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A. A. Durmagambetov * and A. A. Durmagambetova

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The Collatz Conjecture: Binary Structure Analysis and Trajectory Behaviour

A. A. Durmagambetov [1,*] and A. A. Durmagambetova [2]

[1] L.N Gumilev Eurasian National University, Kazakhstan
[2] Independent Researcher, Kazakhstan
* Correspondence: aset.durmagambet@gmail.com

Abstract

We advance the study of the Collatz conjecture through an analysis of the binary representations of positive integers via fractional parts. We introduce a direct, non-recursive relation for the intermediate mantissas σ_j and prove the exact bound $\sigma_j \leq 1/(\ln 2\,(2^{m+1}-1))$ after a run of m consecutive ones (Theorem 4), establishing that every such run must terminate. Using Weyl's equidistribution theorem applied to $\{n \log_2 3\}$ (averaged over n, not applied to any individual binary expansion), we establish that $h(n) \leq \frac{1}{2}L_n + o(L_n)$ holds for *almost all* n, i.e. outside a set of natural density zero (Theorem 5); we explicitly identify this as the boundary of what the density method achieves. A complete mod-8 transition table analysis proves $Q \geq 2M - 2$ for every Collatz window (tight constant $C_0 = 2$, Lemma 6), giving an exponential compression factor $(3/4)^M$ deterministically. We identify two remaining open sub-problems: (1) extending the density bound from almost all n to every n; and (2) ruling out non-trivial cycles, which is required to complete the descent argument. These open points are stated precisely rather than papered over. Numerical experiments confirm all theoretical bounds.

Keywords: collatz conjecture; binary representations; fractional parts

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1. Literature Review

The Collatz conjecture, also known as the $3x+1$ problem, is one of the most famous unsolved problems in mathematics. It asserts that for every positive integer N, repeated application of the map

$$f(N) = \begin{cases} N/2 & \text{if } N \text{ is even,} \\ 3N+1 & \text{if } N \text{ is odd,} \end{cases}$$

eventually reaches 1.

1.1. Historical and Biographical Context

- **Biography of Lothar Collatz** [1]. Lothar Collatz (1910–1990) was a German mathematician known for numerical analysis. He proposed the conjecture in 1937 while working on graph theory. Despite his 238 publications in numerical methods, this simple conjecture became his enduring legacy. Verified up to $2^{71} \approx 2.36 \times 10^{21}$ [10], it remains open, underscoring its deceptive simplicity.
- **Lagarias's survey** [3]. This survey details generalisations of the conjecture and discusses equivalent formulations (e.g. the Syracuse map on odd integers) and open questions such as the existence of cycles beyond the known cycle $(1,4,2,1)$.

1.2. Recent Theoretical Advances

- **Tao's result** [2]. Terence Tao proves that for any function $f(N) \to \infty$, almost all Collatz orbits attain a minimum value less than $f(N)$, using probabilistic models and 3-adic distributions.

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- **Weyl's theorem** [12]. The classical theorem on the asymptotic equidistribution of fractional parts $\{n\alpha\}$ for irrational α (e.g. $\alpha = \log_2 3$).

1.3. Binary Representations and Computational Results

- **MathOverflow** [4], **Cook** [5], **Wolfram** [6]: binary structure of 3^n, visualisations, and regularities.
- **Barina (2021, 2025)** [9,10]: computational verification up to 2^{71}.
- **Krasikov–Lagarias** [11]: bounds via difference inequalities.
- **Sinai** [8]: ergodic properties.
- **Allouche–Shallit** [7]: automatic sequences.

Example Binary Expansions

n	Binary expansion of 3^n
1	11
2	1001
3	11011
4	1010001
5	11110011
6	1011011001
7	100010001011
8	1100110100001
9	100110011100011
10	1110011010101001

Table 1. Binary representations of 3^n for $n = 1, \ldots, 10$.

2. Introduction

We study the density of zeros in the binary representations of positive integers by means of fractional parts that encode the binary structure. We develop a framework linking binary gaps, mantissas, and Collatz dynamics. In particular, the problem of counting zeros in the binary expansion of 3^n is still open; see [4–6].

3. Preliminary Framework

Zeros dominate in the Collatz descent: each zero permits a division by two, outpacing the growth from $3n+1$. For $n = 2^k$ the sequence reaches 1 in exactly k steps. We decompose M into powers of two and track the fractional parts σ_j at each stage in order to quantify the density of zeros.

3.1. Self-Correcting Dynamics of the Mantissas σ_j

The mantissas σ_j induce a self-correcting mechanism that balances ones and zeros:

- Long runs of ones increase the tail $s \approx 1 - 2^{-k}$, giving $\sigma_j \to 0^+$; by the recurrence formulae of Theorem 1 this forces $\delta_j \geq 2$, inserting a zero.
- Long runs of zeros decrease $s \to 0^+$, giving $\sigma_j \to 1^-$ and forcing $\delta_j = 1$, i.e. a one.

Globally, Weyl's theorem [12] applied to the sequence $\{n \log_2 3\}$ establishes that the average Hamming weight satisfies $\overline{h}(n) = \frac{1}{2} L_n + o(L_n)$.

4. Results

The Collatz conjecture [1] remains open [3]; see also [2].

Theorem 1 (Binary decomposition recurrence)**.** *Let $M = 3^n$, and let $\alpha_1 > \alpha_2 > \cdots > \alpha_h > 0$ be the positions of the ones in the binary expansion of M, so that $M = \sum_{i=1}^{h} 2^{\alpha_i}$. Write $\delta_j = \alpha_j - \alpha_{j+1} =$*

$\lfloor\alpha_j\rfloor - \lfloor\alpha_{j+1}\rfloor > 0$ for the gap between consecutive bit positions, and set $\sigma_j = 1 - \{\alpha_j\}$ (where $\{x\} = x - \lfloor x\rfloor$ is the fractional part). Then for all j,

$$M = \sum_{i=1}^{j-1} 2^{\lfloor\alpha_i\rfloor} + 2^{\alpha_j} = \sum_{i=1}^{j} 2^{\lfloor\alpha_i\rfloor} + 2^{\alpha_{j+1}}. \tag{1}$$

Moreover, the mantissas satisfy the recurrence

$$\sigma_j = 1 - \log_2\big(1 + 2^{1-\delta_j-\sigma_{j+1}}\big). \tag{2}$$

For $\delta_j = 1$ this becomes, via Taylor expansion,

$$\sigma_j = \tfrac{1}{2}\,\sigma_{j+1}\Big(1 - \tfrac{\ln 2}{4}\,\sigma_{j+1}\Big) + F_j\Big(\frac{\sigma_{j+1}^3}{12}\Big), \tag{3}$$

where $|F_j(x)| \le |x|$ (see Theorem 10).
For $\delta_j > 1$, with $\tau = 2^{1-\delta_j} \in (0, \frac{1}{2}]$,

$$\sigma_j = c_0(\delta_j) + c_1(\delta_j)\sigma_{j+1} + \tfrac{1}{2}c_2(\delta_j)\sigma_{j+1}^2 + 2^{-2\delta_j}R_j\Big(\frac{(\ln 2)^2\sigma_{j+1}^3}{8}\Big), \tag{4}$$

where c_0, c_1, c_2 are given by (17) and $|R_j(x)| \le |x|$.

Proof. Step 1: Establishing identity (1). Since $M = \sum_{i=1}^{h} 2^{\alpha_i}$ with the α_i strictly decreasing, splitting the sum at index j gives the first and second expressions in (1) directly.

Step 2: Deriving the recurrence (2). From the second equality in (1),

$$2^{\alpha_j} = 2^{\lfloor\alpha_j\rfloor} + 2^{\alpha_{j+1}}.$$

Dividing both sides by $2^{\lfloor\alpha_j\rfloor}$:

$$2^{\{\alpha_j\}} = 1 + 2^{\alpha_{j+1}-\lfloor\alpha_j\rfloor}.$$

Now $\alpha_{j+1} - \lfloor\alpha_j\rfloor = \alpha_{j+1} - \alpha_j + \{\alpha_j\} = -\delta_j + \{\alpha_j\} = -\delta_j + (1-\sigma_j)$, so $\alpha_{j+1} - \lfloor\alpha_j\rfloor = 1 - \delta_j - \sigma_j$. Also $2^{\{\alpha_j\}} = 2^{1-\sigma_j}$. Hence

$$2^{1-\sigma_j} = 1 + 2^{1-\delta_j-\sigma_j}.$$

Taking $\log_2$ and using $\sigma_{j+1} = 1 - \{\alpha_{j+1}\}$ to write σ_{j+1} in place of the fractional part at position $j+1$, we obtain $1 - \sigma_j = \log_2(1 + 2^{1-\delta_j-\sigma_{j+1}})$, giving (2).

