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Leggett-Garg Macro-Realism Inequalities Are Violated by All Dynamical Physical Systems

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Abstract: The Leggett-Garg inequalities (Phys. Rev. Lett., 1985) involving multi-time correlation functions are widely considered as the touchstone for what is defined as Macro-Realism of the physical world, which constitutes two main criteria: a) a macro-real system is in one of the possible definite discrete physical states at any given time, and b) the possibility of measurements without altering a physical state. There are continuing experimental investigations supposedly testing the consequences of macro-realism, reflected in the Leggett-Garg inequalities. I prove the surprising universal result that the Leggett-Garg inequalities are violated *by all dynamical physical systems that respect fundamental conservation laws*, and not merely by microscopic and macroscopic quantum systems. Hence, the inequalities are guaranteed to be violated by any conceivable physical system irrespective and independent of the covering theory. The Leggett-Garg inequalities have no place in the real world where ensemble-averaged expectation values and correlation functions bridge the probabilities of microscopic quantum mechanics and the conservation constraints of the macroscopic world.

Keywords: macro-realism; Leggett-Garg inequalities; quantum dynamics; conservation laws

1. Introduction

Though quantum mechanics is supposed to be a universal theory that is applicable to all physical systems, our experience of the physical world is different in the microscopic and macroscopic regimes. The physical notions and principles developed and understood in the context of a macroscopic 'classical' particle or a physical system are not generally applicable to the dynamics and observations of a microscopic particle with quantized physical states. Nevertheless, the ensemble-averaged expectation values and correlation functions derived from a statistical ensemble of quantum systems have a correspondence with classical expectations.

One widely debated open issue is whether there is an identifiable transition from the microscopic quantum world to the macroscopic world, where the characteristic features of quantum mechanics are no longer applicable. A. Leggett formulated this query in terms of the experimentally accessible criterion of 'macroscopic quantum coherence' and the resulting interference [1,2]. In the dynamical evolution of a quantum system from an initial state I to an observable final state F, if two possible intermediate dynamical paths are denoted as B and C, the probability $P_{I\to(BorC)\to F} - (P_{I\to B\to F} + P_{I\to C\to F}) \equiv K \neq 0$. The magnitude of K is a bilinear interference term that is proportional to $|A^*_{I\to B\to F}A_{I\to C\to F}| \propto \sqrt{P_{I\to B\to F}}\sqrt{P_{I\to C\to F}}$. Clearly, such a term contradicts the hypothesis that in each trial of an ensemble of cases, either the path $I\to B\to F$ or the path $I\to C\to F$ is taken. In other words, when there is a superposition of states and interference, it is not true that $P_{i\to f} = \sum_j P_{i\to j\to f}$.

In this broad context, Leggett and Garg defined the notion of 'macroscopic and real', or 'macro-realism', and derived a class of inequalities that would be obeyed in the time evolution of such macro-real physical systems [3]. These Leggett-Garg inequalities (LGI) are widely considered to be the touchstone accessible to laboratory experiments to test the hypothesis of macro-realism and a breakdown of quantum mechanics in macroscopic systems [4–11].

Following Leggett [2], the refined definition of the notion of a 'macro-realistic' theory consists of three postulates: (1) Macrorealism per se: A macroscopic object which has available to it two or more

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macroscopically distinct states is at any given time in a definite one of those states. (2) Non-invasive measurability: It is possible in principle to determine which of these states the system is in without any effect on the state itself or on the subsequent system dynamics. (3) Induction: The properties of ensembles are determined exclusively by initial conditions (and in particular not by final conditions).

