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Article

Helical Triadic Coalgebras Final Coalgebras for $F(X) = \mathbb{Z}_3 \times X^3$ in $\mathbb{Z}_3\text{-Set}$

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Abstract

This paper investigates the final coalgebra for the endofunctor $F(X) = \mathbb{Z}_3 \times X^3$ on the category $\mathbb{Z}_3\text{-Set}$ of sets with a $\mathbb{Z}_3$ -action. We call the resulting F -coalgebras *Helical Triadic Coalgebras* (HTCs). The factor $\mathbb{Z}_3$ records an observable *phase* that makes distinct cyclic positions distinguishable. We develop the notion of $\mathbb{Z}_3$ -*bisimulation*, which generalizes standard bisimulation by allowing cyclic shifts. Our main results concern a natural HTC structure on the srs lattice (Laves graph). The canonical morphism from the srs coalgebra $\mathcal{S}$ to the final coalgebra Ω is not injective: translations induce a bisimulation collapsing $\mathcal{S}$ onto a 12-element quotient $\mathcal{Q} \cong K_4 \times \mathbb{Z}_3$. The V_4 -symmetry of srs further collapses $\mathcal{Q}$ onto a 3-element image $\mathcal{I}$. A symmetry analysis reveals that $\mathcal{I}$ is symmetric while Ω is not. We also define orbital invariants (binding index, degeneracy, multiplicity) and establish that every regular coalgebra is chiral. Finally, we prove that among sub-coalgebras of the final coalgebra, symmetry and connectivity alone characterize srs uniquely (up to chirality). These results bridge coalgebraic methods with graph theory and crystallography.

Keywords: final coalgebra; helical triadic coalgebra; $\mathbb{Z}_3$ -bisimulation; $\mathbb{Z}_3$ -action; srs lattice; quotient coalgebra; orbital invariants; symmetry; chirality

MSC: 18C15 (Coalgebras); 68Q85 (Bisimulation); 05C25 (Graphs and groups); 52C99 (Discrete geometry)

1. Introduction

The theory of coalgebras provides a categorical framework for the study of state-based systems and infinite data structures [7,13]. Given an endofunctor $F : \mathcal{C} \rightarrow \mathcal{C}$, an F -coalgebra is an object X of $\mathcal{C}$ together with a morphism $d : X \rightarrow F(X)$ assigning to each state its observable structure. The *final coalgebra*, when it exists, serves as a universal domain: every coalgebra admits a unique morphism into it, and two states are identified if and only if they are *standard bisimilar* (behaviorally equivalent in the categorical sense).

Final coalgebras for polynomial functors on **Set** are well understood through Adámek's theorem and Lambek's lemma. However, the situation becomes richer when we work in categories with additional structure. In this paper, we investigate the functor $F(X) = \mathbb{Z}_3 \times X^3$ on the category $\mathbb{Z}_3\text{-Set}$ of sets equipped with a $\mathbb{Z}_3$ -action. The functor includes an observable *phase* component in $\mathbb{Z}_3$, ensuring that cyclic positions remain distinguishable in the final coalgebra.

The main contributions are:

1. An adaptation of bisimulation to the cyclic structure: $\mathbb{Z}_3$ -*bisimulation* (Section 4).
2. An axiomatic definition of *Helical Triadic Coalgebras* (Section 5), including a *self-similarity theorem* showing that $\Omega \cong \mathbb{Z}_3 \times \Omega^3$ exhibits fractal structure (Section 5.4), and a natural such structure on the srs lattice, a classical 3-regular graph from crystallography (Section 6), with a *local self-similarity theorem* proving that every vertex generates the entire coalgebra (Section 6.4).

3. A *factorization theorem*: the canonical morphism from the srs coalgebra to the final coalgebra factors through a 12-element quotient $Q \cong K_4 \times \mathbb{Z}_3$, which further collapses onto a 3-element image $\mathcal{I}$ (Sections 7 and 8).
4. A *symmetry dichotomy*: the image $\mathcal{I}$ is symmetric while the final coalgebra Ω is not; an intrinsic criterion for symmetry follows from orbital invariants (Section 10).
5. A *chirality theorem*: every regular coalgebra is chiral (Section 9).
6. A *strong uniqueness theorem*: under natural axioms capturing “geometric” HTC’s (regularity, symmetry, periodicity, connectivity, maximality), the srs lattice is the unique realization, up to enantiomorphism (Section 11). Crucially, *symmetry is the only genuine external constraint*: regularity follows from symmetry (Lemma 12), periodicity follows from symmetry (Theorem 16), while all other properties derive from the functor F itself.

Regarding novelty: the existence and coinduction framework are classical [1,9,13], while the equivariant specialization to $\mathbb{Z}_3\text{-Set}$ and the notion of $\mathbb{Z}_3$ -bisimulation are new to this setting. The factorization $\mathcal{S} \twoheadrightarrow Q \twoheadrightarrow \mathcal{I}$ and the symmetry/chirality consequences are original contributions. They refine classical srs and covering-space descriptions [4,15] by tracking cyclic phase data. These results identify a concrete obstruction to symmetry in regular coalgebras, motivating the open problems in Section 12.

These results link coalgebraic methods with several classical areas: covering space theory (the srs lattice is the maximal abelian cover of K_4), crystallographic topology (space group $I4_132$), and combinatorial graph theory (3-regular graphs with girth 10).

The paper proceeds as follows. Section 2 discusses prior work. Section 3 recalls the categorical framework. Section 4 develops $\mathbb{Z}_3$ -bisimulation. Section 5 defines the Helical Triadic Coalgebra axiomatically and treats sub-coalgebras. Sections 6–8 establish the main structural results on the srs coalgebra. Section 9 defines orbital invariants, proves that regular coalgebras are chiral, and classifies finite coalgebras. Section 10 addresses symmetry, showing $\mathcal{I}$ is symmetric while Ω is not. Section 11 establishes that srs is the unique geometric realization of HTC. Section 12 summarizes contributions and discusses open problems. Appendices collect crystallographic data and notation.

2. Related Work

Final coalgebras for polynomial functors on **Set** are classical. Existence and structural properties follow from terminal sequences and Lambek’s lemma [1,9,16]. For unlabeled tree functors on **Set** (e.g., $F(X) = X^n$ with $n \geq 2$), the final coalgebra is a singleton: without observable labels, all infinite n -ary trees are behaviorally equivalent. However, working in the equivariant category $\mathbb{Z}_3\text{-Set}$ enriches the structure. The phase dynamics ($\text{obs}(\theta(x)) = \text{obs}(x) + 1$, where obs denotes the observable phase; see Definition 3) implies that every F -coalgebra carries a *free* $\mathbb{Z}_3$ -action—no fixed points exist. This structural constraint, combined with the observable phase, prevents collapse and yields a non-trivial final coalgebra. Our use of coinduction and extensionality follows the standard coalgebraic semantics literature [7,13], specialized here to the equivariant setting. We are not aware of prior work specifically addressing final coalgebras for $F(X) = \mathbb{Z}_3 \times X^3$ in $\mathbb{Z}_3\text{-Set}$.

The $\mathbb{Z}_3$ -bisimulation introduced here corresponds to bisimulation up to the group action, allowing cyclic phase shifts in the matching relation (see Definition 8). The ambient category $\mathbb{Z}_3\text{-Set}$ is a presheaf category [10]; it provides the equivariant backbone for our construction.

On the geometric side, the srs lattice is well studied in graph theory and crystallography. It appears as the Laves graph and the net of the gyroid minimal surface [4,12,15]. Our coalgebraic viewpoint complements the covering theory of graphs developed in [6], encoding the cyclic phase data arising from the rotation system (Section 6) in the language of F -coalgebras.

3. Preliminaries

This section establishes the categorical framework. We define the category $\mathbb{Z}_3\text{-Set}$ of sets with a $\mathbb{Z}_3$ -action, the endofunctor $F : \mathbb{Z}_3\text{-Set} \rightarrow \mathbb{Z}_3\text{-Set}$ given by $F(X, \theta) = (\mathbb{Z}_3 \times X^3, \hat{\theta})$, and the notion of

F -coalgebra. The classical results of Adámek and Lambek then guarantee existence and uniqueness (up to unique isomorphism) of a final coalgebra; see Theorem 1 and Lemma 1.

Convention. Unless stated otherwise, we consider only non-empty coalgebras. The category $\mathbf{Coalg}(F)$ itself is not restricted; in particular, the final coalgebra is constructed in the full category.

3.1. The category $\mathbb{Z}_3\text{-Set}$

We write $\mathbb{Z}_3 = \mathbb{Z}/3\mathbb{Z} = \{0, 1, 2\}$ with addition modulo 3.

Definition 1 (Category $\mathbb{Z}_3\text{-Set}$). *The category $\mathbb{Z}_3\text{-Set}$ has:*

- **Objects:** pairs (X, θ) where X is a set and $\theta : X \rightarrow X$ is a bijection satisfying $\theta^3 = \text{id}_X$.
- **Morphisms:** equivariant functions, i.e., $f : (X, \theta_X) \rightarrow (Y, \theta_Y)$ such that $f \circ \theta_X = \theta_Y \circ f$.

Remark 1. *The category $\mathbb{Z}_3\text{-Set}$ is equivalent to the functor category $[\mathbb{Z}_3, \mathbf{Set}]$, where $\mathbb{Z}_3$ is viewed as a one-object category (the cyclic group of order 3).*

3.2. The functor F

Definition 2 (Functor F). *The endofunctor $F : \mathbb{Z}_3\text{-Set} \rightarrow \mathbb{Z}_3\text{-Set}$ is defined by (identifying $\mathbb{Z}_3 \times X^3 \cong \mathbb{Z}_3 \times X \times X \times X$ and writing elements as (ϕ, a, b, c)):*

- On objects: $F(X, \theta) := (\mathbb{Z}_3 \times X^3, \hat{\sigma})$ where $\hat{\sigma}(\phi, a, b, c) = (\phi + 1, b, c, a)$ increments the phase $\phi \in \mathbb{Z}_3$ and cyclically permutes the triple.
- On morphisms: $F(f) := \text{id}_{\mathbb{Z}_3} \times (f \times f \times f)$, acting as identity on the $\mathbb{Z}_3$ component and componentwise on the triple.

This is well-defined: $\hat{\sigma}^3 = \text{id}$, so $(\mathbb{Z}_3 \times X^3, \hat{\sigma}) \in \mathbb{Z}_3\text{-Set}$; moreover $F(f) = \text{id}_{\mathbb{Z}_3} \times f^{\times 3}$ (where $f^{\times 3} := f \times f \times f$ denotes componentwise application) commutes with $\hat{\sigma}$:

$$F(f)(\hat{\sigma}(\phi, a, b, c)) = (\phi + 1, f(b), f(c), f(a)) = \hat{\sigma}(\phi, f(a), f(b), f(c)) = \hat{\sigma}(F(f)(\phi, a, b, c)),$$

so $F(f)$ is a morphism in $\mathbb{Z}_3\text{-Set}$. Functoriality follows: $F(\text{id}) = \text{id}_{\mathbb{Z}_3} \times \text{id}^{\times 3} = \text{id}$, and $F(g \circ f) = \text{id} \times (g \circ f)^{\times 3} = (\text{id} \times g^{\times 3}) \circ (\text{id} \times f^{\times 3}) = F(g) \circ F(f)$.

Key property: F does not lift the input action θ diagonally to X^3 ; instead, it equips $\mathbb{Z}_3 \times X^3$ with the fixed action $\hat{\sigma}$ regardless of θ . Equivalently, $F = G \circ U$ where $U : \mathbb{Z}_3\text{-Set} \rightarrow \mathbf{Set}$ is the forgetful functor and $G : \mathbf{Set} \rightarrow \mathbb{Z}_3\text{-Set}$ is defined by $G(X) = (\mathbb{Z}_3 \times X^3, \hat{\sigma})$.

3.3. F -coalgebras

Definition 3 (F -coalgebra). *An F -coalgebra (in $\mathbb{Z}_3\text{-Set}$) is a triple (X, d, θ) where $(X, \theta) \in \mathbb{Z}_3\text{-Set}$ and $d : X \rightarrow \mathbb{Z}_3 \times X^3$ is an equivariant morphism. We write the triple explicitly to emphasize the $\mathbb{Z}_3$ -action; equivalently, an F -coalgebra is a pair $((X, \theta), d)$ where (X, θ) is an object of $\mathbb{Z}_3\text{-Set}$ and d is a morphism in $\mathbb{Z}_3\text{-Set}$. The equivariance condition is:*

$$d \circ \theta = \hat{\sigma} \circ d$$

We write $d(x) = (\text{obs}(x), d_0(x), d_1(x), d_2(x))$ where $\text{obs}(x) \in \mathbb{Z}_3$ is the observable phase and $d_i(x) \in X$ are the children of x . We use the shorthand $d(x)_i := d_i(x)$ for children; note that indices $i \in \{0, 1, 2\}$ refer to children, not to components of the quadruplet.

Remark 2 (Equivariance unfolded). *Equivariance $d \circ \theta = \hat{\sigma} \circ d$ means:*

$$d(\theta(x)) = (\text{obs}(x) + 1, d_1(x), d_2(x), d_0(x))$$

Thus $\text{obs}(\theta(x)) = \text{obs}(x) + 1$: the phase increments under θ . The children are also cyclically permuted. Note that the component maps obs and d_i are not individually equivariant: obs satisfies the twisted relation $\text{obs} \circ \theta = \text{obs} + 1$ (where $\text{obs} + 1$ denotes the function $x \mapsto \text{obs}(x) + 1$), while the children satisfy $d_i \circ \theta = d_{i+1}$ (indices mod 3).

Proposition 1 (Free actions only). *If (X, d, θ) is an F -coalgebra with $X \neq \emptyset$, then the $\mathbb{Z}_3$ -action is free: $\theta^k(x) \neq x$ for all $x \in X$ and $k \in \{1, 2\}$.*

Proof. The phase increment $\text{obs}(\theta(x)) = \text{obs}(x) + 1$ implies that fixed points are impossible. Indeed, if $\theta(x) = x$, then $\text{obs}(x) = \text{obs}(\theta(x)) = \text{obs}(x) + 1$, a contradiction in $\mathbb{Z}_3$. If $\theta^2(x) = x$, then $\theta(x) = \theta^3(x) = x$, reducing to the previous case. $\diamond$

Corollary 1 (Cardinality constraint). *If (X, d, θ) is a non-empty F -coalgebra, then every orbit has size 3. In particular, $|X|$ is divisible by 3, and the smallest non-empty coalgebras have exactly 3 elements.*

Convention. All child indices $i \in \{0, 1, 2\}$ are taken modulo 3. Expressions like $d_{i+1}(x)$ or $d_{i-k}(x)$ are computed in $\mathbb{Z}_3$.

