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Scator Holomorphic Functions

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Abstract

Scators form a linear space equipped with a specific non-distributive product. In the elliptic case they can be interpreted as a kind of hypercomplex numbers.The standard definition of holomorphy (requiring the directional derivative to be direction-independent) leads to a generalization of the Cauchy-Riemann equation and to scator holomorphic functions. In this paper we found a complete set of solutions to the generalized Cauchy-Riemann system in the $(1+n)$-dimensional elliptic scator space. For any $n \geqslant 2$ the scator holomorphic functions consist of three classes: components exponential functions, linear functions (of a specific form) and some exceptional solutions parameterized by arbitrary functions of one variable. The obtained family of solutions, although relatively narrow, is greater than analogous functions in the quaternionic or Clifford analysis.

Keywords: scators; hyperholomorphic functions; generalized Cauchy-Riemann equations

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1. Introduction

Scators were introduced and discussed by Manuel Fernández-Guasti [1,2]. They form a linear space with a specific multiplicative structure. In the $(1+n)$-dimensional elliptic case the scator product of two scators, $\overset{o}{a} := (a_0; a_1, \ldots, a_n)$ and $\overset{o}{b} := (b_0; b_1, \ldots, b_n)$, is given by $\overset{o}{u} := (u_0; u_1, \ldots, u_n)$, where

$$\begin{aligned} u_0 &= a_0 b_0 \prod_{k=1}^{n} \left(1 - \frac{a_k b_k}{a_0 b_0}\right), \\ u_k &= \frac{a_k b_0 + b_k a_0}{a_0 b_0 - a_k b_k}\, u_0 \qquad (k = 1, \ldots, n), \end{aligned} \tag{1}$$

provided that $a_0 \neq 0$ and $b_0 \neq 0$ (more general case is presented and discussed in [1]). In the hyperbolic case all minuses should be replaced by pluses in the above formula. Mixed cases can be considered as well. The scator product is non-distributive, although a distributive approach has been recently proposed, see [3,4]. It consists in embedding scators in a larger space, where the multiplication is distributive. By the way, another example of a non-distributive algebra of hypercomplex numbers have been proposed recently [5].

In the hyperbolic case scators have potential physical interpretation and applications related to some deformations of the special relativity [6,7], while the elliptic case is a source of new (non-distributive) hypercomplex numbers and hyperholomorphic functions [2,8].

Any structure of hypercomplex numbers, usually expressed in terms of Clifford numbers (see, e.g., [9]), leads to a natural question of defining analogues of holomorphic functions [10–13]. In this paper, following [8] (see also [14]), we focus on the most straightforward definition of holomorphicity, i.e., the existence of a direction-independent derivative at any point. The scator differentiability of this

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kind implies a system of partial differential equations [8], which can be considered as a generalization of the Cauchy-Riemann equations of the complex analysis:

$$\begin{aligned} &\frac{\partial u_0}{\partial x_0} = \frac{\partial u_j}{\partial x_j}\,, \quad \frac{\partial u_j}{\partial x_0} = -\frac{\partial u_0}{\partial x_j}\,, \\ &\frac{\partial u_0}{\partial x_j}\frac{\partial u_0}{\partial x_m} = -\frac{\partial u_j}{\partial x_j}\frac{\partial u_j}{\partial x_m} \end{aligned} \tag{2}$$

for all $m \neq j$, where m and j take values from 1 to n. The generalized Cauchy-Riemann equations (2) consist of linear equations (looking like n copies of the Cauchy-Riemann equations) and a set of nonlinear equations (for $n > 1$). In the case $n = 1$ the system (2) reduces to the standard Cauchy-Riemann equations because the nonlinear part of the system (2) is missing (both m and j can assume only one value so they cannot be different). Therefore we get standard holomorphic functions on $\mathbb{C}$.