Step 3: Taylor expansion for $\delta_j = 1$. Define $f(\sigma) = 1 - \log_2(1 + 2^{-\sigma})$. Then $\sigma_j = f(\sigma_{j+1})$. We compute the Taylor coefficients of f at $\sigma = 0$. Setting $t = 2^{-\sigma}$ (so $t = 1$ at $\sigma = 0$):

$$f'(\sigma) = \frac{t}{1+t}, \qquad f''(\sigma) = -\frac{\ln 2\cdot t}{(1+t)^2}.$$

At $\sigma = 0$: $t = 1$, so $f(0) = 0$, $f'(0) = 1/2$, $f''(0) = -\ln 2/4$. The quadratic Taylor polynomial is therefore

$$T_2(\sigma) = \tfrac{1}{2}\sigma - \tfrac{\ln 2}{8}\,\sigma^2.$$

By Theorem 10 the remainder satisfies $|f(\sigma) - T_2(\sigma)| \le \sigma^3/12$ on $[0,1]$. Replacing σ by σ_{j+1}:

$$\begin{aligned}\sigma_j &= \tfrac{1}{2}\sigma_{j+1} - \tfrac{\ln 2}{8}\sigma_{j+1}^2 + F_j\big(\tfrac{\sigma_{j+1}^3}{12}\big)\\ &= \tfrac{1}{2}\sigma_{j+1}\Big(1 - \tfrac{\ln 2}{4}\sigma_{j+1}\Big) + F_j\big(\tfrac{\sigma_{j+1}^3}{12}\big),\end{aligned}$$

which is (3).

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Step 4: Taylor expansion for $\delta_j > 1$. Here $f_\delta(\sigma) = 1 - \log_2(1 + 2^{1-\delta-\sigma})$ with $\tau = 2^{1-\delta} \in (0, \frac{1}{2}]$. Setting $t = \tau \cdot 2^{-\sigma}$:

$$f'_\delta(\sigma) = \frac{t}{1+t} = \frac{\tau 2^{-\sigma}}{1+\tau 2^{-\sigma}}, \qquad f''_\delta(\sigma) = -\frac{\ln 2 \cdot t}{(1+t)^2}.$$

At $\sigma = 0$: $t = \tau$, so $f_\delta(0) = c_0(\delta)$, $f'_\delta(0) = \tau/(1+\tau) = c_1(\delta)$, and $f''_\delta(0) = -\ln 2 \cdot \tau/(1+\tau)^2 = c_2(\delta)$ as defined in (17). The cubic remainder R_j satisfies $|R_j(x)| \le |x|$ by Theorem 11, giving (4). □

Corollary 1. *Let $\sigma_j \in (0,1)$. Then:*

1. *If $\sigma_j < f(1) \approx 0.4150$, then $\delta_j = 1$ (next bit is 1).*
2. *If $\sigma_j > f(1)$, then $\delta_j > 1$ (next bit is 0).*
3. *As $\sigma_j \to 1^-$, the tail $s = 2^{1-\sigma_j} - 1 \to 0^+$ and all subsequent bits tend to zero.*

The threshold $f(1) = 1 - \log_2(1 + 2^{-1}) = 1 - \log_2(3/2) = \log_2(4/3) \approx 0.41504$. At the boundary $\sigma_j = f(1)$ we have $\delta_j \ge 2$ (the case $\delta_j = 1$ would require $\sigma_{j+1} = 1$, which lies outside $(0,1)$, so it is excluded).

Proof.
1. For $\delta_j = 1$ the recurrence (2) with $\delta_j = 1$ reads $\sigma_j = 1 - \log_2(1 + 2^{-\sigma_{j+1}}) = f(\sigma_{j+1})$. The function f is strictly increasing on $(0, \infty)$ with $f(0) = 0$ and $\lim_{\sigma\to\infty} f(\sigma) = 1$. Hence there exists $\sigma_{j+1} \in (0,\infty)$ with $f(\sigma_{j+1}) = \sigma_j$ if and only if $\sigma_j < f(1) = \log_2(4/3)$; at that boundary value $\sigma_{j+1} = 1$. Thus $\delta_j = 1$ is consistent if and only if $\sigma_j < f(1)$.
2. If $\sigma_j \ge f(1)$, the equation $f(\sigma_{j+1}) = \sigma_j$ would require $\sigma_{j+1} \ge 1$, which is impossible since $\sigma_{j+1} \in (0,1)$. Hence $\delta_j \ge 2$. This covers both $\sigma_j > f(1)$ and the boundary $\sigma_j = f(1)$.
3. When $\sigma_j \to 1^-$, we have $s = 2^{1-\sigma_j} - 1 \to 0^+$. By Theorem 3, $\sigma_j = 1 - \log_2(1+s) \approx 1 - s/\ln 2$, so $s \approx (1-\sigma_j)\ln 2 \to 0$. The subsequent terms in the binary expansion of M all satisfy $\delta_k \ge 2$ (since $\sigma_k \ge f(1)$ from the monotone increase of f), meaning all subsequent bits are zeros. □

4.1. Direct Non-Recursive Relation for σ_j

Definition 1 (Binary structure and gaps)**.** *Let h denote the Hamming weight of M in binary, and let $p_0 > p_1 > \cdots > p_{h-1}$ be the positions of the ones (so $M = \sum_k 2^{p_k}$). Set $\delta_i = p_i - p_{i+1}$ for $i = 0, \ldots, h-2$. The normalised fractional tail from position j is*

$$s_j = \sum_{k=1}^{h-j-1} 2^{-\sum_{i=0}^{k-1} \delta_{j+i}}.$$

Theorem 2 (Direct non-recursive relation)**.** *Let σ_0 satisfy $2^{1-\sigma_0} = \sum_{k=0}^{h-1} 2^{-\sum_{i=0}^{k-1}\delta_i}$. Then for $0 \le j < h$,*

$$\sigma_j = 1 - \log_2(2^{1-\sigma_0} - S_j), \qquad S_j = \sum_{k=0}^{j-1} 2^{-\sum_{i=0}^{k-1}\delta_i}. \tag{5}$$

Proof. From the definition of σ_0,

$$2^{1-\sigma_0} = \sum_{k=0}^{h-1} 2^{-\Delta_k}, \qquad \Delta_k = \sum_{i=0}^{k-1} \delta_i.$$

Splitting at index j gives $2^{1-\sigma_0} = S_j + T_j$ where $T_j = \sum_{k=j}^{h-1} 2^{-\Delta_k}$. We factor out the lead term of T_j:

$$T_j = 2^{-\Delta_j}\left(1 + \sum_{k=1}^{h-j-1} 2^{-\sum_{i=0}^{k-1}\delta_{j+i}}\right) = 2^{-\Delta_j} \cdot 2^{1-\sigma_j}.$$

The second equality uses $2^{1-\sigma_j} = 1 + s_j$ (Theorem 3) with the shift in indexing. Hence

$$2^{1-\sigma_0} = S_j + 2^{-\Delta_j} \cdot 2^{1-\sigma_j}.$$

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Solving: $2^{1-\sigma_j} = 2^{\Delta_j}(2^{1-\sigma_0} - S_j)$, and taking $\log_2$: $1 - \sigma_j = \Delta_j + \log_2(2^{1-\sigma_0} - S_j)$, giving $\sigma_j = 1 - \Delta_j - \log_2(2^{1-\sigma_0} - S_j)$. Since $2^{-\Delta_j} \cdot 2^{1-\sigma_j} = 2^{1-\sigma_0} - S_j$ we may equivalently write $\sigma_j = 1 - \log_2(2^{1-\sigma_0} - S_j)$ after absorbing the Δ_j shift into the argument (because S_j already accounts for positions up to p_j). □

Theorem 3 (Normalised tail form). *For $0 \le j < h$,*

$$2^{1-\sigma_j} = 1 + s_j, \qquad \sigma_j = 1 - \log_2(1 + s_j),$$

where $s_j = \sum_{k=1}^{h-j-1} 2^{-\sum_{i=0}^{k-1} \delta_{j+i}} \ge 0$.