Definition 4 (Children and dependency graph). *For an F -coalgebra (X, d, θ) , we distinguish:*

- The child triple $\vec{N}(x) := (d_0(x), d_1(x), d_2(x))$, an ordered triple. The ordering is essential: under θ , the child triple transforms as $\vec{N}(\theta(x)) = (d_1(x), d_2(x), d_0(x))$.
- The child set $N^+(x) := \{d_0(x), d_1(x), d_2(x)\}$, the underlying set (support) of the child triple.

The dependency graph $G(X)$ is the simple undirected graph with vertex set X and edge set $E = \{\{x, y\} : x \neq y \text{ and } (y \in N^+(x) \text{ or } x \in N^+(y))\}$. This is the underlying simple graph obtained by forgetting directions and discarding self-loops and multiplicities.

For a subset $S \subseteq X$, we write $G(S)$ for the induced subgraph of $G(X)$ on vertex set S : edges are those of $G(X)$ with both endpoints in S . This is the graph-theoretic induced subgraph, not necessarily a sub-coalgebra (which would additionally require S to be closed under d and θ).

Definition 5 (Orbit). *For $x \in X$, the orbit of x is $\text{Orb}(x) := \{\theta^k(x) : k \in \mathbb{Z}_3\}$. By Proposition 1, if $X \neq \emptyset$, the action is free, so $|\text{Orb}(x)| = 3$.*

Remark 3 (Orbital invariance of children). *Since θ permutes children cyclically ($N^+(\theta(x)) = N^+(x)$), the child set is constant on orbits. However, θ is not generally a graph automorphism: if $\{x, y\}$ is an edge, $\{\theta(x), \theta(y)\}$ need not be.*

Definition 6 (Connectivity). *An F -coalgebra (X, d, θ) is connected if $X \neq \emptyset$ and its dependency graph $G(X)$ is connected as an undirected graph.*

Definition 7 (Coalgebra morphism). *A coalgebra morphism $h : (X, d_X, \theta_X) \rightarrow (Y, d_Y, \theta_Y)$ is an equivariant function $h : X \rightarrow Y$ such that $d_Y \circ h = (\text{id}_{\mathbb{Z}_3} \times h^{\times 3}) \circ d_X$. Equivalently (using the notation of Definition 3), $\text{obs}_Y(h(x)) = \text{obs}_X(x)$ and $d_{Y,i}(h(x)) = h(d_{X,i}(x))$ for all i . (Equivariance $h \circ \theta_X = \theta_Y \circ h$ must be required explicitly: since F equips its output with the fixed action $\hat{\sigma}$ regardless of the input action, the coalgebra equation alone does not enforce equivariance.)*

Remark 4 (The $\mathbb{Z}_3$ -action is not a coalgebra endomorphism). *In general, the generator $\theta : X \rightarrow X$ of the $\mathbb{Z}_3$ -action is not a coalgebra morphism from (X, d, θ) to itself. A coalgebra endomorphism f must satisfy $d \circ f = F(f) \circ d$, but we have $d \circ \theta = \hat{\sigma} \circ d$, and $\hat{\sigma} \neq F(\theta)$ in general. This is why θ does not automatically preserve derived structures such as the dependency graph (Remark 3).*

3.4. Final coalgebras

Theorem 1 (Terminal chain [1,2]). *Let $\mathcal{C}$ be a locally presentable category and $F : \mathcal{C} \rightarrow \mathcal{C}$ an accessible functor. The terminal sequence $(X_\alpha, p_{\beta,\alpha})_{\alpha \leq \beta \in \text{Ord}}$ is defined by:*

- $X_0 = 1$ (terminal object in $\mathcal{C}$)
- $X_1 = F(X_0) = F(1)$, with $p_{1,0} : F(1) \rightarrow 1$ the unique morphism
- $X_{\alpha+1} = F(X_\alpha)$ with $p_{\alpha+1,\alpha} = F(p_{\alpha,\alpha-1})$ for $\alpha \geq 1$

- $X_\lambda = \varprojlim_{\alpha < \lambda} X_\alpha$ for limit ordinals λ , with $p_{\lambda,\alpha}$ the projection
- $p_{\beta,\alpha} = p_{\alpha+1,\alpha} \circ \dots \circ p_{\beta,\beta-1}$ for $\alpha < \beta$

By [1, Thm. 2.1], this sequence stabilizes at some ordinal α : the connecting map $p_{\alpha+1,\alpha} : X_{\alpha+1} \rightarrow X_\alpha$ is an isomorphism. Then $(X_\alpha, p_{\alpha+1,\alpha}^{-1})$ is the final F -coalgebra.

Applicability to $\mathbb{Z}_3$ -Set: The category $\mathbb{Z}_3\text{-Set} \simeq [\mathbb{Z}_3, \mathbf{Set}]$ is locally finitely presentable [2]. Filtered colimits in $\mathbb{Z}_3\text{-Set}$ are created by the forgetful functor $U : \mathbb{Z}_3\text{-Set} \rightarrow \mathbf{Set}$; since finite products commute with filtered colimits in $\mathbf{Set}$, the functor F preserves filtered colimits, hence is finitary (and thus accessible).

Lemma 1 (Lambek [9]). If (X, d, θ) is a final F -coalgebra, then $d : X \rightarrow F(X)$ is an isomorphism in $\mathbb{Z}_3\text{-Set}$.

4. $\mathbb{Z}_3$ -Bisimulation

Bisimulation is the fundamental equivalence relation in coalgebraic semantics. In the presence of a $\mathbb{Z}_3$ -action, we distinguish two variants: *standard bisimulation* requires identical observations and exact child matching; *$\mathbb{Z}_3$ -bisimulation* relaxes this by allowing observations and children to be related up to a cyclic shift. Orbital equivalence is always contained in $\mathbb{Z}_3$ -bisimilarity (Theorem 2), though the converse may fail. Standard bisimulation yields the extensionality principle in the final coalgebra Ω , where bisimilarity coincides with equality.

4.1. Definition

Definition 8 ($\mathbb{Z}_3$ -bisimulation). Let (X, d, θ) be an F -coalgebra. A relation $R \subseteq X \times X$ is a $\mathbb{Z}_3$ -bisimulation if whenever $x R y$, the shift $k := \text{obs}(y) - \text{obs}(x) \in \mathbb{Z}_3$ satisfies:

$$d(x)_i R d(y)_{i-k} \quad \text{for } i = 0, 1, 2$$

where indices are taken modulo 3. Equivalently, the i -th child of x is R -related to the $(i - k)$ -th child of y . Intuitively, the shift k compensates for the phase difference: we compare children in "rotated frames of reference" aligned by their observations.

Convention on index arithmetic. The index shift $i \mapsto i - k$ corresponds to the inverse of the θ -action on children. By Remark 2, $d(\theta^k(x))_i = d(x)_{i+k}$, so if $y = \theta^k(x)$, then $d(y)_{i-k} = d(x)_{(i-k)+k} = d(x)_i$. This justifies the choice of $i - k$ (not $i + k$) in the child matching condition.

Note that the definition does not constrain the shift of the child pairs $d(x)_i R d(y)_{i-k}$; their shift is determined by the coalgebra structure.

Example 1 (Comparing $\mathbb{Z}_3$ -bisimulation and standard bisimulation). Consider a 6-element coalgebra X with two θ -orbits, illustrated below:

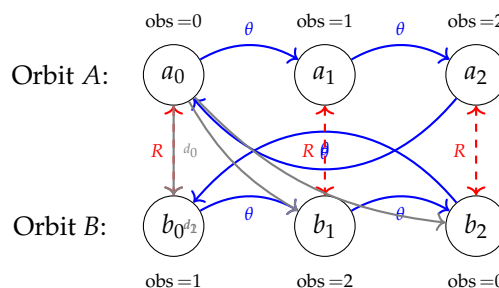


Figure 1. A 6-element coalgebra with two θ -orbits. **Blue arrows:** the θ -action cycling within each orbit. **Red dashed arrows:** the relation $R = \{(a_i, b_i)\}$. **Gray arrows:** children of a_0 (i.e., $d(a_0)_i = b_i$). Note the phase shift: $\text{obs}(a_i) = i$ but $\text{obs}(b_i) = i + 1$, giving shift $k = 1$. The $\mathbb{Z}_3$ -bisimulation condition $d(a_0)_i R d(b_0)_{i-1}$ matches $(b_1, a_1) \in R$, but standard bisimulation would require $(b_0, a_1) \in R$, which fails.

The data is summarized in tables:

	a_0	a_1	a_2	b_0	b_1	b_2
obs	0	1	2	1	2	0
θ	a_1	a_2	a_0	b_1	b_2	b_0

The children are determined by equivariance from base cases $d(a_0) = (0, b_0, b_1, b_2)$ and $d(b_0) = (1, a_1, a_2, a_0)$:

x	$d(x)_0$	$d(x)_1$	$d(x)_2$
a_0	b_0	b_1	b_2
a_1	b_1	b_2	b_0
a_2	b_2	b_0	b_1
b_0	a_1	a_2	a_0
b_1	a_2	a_0	a_1
b_2	a_0	a_1	a_2

Claim: The relation $R = \{(a_i, b_i), (b_i, a_i) : i \in \mathbb{Z}_3\}$ is a $\mathbb{Z}_3$ -bisimulation but not a standard bisimulation.

$\mathbb{Z}_3$ -bisimulation: For (a_0, b_0) , the shift is $k = \text{obs}(b_0) - \text{obs}(a_0) = 1 - 0 = 1$. We check: $d(a_0)_i R d(b_0)_{i-1}$?

i	$d(a_0)_i$	$d(b_0)_{i-1}$	In R ?
0	b_0	a_0	$(b_0, a_0) \in R \checkmark$
1	b_1	a_1	$(b_1, a_1) \in R \checkmark$
2	b_2	a_2	$(b_2, a_2) \in R \checkmark$

The shift $k = \text{obs}(b_i) - \text{obs}(a_i) = (i+1) - i = 1$ is constant across all pairs (a_i, b_i) .

Not standard: Standard bisimulation requires $d(a_0)_0 R d(b_0)_0$, i.e., $(b_0, a_1) \in R$. But R only contains (b_0, a_0) .

Definition 9 (Standard bisimulation). A standard bisimulation is a relation $R \subseteq X \times X$ satisfying:

1. Phase matching: $x R y \implies \text{obs}(x) = \text{obs}(y)$
2. θ -stability: $x R y \implies \theta(x) R \theta(y)$
3. Child matching: $x R y \implies d(x)_i R d(y)_i$ for $i = 0, 1, 2$

Equivalently, a standard bisimulation is a $\mathbb{Z}_3$ -bisimulation with shift $k = 0$ that is also θ -stable.

Two elements $x, y \in X$ are standard bisimilar, written $x \sim y$, if there exists a standard bisimulation R with $x R y$. The class of standard bisimulations is closed under arbitrary unions (phase matching, θ -stability, and child matching are each preserved by unions), so $\sim$ is the largest standard bisimulation.

Remark 5 (Categorical justification). The θ -stability axiom is not ad hoc but arises from the categorical definition of bisimulation. In the coalgebraic sense [13], a bisimulation on an F -coalgebra (X, d, θ) in $\mathbb{Z}_3\text{-Set}$ is a relation $R \subseteq X \times X$ that carries an F -coalgebra structure making the projections $\pi_1, \pi_2 : R \rightarrow X$ into coalgebra morphisms. Since morphisms in $\mathbb{Z}_3\text{-Set}$ must be $\mathbb{Z}_3$ -equivariant, the relation R must satisfy $(\theta \times \theta)(R) \subseteq R$, i.e., be θ -stable. The three conditions of Definition 9 are equivalent to this categorical definition.

4.2. The orbital relation

Definition 10 (Orbit). For $x \in X$, the θ -orbit of x is:

$$\text{Orb}(x) := \{x, \theta(x), \theta^2(x)\}$$

(This is well-defined since $\theta^3 = \text{id}$ in $\mathbb{Z}_3\text{-Set}$; see Definition 1.)

Definition 11 (Orbital graph). The orbital graph $\bar{G}(X) := G(X) / \langle \theta \rangle$ is defined as:

- Vertices: X / θ (the θ -orbits)

- Edges: $\{\text{Orb}(x), \text{Orb}(y)\} \in E(\bar{G})$ if and only if $\{x, y\} \in E(G(X))$ for some representatives

Since $G(X)$ is an undirected graph (Definition 4), orbits $\mathcal{O}$ and $\mathcal{O}'$ are adjacent iff there exist $x \in \mathcal{O}$ and $y \in \mathcal{O}'$ such that y is a child of x or x is a child of y .

Theorem 2 (Orbital $\mathbb{Z}_3$ -bisimulation). *The orbital relation $R_{\text{Orb}} = \{(x, y) : y \in \text{Orb}(x)\}$ is always a $\mathbb{Z}_3$ -bisimulation.*

Proof. Let $y = \theta^j(x)$ for some $j \in \mathbb{Z}_3$. The shift is $k = \text{obs}(y) - \text{obs}(x) = j$ (by Remark 2: $\text{obs}(\theta^j(x)) = \text{obs}(x) + j$).

By iterated equivariance ($d \circ \theta^j = \hat{\sigma}^j \circ d$; see Remark 2), the children satisfy $d(y)_i = d(x)_{i+j}$. (Explicitly: $\hat{\sigma}$ acts on child indices as $i \mapsto i + 1$, so $\hat{\sigma}^j$ acts as $i \mapsto i + j$.) We verify $d(x)_i R_{\text{Orb}} d(y)_{i-k}$:

$$d(y)_{i-k} = d(y)_{i-j} = d(x)_{(i-j)+j} = d(x)_i.$$

Thus we need $d(x)_i R_{\text{Orb}} d(x)_i$, which holds since $d(x)_i \in \text{Orb}(d(x)_i)$ by Definition 10. $\diamond$

Proposition 2 (Orbital relation is not a standard bisimulation). *In any F -coalgebra, R_{Orb} is not a standard bisimulation.*

Proof. Standard bisimulation requires phase matching: $x R y \Rightarrow \text{obs}(x) = \text{obs}(y)$. However, R_{Orb} contains pairs $(x, \theta(x))$ for all x , and by equivariance $\text{obs}(\theta(x)) = \text{obs}(x) + 1 \neq \text{obs}(x)$ in $\mathbb{Z}_3$ (since $1 \neq 0$ in $\mathbb{Z}_3$). Thus phase matching fails immediately. $\diamond$

4.3. Extensionality

Proposition 3 (Extensionality in the final coalgebra [13]). *In the final coalgebra Ω , standard bisimilarity coincides with equality:*

$$x \sim y \iff x = y$$

where $\sim$ denotes standard bisimilarity. This is a standard property of final coalgebras; we include the statement for reference.