In this paper we are going to solve the open problem of finding all solutions of the system (2) in the case of arbitrary n. The case $n = 2$ has been recently already solved [15], while for $n > 2$ two classes of solutions have been earlier reported: linear affine functions [8] and components exponential functions [16,17]. We will show that, similarly as in the case $n = 2$, there exists a third class of solutions.

2. Derivation of Solutions to the Generalized Cauchy-Riemann System

To obtain all solutions, it is necessary to take into account all cases, including exceptional and singular ones. We always assume $n \geqslant 2$, because the case $n = 1$ reduces to the well known classical Cauchy-Riemann equations.

2.1. The case $\dfrac{\partial u_0}{\partial x_0} \neq 0$.

In this case we can easily transform (2) into the following form:

$$\begin{aligned} &\frac{\partial u_j}{\partial x_0} = -\frac{\partial u_0}{\partial x_j}\,, \qquad \frac{\partial u_j}{\partial x_j} = \frac{\partial u_0}{\partial x_0}\,, \qquad (j = 1, \ldots, n), \\ &\frac{\partial u_j}{\partial x_m} = -\frac{\dfrac{\partial u_0}{\partial x_j}\dfrac{\partial u_0}{\partial u_m}}{\dfrac{\partial u_0}{\partial x_0}}\,, \qquad (j, m = 1, \ldots, n), \quad m \neq j\,, \end{aligned} \tag{3}$$

where the right-hand sides depends only on u_0 and its derivatives. The compatibility conditions for the existence of a solution $u_1, \ldots, u_n$ (i.e., $u_{k,\mu\nu} = u_{k,\mu\nu}$ for $k = 1, \ldots n$ and $\mu, \nu = 0, 1, \ldots, n$) are given by:

$$\begin{aligned} &(u_{,j})_{,j} = -(u_{,0})_{,0}\,, \qquad (j = 1, \ldots, n), \\ &(u_{,j})_{,m} = \left(\frac{u_{,j}\, u_{,m}}{u_{,0}}\right)_{,0}\,, \qquad (j, m = 1, \ldots, n), \quad m \neq j\,, \\ &(u_{,0})_{,m} = -\left(\frac{u_{,j}\, u_{,m}}{u_{,0}}\right)_{,j}\,, \qquad (j, m = 1, \ldots, n), \quad m \neq j\,, \end{aligned} \tag{4}$$

where here and in the sequel we use the following notation

$$u := u_0\,, \qquad u_{,0} := \frac{\partial u_0}{\partial x_0}\,, \qquad u_{,j} := \frac{\partial u_0}{\partial x_j}\,, \qquad u_{j,m} := \frac{\partial u_j}{\partial x_m}\,, \tag{5}$$

for $j, m = 1, \ldots, n$. The Einstein summation convention is never used in this paper.

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The compatibility conditions (4) can be easily reduced to the following form

$$
\begin{aligned}
&u_{,jj} + u_{,00} = 0 \qquad (j = 1, \ldots, n)\,, \\
&u_{,jm} = \frac{u_{,j0}\, u_{,m}\, u_{,0} + u_{,m0}\, u_{,j}\, u_{,0} - u_{,00}\, u_{,m}\, u_{,j}}{(u_{,0})^2}\,, \qquad (m, j = 1, \ldots, n), \quad m \neq j, \\
&u_{,0m} = \frac{u_{,0j}\, u_{,j}\, u_{,m} - u_{,jj}\, u_{,m}\, u_{,0} - u_{,mj}\, u_{,j}\, u_{,0}}{(u_{,0})^2}\,, \qquad (m, j = 1, \ldots, n), \quad m \neq j.
\end{aligned}
\tag{6}
$$

The system (6) consists of partial differential equations for only one field, namely $u_0 \equiv u$. Therefore, we plan first to solve the system (6), and then (knowing u) to integrate the linear equations of the first order for the remaining fields u_j $(j = 1, \ldots, n)$.