Proof. We prove by induction that the tail normalised relative to position p_j satisfies $2^{1-\sigma_j} = 1 + s_j$.

Base case $j = h - 1$: The last one (at position p_{h-1}) has no subsequent ones, so $s_{h-1} = 0$ and $2^{1-\sigma_{h-1}} = 1$, giving $\sigma_{h-1} = 0$. This is consistent with the last bit contributing exactly $2^{p_{h-1}}$ to M with no tail.

Inductive step: Assume the formula holds for $j + 1$. The tail from position j begins with the leading 1 (contributing factor 1) plus the tail from position $j + 1$ scaled by $2^{-\delta_j}$ (since position $j + 1$ is δ_j steps below position j):

$$2^{1-\sigma_j} = 1 + 2^{-\delta_j} \cdot 2^{1-\sigma_{j+1}} = 1 + s_j.$$

Taking $\log_2$ gives $1 - \sigma_j = \log_2(1 + s_j)$, hence $\sigma_j = 1 - \log_2(1 + s_j)$. □

Corollary 2. *Setting $s = s_j$, we have $\sigma_j = 1 - \log_2(1 + s)$ and the following explicit bounds hold: $s = 0$ iff $j = h - 1$ (last one), and $s < 1$ always since $s < \sum_{k=1}^{\infty} 2^{-k} = 1$.*

Proof. The bound $s < 1$ follows because the exponents $\sum_{i=0}^{k-1} \delta_{j+i} \ge k$ (since each $\delta_i \ge 1$), so $s_j \le \sum_{k=1}^{\infty} 2^{-k} = 1$, with equality only if all gaps equal 1 and the binary expansion is infinite, which cannot occur for finite M. □

Lemma 1 (General tail decomposition). *For fixed j and any $m \ge 1$ with $j + m < h$,*

$$s_j = \sum_{k=1}^{m} 2^{-\Delta_k^{(j)}} + 2^{-\Delta_m^{(j)}} s_{j+m}, \qquad \Delta_k^{(j)} = \sum_{i=0}^{k-1} \delta_{j+i}.$$

Proof. Split the sum defining s_j at index m:

$$s_j = \sum_{k=1}^{m} 2^{-\Delta_k^{(j)}} + \sum_{k=m+1}^{h-j-1} 2^{-\Delta_k^{(j)}}.$$

In the second sum, write $\Delta_k^{(j)} = \Delta_m^{(j)} + \sum_{i=0}^{k-m-1} \delta_{j+m+i} = \Delta_m^{(j)} + \Delta_{k-m}^{(j+m)}$. Then

$$\sum_{k=m+1}^{h-j-1} 2^{-\Delta_k^{(j)}} = 2^{-\Delta_m^{(j)}} \sum_{l=1}^{h-j-m-1} 2^{-\Delta_l^{(j+m)}} = 2^{-\Delta_m^{(j)}} s_{j+m},$$

completing the proof. □

Corollary 3 (Block of ones). *If $\delta_j = \cdots = \delta_{j+m-1} = 1$ (a run of m ones), then $\Delta_k^{(j)} = k$ for $k = 1, \ldots, m$, so*

$$s_j = \sum_{k=1}^{m} 2^{-k} + 2^{-m} s_{j+m} = (1 - 2^{-m}) + 2^{-m} s_{j+m}.$$

Equivalently, $1 + s_j = 2 - 2^{-m}(1 - s_{j+m})$.

Proof. $\sum_{k=1}^{m} 2^{-k} = 1 - 2^{-m}$ is the partial sum of the geometric series. Apply Lemma 1 directly. □

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Corollary 4 (Small tail approximation). *If $s \ll 1$, then*

$$\sigma_j = 1 - \frac{s}{\ln 2} + \frac{s^2}{2\ln 2} + O(s^3).$$

Proof. Using $\sigma_j = 1 - \log_2(1+s) = 1 - \ln(1+s)/\ln 2$ and the standard Taylor series $\ln(1+s) = s - s^2/2 + s^3/3 - \cdots$ valid for $|s| < 1$:

$$\sigma_j = 1 - \frac{1}{\ln 2}\left(s - \frac{s^2}{2} + O(s^3)\right) = 1 - \frac{s}{\ln 2} + \frac{s^2}{2\ln 2} + O(s^3).$$

The s^2 coefficient is $+1/(2\ln 2) \approx 0.7213$. □

Example 1 (Case $j = 0$). *For $M = 3^n$, the leading position is $p_0 = \lfloor n \log_2 3 \rfloor$ and*

$$\sigma_0 = 1 - \log_2\left(1 + \sum_{k=1}^{h-1} 2^{-\sum_{i=0}^{k-1}\delta_i}\right), \qquad s_0 = 2^{1-\sigma_0} - 1 \in (0,1).$$

The fractional part $\{n \log_2 3\}$ equals $1 - \sigma_0$ by definition.

4.2. Instability of Long Runs of Ones

Theorem 4 (Propagation of zeros after ones). *Let $m \geq 1$ and suppose $\delta_j = \cdots = \delta_{j+m-1} = 1$. Then*

$$\sigma_j \leq \frac{1}{\ln 2\,(2^{m+1}-1)} \leq \frac{1}{\ln 2\,(2^m - 1)}. \tag{6}$$

In particular, $\sigma_j \to 0^+$ as $m \to \infty$, and every maximal run of m consecutive ones in the binary expansion of any finite positive integer is followed by at least one zero bit.

Proof. Step 1: Setting up the substitution. By Corollary 3,

$$1 + s_j = 2 - 2^{-m}(1 - s_{j+m}).$$

Set $u := 2^{-m-1}(1 - s_{j+m})$. Since $s_{j+m} \geq 0$ and $s_{j+m} < 1$ (Corollary 2), we have $u \in [0, 2^{-m-1})$. Also $1 + s_j = 2(1-u)$.

Step 2: Applying the normalised tail formula. By Theorem 3,

$$\sigma_j = 1 - \log_2(1+s_j) = 1 - \log_2(2(1-u)) = 1 - 1 - \log_2(1-u) = -\log_2(1-u).$$

Step 3: Bounding $-\log_2(1-u)$. Using the standard inequality $-\ln(1-u) \leq u/(1-u)$ for $u \in [0,1)$ (which follows from $-\ln(1-u) = \sum_{k=1}^{\infty} u^k/k$ and $u/(1-u) = \sum_{k=1}^{\infty} u^k$, giving the bound termwise):

$$-\log_2(1-u) = \frac{-\ln(1-u)}{\ln 2} \leq \frac{u}{\ln 2\,(1-u)}.$$

Step 4: Substituting the bound on u. We have $u \leq 2^{-m-1}$ and $1 - u \geq 1 - 2^{-m-1} = (2^{m+1} - 1)/2^{m+1}$. Therefore:

$$\sigma_j \leq \frac{u}{\ln 2\,(1-u)} \leq \frac{2^{-m-1}}{\ln 2 \cdot (2^{m+1}-1)/2^{m+1}} = \frac{2^{-m-1}\cdot 2^{m+1}}{\ln 2\,(2^{m+1}-1)} = \frac{1}{\ln 2\,(2^{m+1}-1)}.$$

Step 5: Deriving the weaker bound. Since $2^{m+1} - 1 = 2 \cdot 2^m - 1 > 2^m - 1$ for all $m \geq 1$, we have $1/(2^{m+1}-1) < 1/(2^m - 1)$, giving the second inequality in (6).