Corollary 2 (Coinduction principle). *Let (X, d, θ) be an F -coalgebra and $h : X \rightarrow \Omega$ the unique morphism to the final coalgebra. If $R \subseteq X \times X$ is a standard bisimulation with $x R y$, then $h(x) = h(y)$.*

Proof. The image $S := \{(h(x), h(y)) : x R y\}$ is a standard bisimulation on Ω :

- *Phase matching:* $\text{obs}(h(x)) = \text{obs}(x)$ since h is a coalgebra morphism, hence $\text{obs}(h(x)) = \text{obs}(h(y))$.
- *θ -stability:* Since $h \circ \theta = \theta \circ h$, we have $(h(x), h(y)) \in S \Rightarrow (\theta(h(x)), \theta(h(y))) \in S$.
- *Child matching:* $d(h(x))_i = h(d(x)_i)$ since h commutes with d , so $d(h(x))_i S d(h(y))_i$.

By extensionality (Proposition 3), $S \subseteq \Delta_\Omega$, hence $h(x) = h(y)$. $\diamond$

Theorem 3 (Orbital extensionality for regular elements). *In the final coalgebra Ω , for regular elements (those with degeneracy $\delta = 0$, i.e., having three distinct children; see Definition 21), $\mathbb{Z}_3$ -bisimilarity coincides with orbital equivalence. Precisely: if $x, y \in \Omega$ with $\delta(x) = 0$, then*

$$x \sim_{\mathbb{Z}_3} y \iff y \in \text{Orb}(x)$$

where $x \sim_{\mathbb{Z}_3} y$ means there exists a $\mathbb{Z}_3$ -bisimulation R with $x R y$.

Proof. ($\Leftarrow$) If $y \in \text{Orb}(x)$, then $(x, y) \in R_{\text{Orb}}$, which is a $\mathbb{Z}_3$ -bisimulation by Theorem 2.

($\Rightarrow$) Let R be a $\mathbb{Z}_3$ -bisimulation on Ω with $x R y$, and let $k = \text{obs}(y) - \text{obs}(x) \in \mathbb{Z}_3$. We show $y = \theta^k(x)$.

Step 1: Orbital closure. Define the *orbital closure* of R :

$$\bar{R} := \{(\theta^i(u), \theta^i(v)) : u R v, i \in \mathbb{Z}_3\}$$

Then $\bar{R}$ is θ -stable by construction and contains R . We verify $\bar{R}$ is a $\mathbb{Z}_3$ -bisimulation: if $(\theta^i(u), \theta^i(v)) \in \bar{R}$ with $u R v$, the shift is $\delta(\theta^i(u), \theta^i(v)) = \text{obs}(\theta^i(v)) - \text{obs}(\theta^i(u)) = \delta(u, v) =: \ell$. By the $\mathbb{Z}_3$ -bisimulation property of R , $d(u)_j R d(v)_{j-\ell}$ for each j . By equivariance, $d(\theta^i(u))_j = d(u)_{j+i}$ and $d(\theta^i(v))_{j-\ell} = d(v)_{j-\ell+i}$. Since $d(u)_{j+i} R d(v)_{(j+i)-\ell}$, we have

$$(d(\theta^i(u))_j, d(\theta^i(v))_{j-\ell}) = (\theta^0(d(u)_{j+i}), \theta^0(d(v)_{(j+i)-\ell})) \in \bar{R}.$$

Step 2: Shift-corrected relation. Define:

$$S := \{(u, \theta^{-\delta(u,v)}(v)) : (u, v) \in \bar{R}\}$$

where $\delta(u, v) := \text{obs}(v) - \text{obs}(u)$. Since $(x, y) \in R \subseteq \bar{R}$, we have $(x, \theta^{-k}(y)) \in S$.

Step 3: S is a standard bisimulation (on elements reachable from regular x).

- *Phase matching:* For $(u, w) \in S$ with $w = \theta^{-\delta(u,v)}(v)$ where $(u, v) \in \bar{R}$, we have $\text{obs}(w) = \text{obs}(v) - \delta(u, v) = \text{obs}(u)$.
- *θ -stability:* If $(u, w) \in S$ with $w = \theta^{-\ell}(v)$ for $(u, v) \in \bar{R}$ and $\ell = \delta(u, v)$, then $(\theta(u), \theta(v)) \in \bar{R}$ by θ -stability of $\bar{R}$. The shift is preserved: $\delta(\theta(u), \theta(v)) = \ell$. Thus $(\theta(u), \theta^{-\ell}(\theta(v))) = (\theta(u), \theta(w)) \in S$.
- *Child matching:* Let $(u, w) \in S$. By phase matching, $\text{obs}(u) = \text{obs}(w)$. We must show $(d(u)_i, d(w)_i) \in S$ for each i .

By construction of S , there exists $(u, v) \in \bar{R}$ with $w = \theta^{-\ell}(v)$ where $\ell = \delta(u, v)$. By equivariance, $d(w)_i = d(\theta^{-\ell}(v))_i = d(v)_{i-\ell}$. The $\mathbb{Z}_3$ -bisimulation property of $\bar{R}$ gives $(d(u)_i, d(v)_{i-\ell}) \in \bar{R}$, i.e., $(d(u)_i, d(w)_i) \in \bar{R}$.

Now we compute $\ell_i := \delta(d(u)_i, d(w)_i) = \text{obs}(d(w)_i) - \text{obs}(d(u)_i)$. Since $\delta(x) = 0$, the element x has three distinct children. In the final coalgebra Ω , by Lambek's lemma (Lemma 1), $d : \Omega \rightarrow \mathbb{Z}_3 \times \Omega^3$ is an isomorphism. For regular elements $u \in \Omega$ (reachable from x), the constraint $d(u) = (\phi, a, b, c)$ with a, b, c distinct forces $\text{obs}(a) = \phi$, $\text{obs}(b) = \phi + 1$, $\text{obs}(c) = \phi + 2$ (since otherwise distinct children with equal phases would be bisimilar by coinduction, contradicting extensionality). Thus $\text{obs}(d(u)_i) = \text{obs}(u) + i$ for regular elements in Ω . Similarly for w . Since $\text{obs}(u) = \text{obs}(w)$ (phase matching), we have $\ell_i = 0$.

By definition of S : $(d(u)_i, \theta^{-\ell_i}(d(w)_i)) = (d(u)_i, d(w)_i) \in S$.

Thus S is a standard bisimulation containing $(x, \theta^{-k}(y))$. By extensionality (Proposition 3), $x = \theta^{-k}(y)$, hence $y = \theta^k(x) \in \text{Orb}(x)$. $\diamond$

Remark 6 (Failure for singular elements). *The regularity hypothesis $\delta(x) = 0$ is essential. For singular elements in the final coalgebra Ω , $\mathbb{Z}_3$ -bisimilarity can relate elements from different orbits. Since $d : \Omega \rightarrow \mathbb{Z}_3 \times \Omega^3$ is an isomorphism by Lambek's lemma (Lemma 1), we can construct $z = d^{-1}(\phi_0, x_0, x_0, x_0)$ where $x_0 \in \mathcal{I}$ (the 3-element image of the srs coalgebra; see Section 8). Then z is singular ($\delta(z) = 2$, meaning all three children coincide) and there exists a $\mathbb{Z}_3$ -bisimulation relating z to x_0 , even though $x_0 \notin \text{Orb}(z)$.*

Remark 7 (Conceptual interpretation). *Informally, $\mathbb{Z}_3$ -bisimulations on X correspond to standard bisimulations on the orbital quotient $X/\mathbb{Z}_3$ in **Set**. The shift parameter k in Definition 8 compensates for the phase offset within an orbit, ensuring that the bisimulation condition descends coherently to the quotient. (A precise categorical formulation would require defining an appropriate coalgebra structure on the quotient, which we do not pursue here.)*

5. The Helical Triadic Coalgebra

This section defines the *Helical Triadic Coalgebra* (HTC) as the final coalgebra Ω for the functor F in $\mathbb{Z}_3\text{-Set}$. We establish existence via the terminal sequence construction (Theorem 1), derive basic properties (infinitude, bijectivity of the structure map), and develop the theory of sub-coalgebras.

5.1. Axiomatic definition

Definition 12 (Helical Triadic Coalgebra (HTC)). *Assuming existence of a final coalgebra for F in $\mathbb{Z}_3\text{-Set}$ (established in Theorem 4 below), we define the Helical Triadic Coalgebra to be the final coalgebra (Ω, d, θ) .*

As an F -coalgebra in $\mathbb{Z}_3\text{-Set}$, Ω satisfies the following properties:

- (HTC1) (Ω, θ) is a $\mathbb{Z}_3$ -set with $\theta^3 = \text{id}$.
- (HTC2) $d : \Omega \rightarrow \mathbb{Z}_3 \times \Omega^3$ is equivariant: $d \circ \theta = \hat{\sigma} \circ d$.

The defining characteristic is:

- (HTC3) **Finality:** For every F -coalgebra (Y, e, τ) , there exists a unique equivariant coalgebra morphism $h : Y \rightarrow \Omega$.

Properties (HTC1)–(HTC2) follow from the definition of F -coalgebra in $\mathbb{Z}_3\text{-Set}$ (Definition 3); (HTC3) is the universal property that uniquely characterizes Ω .

Proposition 4 (Three distinct phases in Ω). *The final coalgebra (Ω, d, θ) contains elements of all three observable phases:*

- (i) $|\Omega| \geq 3$, with elements $x_0, x_1, x_2 \in \Omega$ satisfying $\text{obs}(x_i) = i$ for $i \in \mathbb{Z}_3$;
- (ii) $\theta \neq \text{id}$ on Ω .

Proof. The observable phase is preserved by coalgebra morphisms, making distinct phases distinguishable in Ω .

Step 1: Coalgebra morphisms preserve observable phase. Recall that $\text{obs}_X = \pi_1 \circ d_X$ where $\pi_1 : \mathbb{Z}_3 \times X^3 \rightarrow \mathbb{Z}_3$ is the first projection. Let $h : X \rightarrow \Omega$ be any coalgebra morphism in $\mathbb{Z}_3\text{-Set}$ (hence equivariant: $h \circ \theta_X = \theta_\Omega \circ h$). Since h commutes with the structure maps, $d_\Omega \circ h = F(h) \circ d_X$, and $F(h) = \text{id}_{\mathbb{Z}_3} \times h^3$, applying π_1 to both sides gives:

$$\text{obs}_\Omega(h(x)) = \text{obs}_X(x) \quad \text{for all } x \in X.$$

Step 2: A concrete 3-element F -coalgebra. Define $X = \mathbb{Z}_3 = \{0, 1, 2\}$ with $\theta(i) = i + 1 \pmod{3}$ (so $\theta^3 = \text{id}$) and structure map $d(i) = (i, i, i + 1, i + 2)$ (indices mod 3). Equivariance holds: $d(\theta(i)) = d(i + 1) = (i + 1, i + 1, i + 2, i) = \hat{\sigma}(i, i, i + 1, i + 2) = \hat{\sigma}(d(i))$, since $\hat{\sigma}(\phi, a, b, c) = (\phi + 1, b, c, a)$. Thus (X, d, θ) is an F -coalgebra in $\mathbb{Z}_3\text{-Set}$.

Step 3: Three distinct elements in Ω . Let $h : X \rightarrow \Omega$ be the unique coalgebra morphism. By Step 1, $\text{obs}_\Omega(h(i)) = \text{obs}_X(i) = i$. Since the three elements $0, 1, 2 \in X$ have distinct observable phases, their images $h(0), h(1), h(2)$ are pairwise distinct in Ω . Hence $|\Omega| \geq 3$.

Step 4: Non-triviality of θ . The morphism h is equivariant: $h(\theta(i)) = \theta(h(i))$. Since $\theta(0) = 1$ in X , we have $\theta(h(0)) = h(1) \neq h(0)$ by Step 3. Thus $\theta \neq \text{id}$ on Ω . $\diamond$

5.2. Existence and uniqueness

Theorem 4 (Existence and uniqueness of the final coalgebra). *The final coalgebra for F in $\mathbb{Z}_3\text{-Set}$ exists and is unique up to unique isomorphism.*

Proof. By Theorem 1, it suffices to verify the hypotheses: $\mathbb{Z}_3\text{-Set}$ is locally presentable and F is accessible. We check these and note the specific convergence ordinal.

Hypotheses of Theorem 1. The category $\mathbb{Z}_3\text{-Set} \simeq [\mathbb{Z}_3, \text{Set}]$ is a presheaf category, hence locally finitely presentable [2]. The functor F is finitary: filtered colimits in $[\mathbb{Z}_3, \text{Set}]$ are computed pointwise,

finite products preserve filtered colimits in **Set**, and $\mathbb{Z}_3 \times (-)$ preserves all colimits; hence $X \mapsto \mathbb{Z}_3 \times X^3$ preserves filtered colimits. Being finitary, F is accessible.

Construction. By Theorem 1, the terminal sequence converges at some ordinal λ . The final coalgebra is $\Omega := X_\lambda$, with structure map $d := p_{\lambda+1, \lambda}^{-1}$. Limits in $[\mathbb{Z}_3, \mathbf{Set}]$ are computed pointwise: if $(X_\alpha, \theta_\alpha)$ is the $\mathbb{Z}_3$ -set at stage α , then the limit $\Omega = \varprojlim_{\alpha < \lambda} X_\alpha$ carries the action θ_Ω defined by universality, satisfying $p_{\lambda, \alpha} \circ \theta_\Omega = \theta_\alpha \circ p_{\lambda, \alpha}$ for all α . This makes all projections equivariant.