A typical example considered by Leggett and Garg is the dynamics of a macro-real system (as per their definition) with two definite states, with observable values ± 1. The transition between the two states is by some unspecified dynamical process. Because of the dynamics, the observables will be time-dependent, and a measurement at a particular time will return one of the results ± 1, for a particular observable Q. Thus, two measurements separated by a dynamical parameter like a time interval Δt (for any two-level system), or an angle $\Delta\theta$ (for a spinorial system), can give results that systematically depend on the parameter Δt. All two-level dynamical systems can be mapped to a spinorial system with the two levels coupled by a fixed process. Now consider two independent measurements of Q on the same macro-real system at two times, with values q_i at t_i and q_j at t_j, $t_i < t_j$. Repetitions of the same protocol, with $t_j - t_i$ fixed, result in statistical ensembles of pairs of values. Then an ensemble-averaged two-time correlation can be defined for each ensemble as $C_{ij} = \langle q_i q_j \rangle = \sum_k q_{ik} q_{jk} p_{ij}$. We have C_{12} for measurements at times t_1 and t_2, C_{23} for times t_2 and t_3, C_{34} for times t_3 and t_4 and, C_{14} for times t_1 and t_4 ($t_1 < t_2 < t_3 < t_4$). Leggett and Garg showed that any macro-realistic theory predicts the inequality

$$L_4 \equiv C_{12} + C_{23} + C_{34} - C_{14} \leq 2 \tag{1}$$

for any values of the t_i. Another form involving three correlations is $L_3 \equiv C_{12} + C_{23} - C_{13} \leq 1$. These inequalities are used as the criterion to test Leggett-Garg (LG) macro-realism against quantum mechanical traits, in different physical systems [4,5,7–11].

I will now show the surprising result that the LGI are universally violated by all genuine dynamical physical systems, even when they are notionally macroscopic, but with discrete physical states. The LGI are unsuitable for probing macro-realism in the real world because the formulation of the LGI is not compatible with the physical constraints arising from the fundamental conservation laws that statistical ensembles of any physical system should obey, irrespective of whether the system is microscopic, macroscopic, or macro-real. In other words, the LGI were formulated without paying attention to the physical constraints necessarily obeyed by ensembles of physical systems, reflected only in ensemble-averaged quantities like correlation functions.

I note here that the LGI are obeyed trivially in systems like a tossed coin or a dice, because the discrete-valued states in those cases are not constrained by any physical dynamics. In fact, the two-time correlation functions C_{ij} of random outcomes in coin-tossing are identically zero. Though this is obvious, an explicit mention is made to avoid any confusion.

2. Two-Time Correlation in Any Two-State Physical System

I now prove the main result that the two-time correlations of the discrete values of dynamical variables have a specific form that violates the Leggett-Garg inequalities universally, irrespective of whether the physical system is microscopic, macroscopic, quantum mechanical, classical etc. The only fact I use in the proof is the validity of fundamental conservation laws, like the conservation of angular momentum or the conservation of the total probability, for ensemble-averaged expectation values and correlations. In other words, physical systems and their theories that obey such conservation constraints violate the LGI. *Since these are the only legitimate physical systems, any compliance with the Leggett-Garg inequalities in the real world is impossible.*

Following Leggett and Garg [3], consider an archetypical two-state system with two discrete physical states with the values of the observable $+1$, and -1. The choice of the system is completely general; it can be microscopic or macroscopic. The scenario is that of an ensemble of a large number of measurements of the values ± 1 of the observable Q at different times t_1, t_2, t_3, and t_4. The probability to get the result $+1$ (and -1) varies with some parameter that can be chosen for the measurements. *I*

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emphasize that I will not invoke the peculiarities of quantum mechanics, like the superposition of states, to be consistent with the formulation of Leggett's macro-realism. For a two-state system for which the observable is the value of a component of the spin along a chosen direction (a two-valued spinor), for example, the variable parameter is an angle relative to the initial direction, and the measurements times t_1, t_2, t_3, and t_4 are at the parameter values $\hat{n}_{\theta 1}$, $\hat{n}_{\theta 2}$, $\hat{n}_{\theta 3}$, and $\hat{n}_{\theta 4}$. The ensemble-averaged two-time correlations are calculated as $C_{ij} \equiv \langle q_i q_j \rangle = \frac{1}{N}\sum_k^N q_{ik}q_{jk}$, $i,j = 1,2,3,4$, $i < j$. The total number of the pairs of measured values in an ensemble is denoted as $N = N_+ + N_-$, where N_+ (N_-) is the number of $+1$ (-1) outcomes. An important ensemble quantity is the 'population difference', $M \equiv N_+ - N_-$. In the language of spins, M is the 'polarization', or the 'ensemble angular momentum' $\vec{J}_n = \hat{n}(N_+ - N_-)_n$.