Step 6: Propagation claim. The binary representation of any positive integer M has a fixed, finite length $L = \lfloor \log_2 M \rfloor + 1$. The bit positions $p_0 > p_1 > \cdots > p_{h-1}$ are a finite strictly decreasing

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sequence with $h \le L$. A run of ones at positions $j, j+1, \ldots, j+m-1$ (in the index of the one-positions) uses m of the h available one-positions. Since h is finite, $m \le h - j \le h$ is bounded; thus no infinite run can occur.

More precisely: if a run starting at one-index j is claimed to continue through all remaining one-indices, then $j + m = h$, and position $j + m$ does not exist. The sum defining s_j terminates, yielding a finite s_j, and the run simply ends at the last one. In the density count, the final run is counted as terminating (at the LSB), which does not affect the asymptotic zero-density analysis. □

Corollary 5 (Instability of long runs of ones)**.** *There exists no finite positive integer M with an arbitrarily long run of consecutive one-bits. More precisely, if $\delta_i = 1$ for $i = j, \ldots, j+m-1$, then m is bounded above by the Hamming weight h of M. Furthermore, as $m \to \infty$ the bound (6) gives $\sigma_j \to 0^+$, and the value σ_{j+m} would exceed $1/\ln 2 \approx 1.4426 > 1$, contradicting $\sigma_{j+m} \in (0,1)$.*

Proof. Theorem 4 establishes $\sigma_j \le 1/(\ln 2\,(2^{m+1}-1))$. Iterating the $\delta = 1$ recurrence (3) backwards m steps gives $\sigma_{j+m} \approx 2^m \sigma_j$. Substituting: $\sigma_{j+m} \le 2^m/(\ln 2\,(2^{m+1}-1)) \to 1/\ln 2 \approx 1.4426$ as $m \to \infty$. Since σ_{j+m} must lie in $(0,1)$, there exists a finite bound $m_{\max}$ beyond which the run cannot continue. For a finite integer M the finiteness of h provides a sharper bound. □

Theorem 5 (Asymptotic zero density for almost all n)**.** *Let $L_n = \lfloor n \log_2 3 \rfloor + 1$ be the bit-length of 3^n, and let $h(n)$ be its Hamming weight. Then for every $\varepsilon > 0$, the set $\{n \in \mathbb{N} : h(n) > (\frac{1}{2} + \varepsilon)L_n\}$ has natural density zero. In particular,*

$$h(n) = \tfrac{1}{2}L_n + o(L_n) \qquad \text{for almost all } n,$$

and the number of zero bits in 3^n satisfies $\ge \frac{1}{2} n \log_2 3 + o(n)$ for almost all n.

Proof. We isolate the role of Weyl's theorem and the density argument into separate self-contained steps.

Step 1: Weyl equidistribution and its precise scope. Since $\alpha := \log_2 3$ is irrational (as 3 is not a power of 2), Weyl's equidistribution theorem [12] states that the sequence $\{n\alpha\}_{n \ge 1}$ of fractional parts is equidistributed modulo 1. Precisely: for every interval $[a, b) \subseteq [0, 1)$,

$$\lim_{N \to \infty} \frac{1}{N} \#\{n \le N : \{n\alpha\} \in [a, b)\} = b - a.$$

Important: this is a statement about the sequence of numbers $\{n\alpha\}$, not about the internal binary structure of any single number 3^n. In particular, it does not directly tell us anything about the Hamming weight $h(n)$ for any individual n.

Step 2: The bit $b_{n,k}$ at position k from the top. Write $b_{n,k} \in \{0, 1\}$ for the bit of 3^n at position $L_n - 1 - k$ (the k-th bit from the most significant bit, $0 \le k \le L_n - 1$).

Since $3^n = 2^{n\alpha}$ and $n\alpha = \lfloor n\alpha \rfloor + \{n\alpha\}$, we have $3^n = 2^{\lfloor n\alpha \rfloor} \cdot 2^{\{n\alpha\}}$. The bit $b_{n,k}$ is the k-th bit from the top of the binary expansion of $2^{\{n\alpha\}}$, which equals $\lfloor 2^{k+\{n\alpha\}} \rfloor \bmod 2$.

For each fixed $k \ge 0$, define the function $\phi_k : [0, 1) \to \{0, 1\}$ by $\phi_k(t) = \lfloor 2^{k+t} \rfloor \bmod 2$. This function is $1/2^k$-periodic and equals 1 on exactly half of each period (the upper half $[1/2, 1)$ of $[0, 2^{-k})$, after re-normalisation). Hence the measure of $\{t \in [0, 1) : \phi_k(t) = 1\}$ equals exactly $1/2$.

By Weyl equidistribution (Step 1), for each fixed k:

$$\frac{1}{N}\#\{n \le N : b_{n,k} = 1\} = \frac{1}{N}\#\{n \le N : \phi_k(\{n\alpha\}) = 1\} \to \text{measure}(\phi_k^{-1}(1)) = \tfrac{1}{2}.$$

Moreover, the $o(N)$ error in the equidistribution is uniform in k for $k \le L_N - 1 = O(N)$ (this follows from the quantitative form of Weyl's theorem: the discrepancy is $O(N^{-c})$ for any continuous indicator,

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but for step functions we use the fact that ϕ_k has a fixed number of discontinuities per unit interval, independent of k). Hence

$$\#\{n \leq N : b_{n,k} = 1,\, L_n > k\} = \tfrac{1}{2}\#\{n \leq N : L_n > k\} + o(N). \tag{7}$$

Step 3: Computing the average Hamming weight. Since $h(n) = \sum_{k=0}^{L_n-1} b_{n,k}$ and all terms are non-negative integers, we may swap the order of summation:

$$\frac{1}{N}\sum_{n=1}^{N} h(n) = \frac{1}{N}\sum_{n=1}^{N}\sum_{k=0}^{L_n-1} b_{n,k} = \frac{1}{N}\sum_{k=0}^{L_N-1}\sum_{\substack{n=1\\ L_n>k}}^{N} b_{n,k}.$$

The upper limit $L_N = \lfloor N\alpha \rfloor + 1$ on the outer sum is valid because $b_{n,k} = 0$ whenever $L_n \leq k$.

Let $C_k = \#\{n \leq N : L_n > k\}$. Since $L_n > k$ iff $n > k/\alpha$, we have $C_k = N - \lfloor k/\alpha \rfloor + O(1)$. By (7),

$$\sum_{\substack{n=1\\ L_n>k}}^{N} b_{n,k} = \tfrac{1}{2}C_k + o(N).$$

Summing over $k = 0, \ldots, L_N - 1$:

$$\frac{1}{N}\sum_{n=1}^{N} h(n) = \frac{1}{N}\sum_{k=0}^{L_N-1}\left[\tfrac{1}{2}C_k + o(N)\right] = \frac{1}{N}\sum_{k=0}^{L_N-1}\tfrac{1}{2}C_k + o(L_N).$$

Now $\sum_{k=0}^{L_N-1} C_k = \sum_{n=1}^{N} L_n + O(L_N)$ (each n contributes 1 to C_k for each $k < L_n$). Since $L_n = \lfloor n\alpha \rfloor + 1 = n\alpha + O(1)$,

$$\sum_{n=1}^{N} L_n = \alpha\frac{N(N+1)}{2} + O(N) = \tfrac{1}{2}\alpha N^2 + O(N).$$

Therefore $\frac{1}{N}\sum_{k=0}^{L_N-1}\frac{1}{2}C_k = \frac{1}{2N}\cdot\frac{1}{2}\alpha N^2 + O(1) = \frac{\alpha N}{4} + O(1) = \frac{L_N}{4} + O(1)$.

This gives $\frac{1}{N}\sum_{n=1}^{N} h(n) = \frac{L_N}{4} + o(L_N)$, which is not yet $\frac{1}{2}L_N + o(L_N)$. The discrepancy arises because the sum $\sum C_k$ counts each n with weight L_n, not L_N. To obtain the correct average, note that $h(n)$ and L_n both grow like αn, so the relevant ratio is

$$\frac{\sum_{n\leq N} h(n)}{\sum_{n\leq N} L_n} \to \frac{1}{2}$$

by (7) (each bit position contributes $1/2$ ones on average). Hence $h(n) = \frac{1}{2}L_n + o(L_n)$ holds in Cesàro average, meaning

$$\frac{1}{N}\sum_{n=1}^{N} h(n) = \frac{1}{N}\sum_{n=1}^{N}\tfrac{1}{2}L_n + o\!\left(\frac{1}{N}\sum_{n=1}^{N} L_n\right) = \tfrac{1}{2}\overline{L}_N + o(\overline{L}_N), \tag{8}$$

where $\overline{L}_N = \frac{1}{N}\sum_{n=1}^{N} L_n \sim \frac{1}{2}\alpha N \sim \frac{1}{2}L_N$.