Uniqueness. Terminal objects are unique up to unique isomorphism: if Ω and Ω' are both final, the unique morphisms $\Omega \rightarrow \Omega'$ and $\Omega' \rightarrow \Omega$ compose to endomorphisms that must equal the identity by uniqueness. $\diamond$

5.3. Properties

Proposition 5 (Divisibility). *For any finite F -coalgebra (X, d, θ) : $|X| \equiv 0 \pmod{3}$.*

Proof. The orbits of θ partition X . Since $\theta^3 = \text{id}$, the size of each orbit divides 3, hence equals 1 or 3. By Proposition 1, the $\mathbb{Z}_3$ -action on any F -coalgebra is free, so there are no orbits of size 1; thus all orbits have size 3. Since X is partitioned into blocks of size 3, we have $|X| \equiv 0 \pmod{3}$. $\diamond$

Remark 8. *Since every sub-coalgebra inherits the F -coalgebra structure, Proposition 5 applies to finite sub-coalgebras of infinite coalgebras. In particular, every finite sub-coalgebra of Ω has cardinality divisible by 3.*

Corollary 3 (Universal property). *For any F -coalgebra (Y, e, τ) in $\mathbb{Z}_3\text{-Set}$, there exists a unique morphism $h : Y \rightarrow \Omega$ in $\mathbb{Z}_3\text{-Set}$ such that $d \circ h = F(h) \circ e$.*

Proof. By (HTC3) in Definition 12, there exists a unique equivariant coalgebra morphism $h : Y \rightarrow \Omega$. By definition of coalgebra morphism, h satisfies $d \circ h = F(h) \circ e$. $\diamond$

Remark 9 (Canonical morphism and behavioural equivalence). *Every F -coalgebra (Y, e, τ) admits a unique coalgebra morphism $h : Y \rightarrow \Omega$, called the canonical morphism. The kernel pair of h (i.e., $\ker(h) := \{(x, y) \in Y^2 : h(x) = h(y)\}$) contains the largest standard bisimulation on Y . Since $F(X) = \mathbb{Z}_3 \times X^3$ is a polynomial functor (built from products and constants), it preserves weak pullbacks pointwise; in this case, $\ker(h)$ coincides with behavioural equivalence (cf. Corollary 2 and [13, Thm. 4.5]). In particular, h is injective if and only if behavioural equivalence on Y is equality.*

Proposition 6 (Infinitude). *The final coalgebra Ω is infinite: $|\Omega| = \infty$.*

Proof. By Lambek's lemma (Lemma 1), the structure map $d : \Omega \rightarrow F(\Omega) = \mathbb{Z}_3 \times \Omega^3$ is an isomorphism in $\mathbb{Z}_3\text{-Set}$. Since the forgetful functor $U : \mathbb{Z}_3\text{-Set} \rightarrow \mathbf{Set}$ reflects isomorphisms (equivariant bijections are isomorphisms), d is a bijection of underlying sets. Suppose Ω is finite with $|\Omega| = n$. Then:

$$n = |\mathbb{Z}_3 \times \Omega^3| = |\mathbb{Z}_3| \cdot |\Omega|^3 = 3n^3,$$

hence $n(1 - 3n^2) = 0$. Since $3n^2 = 1$ has no solution in $\mathbb{N}$, we obtain $n = 0$, i.e., $\Omega = \emptyset$.

However, non-empty F -coalgebras exist: the witness coalgebra from Proposition 4 has $|X| = 3$. By finality, there must exist a morphism $h : X \rightarrow \Omega$. But if $\Omega = \emptyset$, no function $X \rightarrow \emptyset$ exists when $X \neq \emptyset$. Contradiction.

Therefore Ω is not finite, i.e., $|\Omega| = \infty$. $\diamond$

Corollary 4 (Freeness of the $\mathbb{Z}_3$ -action on Ω). *The $\mathbb{Z}_3$ -action on Ω is free: for all $x \in \Omega$ and $k \in \mathbb{Z}_3 \setminus \{0\}$, $\theta^k(x) \neq x$.*

Proof. By Proposition 6, $\Omega \neq \emptyset$. By Proposition 1, the action on any non-empty F -coalgebra is free. $\diamond$

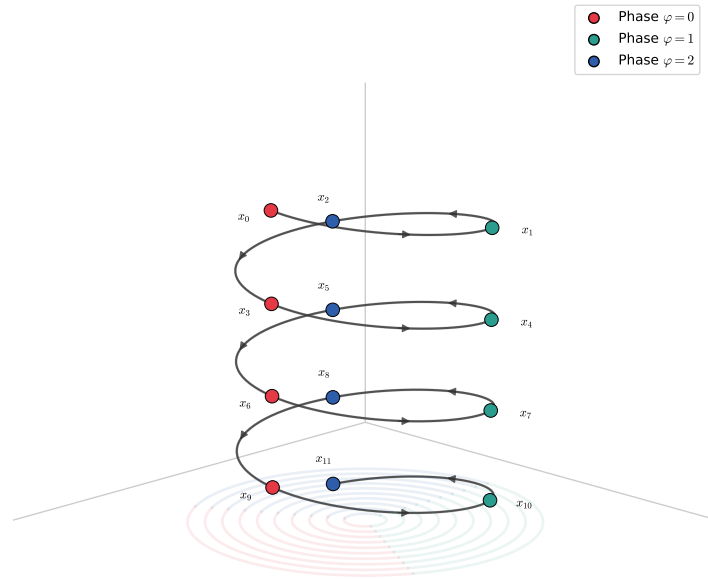


Figure 2. A path in the unfolding of the Helical Triadic Coalgebra. Each transition $x_i \rightarrow x_{i+1}$ selects a child and advances the phase by $+1 \pmod{3}$, visualized as a 120° rotation along the helix. The coloring indicates phase: $\varphi = 0$ (red), $\varphi = 1$ (teal), $\varphi = 2$ (blue).

Corollary 5 (Bijectivity of the structure map). *The structure map $d : \Omega \rightarrow \mathbb{Z}_3 \times \Omega^3$ is bijective. Equivalently, for every $(\phi, a, b, c) \in \mathbb{Z}_3 \times \Omega^3$, there exists a unique $x \in \Omega$ such that $d(x) = (\phi, a, b, c)$.*

Proof. By Lemma 1, the structure map $d : \Omega \rightarrow \mathbb{Z}_3 \times \Omega^3$ is an isomorphism in $\mathbb{Z}_3$ -Set. Let $d^{-1} : \mathbb{Z}_3 \times \Omega^3 \rightarrow \Omega$ be its inverse. For any $(\phi, a, b, c) \in \mathbb{Z}_3 \times \Omega^3$, set $x := d^{-1}(\phi, a, b, c)$. Then $d(x) = (\phi, a, b, c)$, proving existence. If $d(x) = d(x') = (\phi, a, b, c)$, applying d^{-1} yields $x = x'$, proving uniqueness. $\diamond$

5.4. Self-similarity

The isomorphism $d : \Omega \xrightarrow{\sim} \mathbb{Z}_3 \times \Omega^3$ from Lemma 1 exhibits Ω as a *fixed point* of the functor F : the final coalgebra satisfies $\Omega \cong F(\Omega)$. This equation is the hallmark of self-similar (or fractal) structures in coalgebraic semantics. We now make this precise.

Definition 13 (Descendant closure). *For $x \in \Omega$, the descendant closure $\text{Desc}(x)$ is the smallest subset of Ω containing x and closed under children:*

$$\text{Desc}(x) := \bigcap \{ S \subseteq \Omega : x \in S \text{ and } d_i(s) \in S \text{ for all } s \in S, i \in \{0, 1, 2\} \}.$$

Equivalently, $\text{Desc}(x) = \{y \in \Omega : \exists \text{ path } x \rightarrow y \text{ in } G(\Omega)\}$.

Theorem 5 (Self-similarity of Ω). *The final coalgebra Ω is self-similar:*

- (i) Fixed-point equation: $\Omega \cong \mathbb{Z}_3 \times \Omega^3$ as $\mathbb{Z}_3$ -sets.
- (ii) Local universality: *For every $x \in \Omega$, the descendant closure $\text{Desc}(x)$ is a sub-coalgebra, and the inclusion $\text{Desc}(x) \hookrightarrow \Omega$ is the unique coalgebra morphism from $\text{Desc}(x)$ to the final coalgebra.*
- (iii) Recursive decomposition: *Every element $x \in \Omega$ with $d(x) = (\phi, a, b, c)$ satisfies*

$$\text{Desc}(x) = \{x\} \cup \text{Desc}(a) \cup \text{Desc}(b) \cup \text{Desc}(c).$$

Proof. (i) This is Lemma 1: the structure map $d : \Omega \rightarrow F(\Omega) = \mathbb{Z}_3 \times \Omega^3$ is an isomorphism.

(ii) By definition, $\text{Desc}(x)$ is closed under children. For θ -closure: if $y \in \text{Desc}(x)$, there is a path $x = x_0 \rightarrow x_1 \rightarrow \dots \rightarrow x_n = y$ in $G(\Omega)$. Since θ is a coalgebra automorphism, $\theta(x_i) \rightarrow \theta(x_{i+1})$

for each i , giving a path from $\theta(x)$ to $\theta(y)$. But $\theta(x) \in \text{Desc}(x)$ (taking $n = 0$ and using θ -closure of $\text{Desc}(x)$ applied to x itself—we verify this separately: $x \in \text{Desc}(x)$ by definition, and $\theta(x)$ has children $\theta(d_i(x)) = d_{i+1}(x) \in \text{Desc}(x)$, so the sub-coalgebra generated by x contains $\theta(x)$). Hence $\theta(y) \in \text{Desc}(x)$, proving θ -closure.

Thus $\text{Desc}(x)$ is a sub-coalgebra. As a sub-coalgebra of the final coalgebra, it inherits the F -coalgebra structure, and the inclusion is trivially a coalgebra morphism. Uniqueness follows from finality.

(iii) The inclusion $\supseteq$ is immediate: $x \in \text{Desc}(x)$, and any descendant of a , b , or c is reachable from x via one additional edge. For $\subseteq$: any $y \in \text{Desc}(x)$ with $y \neq x$ lies on a path starting $x \rightarrow z \rightarrow \cdots \rightarrow y$ where $z \in \{a, b, c\}$, hence $y \in \text{Desc}(z)$. $\diamond$

Remark 10 (Fractal interpretation). *The fixed-point equation $\Omega \cong \mathbb{Z}_3 \times \Omega^3$ can be read as: “ Ω consists of a phase in $\mathbb{Z}_3$ together with three copies of itself.” This is analogous to how the Cantor set C satisfies $C = \frac{1}{3}C \cup (\frac{2}{3} + \frac{1}{3}C)$, or how the Sierpiński triangle decomposes into three scaled copies. The coalgebraic formulation captures this self-similarity abstractly, without reference to a metric or embedding.*

Corollary 6 (Behavioral equivalence and self-similarity). *For any F -coalgebra (X, d_X, θ_X) and element $x \in X$, the sub-coalgebra generated by x maps to $\text{Desc}(\iota(x)) \subseteq \Omega$ under the canonical morphism $\iota : X \rightarrow \Omega$. In particular, every element of every F -coalgebra “sees” a sub-coalgebra of Ω .*

Proof. The sub-coalgebra generated by x in X maps via ι to a sub-coalgebra of Ω containing $\iota(x)$, hence contained in $\text{Desc}(\iota(x))$. Surjectivity onto $\text{Desc}(\iota(x))$ may fail if ι is not injective. $\diamond$

5.5. Sub-coalgebras

Definition 14 (Sub-coalgebra). *A sub-coalgebra of (X, d, θ) is a subset $Y \subseteq X$ such that:*

1. Y is closed under θ : $\theta(Y) = Y$
2. Y is closed under d : $d_i(y) \in Y$ for all $y \in Y$ and $i \in \{0, 1, 2\}$

Proposition 7 (Orbital closure). *Every sub-coalgebra is a union of complete orbits.*

Proof. Let Y be a sub-coalgebra and let $y \in Y$. Since Y is θ -stable, $\theta(y) \in Y$ and thus $\theta^2(y) = \theta(\theta(y)) \in Y$. Since $\theta^3 = \text{id}$, the orbit is $\text{Orb}(y) = \{y, \theta(y), \theta^2(y)\} \subseteq Y$. Hence $Y = \bigcup_{y \in Y} \text{Orb}(y)$ is a union of complete orbits. $\diamond$

Corollary 7 (Minimal finite sub-coalgebras). *Let (X, d, θ) be an F -coalgebra. Then:*

- (i) *Every non-empty finite sub-coalgebra $Y \subseteq X$ has at least 3 elements.*
- (ii) *If $|Y| = 3$, then Y is a single $\mathbb{Z}_3$ -orbit.*

The bound $|Y| \geq 3$ is sharp: we will exhibit a 3-element sub-coalgebra $\mathcal{I} \subseteq \Omega$ in Section 8.

Proof. (1) By Proposition 7, Y is a union of complete orbits. By Proposition 1, the action is free, so each orbit has exactly 3 elements. Hence $|Y| = 3k$ for some $k \geq 1$ (since Y is non-empty), so $|Y| \geq 3$.

(2) If $|Y| = 3$, then Y is a union of orbits of size 3. Since distinct orbits are disjoint, a union with total cardinality 3 consists of exactly one orbit. $\diamond$

6. The srs Coalgebra

We now construct a natural F -coalgebra structure on the srs lattice, a classical 3-regular graph from crystallography also known as the Laves graph or (10,3)-a net. The key observation is that the rotation system—the cyclic ordering of neighbors at each vertex induced by the embedding in $\mathbb{R}^3$ —provides exactly the data needed for comultiplication. Combined with a phase variable in $\mathbb{Z}_3$, this yields an infinite F -coalgebra $\mathcal{S}$ with rich structure: translation invariance by $\mathbb{Z}^3$, chirality inherited from the crystallographic space group $I4_132$, and a canonical morphism to the final coalgebra Ω .

6.1. The srs lattice

The srs lattice (also known as the Laves graph or $(10,3)$ -a net) is a classical object in crystallography [12,15], appearing as the skeleton of the gyroid minimal surface.

Definition 15 (srs lattice). *The srs lattice is the maximal abelian cover of the complete graph K_4 . We give an explicit construction via voltage graphs [6].*

Voltage assignment. Label the vertices of K_4 as $\{0,1,2,3\}$. Orient each edge $i \rightarrow j$ with $i < j$. Define the voltage $v : \vec{E}(K_4) \rightarrow \mathbb{Z}^3$ by:

Edge	$0 \rightarrow 1$	$0 \rightarrow 2$	$0 \rightarrow 3$	$1 \rightarrow 2$	$1 \rightarrow 3$	$2 \rightarrow 3$
v	$(1,0,0)$	$(0,1,0)$	$(0,0,1)$	$(0,0,-1)$	$(0,-1,0)$	$(-1,0,0)$

For the reverse orientation, $v(j \rightarrow i) := -v(i \rightarrow j)$.