2.1.1. Solving the Compatibility Conditions (6)

Substituting $u_{,jj}$ and $u_{,jm}$ from the first two equations of (6) into the last equation (6) we get

$$
u_{,0m} = \frac{u_{,j0}\, u_{,j}\, u_{,m} + u_{,00}\, u_{,m}\, u_{,0}}{(u_{,0})^2} - \frac{u_{,j}}{u_{,0}} \left(\frac{u_{,j0}\, u_{,m}\, u_{,0} + u_{,m0}\, u_{,j}\, u_{,0} - u_{,00}\, u_{,m}\, u_{,j}}{(u_{,0})^2} \right), \tag{7}
$$

for $m, j = 1, \ldots, n$ $(m \neq j)$. An elementary computation yields

$$
u_{,0m} = \frac{u_{,00}\, u_{,m}}{u_{,0}} - \frac{(u_{,j})^2}{u_{,0}} \left(\frac{u_{,m0}\, u_{,0} - u_{,00}\, u_{,m}}{(u_{,0})^2} \right), \tag{8}
$$

which is equivalent to

$$
\left(1 + \left(\frac{u_{,j}}{u_{,0}} \right)^2 \right) u_{,0m} = \frac{u_{,00}\, u_{,m}}{u_{,0}} \left(1 + \left(\frac{u_{,j}}{u_{,0}} \right)^2 \right). \tag{9}
$$

Hence $u_{,0m}\, u_{,0} = u_{,00}\, u_{,m}$ and

$$
\left(\frac{u_{,m}}{u_{,0}} \right)_{,0} = 0\,. \tag{10}
$$

Thus, as a necessary consequence of the assumption $u_{,0} \neq 0$, we get the following useful equations:

$$
u_{,m} = u_{,0}\, F_m(x_1, \ldots, x_n) \qquad (m = 1, \ldots, n), \tag{11}
$$

where F_m are functions which do not depend on x_0. Hence also

$$
\begin{aligned}
&u_{,m0} = u_{,00}\, F_m \qquad (m = 1, \ldots, n), \\
&u_{,mj} = u_{,00}\, F_j F_m + u_{,0}\, F_{m,j} \qquad (j, m = 1, \ldots, n),
\end{aligned}
\tag{12}
$$

including the case $m = j$. Now, we can simplify the system (6) using (11) and (12):

$$
\begin{aligned}
&u_{,00}\,(1 + F_j^2) + u_{,0}\, F_{j,j} = 0 \qquad (j = 1, \ldots, n)\,, \\
&u_{,0}\, F_{j,m} = 0 \qquad (m, j = 1, \ldots, n), \quad m \neq j, \\
&0 = -u_{,0}\, F_j F_{m,j} \qquad (m, j = 1, \ldots, n), \quad m \neq j.
\end{aligned}
\tag{13}
$$

We point out that in the second equation four terms turned out to be identical ($u_{,00}\, F_m F_j$, up to a sign) and canceled out. In the third equation all terms by $u_{,00}$ canceled out, as well.

Finally, as a consequence of the generalized Cauchy-Riemann system (6) we get

$$
\frac{F_{j,j}}{1 + F_j^2} = -\frac{u_{,00}}{u_{,0}}\,, \qquad F_j = F_j(x_j)\,, \qquad u,j = u_{,0}\, F_j \qquad (j = 1, \ldots, n)\,, \tag{14}
$$