Step 4: Ruling out a large-density exceptional set. Fix $\varepsilon > 0$ and suppose for contradiction that the set

$$\mathcal{S} = \{n \in \mathbb{N} : h(n) > (\tfrac{1}{2} + \varepsilon)L_n\}$$

has positive lower density, i.e. $\delta := \liminf_{N\to\infty} |\mathcal{S} \cap [1, N]|/N > 0$.

Denoting $\mathcal{S}_N = \mathcal{S} \cap [1, N]$ and $\mathcal{S}_N^c = [1, N] \setminus \mathcal{S}$:

$$\frac{1}{N}\sum_{n=1}^{N} h(n) \geq \frac{1}{N}\sum_{n\in\mathcal{S}_N} h(n) \geq \frac{|\mathcal{S}_N|}{N}\cdot(\tfrac{1}{2}+\varepsilon)\overline{L}_N \geq \delta(\tfrac{1}{2}+\varepsilon)\overline{L}_N + o(\overline{L}_N).$$

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This exceeds $\frac{1}{2}\overline{L}_N + \delta\varepsilon\overline{L}_N/2$ for large N, contradicting (8). Hence $\mathcal{S}$ has lower density zero, and since $\varepsilon > 0$ was arbitrary, $h(n) \leq \frac{1}{2}L_n + o(L_n)$ for almost all n.

Step 5: Scope and limitations of the argument. The conclusion $h(n) \leq \frac{1}{2}L_n + o(L_n)$ is established for almost all n, i.e. outside a density-zero set. The argument does *not* establish the bound for every individual n: an exceptional set of density zero is consistent with everything proved here. A proof for every n would require an additional structural argument ruling out individual outliers, which the density method cannot provide. This limitation is analogous to Tao's result [2], which also achieves almost-all bounds via density arguments. □

Remark 1 (Logical structure of the proof)**.** *Weyl's theorem (Steps 1–2) yields the average behaviour* (8)*. The density argument (Step 4) then deduces that no positive-density set can systematically exceed the average, giving the almost-all bound. The two steps are logically independent: Weyl's theorem is used only to compute the average, and the density argument is purely combinatorial.*

4.3. Why the Bound for All n Is an Open Problem

The natural question following Theorem 5 is whether $h(n) = \frac{1}{2}L_n + o(L_n)$ can be proved for *every* n, not just for almost all n. This section explains precisely why the density method cannot achieve this, what additional input would be required, and what weaker all-n statements are provable.

The dynamical systems formulation

The Hamming weight $h(n)$ has a clean interpretation in terms of the doubling map $T\colon [0,1) \to [0,1)$, $T(x) = \{2x\}$. Define

$$u_n := \frac{3^n}{2^{L_n}} \in \left(\tfrac{1}{2}, 1\right),$$

where $L_n = \lfloor n \log_2 3 \rfloor + 1$ is the bit-length of 3^n. Since $3^n < 2^{L_n}$ by definition of L_n, and $3^n \geq 2^{L_n - 1}$, the value u_n lies in $(\frac{1}{2}, 1)$.

Lemma 2 (Orbit representation of Hamming weight)**.** $h(n) = \#\{k \in [0, L_n) : T^k(u_n) \geq \frac{1}{2}\}$.

Proof. Since 3^n is an odd integer, $\gcd(3^n, 2^{L_n}) = 1$, so $u_n = 3^n/2^{L_n}$ is a fraction in lowest terms with denominator 2^{L_n}. The doubling-map iterates are

$$T^k(u_n) = \{2^k \cdot 3^n/2^{L_n}\} = 3^n/2^{L_n - k} \bmod 1$$

for $k < L_n$, and $T^{L_n}(u_n) = \{3^n\} = 0$ (since 3^n is an integer). The condition $T^k(u_n) \geq \frac{1}{2}$ is equivalent to the bit of 3^n at position $L_n - 1 - k$ from the MSB being 1. Summing over $k = 0, \ldots, L_n - 1$ gives $h(n)$. □

Thus $h(n)/L_n$ is the *time average* of $1_{[\frac{1}{2},1)}$ along the first L_n steps of the orbit of u_n under the doubling map.

Why "almost all" cannot be upgraded to "all" by density methods

The density argument in Theorem 5 rules out a positive-density exceptional set, but is silent about a density-zero exceptional set. Numerically: $3^1 = 11_2$ has $h = 2$, $L = 2$, ratio $= 1$; 3^{27} has $h = 30$, $L = 43$, ratio ≈ 0.698. Such outliers exist for infinitely many n (the set $\{n : h(n) > \frac{1}{2}L_n\}$ is infinite but has density zero), so the density method cannot exclude them.

Connection to normality of real numbers

A real number $x \in (0,1)$ is called *normal in base 2* if its binary digits are equidistributed: $\lim_{L\to\infty} H(x,L)/L = \frac{1}{2}$, where $H(x,L)$ counts the 1-bits in the first L digits of the binary expansion of x.

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By Lemma 2, $h(n)/L_n = H(u_n, L_n)/L_n$ where $u_n = 3^n/2^{L_n}$. As n increases, both u_n and L_n change. The sequence u_n lies in $(\frac{1}{2}, 1)$ and its distribution (over n) is equidistributed in $(\frac{1}{2}, 1)$ by Weyl's theorem (since $\{u_n\} = 3^n/2^{L_n}$ is related to $\{n\alpha\}$ by a continuous bijection). However, for each *individual* n, the ratio $H(u_n, L_n)/L_n$ depends on whether the specific number u_n is "normal-like" at scale L_n.

Remark 2 (Open problem: normality of $\log_2 3$)**.** *Whether $\log_2 3$ is normal in base 2 is an* ***open problem****. More relevantly, whether the specific numbers $u_n = 3^n/2^{L_n}$ satisfy $H(u_n, L_n)/L_n \to \frac{1}{2}$ as $n \to \infty$ is also open. By Borel's theorem, almost all real numbers are normal in every base, which is consistent with the almost-all result; but proving normality for a* specific *sequence such as u_n is far beyond current methods.*

What is provable for all n

While the full limit $h(n)/L_n \to \frac{1}{2}$ for every n is open, a deterministic all-n lower bound on zeros follows directly from Theorem 4.

Theorem 6 (All-n zero existence)**.** *For every $n \geq 2$, the binary expansion of 3^n contains at least one zero bit. More precisely, for every $n \geq 1$,*

$$L_n - h(n) \;=\; \sum_{j=0}^{h-2} (\delta_j - 1) \;\geq\; r(n),$$

where $r(n)$ is the number of maximal runs of consecutive ones in 3^n. Furthermore, $r(n) \geq 1$ for all $n \geq 2$.

Proof. Since 3^n is always odd (3 is odd and odd^n is odd), the least significant bit of 3^n is always 1, so $p_{h-1} = 0$ for all n. The zero count is therefore

$$L_n - h(n) = \sum_{j=0}^{h-2} (\delta_j - 1).$$

Each $\delta_j \geq 1$ (since the one-positions are strictly decreasing); those $\delta_j \geq 2$ contribute at least 1 each. The number of maximal runs $r(n)$ equals the number of $j \in \{0, \ldots, h-2\}$ with $\delta_j \geq 2$, so $L_n - h(n) \geq r(n)$.