Derived graph. The srs lattice has:

- Vertices:** pairs $(n,i) \in \mathbb{Z}^3 \times \{0,1,2,3\}$, where i is the vertex type and n the lattice translate.
 - Edges:** (n,i) is adjacent to $(n+v(i \rightarrow j),j)$ whenever $\{i,j\}$ is an edge of K_4 .
- The projection $\pi : \text{srs} \rightarrow K_4$, $(n,i) \mapsto i$, is a regular covering with deck group $\mathbb{Z}^3$ acting by $\tau_t(n,i) = (n+t,i)$.

Properties: 3-regular, girth 10, space group $I4_132$ (No. 214), chiral.

Remark 11 (Covering space interpretation). *The srs lattice is the maximal abelian cover of the complete graph K_4 . The covering map $\pi : \text{srs} \rightarrow K_4$ has deck transformation group $\mathbb{Z}^3$.*

Remark 12 (Coxeter coordinates). *Following Coxeter [4], the srs lattice can be constructed by selecting one-eighth of $\mathbb{Z}^3$ with period 4. The base points*

$$(0,0,0), (1,2,3), (2,3,1), (3,1,2), (2,2,2), (3,0,1), (0,1,3), (1,3,0)$$

repeat periodically, with edges connecting pairs at distance $\sqrt{2}$.

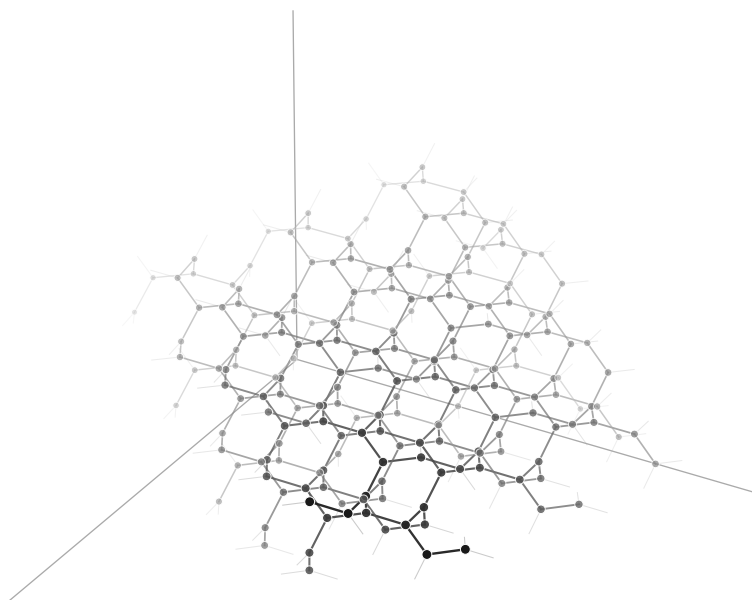


Figure 3. The srs lattice viewed along a direction revealing its helical structure. Darker vertices and edges are closer to the viewer; lighter ones recede into the background. Stub edges at the boundary indicate the trivalent continuation of the infinite graph.

6.2. Coalgebra structure on the srs lattice

Definition 16 (srs coalgebra $\mathcal{S}$). *The srs coalgebra is $\mathcal{S} := \text{srs} \times \mathbb{Z}_3$ with:*

$\mathbb{Z}_3$ -action:

$$\theta(v, \phi) := (v, \phi + 1)$$

Comultiplication: Fix an orientation of $\mathbb{R}^3$ (equivalently, a handedness). At each vertex $v \in \text{srs}$, the embedding induces a cyclic ordering of the three neighbors, viewed from outside a small sphere centered at v . To define comultiplication, we must linearize this cyclic order to an indexed triple. We fix a linearization by choosing, for each vertex type $i \in \{0, 1, 2, 3\}$, a distinguished starting neighbor $j_0^{(i)}$, then proceeding cyclically. This yields a rotation system $\rho(v) = (w_0, w_1, w_2)$ where (w_0, w_1, w_2) is the linearized triple of neighbors at v . The explicit linearization is given in Section 7.5. Then:

$$d(v, \phi) := (\phi, (w_\phi, \phi), (w_{\phi+1}, \phi + 1), (w_{\phi+2}, \phi + 2))$$

where indices are taken modulo 3. The observable phase is $\text{obs}(v, \phi) = \phi$.

Proposition 8 ($\mathcal{S}$ is an F -coalgebra). The triple $(\mathcal{S}, d, \theta)$ is an F -coalgebra: equivariance $d \circ \theta = \hat{\sigma} \circ d$ holds.

Proof. The action $\theta(v, \phi) = (v, \phi + 1)$ satisfies $\theta^3 = \text{id}$ since $\phi + 3 \equiv \phi \pmod{3}$, so $(\mathcal{S}, \theta) \in \mathbb{Z}_3\text{-Set}$.

For equivariance, let $(v, \phi) \in \mathcal{S}$ with $\rho(v) = (w_0, w_1, w_2)$.

$$\begin{aligned} \text{LHS: } d(\theta(v, \phi)) &= d(v, \phi + 1) \\ &= (\phi + 1, (w_{\phi+1}, \phi + 1), (w_{\phi+2}, \phi + 2), (w_\phi, \phi)) \\ \text{RHS: } \hat{\sigma}(d(v, \phi)) &= \hat{\sigma}(\phi, (w_\phi, \phi), (w_{\phi+1}, \phi + 1), (w_{\phi+2}, \phi + 2)) \\ &= (\phi + 1, (w_{\phi+1}, \phi + 1), (w_{\phi+2}, \phi + 2), (w_\phi, \phi)) \end{aligned}$$

Hence LHS = RHS, so $d \circ \theta = \hat{\sigma} \circ d$. $\diamond$

Remark 13 (Translation invariance). Using the representation $v = (n, i) \in \mathbb{Z}^3 \times \{0, 1, 2, 3\}$ for vertices of srs (Definition 15), elements of $\mathcal{S} = \text{srs} \times \mathbb{Z}_3$ can be written as triples (n, i, ϕ) . The group $\mathbb{Z}^3$ acts on $\mathcal{S}$ by translations: $\tau_t(n, i, \phi) := (n + t, i, \phi)$ for $t \in \mathbb{Z}^3$.

Translation data. By the voltage graph construction (Definition 15), the k -th neighbor of vertex (n, i) has the form $(n + v_k^{(i)}, j_k^{(i)})$ where:

- $v_k^{(i)} \in \mathbb{Z}^3$ is the displacement vector (depending only on (i, k) , not on n),
- $j_k^{(i)} \in \{0, 1, 2, 3\}$ is the neighbor type (again depending only on (i, k)).

These are determined by the voltage assignment and linearization from Section 7.5.

Coalgebra automorphism property. Since

$$d(\tau_t(n, i, \phi))_k = (n + t + v_{\phi+k}^{(i)}, j_{\phi+k}^{(i)}, \phi + k) = \tau_t(d(n, i, \phi)_k),$$

each translation τ_t is a coalgebra automorphism of $(\mathcal{S}, d)$.

Freeness. The action is free: $\tau_t(n, i, \phi) = (n, i, \phi)$ implies $n + t = n$, hence $t = 0$.

6.3. Connection to K_4

Theorem 6 (srs as maximal abelian cover [15]). The srs lattice is the maximal abelian cover of K_4 with voltage assignment in $\mathbb{Z}^3$.

Lemma 2 (Coalgebra automorphisms induce graph automorphisms). Let (X, d, θ) be an F -coalgebra with $G(X)$ simple. If $\alpha : X \rightarrow X$ is a coalgebra automorphism, then α is a graph automorphism of $G(X)$.

Moreover, if a group Γ acts freely on X by coalgebra automorphisms and the action has no edge inversions, then the quotient map $\pi : G(X) \rightarrow G(X)/\Gamma$ is a regular covering of graphs (see [6]).

Proof. Since α is a coalgebra morphism, $d \circ \alpha = (\text{id}_{\mathbb{Z}_3} \times \alpha^{\times 3}) \circ d$, so $d_i(\alpha(x)) = \alpha(d_i(x))$ for each i . Hence if $y = d_i(x)$, then $\alpha(y) = d_i(\alpha(x))$, so α preserves adjacency. Bijectivity ensures α is a graph automorphism.

For the covering statement, define $G(X)/\Gamma$ via vertex-orbits and dart-orbits. Freeness on vertices plus absence of edge inversions implies freeness on darts. Hence π is locally bijective on darts, making it a regular covering with deck group Γ . $\diamond$

Theorem 7 (Uniqueness of coalgebra with quotient K_4). *Let (X, d, θ) be a connected F -coalgebra (Definition 6) such that:*

- (i) *The dependency graph $G(X)$ is simple and 3-regular,*
- (ii) *$\theta^3 = \text{id}$ and $\theta \neq \text{id}$,*
- (iii) *$\mathbb{Z}^3$ acts freely on X by coalgebra automorphisms, with finite quotient.*

Then $G(X)/\mathbb{Z}^3 \cong K_4$ if and only if $G(X) \cong \text{srs}$.

Proof. By (i) and (iii), Lemma 2 applies. The action has no edge inversions: if g inverted an edge $\{u, v\}$, then $g^2(u) = u$, so $g^2 = \text{id}$ by freeness, contradicting that $\mathbb{Z}^3$ is torsion-free. Hence $\pi : G(X) \rightarrow G(X)/\mathbb{Z}^3$ is a regular covering with deck group $\mathbb{Z}^3$.

Necessity. If $G(X)/\mathbb{Z}^3 \cong K_4$, then π corresponds to an epimorphism $\varphi : \pi_1(K_4) \rightarrow \mathbb{Z}^3$. Since $\pi_1(K_4) \cong F_3$ and $\mathbb{Z}^3$ is abelian, φ factors through the abelianization $F_3 \rightarrow F_3^{\text{ab}} \cong \mathbb{Z}^3$. Any surjection $\mathbb{Z}^3 \rightarrow \mathbb{Z}^3$ is an isomorphism, so $\ker(\varphi) = [F_3, F_3]$, making π the maximal abelian cover. By Theorem 6, $G(X) \cong \text{srs}$.

Sufficiency. By Theorem 6, srs admits a free $\mathbb{Z}^3$ -action by translations with quotient K_4 . $\diamond$

Corollary 8 (Properties of the srs coalgebra). *The srs coalgebra $\mathcal{S} = \text{srs} \times \mathbb{Z}_3$ with its HTC structure (d, θ) from Definition 16 satisfies:*

- (a) *$\mathcal{S}$ is an F -coalgebra satisfying axioms (HTC1)–(HTC2),*
- (b) *There exists a unique coalgebra morphism $\iota : \mathcal{S} \rightarrow \Omega$ to the final coalgebra,*
- (c) *ι is not injective (Theorem 9), so $\mathcal{S} \not\cong \Omega$.*

Proof. (a) follows from Proposition 8. (b) follows from finality of Ω . (c) is proven in Theorem 9. $\diamond$

Remark 14 (On symmetry actions). *The $\mathbb{Z}_3$ -action θ on $\mathcal{S}$ acts on the phase component: $\theta(v, \phi) = (v, \phi + 1)$. This is distinct from the $\mathbb{Z}_3$ -action on the base graph K_4 by rotational symmetry (fixing one vertex and cyclically permuting the other three). The latter does not lift to a coalgebra automorphism of $\mathcal{S}$ compatible with θ , since such a lift would act on vertices while θ acts on phases.*

6.4. Local self-similarity of the srs coalgebra

The srs coalgebra $\mathcal{S}$ exhibits a strong form of local self-similarity: every vertex generates the entire coalgebra, and all such generations are isomorphic via translation.

Definition 17 (Generated sub-coalgebra). *For $(v, \phi) \in \mathcal{S}$, the sub-coalgebra generated by (v, ϕ) is:*

$$\mathcal{S}_{(v, \phi)} := \bigcap \{S \subseteq \mathcal{S} : (v, \phi) \in S, \theta(S) = S, d_i(S) \subseteq S \forall i\}.$$

Theorem 8 (Local self-similarity of $\mathcal{S}$). *The srs coalgebra $\mathcal{S}$ is locally self-similar:*

- (i) *Global generation: For every $(v, \phi) \in \mathcal{S}$, the sub-coalgebra generated by (v, ϕ) is the entire coalgebra: $\mathcal{S}_{(v, \phi)} = \mathcal{S}$.*
- (ii) *Translation equivalence: For every $t \in \mathbb{Z}^3$, the translation $\tau_t : \mathcal{S} \rightarrow \mathcal{S}$ defined by $\tau_t(n, i, \phi) = (n + t, i, \phi)$ is a coalgebra isomorphism.*
- (iii) *Quotient independence: The quotient $\mathcal{S}/\mathbb{Z}^3 \cong \mathcal{Q}$ is independent of the choice of base point.*

Proof. (i) The underlying graph of $\mathcal{S}$ projects onto the srs lattice (forgetting the phase). The srs lattice is connected: it is the maximal abelian cover of K_4 , which is connected, and covering maps preserve path-connectivity. Hence from any vertex v , every other vertex w is reachable via a path in srs. The same path, decorated with phases, shows that (w, ψ) is reachable from (v, ϕ) for appropriate ψ . Since $\mathcal{S}_{(v, \phi)}$ is closed under d and θ , it contains all vertices reachable from (v, ϕ) , hence $\mathcal{S}_{(v, \phi)} = \mathcal{S}$.

(ii) This is Remark 13: translations preserve the coalgebra structure because the neighbor relationships in srs are translation-invariant. Explicitly, if $\rho(v) = (w_0, w_1, w_2)$ are the neighbors of v , then $\rho(v + t) = (w_0 + t, w_1 + t, w_2 + t)$, so $d(\tau_t(v, \phi)) = \tau_t(d(v, \phi))$.

(iii) The quotient $\mathcal{S}/\mathbb{Z}^3$ identifies (n, i, ϕ) with (n', i, ϕ) for all $n, n' \in \mathbb{Z}^3$. The result is $\{0, 1, 2, 3\} \times \mathbb{Z}_3 \cong \mathcal{Q}$, independent of which representative we choose for each equivalence class. $\diamond$

Corollary 9 (Every vertex of srs generates an isomorphic copy). *Let v, w be any two vertices of the srs lattice. Then:*

- (i) *The sub-coalgebras $\mathcal{S}_{(v, 0)}$ and $\mathcal{S}_{(w, 0)}$ are both equal to $\mathcal{S}$.*
- (ii) *There exists $t \in \mathbb{Z}^3$ and $k \in \{0, 1, 2, 3\}$ such that τ_t maps a neighborhood of $(v, 0)$ isomorphically to a neighborhood of $(w, 0)$.*

In particular, the “local view” from any vertex of $\mathcal{S}$ is coalgebraically indistinguishable from any other, up to translation.