In an ensemble of N elements, $M = N_+ - N_-$ varies from $+N$ to $-N$. The conservation of total probability imposes the constraint,

$$N^2 = (N_+ + N_-)^2 = N_+^2 + N_-^2 + 2N_+N_- = M^2 + 4N_+N_- \equiv M^2 + C^2 \tag{2}$$

We have $M = 0$ and $C = N$ when $N_+ = N_- = N/2$. The ensemble-averaged quantities are $m \equiv (N_+ - N_-)/N$ and $c \equiv 2\sqrt{N_+N_-}/N$, with $m^2 + c^2 = 1$. Therefore, *the ensemble conservation constraint in any two-state system implies* $m = \cos\theta$ *and* $c = \sin\theta$, where θ is a variable parameter in dynamics, like time, angle etc. In the language of two-state spins, the ensemble-averaged spatial components of the conserved ensemble angular momentum obey $S^2 = S_x^2 + S_y^2 + S_z^2$ or

$$\vec{S} = \hat{x}S_x + \hat{y}S_y + \hat{z}S_z \tag{3}$$

which reduces to $\langle S_z(\theta)\rangle = \cos\theta$ and $\langle S_x(\theta)\rangle = \sin\theta$ in the x-z plane ($\varphi = 0$).

Now I derive a crucial result, that *the ensemble-averaged two-time correlation function is identical to the ensemble-averaged population difference* (or ensemble polarization). The total ensemble consists of two subensembles, one with the initial state at t_i as $+1$ and another with the initial state at t_i as -1. The two-time correlation is

$$\begin{aligned} C_{ij}(t_j - t_i) &= \frac{1}{N}\sum_{k=1}^{N} q_{ik}q_{jk} = \frac{1}{N}\left((+1)_i \sum_{k:q_i=+1} q_{jk} + (-1)_i \sum_{k:q_i=-1} q_{jk}\right) \\ &= \frac{1}{N}\left(+1[N_{+j} - N_{-j}] + (-1)[N_{-j} - N_{+j}]\right) = \langle N_{+j} - N_{-j}\rangle = \langle M_j \rangle \end{aligned} \tag{4}$$

I have established the important relation between the two-time correlation and the ensemble-averaged population difference that obeys the conservation constraint $N^2 = M^2 + C^2$ (or $S^2 = S_x^2 + S_z^2$). The result gives the generic form of the two-time correlation in any dynamical physical ensemble of measurements that takes values only ± 1, irrespective of whether they are macro-real (the LG criteria) or not. We have

$$C_{ij}(t_i, t_j) = \langle M_j \rangle \equiv m_j = \cos[a(t_j - t_i)] \equiv \cos(\theta_j - \theta_i) \tag{5}$$

where $1/a$ is a characteristic dynamical time scale, depending on the physical system. For the two-state system mapped to the two-valued spin, the terms $(+1\sum_{k:q_i=+1} q_{jk})$ (and $(-1\sum_{k:q_i=-11} q_{jk})$) in the two-time correlation function C_{ij} are exactly the ensemble-averaged components of the angular momentum in the direction $\hat{n}_j$, when the initial state has the average angular momentum $+1$ (and -1) in the direction $\hat{n}_i$. Assuming only the validity of the conservation constraint on the spatial components of the ensemble angular momentum, we have again $C_{ij} = \langle S \rangle_\theta = \cos\theta$. *Remarkably, these general results are sufficient to show that all dynamical physical systems with two-valued (± 1) observables violate the Leggett-Garg inequalities in a range of parameter values ($\Delta t = t_j - t_i$), irrespective of whether they are microscopic or macroscopic.*

When the dynamical parameter $\Delta\theta$ (or Δt) is very small, $C_{ij} = \cos\theta \approx 1 - \theta^2/2$. It is clear that we cannot have C_{ij} varying linear in θ (for small θ) because then we get $m = 1 - a\theta$ and $m^2 \approx 1 - 2a\theta$.