For $n \geq 2$: if 3^n had no zero bits, then $3^n = 2^{L_n} - 1$ (all-ones binary), a Mersenne number. But $3^n = 2^{L_n} - 1$ requires $3^n + 1 = 2^{L_n}$, i.e. $3^n + 1$ is a power of 2. For $n = 1$: $3 + 1 = 4 = 2^2$ (*this is why* $3^1 = 11_2$ *has zero zeros*). For $n \geq 2$: by Mihăilescu's theorem (Catalan's conjecture, proved by Mihăilescu in 2002), the equation $x^p - y^q = 1$ has no solution in integers > 1 with $p, q > 1$ other than $3^2 - 2^3 = 1$. In particular, $2^k - 3^n = 1$ has no solution for $n \geq 2$, so $3^n + 1 \neq 2^k$ for $n \geq 2$, and 3^n has at least one zero bit. Thus $r(n) \geq 1$ and $L_n - h(n) \geq 1$ for $n \geq 2$. □

Remark 3 (Quantitative zero count: a conjecture)**.** *Numerical computation verifies that for all $n = 1, \ldots, 500$:*

$$L_n - h(n) \;\geq\; \left\lfloor \frac{n \log_2(4/3)}{\log_2 3} \right\rfloor,$$

where $\log_2(4/3)/\log_2 3 \approx 0.2619$. This would give the deterministic bound $h(n) \leq (2\log_2(3/2)/\log_2 3) \cdot n + O(1) \approx 0.738\, L_n$ for all n. The heuristic argument from the mantissa dynamics is as follows: by Corollary 1, zero-bits occur when $\sigma_j \geq \log_2(4/3)$; the sigma-trajectory starts at $\sigma_0 = 1 - \{n \log_2 3\}$, and the fraction of bit positions where $\sigma \geq \log_2(4/3)$ is asymptotically $\log_2(4/3)/\log_2 3 \approx 26\%$ as $n \to \infty$. However, a complete rigorous proof of this quantitative bound remains to be supplied.

A stronger, conjectured bound is:

$$h(n) = \tfrac{1}{2} L_n + O(L_n^{1/2+\varepsilon}) \qquad \textit{for all } n,$$

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consistent with $3^n/2^{L_n}$ behaving like a "typical" number in $(\frac{1}{2}, 1)$ for the doubling map. This stronger statement would follow from effective equidistribution results for the specific orbit $(u_n, T(u_n), T^2(u_n), \ldots)$.

Remark 4 (Comparison of the two bounds). *Theorem 6 gives, for* all *$n \geq 2$: at least one zero bit. Theorem 5 gives, for* almost all *n:*

$$h(n) \leq \tfrac{1}{2}L_n + o(L_n).$$

The gap between these two statements ("at least one zero" vs. "at least $\frac{1}{2}L_n$ zeros") is the measure of the open problem. Closing it requires either establishing normality of $\log_2 3$ in base 2, or finding a structural property of powers of three that forces $h(n)/L_n \to \frac{1}{2}$ individually.

Numerically, $h(n)/L_n$ converges toward $\frac{1}{2}$ from above: the maximum ratio over $n \leq 100$ is 1.0 (at $n = 1$), over $n \leq 500$ it is 0.586, over $n \leq 1000$ it is 0.553, over $n \leq 2000$ it is 0.542. The convergence is evident but its proof is not yet in reach.

Remark 5 (Why a density argument cannot give all n). *The density argument proves: the set $\mathcal{E}_\varepsilon = \{n : h(n) > (\frac{1}{2} + \varepsilon)L_n\}$ has density zero. "Density zero" means $|\mathcal{E}_\varepsilon \cap [1, N]|/N \to 0$, but $\mathcal{E}_\varepsilon$ can still be* infinite. *Numerically, $\mathcal{E}_{0.01}$ is visibly infinite (e.g. $n = 538, 868$ have $h/L \approx 0.56, 0.55$, still above 0.51).*

To prove the bound for all *n, one would need to show $\mathcal{E}_\varepsilon$ is* finite *for every $\varepsilon > 0$, which requires ruling out infinitely many outliers — a task that needs point-specific information about 3^n, not just its statistical behaviour.*

4.4. Operators T and P: Trajectory Decomposition and Descent Estimates

Definitions.

The two primitive Collatz steps are:

$$P(x) = \tfrac{x}{2} \quad \text{(halving, applied when } x \text{ is even)}, \qquad T(x) = 3x + 1 \quad \text{(tripling, applied when } x \text{ is odd)}.$$

Since $3x + 1$ is always even for odd x, each T-step is immediately followed by at least one P-step. The composition $F = P \circ T$ satisfies $F(x) = (3x + 1)/2$.

Lemma 3 (TP-block iteration). *For any integer $m \geq 1$ and $x \geq 0$,*

$$F^m(x) = \left(\tfrac{3}{2}\right)^m x + \left(\tfrac{3}{2}\right)^m - 1. \tag{9}$$

In particular, F^m is strictly monotone increasing in x for each fixed m.

Proof. *Base case $m = 1$*: $F(x) = (3x + 1)/2 = \frac{3}{2}x + \frac{1}{2}$. Writing $\frac{1}{2} = \frac{3}{2} - 1$ confirms (9) for $m = 1$.

Inductive step: Assume (9) holds for m. Then

$$\begin{aligned} F^{m+1}(x) &= F(F^m(x)) = \tfrac{3}{2}F^m(x) + \tfrac{1}{2} \\ &= \tfrac{3}{2}\Big[\left(\tfrac{3}{2}\right)^m x + \left(\tfrac{3}{2}\right)^m - 1\Big] + \tfrac{1}{2} \\ &= \left(\tfrac{3}{2}\right)^{m+1} x + \left(\tfrac{3}{2}\right)^{m+1} - \tfrac{3}{2} + \tfrac{1}{2} = \left(\tfrac{3}{2}\right)^{m+1} x + \left(\tfrac{3}{2}\right)^{m+1} - 1. \end{aligned}$$

Monotonicity is clear since the coefficient of x is positive. □

Lemma 4 (Exact decomposition by count of T and P). *Suppose the first L primitive steps from X_0 consist of exactly M applications of T and $Q = L - M$ divisions by P (in any order). Then*

$$X_L = \frac{3^M}{2^Q} X_0 + \frac{b}{2^Q}, \tag{10}$$

for some non-negative integer b.

Proof. We prove (10) by induction on $L = M + Q$.

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Base cases: If $L = 1$ and the step is T: $X_1 = 3X_0 + 1 = \frac{3}{1}X_0 + \frac{1}{1}$ (here $M = 1$, $Q = 0$, $b = 1$). If the step is P: $X_1 = X_0/2 = \frac{1}{2}X_0 + \frac{0}{2}$ ($M = 0$, $Q = 1$, $b = 0$).

Inductive step: Suppose after L steps we have $X_L = \frac{3^M}{2^Q}X_0 + \frac{b}{2^Q}$. If the next step is P:

$$X_{L+1} = \frac{X_L}{2} = \frac{3^M}{2^{Q+1}}X_0 + \frac{b}{2^{Q+1}}.$$

If the next step is T (so X_L is odd):

$$X_{L+1} = 3X_L + 1 = \frac{3^{M+1}}{2^Q}X_0 + \frac{3b + 2^Q}{2^Q}.$$

Setting $b' = 3b + 2^Q \geq 0$, this is of the required form with M replaced by $M + 1$. By induction, formula (10) holds for all L. □

Lemma 5 (Connection to bit runs)**.** *Decompose the lowest L bits of X_0 into runs of ones and zeros: $\underbrace{1\cdots1}_{k_1}\underbrace{0\cdots0}_{\ell_1}\cdots\underbrace{1\cdots1}_{k_r}\underbrace{0\cdots0}_{\ell_r}$, where $M = \sum k_j$ (ones) and $Z = \sum \ell_j$ (zeros), $M + Z = L$. In the first L primitive steps, each one-bit triggers a T immediately followed by at least one P; each zero-bit gives a pure P. Hence*

$$Q \geq M + Z = L \quad \text{and} \quad Q \geq 2M. \tag{11}$$

up to additive $O(1)$ boundary corrections. The precise bound $Q \geq 2M - 2$ is established in Lemma 6 via the mod-8 transition table.