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which means that
$$\frac{F_{j,j}}{1+F_j^2} = -\omega_0 \,, \quad (\log|u_{,0}|)_{,0} = \omega_0 \quad (\omega_0 = \text{const}). \tag{15}$$
The above equations for F_j and u can be easily integrated. If $\omega_0 \neq 0$, then
$$\begin{aligned} F_j &= -\tan(\omega_0 x_j + a_j) \quad (a_j = \text{const}) \quad j = 1, \ldots, n \,, \\ u &= \frac{1}{\omega_0}\, e^{\omega_0 x_0}\, g(x_1, \ldots, x_n) + h(x_1, \ldots, x_n) \end{aligned} \tag{16}$$
where g and h are functions of n variables. The last equation (for $u_{,j}$) of (14) yields
$$\frac{1}{\omega_0}\, e^{\omega_0 x_0}\, g_{,j} + h_{,j} = -e^{\omega_0 x_0} g \,\tan(\omega_0 x_j + a_j) \,. \tag{17}$$
Hence $h_{,j} = 0$ (for $j = 1, \ldots, n$), i.e.,
$$h(x_1, \ldots, x_n) = d_0 \,, \qquad (d_0 = \text{const}), \tag{18}$$
and
$$g_{,j} = -\omega_0 g \tan(\omega_0 x_j + a_j) \qquad (j = 1, \ldots, n), \tag{19}$$
which is equivalent to
$$\frac{\partial}{dx_j}\left(\frac{g}{\cos(\omega_0 x_j + a_j)}\right) = 0 \qquad (j = 1, \ldots, n). \tag{20}$$
Therefore
$$g = g_0 \,\cos(\omega_0 x_1 + a_1)\cos(\omega_0 x_2 + a_2) \ldots \cos(\omega_0 x_n + a_n), \tag{21}$$
where g_0 is a constant, and, finally
$$\begin{aligned} u &= d_0 + \frac{g_0}{\omega_0}\, e^{\omega_0 x_0} \prod_{j=1}^{n} \cos(\omega_0 x_j + a_j), \\ u_{,0} &= e^{\omega_0 x_0} g \,. \end{aligned} \tag{22}$$
If $\omega_0 = 0$, then $F_j = c_j = \text{const}$ and
$$u = x_0\, g(x_1, \ldots, x_n) + h(x_1, \ldots, x_n) \,. \tag{23}$$
Similarly as in the generic case, constraints on g and h follow from the last equation of (14):
$$x_0\, g_{,j} + h_{,j} = g\, c_j \qquad (j = 1, \ldots, n). \tag{24}$$
' Hence $g = g_0 = \text{const}$ and h is linear with respect to all variables:
$$u = d_0 + g_0(x_0 + c_1 x_1 + \ldots + c_n x_n) \,, \tag{25}$$
where d_0, g_0 and c_j ($j = 1, \ldots, n$) are constants.

2.1.2. Solving the Full System (3)

First, we consider the case $u_{,0} \neq 0$ (remember also that $u_0 \equiv u$)
$$\begin{aligned} u_{j,0} &= -u_{,0}\, F_j \,, \\ u_{j,j} &= u_{,0} \,, \\ u_{j,m} &= -u_{,0}\, F_j F_m \,, \end{aligned} \tag{26}$$

where $u_{,0}$ is given by (22). The first equation can be easily integrated

$$u_j = -uF_j + D_j(x_1, \ldots, x_n) \tag{27}$$

Inserting (27) into the third equation of (26) we get $D_{j,m} = 0$ (for $m \neq j$), i.e.,

$$D_j = D_j(x_j) \,. \tag{28}$$

Then the second equation of (26) reduces to

$$-u_{,0}\, F_j^2 - uF_{j,j} + D_{j,j} = u_{,0} \ , \tag{29}$$

and, taking into account $u = d_0 - (\omega_0)^{-1} u_{,0}$ and $F_{j,j} = \omega_0(1 + F_j^2)$ (see (15) and (22)), we get

$$d_0 F_{j,j} + D_{j,j} = 0 \,. \tag{30}$$

Thus

$$D_j = d_0 F_j + d_j \qquad (j = 1, \ldots, n), \tag{31}$$

where d_j are constants. Thus, by (27), we obtain

$$u_j = (d_0 - u) F_j + d_j \,. \tag{32}$$

Therefore, in the case $\omega_0 \neq 0$ we have

$$u_j = d_j + \frac{g_0}{\omega_0}\, e^{\omega_0 x_0}\, \tan(\omega_0 x_j + a_j) \prod_{k=1}^{n} \cos(\omega_0 x_k + a_k). \tag{33}$$

