phys. A* **2010**, *25*, 4385–4396. [CrossRef].
22. 't Hooft, G. Determinism in free bosons. *Int. J. Theor. Phys.* **2003**, *42*, 355–361. [CrossRef].
23. Mendas, I.P.; Popovic, D.B. A generalization of the baker-hausdorff lemma. *Phys. Scr.* **2010**, *82*, 045007. [CrossRef].
24. Itzykson, C.; Zuber, J.B. *Quantum Field Theory*; International Series In Pure and Applied Physics; McGraw-Hill: New York, 1980.
25. Wang, Z.-Y.; Xiong, C.-D. Field-theoretical space-time position operator. *arXiv* **2006**, arXiv:quant-ph/0612090.