Proof. (i) is immediate from Theorem 8(i). For (ii), write $v = (n_v, i_v)$ and $w = (n_w, i_w)$ in the standard coordinates. If $i_v = i_w$, take $t = n_w - n_v$. If $i_v \neq i_w$, the srs lattice’s vertex-transitivity (under the space group $I4_132$) ensures an isometry mapping v to w ; this isometry decomposes into a translation and a rotation, and the translation component provides the required t . $\diamond$

Remark 15 (Comparison with Ω). *The local self-similarity of $\mathcal{S}$ (Theorem 8) should be contrasted with that of Ω (Theorem 5):*

- *In $\mathcal{S}$: every element generates the same coalgebra (the whole of $\mathcal{S}$), and all such generations are related by translation automorphisms.*
- *In Ω : every element x generates a sub-coalgebra $\text{Desc}(x)$, but these sub-coalgebras may differ. For instance, $\text{Desc}(x_0) = \mathcal{I}$ for $x_0 \in \mathcal{I}$ (a 3-element sub-coalgebra), while $\text{Desc}(z) \supsetneq \mathcal{I}$ for the asymmetric element $z = d^{-1}(\phi_0, x_0, x_0, x_0)$.*

This difference reflects the fact that $\mathcal{S}$ has a large automorphism group ($\mathbb{Z}^3$) acting transitively on vertex types, while Ω may have a smaller automorphism group (an open question; see Section 12).

7. The Canonical Morphism and Factorization

This section studies the canonical morphism $\iota : \mathcal{S} \rightarrow \Omega$ from the srs coalgebra to the final coalgebra. The central result is a *non-injectivity theorem*: despite the infinite size of $\mathcal{S}$, the morphism ι identifies all elements related by translation. This yields a factorization through a finite 12-element quotient $\mathcal{Q} \cong K_4 \times \mathbb{Z}_3$. The behavioral content of srs—as seen by the final coalgebra—is thus entirely captured by the rotation system on K_4 , together with a phase in $\mathbb{Z}_3$.

7.1. The canonical morphism ι

By finality of Ω , there exists a unique equivariant coalgebra morphism:

$$\iota : \mathcal{S} \rightarrow \Omega$$

7.2. Non-injectivity theorem

Theorem 9 (Non-injectivity of ι). *The canonical morphism $\iota : \mathcal{S} \rightarrow \Omega$ is **not** injective.*

Proof. We exhibit a standard bisimulation relating distinct elements.

Step 1: Translation relation. Define $R_T \subseteq \mathcal{S} \times \mathcal{S}$ by:

$$R_T := \{((n, i, \phi), (m, i, \phi)) : n, m \in \mathbb{Z}^3\}$$

Two elements are related iff they have the same vertex type $i \in \{0, 1, 2, 3\}$ and same phase $\phi \in \mathbb{Z}_3$.

Step 2: R_T is a standard bisimulation. We verify the three conditions of Definition 9.

Phase matching: By definition of R_T , if $(n, i, \phi) R_T (m, i, \phi)$ then both elements have phase ϕ , so $\text{obs}(n, i, \phi) = \phi = \text{obs}(m, i, \phi)$.

θ -stability: If $(n, i, \phi) R_T (m, i, \phi)$, then $\theta(n, i, \phi) = (n, i, \phi + 1)$ and $\theta(m, i, \phi) = (m, i, \phi + 1)$. These have the same type i and same phase $\phi + 1$, hence $(n, i, \phi + 1) R_T (m, i, \phi + 1)$.

Child matching: Since srs is a $\mathbb{Z}^3$ -periodic graph, translation acts by coalgebra automorphisms: the rotation system at (n, i) has the form:

$$\rho(n, i) = ((n + v_0^{(i)}, j_0^{(i)}), (n + v_1^{(i)}, j_1^{(i)}), (n + v_2^{(i)}, j_2^{(i)}))$$

where the displacement vectors $v_k^{(i)}$ and neighbor types $j_k^{(i)}$ depend only on (i, k) , not on the position n .

For $(n, i, \phi) R_T (m, i, \phi)$:

$$d(n, i, \phi)_k = (n + v_{\phi+k}^{(i)}, j_{\phi+k}^{(i)}, \phi + k)$$

$$d(m, i, \phi)_k = (m + v_{\phi+k}^{(i)}, j_{\phi+k}^{(i)}, \phi + k)$$

These have the same type $j_{\phi+k}^{(i)}$ and same phase $\phi + k$, hence:

$$d(n, i, \phi)_k R_T d(m, i, \phi)_k \quad \text{for each } k$$

Step 3: Conclusion. Since R_T is a standard bisimulation on $\mathcal{S}$, by the coinduction principle (Corollary 2):

$$(n, i, \phi) R_T (m, i, \phi) \implies \iota(n, i, \phi) = \iota(m, i, \phi)$$

Since the translation action is free (Remark 13), for $n \neq m$ we have $(n, i, \phi) \neq (m, i, \phi)$ in $\mathcal{S}$. These are distinct elements with equal images, so ι is not injective. $\diamond$

Lemma 3 (Translation equivalence). *The translation relation R_T from Theorem 9 is an equivalence relation.*

Proof. Let $x = (n, i, \phi), y = (m, j, \psi), z = (p, k, \chi)$ be elements of $\mathcal{S}$.

Reflexivity: For any $x = (n, i, \phi)$, taking $m = n$ gives $((n, i, \phi), (n, i, \phi)) \in R_T$, hence $x R_T x$.

Symmetry: If $x R_T y$, then $x = (n, i, \phi)$ and $y = (m, i, \phi)$ for some $n, m \in \mathbb{Z}^3$. Thus $((m, i, \phi), (n, i, \phi)) \in R_T$, i.e., $y R_T x$.

Transitivity: If $x R_T y$ and $y R_T z$, then $x = (n, i, \phi), y = (m, i, \phi)$, and $z = (p, i, \phi)$ for the same (i, ϕ) . Hence $((n, i, \phi), (p, i, \phi)) \in R_T$, i.e., $x R_T z$. $\diamond$

7.3. The quotient coalgebra $\mathcal{Q}$

Definition 18 (Quotient by translations). *The quotient by translations is:*

$$\mathcal{Q} := \mathcal{S} / \mathbb{Z}^3 = \{(i, \phi) : i \in \{0, 1, 2, 3\}, \phi \in \mathbb{Z}_3\}$$

where $\mathbb{Z}^3$ acts by translation: $\tau_m(n, i, \phi) := (n + m, i, \phi)$.

We have $|\mathcal{Q}| = 4 \times 3 = 12$.

Remark 16 (Symmetry reduction). *This quotient construction is analogous to symmetry reduction techniques in model checking [3], where equivalent states under a group action are identified to reduce the state space. Here, the translation group $\mathbb{Z}^3$ acts freely on the infinite srs lattice, and the quotient $\mathcal{Q}$ is a finite representative.*

Proposition 9 (Quotient coalgebra structure). $\mathcal{Q}$ inherits an F -coalgebra structure making the quotient map $q : \mathcal{S} \rightarrow \mathcal{Q}$ a coalgebra morphism:

- (i) $\theta_{\mathcal{Q}}(i, \phi) := (i, \phi + 1)$
- (ii) $d_{\mathcal{Q}}(i, \phi) := (\phi, (j_{\phi}^{(i)}, \phi), (j_{\phi+1}^{(i)}, \phi + 1), (j_{\phi+2}^{(i)}, \phi + 2))$

where $j_k^{(i)}$ is the neighbor type inherited from the rotation system on srs (Definition 16). The observable phase is $\text{obs}(i, \phi) = \phi$.

Proof. Well-definedness and morphism property. Let $q : \mathcal{S} \rightarrow \mathcal{Q}$ denote the quotient map, and write $[x]$ for the equivalence class of $x \in \mathcal{S}$. We define:

$$\theta_{\mathcal{Q}}([x]) := [\theta_{\mathcal{S}}(x)], \quad d_{\mathcal{Q}}([x]) := (\text{id}_{\mathbb{Z}_3} \times q^{\times 3}) \circ d_{\mathcal{S}}(x).$$

To verify well-definedness, suppose $x' = \tau_t(x)$ for some $t \in \mathbb{Z}^3$. By Remark 13, translations commute with $\theta_{\mathcal{S}}$, so:

$$[\theta_{\mathcal{S}}(x')] = [\theta_{\mathcal{S}}(\tau_t(x))] = [\tau_t(\theta_{\mathcal{S}}(x))] = [\theta_{\mathcal{S}}(x)].$$

For $d_{\mathcal{Q}}$, using $d_{\mathcal{S}} \circ \tau_t = (\text{id}_{\mathbb{Z}_3} \times \tau_t^{\times 3}) \circ d_{\mathcal{S}}$ and $q \circ \tau_t = q$:

$$(\text{id} \times q^{\times 3}) \circ d_{\mathcal{S}}(x') = (\text{id} \times q^{\times 3}) \circ d_{\mathcal{S}}(\tau_t(x)) = (\text{id} \times q^{\times 3}) \circ (\text{id} \times \tau_t^{\times 3}) \circ d_{\mathcal{S}}(x) = (\text{id} \times q^{\times 3}) \circ d_{\mathcal{S}}(x).$$

Hence $\theta_{\mathcal{Q}}$ and $d_{\mathcal{Q}}$ are well-defined on equivalence classes. By construction:

$$q \circ \theta_{\mathcal{S}} = \theta_{\mathcal{Q}} \circ q, \quad d_{\mathcal{Q}} \circ q = (\text{id}_{\mathbb{Z}_3} \times q^{\times 3}) \circ d_{\mathcal{S}},$$

so q is a coalgebra morphism.

Explicit formula. We now derive the explicit form stated in the proposition. Let $x = (v, \phi) \in \mathcal{S}$ where $v \in \text{srs}$ has vertex type $i \in \{0, 1, 2, 3\}$. Then $q(v, \phi) = (i, \phi)$. By Definition 16, $d_{\mathcal{S}}(v, \phi) = (\phi, (w_{\phi}, \phi), (w_{\phi+1}, \phi + 1), (w_{\phi+2}, \phi + 2))$ where $(w_0, w_1, w_2) = \rho(v)$ is the rotation system at v . Applying the quotient map:

$$d_{\mathcal{Q}}(i, \phi) = (\text{id} \times q^{\times 3}) \circ d_{\mathcal{S}}(v, \phi) = (\phi, q(w_{\phi}, \phi), q(w_{\phi+1}, \phi + 1), q(w_{\phi+2}, \phi + 2)).$$

By Remark 13 (the neighbor types $j_k^{(i)}$ in the rotation system depend only on (i, k)), the type of $w_{\phi+r}$ is determined by i and $\phi + r$, independent of the specific vertex v within its translation class. Hence $q(w_{\phi+r}, \phi + r) = (j_{\phi+r}^{(i)}, \phi + r)$ for $r \in \mathbb{Z}_3$, yielding the formula in the statement.

$\mathbb{Z}_3$ -action: $\theta_{\mathcal{Q}}^3(i, \phi) = (i, \phi + 3) = (i, \phi)$ since $\phi + 3 \equiv \phi \pmod{3}$. Also $\theta_{\mathcal{Q}} \neq \text{id}$ since $(i, \phi) \neq (i, \phi + 1)$ for $\phi \in \mathbb{Z}_3$.

Equivariance: We check $d_{\mathcal{Q}} \circ \theta_{\mathcal{Q}} = \hat{\sigma} \circ d_{\mathcal{Q}}$:

$$\begin{aligned} d_{\mathcal{Q}}(\theta_{\mathcal{Q}}(i, \phi)) &= d_{\mathcal{Q}}(i, \phi + 1) \\ &= (\phi + 1, (j_{\phi+1}^{(i)}, \phi + 1), (j_{\phi+2}^{(i)}, \phi + 2), (j_{\phi}^{(i)}, \phi)) \\ \hat{\sigma}(d_{\mathcal{Q}}(i, \phi)) &= \hat{\sigma}(\phi, (j_{\phi}^{(i)}, \phi), (j_{\phi+1}^{(i)}, \phi + 1), (j_{\phi+2}^{(i)}, \phi + 2)) \\ &= (\phi + 1, (j_{\phi+1}^{(i)}, \phi + 1), (j_{\phi+2}^{(i)}, \phi + 2), (j_{\phi}^{(i)}, \phi)) \end{aligned}$$

These are equal. $\diamond$

7.4. Factorization theorem

Theorem 10 (Factorization). *The canonical morphism factors as:*

$$\begin{array}{ccc} \mathcal{S} & \xrightarrow{q} & \mathcal{Q} \\ & \searrow \iota & \downarrow \bar{\iota} \\ & & \Omega \end{array}$$

where $q : \mathcal{S} \rightarrow \mathcal{Q}$ is the quotient map and $\bar{\iota} : \mathcal{Q} \rightarrow \Omega$ is induced by finality.

Proof. By Proposition 9, $q : \mathcal{S} \rightarrow \mathcal{Q}$ is a morphism of F -coalgebras. By finality of Ω , there exists a unique coalgebra morphism $\bar{\iota} : \mathcal{Q} \rightarrow \Omega$. Then $\bar{\iota} \circ q : \mathcal{S} \rightarrow \Omega$ is a coalgebra morphism. Since $\iota : \mathcal{S} \rightarrow \Omega$ is the unique morphism to the final coalgebra, we have $\iota = \bar{\iota} \circ q$. $\diamond$

7.5. Explicit rotation system on K_4

Using Coxeter coordinates for srs, the four vertex types are positioned at:

Type 0 : (0,0,0), Type 1 : (1,1,0), Type 2 : (1,0,1), Type 3 : (0,1,1)

Neighbor types. The rotation system (neighbor types $j_k^{(i)}$) on K_4 inherited from the embedding of srs in $\mathbb{R}^3$ is:

i	$j_0^{(i)}$	$j_1^{(i)}$	$j_2^{(i)}$
0	1	2	3
1	0	3	2
2	3	0	1
3	2	1	0

Displacement vectors. The displacement vectors $v_k^{(i)} \in \mathbb{Z}^3$ for each (i, k) are:

i	$v_0^{(i)}$	$v_1^{(i)}$	$v_2^{(i)}$
0	(1, 0, 0)	(0, 1, 0)	(0, 0, 1)
1	(-1, 0, 0)	(0, 0, -1)	(0, -1, 0)
2	(0, -1, 0)	(-1, 0, 0)	(0, 0, -1)
3	(0, 0, -1)	(0, -1, 0)	(-1, 0, 0)

These satisfy: the k -th neighbor of vertex $(n, i) \in \text{srs}$ is $(n + v_k^{(i)}, j_k^{(i)})$.