For $\omega_0 = 0$ everything simplifies: $u_{,0} = g_0$, $F_j = c_j$ and the system (26) reduces to

$$\begin{aligned} u_{j,0} &= -g_0 c_j \ , \\ u_{j,j} &= g_0 \ , \\ u_{j,m} &= -g_0 c_j c_m \ , \end{aligned} \tag{34}$$

and its general solution is given by

$$u_j = d_j + g_0 \left(-c_j x_0 + x_j - c_j \sum_{\substack{k=1 \\ k \neq j}}^{n} c_k x_k \right). \tag{35}$$

In other words, the scator holomorphic function given by (25) and (35) is just the affine map

$$u_\mu = d_\mu - g_0 \sum_{\nu=0}^{n} J_{\mu\nu} x_\nu \qquad (\mu = 0, 1, \ldots, n), \tag{36}$$

where $J_{\mu\nu}$ are the entries of the following square matrix J of size $(n+1)$:

$$J = \begin{pmatrix} -1 & -c_1 & -c_2 & -c_3 & \ldots & -c_n \\ c_1 & -1 & c_1 c_2 & c_1 c_3 & \ldots & c_1 c_n \\ c_2 & c_2 c_1 & -1 & c_2 c_3 & \ldots & c_2 c_n \\ c_3 & c_3 c_1 & c_3 c_2 & -1 & \ldots & c_3 c_n \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ c_n & c_n c_1 & c_n c_2 & c_n c_3 & \ldots & -1 \end{pmatrix}, \tag{37}$$

which actually is the Jacobi matrix of the map (36). While the symmetric form of the Jacobi matrix J is intriguing, an interpretation and implications of this phenomenon are not yet clear.

2.2. The Case $u_{,0} = 0$

In the case $u_{,0} = 0$ we have to come back to the original system (2) taking into account that $u_{j,j} = u_{,0} = 0$. Unlike the previous subsection, we do not consider the compatibility conditions first, but immediately solve the full system.

$$\begin{aligned} & u_{j,j} = 0 \, , \\ & u_{j,0} = -u_{,j} \, , \\ & u_{,j} \, u_{,m} = 0 \end{aligned} \tag{38}$$

The last equation of (38) implies that $u_{,j}$ is non-zero for at most one value of j, say $j = m$. In other words u depends only on one variable: $u = \rho_m(x_m)$. Then the second equation yields:

$$\begin{aligned} & u_{m,0} = -\rho_m'(x_m) \, , \\ & u_{j,0} = 0 \qquad (j \neq m) \, , \end{aligned} \tag{39}$$

which means that

$$\begin{aligned} u_m &= -x_0 \, \rho_m'(x_m) + \psi_m(x_1, \ldots, x_n), \\ u_j &= \phi_j(x_1, \ldots, x_n) \qquad (j \neq m) \, , \end{aligned} \tag{40}$$

and, applying the first equation of (38), we get $\rho_m'' = 0$ and $\psi_{m,m} = 0$, which means that for any $m \leqslant n$ we have:

$$\begin{aligned} u &= c_1 + c_0 x_m \, , \\ u_m &= -c_0 x_0 + \psi_m(x_1, \ldots, \widehat{x_m}, \ldots, x_n) \, , \\ u_j &= \phi_{mj}(x_1, \ldots, \widehat{x_j}, \ldots, x_n) \quad (j \neq m), \end{aligned} \tag{41}$$

where c_0 and c_1 are constants, ψ_m and ϕ_{mj} are arbitrary functions of $n-1$ variables, and the notation $\widehat{x_m}$ means that ψ_m does not depend on x_m (and, in our case, can depend on all other variables). Two indices by ϕ in the last line (ϕ_{mj}) underline that for each m we have a solution $(u_0, u_1, \ldots, u_n)$, expressed by a different set of arbitrary functions. It is convenient to denote $\phi_{mm} = \psi_m$.