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Then the conservation constraint $m^2 + c^2 = 1$ cannot be satisfied with $c \propto \theta$ (we cannot have $c \propto \sqrt{\theta}$ because $dc/d\theta$ should remain finite when $\lim \theta \to 0$). We have proved that for all dynamical physical systems, including macro-real systems, $C_{ij}(\Delta t) = 1 - a^2(\Delta t)^2$. With $\Delta t_{12} = \Delta t_{23} = \Delta t_{34} = \Delta t$, we get $\Delta t_{13} = 2\Delta t$ and $\Delta t_{14} = 3\Delta t$. Then,

$$L_3 = C_{12} + C_{23} - C_{13} = 1 - a^2[(\Delta t_{12})^2 + (\Delta t_{23})^2 - (\Delta t_{13})^2] = 1 + 2a^2(\Delta t)^2 > 1 \quad (6)$$

$$L_4 = C_{12} + C_{23} + C_{34} - C_{14} = 1 - a^2[(\Delta t_{12})^2 + (\Delta t_{23})^2 + (\Delta t_{34})^2 - (\Delta t_{14})^2] = 1 + 6a^2(\Delta t)^2 > 1 \quad (7)$$

The Leggett-Garg inequalities are universally violated, by all physical systems because of the necessity for ensemble-averaged quantities to obey the conservation constraints. The LG criteria are not sufficient to characterise a macro-real dynamical system. *This result is proved without any reference to quantum mechanics, or any particular theory*. The obvious implication is that all experimental tests searching for any compliance with macro-realism, with any conceivable physical system, are guaranteed to see a violation of the inequalities, proving that the inequalities are physically ineffective.

For larger intermediate values of $\Delta\theta$ (or $a\Delta t$), the inequality L_3 and L_4 are violated for a range of parameter values. With equally spaced relative angles $\theta_j = (j-1)\pi/6$ for example, we have $C_{12} = C_{23} = C_{34} = 0.866$, $C_{13} = 0.5$, and $C_{14} = 0$. Then $L_4 = 2.6 > 2$, and $L_3 = 1.23 > 1$, violating the Leggett-Garg inequalities.

3. Discussion

Ensembles of all physical systems with discretely valued observables violate the LGI because the LG criterion of macro-realism is not compatible with the necessary physical constraints arising in the conservation of dynamical quantities, applied to a statistical ensemble of measurements. Analogous results have been obtained earlier in the context of Bell's inequalities derived for local hidden variable theories [12,13]. It is natural to ask the question how exactly physical systems manage to comply with the conservation laws when a single system can occupy only a few discrete distinct states. More specifically, a microscopic spinorial quantum system has only two distinct states of spin with observable values ± 1, *when measured in any direction*. So, clearly, a single system prepared with its spin along an axis cannot return a value $+\cos\theta$ if measured along another axis at a relative angle θ. Yet, for an ensemble of measurements, we get the average value as $+\cos\theta$, fully consistent with the conservation constraint and the vectorial nature of angular momentum.

This is possible because the spinorial probability to be in the state $|\uparrow\rangle$, when measured along an axis at the relative angle θ, is $p(+1) = \cos^2(\theta/2)$, and the probability to be in the state $|\downarrow\rangle$ is $p(-1) = \sin^2(\theta/2)$. Therefore, the ensemble average is $^{+}1 \times p(+1) + (^{-}1) \times p(-1) = \cos^2(\theta/2) - \sin^2(\theta/2) = \cos\theta$. The mathematical identity $\cos^2\frac{\theta}{2} - \sin^2\frac{\theta}{2} = \cos\theta$ represents the relation between the spinorial probabilities, which is determined by quantum mechanics, and the conserved vectorial bulk angular momentum in the ensemble.