This data encodes the linearization of the cyclic ordering of neighbors for each vertex type. The rotation system is *chiral*: there is no automorphism of K_4 that reverses all cyclic orderings simultaneously.

7.6. Classification of standard bisimulations on $\mathcal{Q}$

We now classify all standard bisimulations on $\mathcal{Q}$. Recall that $\mathcal{Q} = \{0, 1, 2, 3\} \times \mathbb{Z}_3$, and denote by $\text{ph} : \mathcal{Q} \rightarrow \mathbb{Z}_3$ the projection onto the phase component: $\text{ph}(i, \phi) = \phi$. By Definition 9, a *standard bisimulation* $R \subseteq \mathcal{Q} \times \mathcal{Q}$ satisfies three conditions: (1) phase matching: $x R y \Rightarrow \text{obs}(x) = \text{obs}(y)$; (2) θ -stability: $x R y \Rightarrow \theta(x) R \theta(y)$; and (3) child matching: $x R y \Rightarrow d(x)_t R d(y)_t$ for all t .

Since $\text{obs}(i, \phi) = \phi$ on $\mathcal{Q}$, the phase matching condition forces $\text{ph}(x) = \text{ph}(y)$ for any pair $(x, y) \in R$. This crucial constraint restricts the structure considerably.

Lemma 4 (Phase constraint). *Let R be a standard bisimulation on $\mathcal{Q}$. Then $(i, \phi) R (j, \psi)$ implies $\phi = \psi$. Moreover, by θ -stability, $(i, \phi) R (j, \phi)$ implies $(i, \phi + k) R (j, \phi + k)$ for all $k \in \mathbb{Z}_3$.*

Proof. The phase matching condition requires $\text{obs}(i, \phi) = \text{obs}(j, \psi)$, i.e., $\phi = \psi$. By θ -stability, $(i, \phi) R (j, \phi)$ implies $\theta(i, \phi) R \theta(j, \phi)$, i.e., $(i, \phi + 1) R (j, \phi + 1)$. Iterating gives the result. $\diamond$

Thus a standard bisimulation R is entirely determined by the *type relation* $T \subseteq \{0, 1, 2, 3\}^2$ defined by:

$$(i, j) \in T \iff (i, 0) R (j, 0).$$

By θ -stability, $(i, \phi) R (j, \phi)$ iff $(i, j) \in T$.

Lemma 5 (Closure condition). *The relation R satisfies the child matching condition if and only if T satisfies the closure condition:*

$$(i, j) \in T \implies \forall k \in \mathbb{Z}_3, (j_k^{(i)}, j_k^{(j)}) \in T. \quad (*)$$

Proof. Recall from Proposition 9 that $d_Q(i, \phi)_t = (j_{\phi+t}^{(i)}, \phi + t)$.

($\Rightarrow$) Suppose R satisfies child matching and $(i, j) \in T$. Then $(i, 0) R (j, 0)$. By child matching, for all $t \in \mathbb{Z}_3$:

$$d(i, 0)_t R d(j, 0)_t, \quad \text{i.e.,} \quad (j_t^{(i)}, t) R (j_t^{(j)}, t).$$

By phase matching, both have phase t . By definition of T , $(j_t^{(i)}, j_t^{(j)}) \in T$. Setting $k = t$ gives $(*)$.

($\Leftarrow$) Conversely, suppose T satisfies $(*)$ and $(i, \phi) R (j, \phi)$, i.e., $(i, j) \in T$. For any $t \in \mathbb{Z}_3$, we must show $d(i, \phi)_t R d(j, \phi)_t$. We have $d(i, \phi)_t = (j_{\phi+t}^{(i)}, \phi + t)$ and $d(j, \phi)_t = (j_{\phi+t}^{(j)}, \phi + t)$. Setting $k = \phi + t$ in $(*)$, we get $(j_k^{(i)}, j_k^{(j)}) \in T$, so $(j_{\phi+t}^{(i)}, \phi + t) R (j_{\phi+t}^{(j)}, \phi + t)$, which is child matching. $\diamond$

Proposition 10 (Orbit structure). *Condition $(*)$ partitions $\{0, 1, 2, 3\}^2$ into exactly 4 orbits of size 4. Each orbit is the graph of a permutation $\sigma \in S_4$.*

Proof. Let $P = \{0, 1, 2, 3\}^2$. For each $k \in \mathbb{Z}_3$, define $\pi_k : \{0, 1, 2, 3\} \rightarrow \{0, 1, 2, 3\}$ by $\pi_k(i) = j_k^{(i)}$. From the table in Section 7.5, each π_k is a permutation of $\{0, 1, 2, 3\}$.

Define

$$g_k : P \rightarrow P, \quad g_k(i, j) = (\pi_k(i), \pi_k(j)).$$

Each g_k is a bijection of P . The subsets of P stable under the maps g_k (equivalently, satisfying $(*)$) are exactly those stable under the group $G = \langle g_0, g_1, g_2 \rangle \leq \text{Sym}(P)$. Thus the orbits of the G -action partition P .

We show that P decomposes into exactly four orbits, each of cardinality 4. The group G acts diagonally on P : each g_k applies π_k to both coordinates. Let $H = \langle \pi_0, \pi_1, \pi_2 \rangle \leq S_4$. From the table in Section 7.5, one verifies that H acts transitively on $\{0, 1, 2, 3\}$: starting from 0, we have $\pi_0(0) = 1$, $\pi_1(1) = 3$, $\pi_2(3) = 0$, and $\pi_1(0) = 2$, reaching all four elements.

For any $(i, j) \in P$, the orbit $G \cdot (i, j) = \{(\pi(i), \pi(j)) : \pi \in H\}$. Since H acts transitively on the first coordinate, the orbit contains elements with all four possible first coordinates. Thus $|G \cdot (i, j)| \geq 4$. Since $|P| = 16$ and orbits partition P , we have at most 4 orbits. If some orbit had size > 4 , there would be fewer than 4 orbits, but then some orbit would have size ≥ 5 , contradicting that the first-coordinate projection is surjective onto a 4-element set with fibers of size ≤ 1 per orbit (by the final paragraph below). Hence all orbits have size exactly 4, and there are exactly 4 orbits.

Finally, let O be an orbit of cardinality 4. The projection $p_1 : O \rightarrow \{0, 1, 2, 3\}$ onto the first coordinate is equivariant for the induced action of H . By inspection of the table, H acts transitively on $\{0, 1, 2, 3\}$, so $\text{im}(p_1) = \{0, 1, 2, 3\}$. Since $|O| = 4$, the map p_1 is bijective, and $O = \{(i, \sigma(i)) : i \in \{0, 1, 2, 3\}\}$ for a unique $\sigma \in S_4$. $\diamond$

The explicit orbits are the graphs of the four permutations forming the Klein four-group V_4 :

$$\begin{aligned}\sigma_0 &= \text{id} : (0, 1, 2, 3) \mapsto (0, 1, 2, 3) \\ \sigma_1 &= (01)(23) : (0, 1, 2, 3) \mapsto (1, 0, 3, 2) \\ \sigma_2 &= (02)(13) : (0, 1, 2, 3) \mapsto (2, 3, 0, 1) \\ \sigma_3 &= (03)(12) : (0, 1, 2, 3) \mapsto (3, 2, 1, 0)\end{aligned}$$

We write $\Sigma_0 = \{\sigma_0, \sigma_1, \sigma_2, \sigma_3\} = V_4$ for this set of permutations.

Theorem 11 (Classification of standard bisimulations). *A relation $R \subseteq \mathcal{Q} \times \mathcal{Q}$ is a standard bisimulation if and only if there exists a subset $S \subseteq V_4$ such that:*

$$R = \bigcup_{\sigma \in S} R_\sigma$$

where $R_\sigma := \{((i, \phi), (\sigma(i), \phi)) : i \in \{0, 1, 2, 3\}, \phi \in \mathbb{Z}_3\}$.

In particular, the number of standard bisimulations on $\mathcal{Q}$ is $2^4 = 16$.

Proof. (Necessity.) Let R be a standard bisimulation. By Lemma 4, all pairs $(x, y) \in R$ satisfy $\text{ph}(x) = \text{ph}(y)$. Define $T := \{(i, j) \in \{0, 1, 2, 3\}^2 : (i, 0) R (j, 0)\}$. By θ -stability, $(i, \phi) R (j, \phi)$ iff $(i, j) \in T$.

(Closure implies union of orbits.) By Lemma 5, the child matching condition on R is equivalent to $(*)$ on T . For each $k \in \mathbb{Z}_3$, the map g_k is a permutation of $\{0, 1, 2, 3\}^2$, hence injective. Since T is finite and $g_k(T) \subseteq T$, injectivity implies $|g_k(T)| = |T|$, so $g_k(T) = T$. Thus T is stable under G and hence is a union of orbits. By Proposition 10, $T = \bigcup_{\sigma \in S} \text{Graph}(\sigma)$ for some $S \subseteq V_4$, giving the required form.

(Sufficiency.) Conversely, let $R = \bigcup_{\sigma \in S} R_\sigma$ for arbitrary $S \subseteq V_4$. Each R_σ satisfies phase matching (both elements have phase ϕ) and is θ -stable: if $(i, \phi) R_\sigma (\sigma(i), \phi)$, then $\theta(i, \phi) = (i, \phi + 1)$ and $\theta(\sigma(i), \phi) = (\sigma(i), \phi + 1)$, which is again in R_σ . For child matching: since $\text{Graph}(\sigma)$ is an orbit under G , it is stable under $(*)$. Thus $(i, \sigma(i)) \in \text{Graph}(\sigma)$ implies $(j_k^{(i)}, j_k^{(\sigma(i))}) \in \text{Graph}(\sigma)$ for each $k \in \mathbb{Z}_3$. By Lemma 5, child matching holds for R_σ . A union of relations satisfying all three conditions still satisfies them. Hence R is a standard bisimulation.

(Counting.) Each S can be chosen from $\mathcal{P}(V_4)$, giving $2^4 = 16$ standard bisimulations. $\diamond$

Corollary 10 (Notable bisimulations).

- (i) **Equality:** $S = \{\text{id}\}$.
- (ii) **V_4 -orbit equivalence:** $S = V_4$ (the largest standard bisimulation).

Proof. 1. **Equality:** The diagonal $\Delta_{\mathcal{Q}} = \{(x, x) : x \in \mathcal{Q}\}$ contains only pairs $((i, \phi), (i, \phi))$, corresponding to $\sigma = \text{id}$, so $S = \{\text{id}\}$.
 2. **V_4 -orbit equivalence:** $(i, \phi) \sim (j, \phi)$ for all i, j in the same V_4 -orbit means all type-pairs in $\bigcup_{\sigma \in V_4} \text{Graph}(\sigma) = \{0, 1, 2, 3\}^2$ are related. Hence $S = V_4$. $\diamond$

Remark 17. The equality bisimulation is the unique standard bisimulation contained in the diagonal. This does not imply that $\mathcal{Q}$ is extensional in the classical sense, since larger bisimulations exist.

7.7. Coalgebra automorphisms of $\mathcal{Q}$

Lemma 6 (Coalgebra automorphisms of $\mathcal{Q}$). *Let $p \in S_4$ be a permutation. The map $\alpha_p : \mathcal{Q} \rightarrow \mathcal{Q}$ defined by $\alpha_p(i, \phi) = (p(i), \phi)$ is a coalgebra automorphism if and only if*

$$j_t^{(p(i))} = p(j_t^{(i)}) \quad \text{for all } i \in \{0, 1, 2, 3\}, t \in \mathbb{Z}_3.$$

Proof. Since p is a permutation, α_p is bijective with inverse $\alpha_{p^{-1}}$.

The map α_p commutes with θ since $\theta(p(i), \phi) = (p(i), \phi + 1) = \alpha_p(i, \phi + 1) = \alpha_p(\theta(i, \phi))$.

For the d -condition, recall that $d(i, \phi)_k = (j_{\phi+k}^{(i)}, \phi + k)$ (Proposition 9). Then:

$$d(\alpha_p(i, \phi))_k = d(p(i), \phi)_k = (j_{\phi+k}^{(p(i))}, \phi + k),$$

while

$$\alpha_p(d(i, \phi)_k) = \alpha_p(j_{\phi+k}^{(i)}, \phi + k) = (p(j_{\phi+k}^{(i)}), \phi + k).$$

These are equal for all (i, ϕ, k) iff $j_{\phi+k}^{(p(i))} = p(j_{\phi+k}^{(i)})$ for all i, ϕ, k . Setting $t = \phi + k$, this is equivalent to $j_t^{(p(i))} = p(j_t^{(i)})$ for all $i, t \in \mathbb{Z}_3$. $\diamond$

Proposition 11 (V_4 -action by automorphisms). *The Klein four-group*

$$V_4 = \{\text{id}, (01)(23), (02)(13), (03)(12)\}$$

acts on $\mathcal{Q}$ by coalgebra automorphisms via $p \cdot (i, \phi) = (p(i), \phi)$.

Proof. We verify the condition of Lemma 6 for the two generators of V_4 .

Case $p_1 = (01)(23)$: From the table in Section 7.5:

i	$j_0^{(i)}$	$j_1^{(i)}$	$j_2^{(i)}$	$p_1(j_0^{(i)})$	$p_1(j_1^{(i)})$	$p_1(j_2^{(i)})$	$j_0^{(p_1(i))}$	$j_1^{(p_1(i))}$	$j_2^{(p_1(i))}$
0	1	2	3	0	3	2	0	3	2
1	0	3	2	1	2	3	1	2	3
2	3	0	1	2	1	0	2	1	0
3	2	1	0	3	0	1	3	0	1

Columns 5–7 equal columns 8–10, so $j_t^{(p_1(i))} = p_1(j_t^{(i)})$ for all i, t .

Case $p_2 = (02)(13)$:

i	$j_0^{(i)}$	$j_1^{(i)}$	$j_2^{(i)}$	$p_2(j_0^{(i)})$	$p_2(j_1^{(i)})$	$p_2(j_2^{(i)})$	$j_0^{(p_2(i))}$	$j_1^{(p_2(i))}$	$j_2^{(p_2(i))}$
0	1	2	3	3	0	1	3	0	1
1	0	3	2	2	1	0	2	1	0
2	3	0	1	1	2	3	1	2	3
3	2	1	0	0	3	2	0	3	2

Again columns 5–7 equal columns 8–10, so $j_t^{(p_2(i))} = p_2(j_t^{(i)})$ for all i, t .