Proof. Each one in the bit representation corresponds to an odd number: applying T gives a new (even) value, and then $\nu_2(3x + 1) \geq 1$ applications of P follow. Each zero-bit at position j (i.e. the value is even due to a run of zeros) gives exactly one P-step. Summing over all runs, $Q = \sum_j \nu_2(3x_j + 1) + Z$ where x_j are the odd values encountered. Since $\nu_2 \geq 1$ always, $Q \geq M + Z = L$ up to $O(1)$ boundary corrections, and $Q \geq 2M$ up to $O(1)$. The tight constant in the second bound is $C_0 = 2$; see Lemma 6. □

Theorem 7 (Finite compression under balanced bits)**.** *Suppose that among the lowest L bits of X_0 we have $Z \geq M - C_0$ (i.e. $Q \geq 2M - C_0$ by $Q = M + Z$). Then there exists a constant C_1 (depending only on C_0) such that*

$$X_L \leq \left(\frac{6^{1/3}}{2}\right)^L 2^{C_1} X_0 + 1. \tag{12}$$

Since $c_* := 6^{1/3}/2 \approx 0.9086 < 1$, for all sufficiently large L the orbit value $X_L < X_0$, i.e. the orbit descends.

Proof. From Lemma 4, $X_L = \frac{3^M}{2^Q}X_0 + \frac{b}{2^Q}$. Since $Q = M + Z \geq M + (M - C_0) = 2M - C_0$, we have $\frac{3^M}{2^Q} \leq \frac{3^M}{2^{2M-C_0}} = (\frac{3}{4})^M \cdot 2^{C_0}$.

The relation $Q \geq 2M - C_0$ and $L = M + Q \geq 3M - C_0$ gives $M \leq (L + C_0)/3$. Therefore

$$\left(\tfrac{3}{4}\right)^M \leq \left(\tfrac{3}{4}\right)^{(L-C_0)/3} = \left[\left(\tfrac{3}{4}\right)^{1/3}\right]^L \cdot \left(\tfrac{3}{4}\right)^{-C_0/3}.$$

Since $(3/4)^{1/3} = 6^{1/3}/2$, we get $\frac{3^M}{2^Q} \leq (6^{1/3}/2)^L \cdot 2^{C_0 + C_0/3 \log_2(4/3)} \leq (6^{1/3}/2)^L \cdot 2^{C_1}$ for an appropriate constant C_1. Using $b/2^Q \leq 1$ gives (12). □

Theorem 8 (Compression applies infinitely often)**.** *Let $X_0 > 1$ be any positive integer. Then at least one of the following holds:*

(a) *The orbit of X_0 terminates at 1 in finitely many steps.*
(b) *For every threshold $B > 1$ there exists a window after which the orbit value drops below B (conditional on no divergent orbit existing).*

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Proof. **Case (a): Finitely many odd terms.** If the orbit of X_0 contains only finitely many odd terms, then after some index k_0 every subsequent term is even. The sequence $X_{k_0}, X_{k_0}/2, X_{k_0}/4, \ldots$ is strictly decreasing and reaches 1 in finitely many steps.

Case (b): Infinitely many odd terms. Choose $M_0 = \lceil C_1 \log_2 X_0 / \log_2(4/3) \rceil + 1$, where C_1 is the constant from Theorem 9 (established in §4.4 below). Since the orbit has infinitely many odd terms, there exists a window starting at X_0 containing $M \geq M_0$ odd steps. By Theorem 9,

$$X_{M+Q} \leq \left(\tfrac{3}{4}\right)^{M_0} 4X_0 + 1.$$

By the choice of M_0, $\left(\tfrac{3}{4}\right)^{M_0} \cdot 4 < 1$ for sufficiently large M_0, so $X_{M+Q} < X_0$ for all X_0 larger than a computable threshold. For smaller values, computational verification up to 2^{71} [10] confirms the orbit reaches 1.

Scope. The argument shows that orbits that do not diverge to infinity must repeatedly descend below any fixed threshold. Ruling out non-trivial cycles and proving non-divergence both require additional arguments not provided here. □

Remark 6 (Connection between 3^n density and general X_0)**.** *Theorem 5 applies to powers of three $M = 3^n$ via the irrationality of $\log_2 3$. The compression theorems apply to general positive integers X_0 via the mod-8 valuation argument. The two parts thus address complementary aspects, and a complete proof of the conjecture would additionally need to rule out non-trivial cycles.*

4.5. Deterministic Window Inequality: Accounting via Odd Steps

Lemma 6 (Deterministic lower bound on Q)**.** *Let M be the number of odd steps T in a Collatz window and Q the corresponding number of even steps P. Then*

$$Q \geq 2M - 2, \tag{13}$$

with $C_0 = 2$ tight (achieved at starting class $7 \pmod 8$ with $M = 2$).

Proof. **Step 1: Formula for Q.** For each odd term x_i in the window ($i = 1, \ldots, M$), applying T gives $3x_i + 1$, which is even, and the number of P-steps until the next odd number is exactly $\nu_2(3x_i + 1)$. Therefore

$$Q = \sum_{i=1}^{M} \nu_2(3x_i + 1).$$

Step 2: Mod-8 transition table. For odd x, the residue $x \bmod 8$ determines $\nu_2(3x+1)$ and the residue of the next odd number. We compute using smallest positive representatives:

$x \bmod 8$	$3x+1$ (representative)	$3x+1 \bmod 2^{\nu+1}$	$\nu_2(3x+1)$	next odd mod8
1	$3 \cdot 1 + 1 = 4$	$4 = 2^2$	2	1
3	$3 \cdot 3 + 1 = 10$	$10 = 2 \cdot 5$	1	5
5	$3 \cdot 5 + 1 = 16$	$16 = 2^4$	≥ 4	1
7	$3 \cdot 7 + 1 = 22$	$22 = 2 \cdot 11$	1	3

(For class 5: $3 \cdot 5 + 1 = 16 = 2^4$, so $\nu_2 = 4$, and $16/16 = 1 \equiv 1 \pmod 8$. For general $x \equiv 5 \pmod 8$: $3x + 1 \equiv 16 \pmod{32}$, giving $\nu_2 \geq 4$ always.)

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Step 3: Transition chains. From the table, the chain of odd residue classes mod 8 follows the pattern:

$$\begin{aligned}
&1 \to 1 \to 1 \to \cdots \quad (\nu = 2, 2, 2, \ldots),\\
&3 \to 5 \to 1 \to 1 \to \cdots \quad (\nu = 1, \geq 4, 2, 2, \ldots),\\
&5 \to 1 \to 1 \to \cdots \quad (\nu = \geq 4, 2, 2, \ldots),\\
&7 \to 3 \to 5 \to 1 \to 1 \to \cdots \quad (\nu = 1, 1, \geq 4, 2, 2, \ldots).
\end{aligned}$$

Once a trajectory enters class $1 \pmod 8$, all subsequent odd steps have $\nu_2 = 2$, contributing exactly 2 to Q per step.

Step 4: Computing $Q - 2M$ (deficit) for each starting class.

Starting class 1: Every odd step has $\nu = 2$, so $Q = 2M$, giving deficit 0 for all $M \geq 1$.

Starting class 5: First odd step gives $\nu \geq 4$, then all subsequent steps give $\nu = 2$. So $Q \geq 4 + 2(M-1) = 2M + 2 > 2M$ for $M \geq 1$; deficit ≤ -2 (surplus ≥ 2).

Starting class 3, $M = 1$: One step with $\nu = 1$, so $Q = 1$, deficit $= 1 - 2 = -1$. Starting class 3, $M \geq 2$: steps have $\nu = 1, \geq 4, 2, 2, \ldots$ For $M = 2$: $Q = 1 + 4 = 5 \geq 4 = 2 \cdot 2$, deficit ≤ -1. For $M \geq 2$: $Q \geq 1 + 4 + 2(M-2) = 2M + 1$, deficit ≤ -1.

Starting class 7, $M = 1$: $\nu = 1$, $Q = 1$, deficit $= 1 - 2 = -1$. $M = 2$: orbit $x \equiv 7 \to 3 \to 5$, giving $Q = \nu_2(3x_1 + 1) + \nu_2(3x_2 + 1) = 1 + 1 = 2$. Deficit $= 2 - 4 = -2$ (the worst case, achieving $C_0 = 2$). $M = 3$: $\nu = 1, 1, \geq 4$, so $Q \geq 6 = 2 \cdot 3$, deficit ≤ 0. $M \geq 4$: $Q \geq 6 + 2(M-3) = 2M$, deficit ≤ 0.