When we consider a macroscopic system, the crucial point regarding its physical constitution is that it is necessarily composed of a very large number of elementary microscopic systems. All macroscopic objects are composed of elementary particles and their atomic composites. Therefore, the Leggett-Garg scenario of just two distinct states for a macroscopic object is a drastic extreme, essentially impossible in nature considering the conditions demanded by it. This can be illustrated easily. Consider the macroscopic object consisting of a very large number ($N \approx 10^9$) of elementary spinorial constituents. A general polarised macrostate is represented as the product state $|S\rangle = |\uparrow\rangle^{\otimes N}$. This state has the robust average value $+1$ for the dichotomic observable at any time (conservation). To display either $+1$ or -1 randomly when observed, the composite system needs to be prepared in the entangled state,

$$|S\rangle_E = \frac{1}{\sqrt{2}}(|\uparrow_1\uparrow_2 \dots \uparrow_N\rangle + |\downarrow_1\downarrow_2 \dots \downarrow_N\rangle) \equiv \frac{1}{\sqrt{2}}(|\uparrow\rangle^{\otimes N} + e^{i\phi}|\downarrow\rangle^{\otimes N}) \quad (8)$$

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The probability for this state occurring 'naturally' is a miniscule $1/2^{10^9}$. This negligible number indicates also the near impossibility of preparing such a state.

A necessary consequence of the 'atomic compositeness' of every macroscopic system is that the number of distinct quantised states is of the same order as the number of constituent microscopic systems. In fact, A. Leggett had categorically stated [1] the drastic difference between a quantum state in phenomena like superfluidity and Josephson currents, with the wavefunction of the form $(a\psi_1 + b\psi_2)^N$, and a truly macroscopic quantum state, with a wavefunction of the form $a\psi_1^N + b\psi_2^N$, as the state in Equation (8). In any reasonable notion of being 'macroscopic', N is gigantic, and the preparation of a state of this kind is nearly impossible. While the primary LG criterion that a macroscopic system is in one of the discrete states at any given time could be approximately true, those states are numerous, closely spaced and quasi-continuous, differing only by a Planck action. This facilitates the faithful compliance with the constraints of fundamental conservation laws. A nearly non-invasive measurement can be done under this condition because a measurement interaction needs to change the action only by one or a few units of the Planck action, which does not perturb significantly the macroscopic physical state. Yet the LGI are violated, as I showed, because of the necessity to comply with the conservation constraints. The Leggett-Garg inequalities have no place in the real world where ensemble-averaged expectation values and correlation functions bridge the probabilities of microscopic quantum mechanics and the conservation constraints of the macroscopic world.

References

1. A. J. Leggett, Progress of Theoretical Physics Supplement **69**, 80 (1980).
2. A. J. Leggett, J. Phys.: Condens. Matter **14**, R415 (2002).
3. A. J. Leggett and A. Garg, Phys. Rev. Lett. **54**, 857 (1985).
4. C. Emary, N. Lambert and F. Nori, Rep. Prog. Phys. **77**, 016001 (2014).
5. G. Vitagliano and C. Budroni, Phys. Rev. A **107**, 040101 (2023)
6. A. J. Leggett, Rep. Prog. Phys. **71**, 022001 (2008).
7. N. Lambert, K. Debnath, A. F. Kockum, G. C. Knee, W. J. Munro, and F. Nori, Phys. Rev. A **94**, 012105 (2016).
8. C. Budroni, G. Vitagliano, G. Colangelo, R. J. Sewell, O. Gühne, G. Tóth, and M.W. Mitchell, Phys. Rev. Lett. 115, 200403 (2015).
9. L. Rosales-Zárate, B. Opanchuk, Q. Y. He, and M. D. Reid, Phys. Rev. A **97**, 042114 (2018).
10. F. Fröwis, P. Sekatski, W. Dür, N. Gisin, and N. Sangouard, Rev. Mod. Phys. **90**, 025004 (2018).
11. G. C. Knee, K. Kakuyanagi, M.-C. Yeh, Y. Matsuzaki, H. Toida, H. Yamaguchi, S. Saito, A. J. Leggett, and W. J. Munro, Nat. Commun. **7**, 13253 (2016).
12. C. S. Unnikrishnan, Europhys. Lett. **69**, 489 (2005).
13. C. S. Unnikrishnan, Pramana-Jl. Phys. **65**, 359 (2005).