3. Final Results and Classification of Solutions

The results of previous section, expressed by the formulas (22), (33), (36) and (41), can be summarized by the following theorem.

Theorem 1. *The full set of solutions to the generalized Cauchy-Riemann equations (2) consists of three families.*

- *Components exponential functions*

$$\begin{aligned} u_0 &= d_0 + \lambda_0 \, e^{\omega_0 x_0} \prod_{j=1}^{n} \cos(\omega_0 x_j + a_j), \\ u_k &= d_k + \lambda_0 \, e^{\omega_0 x_0} \, \tan(\omega_0 x_k + a_k) \prod_{j=1}^{n} \cos(\omega_0 x_j + a_j), \quad (k = 1, \ldots, n), \end{aligned} \tag{42}$$

  where λ_0, a_μ and d_μ ($\mu = 0, 1, \ldots, n$) are real constants.

- *Affine functions of the special form:*

$$u_\mu = d_\mu + \lambda_0 \sum_{\nu=0}^{n} J_{\mu\nu} x_\nu \qquad (\mu = 0, 1, \ldots, n), \tag{43}$$

where λ_0 and d_μ are real constants and $J_{\mu\nu}$ are entries of the constant matrix J given by (37), which is expressed by constants c_μ ($\mu = 0, 1, \ldots, n$).

- *Exceptional solutions parameterized by n^2 arbitrary functions:*

$$
\begin{aligned}
u_0 &= c_0 x_m + c_1 \,, \\
u_j &= -c_0 x_0\, \delta_{jm} + \phi_{mj}(x_1, \ldots, \widehat{x}_j, \ldots, x_n) \,,
\end{aligned} \tag{44}
$$

where c_0 and c_1 are real constants, δ_{mj} is Kronecker's delta, ϕ_{mj} are arbitrary functions of $n-1$ variables, and the notation $\widehat{x}_j$ means that ϕ_{mj} depends on all variables $x_1, \ldots, x_n$ except x_j.

The structure of the solution space of the generalized Cauchy–Riemann equations is essentially the same for all $n \geqslant 2$. In particular, for $n = 2$ the results of [15], although obtained in a little bit different way, coincide with the results presented above. It should be noted that the paper [15] contains misprints in formulas (7), (32), and (33). In those formulas, the functions V_1 and V_2 should depend only on z while the functions W_1 and W_2 should depend only on y.

The system (2) is nonlinear (for $n \geqslant 2$) so linear combinations of solutions usually do not satisfy this system. However, one can easily show that homogeneous affine transformations (translations and homogeneous dilations) are symmetries of the system (2).

Theorem 2. *The generalized Cauchy-Riemann equations (2) are invariant with respect to homogeneous affine transformations both in dependent and independent variables:*

$$
\tilde{u}_\mu = d_\mu + \lambda_0\, u_\mu \,, \qquad \tilde{x}_\mu = a_\mu + \omega_0\, x_\mu \,, \tag{45}
$$

where $\lambda_0, \omega_0, a_\mu, d_\mu \in \mathbb{R}$ (for $\mu = 0, 1, \ldots, n$), with $\lambda_0 \neq 0$ and $\omega_0 \neq 0$.

Many constants used in Theorem 1 arise from these symmetries. We use similar notation in both theorems to emphasize this relationship.

4. Conclusions

We derived the complete set of solutions to the generalized Cauchy-Riemann equations (2) in the elliptic scator space of any dimension. In other words, we found all scator holomorphic functions. The obtained set of solutions is not very rich, but in the case of quaternionic analysis the analogous set is much smaller, consisting only of linear affine functions [10,11]. Therefore, in the Clifford analysis, including the quaternionic analysis, other definitions of holomorphicity are widely used, like Clifford-holomorphic or monogenic functions, see, e.g., [12]. It would be interesting to investigate such possibilities also in the case of the scator spaces.

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