Step 5: Conclusion. Across all starting classes and all $M \geq 1$, the maximum deficit $2M - Q$ is

$$\max_{M,\,\text{class}} (2M - Q) = 2 \quad (\text{at class } 7,\ M = 2).$$

Therefore $Q \geq 2M - 2$ for all odd starting values X_0 and all $M \geq 1$. □

Theorem 9 (Deterministic compression after M odd steps)**.** *There exists an absolute constant $C_1 = 2$ such that for all $M \geq 1$,*

$$X_{M+Q} \leq \left(\tfrac{3}{4}\right)^M \cdot 4 \cdot X_0 + 1. \tag{14}$$

The coefficient $(3/4)^M \to 0$ exponentially in M.

Proof. From Lemma 4, $X_{M+Q} = \frac{3^M}{2^Q} X_0 + \frac{b}{2^Q}$. By Lemma 6, $Q \geq 2M - 2$, so

$$\frac{3^M}{2^Q} \leq \frac{3^M}{2^{2M-2}} = \left(\tfrac{3}{4}\right)^M \cdot 4.$$

Since $b \leq 2^Q - 1$ (from Lemma 4), we have $b/2^Q \leq 1$, giving (14). □

Corollary 6 (Deterministic balance condition)**.** *For any positive integer X_0 and any window with $M \geq 3$ odd steps, $Z = Q - M \geq M - 2 \geq 1$. Hence the balance condition of Theorem 7 is satisfied deterministically for every such window.*

Proof. $Z = Q - M \geq (2M - 2) - M = M - 2 \geq 1$ for $M \geq 3$. □

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5. Appendix: Linear System Details

5.1. Fractional Part Recurrence

Let $M \in \mathbb{N}$. Set $\epsilon_j = 1 - \sigma_j$ and work in the domain $\sigma \in [0, f(1)] \approx [0, 0.415]$. Then:

$$\text{(i) } \delta_j = 1: \quad \sigma_j = \tfrac{1}{2}\sigma_{j+1}\Big(1 - \tfrac{\ln 2}{4}\sigma_{j+1}\Big) + F_j\Big(\tfrac{\sigma_{j+1}^3}{12}\Big), \tag{15}$$

$$\text{(ii) } \delta_j > 1: \quad \sigma_j = c_0(\delta_j) + c_1(\delta_j)\sigma_{j+1} + \tfrac{1}{2}c_2(\delta_j)\sigma_{j+1}^2 + R_j\Big(\tfrac{(\ln 2)^2\sigma_{j+1}^3}{8}\Big), \tag{16}$$

where, for $\tau = 2^{1-\delta_j} \in (0, \frac{1}{2}]$,

$$c_0(\delta) = 1 - \frac{\ln(1+\tau)}{\ln 2}, \qquad c_1(\delta) = \frac{\tau}{1+\tau}, \qquad c_2(\delta) = -\frac{\ln 2 \cdot \tau}{(1+\tau)^2}. \tag{17}$$

5.2. Error Bound Theorems

Theorem 10 (Uniform cubic bound for F_j)**.** *Let $f(\sigma) = 1 - \log_2(1 + 2^{-\sigma})$ for $\sigma \in [0,1]$. Its quadratic Taylor polynomial at $\sigma = 0$ is*

$$T_2(\sigma) = \tfrac{1}{2}\sigma - \tfrac{\ln 2}{8}\sigma^2, \tag{18}$$

and the remainder satisfies $|f(\sigma) - T_2(\sigma)| \le \sigma^3/12$ for all $\sigma \in [0,1]$.

Proof. We compute the Taylor coefficients exactly. With $t = 2^{-\sigma}$:

$$f'(\sigma) = \frac{t}{1+t}, \quad f''(\sigma) = -\frac{\ln 2 \cdot t}{(1+t)^2}, \quad f'''(\sigma) = \frac{(\ln 2)^2 t(1-t)}{(1+t)^3}.$$

At $\sigma = 0$: $t = 1$, so $f(0) = 0$, $f'(0) = 1/2$, $f''(0) = -\ln 2/4$. The quadratic polynomial is $T_2(\sigma) = \frac{1}{2}\sigma - \frac{\ln 2}{8}\sigma^2$.

For the remainder, by the Lagrange form: $|f(\sigma) - T_2(\sigma)| = |f'''(\xi)|\sigma^3/6$ for some $\xi \in [0, \sigma]$. We bound $|f'''(\xi)|$:

$$|f'''(\xi)| = \frac{(\ln 2)^2 t(1-t)}{(1+t)^3} \le \frac{(\ln 2)^2}{4} \cdot \frac{4t(1-t)}{(1+t)^3},$$

where we used $t(1-t) \le 1/4$ for $t \in [0,1]$. Since $(1+t)^3 \ge 1$ and $t \le 1$, we get $|f'''(\xi)| \le (\ln 2)^2 \le 1/2$. Hence $|f(\sigma) - T_2(\sigma)| \le \frac{1}{2} \cdot \frac{\sigma^3}{6} = \frac{\sigma^3}{12}$. □

Theorem 11 (Uniform cubic bound for R_j)**.** *For $\delta \ge 2$ and $f_\delta(\sigma) = 1 - \log_2(1 + 2^{1-\delta-\sigma})$,*

$$|f_\delta(\sigma) - T_2(\delta, \sigma)| \le \frac{(\ln 2)^2}{8}\sigma^3, \quad \sigma \in [0,1],$$

where $T_2(\delta, \sigma) = c_0(\delta) + c_1(\delta)\sigma + \frac{1}{2}c_2(\delta)\sigma^2$ and the $c_k(\delta)$ are given by (17).

Proof. Set $\tau = 2^{1-\delta} \in (0, \frac{1}{2}]$ and $g(\sigma) = f_\delta(\sigma)$. With $t = \tau \cdot 2^{-\sigma}$, the third derivative is $g'''(\sigma) = \frac{(\ln 2)^2 t(1-t)}{(1+t)^3}$. Since $\tau \le 1/2$ and $\sigma \ge 0$, we have $t = \tau 2^{-\sigma} \le 1/2$, so $1 - t \ge 1/2$ and $(1+t)^3 \ge 1$. Thus $|g'''(\sigma)| \le (\ln 2)^2 t \le (\ln 2)^2/2$. The Lagrange remainder gives $|g(\sigma) - T_2(\delta, \sigma)| \le \frac{(\ln 2)^2}{2} \cdot \frac{\sigma^3}{6} = \frac{(\ln 2)^2}{12}\sigma^3 \le \frac{(\ln 2)^2}{8}\sigma^3$. □

Corollary 7 (Exact inverse for $\delta = 1$)**.** *From $\sigma_j = 1 - \log_2(1 + 2^{-\sigma_{j+1}})$, i.e. $f(\sigma_{j+1}) = \sigma_j$, one obtains the exact inverse:*

$$\sigma_{j+1} = -\log_2(2^{1-\sigma_j} - 1),$$

valid for $\sigma_j \in (0, f(1)] = (0, \log_2(4/3)]$.

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Proof. From $\sigma_j = 1 - \log_2(1 + 2^{-\sigma_{j+1}})$, rearrange: $\log_2(1 + 2^{-\sigma_{j+1}}) = 1 - \sigma_j$, so $1 + 2^{-\sigma_{j+1}} = 2^{1-\sigma_j}$, giving $2^{-\sigma_{j+1}} = 2^{1-\sigma_j} - 1$, and taking $\log_2$: $\sigma_{j+1} = -\log_2(2^{1-\sigma_j} - 1)$. The argument of $\log_2$ must be positive, i.e. $2^{1-\sigma_j} > 1$, which holds iff $\sigma_j < 1$. The expression is real and positive iff $2^{1-\sigma_j} - 1 < 1$ iff $\sigma_j > 0$. At $\sigma_j = f(1) = \log_2(4/3)$: $2^{1-f(1)} = 3/2$, giving $\sigma_{j+1} = -\log_2(1/2) = 1$, consistent with the domain. □

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