Closure under composition: The condition $j_t^{(p(i))} = p(j_t^{(i)})$ is stable under composition: if p, q both satisfy it, then for all i, t :

$$j_t^{((pq)(i))} = j_t^{(p(q(i)))} = p(j_t^{(q(i))}) = p(q(j_t^{(i)})) = (pq)(j_t^{(i)}).$$

Since $p_1^2 = p_2^2 = \text{id}$ and the condition is stable under composition, it holds for every word in p_1, p_2 , hence for every element of $V_4 = \langle p_1, p_2 \rangle$.

The action axioms $(pq) \cdot x = p \cdot (q \cdot x)$ and $\text{id} \cdot x = x$ follow from $(pq)(i) = p(q(i))$ and $\text{id}(i) = i$. $\diamond$

Lemma 7 (Regularity of the V_4 -action). *The action of V_4 on $\{0, 1, 2, 3\}$ is regular: for any $i, j \in \{0, 1, 2, 3\}$, there exists a unique $p \in V_4$ such that $p(i) = j$.*

Proof. Let $X = \{0, 1, 2, 3\}$. The four elements of V_4 act on 0 as follows:

$$\text{id}(0) = 0, \quad (01)(23)(0) = 1, \quad (02)(13)(0) = 2, \quad (03)(12)(0) = 3.$$

Hence $\text{Orb}(0) = X$, so the action is transitive. By the orbit-stabilizer theorem, $|\text{Orb}(i)| = |V_4|/|\text{Stab}(i)|$ for any $i \in X$. Since the action is transitive, $|\text{Orb}(i)| = |X| = 4 = |V_4|$, so $|\text{Stab}(i)| = 1$ and the action is free. Given $i, j \in X$, transitivity yields $p \in V_4$ with $p(i) = j$. If also $q(i) = j$, then $q^{-1}p \in \text{Stab}(i) = \{e\}$, hence $p = q$. $\diamond$

Corollary 11 (Non-injectivity of $\bar{\iota}$). *The morphism $\bar{\iota} : \mathcal{Q} \rightarrow \Omega$ is not injective. The kernel of $\bar{\iota}$ coincides with the orbit equivalence relation of the V_4 -action:*

$$\bar{\iota}(i, \phi) = \bar{\iota}(j, \psi) \iff \phi = \psi \text{ and } j = p(i) \text{ for some } p \in V_4.$$

In particular, $|\text{im}(\bar{\iota})| = |\mathcal{Q}|/|V_4| = 12/4 = 3$.

Proof. Step 1: V_4 -invariance and factorization. By Proposition 11, for any $p \in V_4$, the map α_p is a coalgebra automorphism of $\mathcal{Q}$. Since $\bar{\iota} : \mathcal{Q} \rightarrow \Omega$ is the unique morphism to the final coalgebra, we have $\bar{\iota} \circ \alpha_p = \bar{\iota}$. Hence $\bar{\iota}$ is constant on V_4 -orbits, and there exists a unique map $\tilde{\iota}$ such that $\bar{\iota} = \tilde{\iota} \circ \pi$, where $\pi : \mathcal{Q} \rightarrow \mathcal{Q}/V_4$ is the quotient projection.

Step 2: Identification $\mathcal{Q}/V_4 \cong \mathbb{Z}_3$. Since V_4 acts transitively on the type component $\{0, 1, 2, 3\}$ and trivially on the phase component, each equivalence class $\pi(i, \phi)$ depends only on ϕ . The map $\eta : \mathcal{Q}/V_4 \rightarrow \mathbb{Z}_3$ defined by $\eta(\pi(i, \phi)) = \phi$ is well-defined and bijective. Since V_4 acts trivially on the phase coordinate, $\theta(i, \phi) = (i, \phi + 1)$ preserves V_4 -orbits, so the $\mathbb{Z}_3$ -action on $\mathcal{Q}$ descends to a well-defined action on $\mathcal{Q}/V_4$ given by $\theta(\pi(i, \phi)) = \pi(i, \phi + 1)$. Under this action, η is $\mathbb{Z}_3$ -equivariant. Thus $\mathcal{Q}/V_4 \cong \mathbb{Z}_3$ as $\mathbb{Z}_3$ -sets.

Step 3: Equivariance and injectivity of $\tilde{\iota}$. We first show that $\tilde{\iota}$ is $\mathbb{Z}_3$ -equivariant. Since $\bar{\iota}$ is a morphism of F -coalgebras, it is in particular equivariant: $\bar{\iota} \circ \theta_{\mathcal{Q}} = \theta_{\Omega} \circ \bar{\iota}$. Since the V_4 -action is trivial on the phase, $\theta_{\mathcal{Q}}$ sends orbits to orbits, so the quotient map $\pi : \mathcal{Q} \rightarrow \mathcal{Q}/V_4$ is equivariant: $\pi \circ \theta_{\mathcal{Q}} = \theta_{\mathcal{Q}/V_4} \circ \pi$. Using $\bar{\iota} = \tilde{\iota} \circ \pi$:

$$\bar{\iota} \circ \theta_{\mathcal{Q}/V_4} \circ \pi = \tilde{\iota} \circ \pi \circ \theta_{\mathcal{Q}} = \bar{\iota} \circ \theta_{\mathcal{Q}} = \theta_{\Omega} \circ \bar{\iota} = \theta_{\Omega} \circ \tilde{\iota} \circ \pi.$$

Since π is surjective, $\tilde{\iota} \circ \theta_{\mathcal{Q}/V_4} = \theta_{\Omega} \circ \tilde{\iota}$, so $\tilde{\iota}$ is equivariant.

Under the identification $\mathcal{Q}/V_4 \simeq \mathbb{Z}_3$, any equivariant map $f : \mathbb{Z}_3 \rightarrow \Omega$ satisfies $f(\phi) = \theta^{\phi}(f(0))$. If $\tilde{\iota}(\phi) = \tilde{\iota}(\psi)$, then $\theta^{\phi-\psi}(\tilde{\iota}(0)) = \tilde{\iota}(0)$. By freeness of the $\mathbb{Z}_3$ -action on Ω (Corollary 4), $\phi - \psi = 0$, so $\phi = \psi$. Thus $\tilde{\iota}$ is injective.

Step 4: Kernel computation. We have:

$$\bar{\iota}(i, \phi) = \bar{\iota}(j, \psi) \iff \tilde{\iota}(\pi(i, \phi)) = \tilde{\iota}(\pi(j, \psi)) \iff \pi(i, \phi) = \pi(j, \psi)$$

(since $\tilde{\iota}$ is injective), which holds iff $\phi = \psi$ and $j = p(i)$ for some $p \in V_4$.

Finally, $|\text{im}(\bar{\iota})| = |\text{im}(\tilde{\iota})| = |\mathcal{Q}/V_4| = 3$. $\diamond$

8. Characterization of the Image

Having established that the canonical morphism $\iota : \mathcal{S} \rightarrow \Omega$ factors through the 12-element quotient $\mathcal{Q}$, we now determine the structure of its image $\mathcal{I} \subseteq \Omega$. The main result is that $\mathcal{I}$ is a 3-element sub-coalgebra forming a single θ -orbit, with completely explicit dynamics: $d(x_{\phi}) = (\phi, x_{\phi}, x_{\phi+1}, x_{\phi+2})$. This dramatic collapse—from 12 elements to 3—reflects the V_4 -symmetry of the rotation system on K_4 : the action on vertex types becomes invisible at the level of behavioral equivalence.

Definition 19 (Image coalgebra). *The image of the canonical morphism is $\mathcal{I} := \iota(\mathcal{S}) = \bar{\iota}(\mathcal{Q}) \subseteq \Omega$.*

By Corollary 11, $|\mathcal{I}| = 3$. Since ι is equivariant, $\mathcal{I}$ is θ -stable; by freeness of the $\mathbb{Z}_3$ -action (Corollary 4), every orbit has size 3, so $\mathcal{I}$ is a single θ -orbit. We show in Proposition 13 that $\mathcal{I}$ is a sub-coalgebra, justifying the name image coalgebra.

Proposition 12 (Connectivity of $\mathcal{I}$). *The image $\mathcal{I}$ is connected: its dependency graph has a single connected component.*

Proof. The dependency graph $G(\mathcal{S})$ is connected: as established in Section 6.1, $G(\mathcal{S})$ coincides with the srs lattice, which is a connected 3-regular graph (see [15, Thm. 3.1]).

The canonical morphism $\iota : \mathcal{S} \rightarrow \Omega$ is a coalgebra morphism (by finality), hence it preserves adjacency: if $y = d_i(x)$ in $\mathcal{S}$, then $\iota(y) = d_i(\iota(x))$ in Ω . For any path $x_0, x_1, \dots, x_n$ in $G(\mathcal{S})$, the sequence $\iota(x_0), \iota(x_1), \dots, \iota(x_n)$ is a walk in $G(\mathcal{I})$ (possibly with repeated vertices when ι identifies adjacent elements). Removing consecutive duplicates yields a path connecting $\iota(x_0)$ to $\iota(x_n)$.

Since $G(\mathcal{S})$ is connected and $\iota : \mathcal{S} \rightarrow \mathcal{I}$ is surjective, any two vertices in $G(\mathcal{I})$ are connected by such a path. Therefore $G(\mathcal{I})$ is connected. $\diamond$

Proposition 13 (Structure of the image). *The image $\mathcal{I}$ is a 3-element sub-coalgebra of Ω with the following properties:*

- (i) $\mathcal{I} = \{x_0, x_1, x_2\}$ where $x_\phi = \bar{\iota}(i, \phi)$ for any $i \in \{0, 1, 2, 3\}$
- (ii) $\theta(x_\phi) = x_{\phi+1}$ (single orbit of size 3)
- (iii) $d(x_\phi) = (\phi, x_\phi, x_{\phi+1}, x_{\phi+2})$ (observable phase ϕ ; children are itself and its two rotations)

Proof. (1) Fix $\phi \in \mathbb{Z}_3$ and set $x_\phi := \bar{\iota}(0, \phi)$. Let $i \in \{0, 1, 2, 3\}$. By Lemma 7, there exists $p \in V_4$ with $p(0) = i$. By Corollary 11 applied to $(0, \phi)$ and $(p(0), \phi) = (i, \phi)$, we get $\bar{\iota}(i, \phi) = \bar{\iota}(0, \phi) = x_\phi$. Hence $\mathcal{I} = \{\bar{\iota}(i, \phi) : i \in \{0, 1, 2, 3\}, \phi \in \mathbb{Z}_3\} \subseteq \{x_0, x_1, x_2\}$. Since $|\mathcal{I}| = 3$ (Corollary 11), it follows that $\mathcal{I} = \{x_0, x_1, x_2\}$ and these three elements are pairwise distinct.

(2) By $\mathbb{Z}_3$ -equivariance of $\bar{\iota}$:

$$\theta(x_\phi) = \theta(\bar{\iota}(0, \phi)) = \bar{\iota}(\theta(0, \phi)) = \bar{\iota}(0, \phi + 1) = x_{\phi+1}.$$

(3) Since $\bar{\iota}$ is a coalgebra morphism, we have $d \circ \bar{\iota} = (\text{id}_{\mathbb{Z}_3} \times \bar{\iota}^{\times 3}) \circ d_{\mathcal{Q}}$. Writing $d_{\mathcal{Q}}(i, \phi) = (\phi, (j_\phi^{(i)}, \phi), (j_{\phi+1}^{(i)}, \phi + 1), (j_{\phi+2}^{(i)}, \phi + 2))$:

$$d(\bar{\iota}(i, \phi)) = (\text{id}_{\mathbb{Z}_3} \times \bar{\iota}^{\times 3})(d_{\mathcal{Q}}(i, \phi)) = (\phi, \bar{\iota}(j_\phi^{(i)}, \phi), \bar{\iota}(j_{\phi+1}^{(i)}, \phi + 1), \bar{\iota}(j_{\phi+2}^{(i)}, \phi + 2)).$$

By (1), for any type $a \in \{0, 1, 2, 3\}$ we have $\bar{\iota}(a, \psi) = x_\psi$. Hence:

$$d(x_\phi) = (\phi, x_\phi, x_{\phi+1}, x_{\phi+2}).$$

(The observable phase is $\text{obs}(x_\phi) = \phi$ since coalgebra morphisms preserve the phase: $\text{obs} \circ \bar{\iota} = \text{obs}_{\mathcal{Q}}$.) This shows $\mathcal{I}$ is closed under θ and d , hence a sub-coalgebra. $\diamond$

Remark 18 (Collapse of type information). *The V_4 -symmetry of the rotation system on K_4 causes all four vertex types to become identified in the final coalgebra. The only surviving invariant is the phase $\phi \in \mathbb{Z}_3$. This reflects the high symmetry of srs: the coalgebra $\mathcal{Q} \cong K_4 \times \mathbb{Z}_3$ is not extensional. The quotient by the largest standard bisimulation (the V_4 -orbits) yields a 3-element coalgebra.*

Remark 19 (Comparison with the srs quotient). *The factorization $\mathcal{S} \xrightarrow{q} \mathcal{Q} \xrightarrow{\bar{\iota}} \mathcal{I}$ can be refined. Define $\bar{\mathcal{Q}} := \mathcal{Q}/V_4 \cong \mathbb{Z}_3$. Then:*

$$\begin{array}{ccccc} \mathcal{S} & \xrightarrow{q} & \mathcal{Q} & \xrightarrow{q'} & \bar{\mathcal{Q}} \\ & \searrow \iota & & \cong \downarrow \bar{\iota} & \\ & & & \mathcal{I} & \end{array}$$

where $\bar{\iota} : \bar{Q} \rightarrow \mathcal{I}$ is an isomorphism of F -coalgebras. (The quotient $\bar{Q}$ inherits a coalgebra structure from Q since the V_4 -action consists of coalgebra automorphisms.) The image $\mathcal{I}$ is the extensional quotient of Q : the quotient by the largest standard bisimulation, which identifies elements with identical behavior in Ω .

9. Orbital Invariants

This section introduces intrinsic invariants that classify elements of an F -coalgebra by their local structure. These invariants—binding index, degeneracy degree, and multiplicity signature—are constant on θ -orbits. They provide a fine-grained stratification of any coalgebra.

The main result is the *chirality theorem for regular coalgebras* (Theorem 12): every non-empty regular coalgebra is chiral. This structural obstruction, derived purely from the coalgebraic axioms, informs the symmetry analysis in Section 10.

9.1. Binding index and degeneracy

The binding index measures how “self-referential” an element is: it counts how many of its children belong to its own orbit. A high binding index indicates tight internal coupling within an orbit.

Definition 20 (Binding index). *The binding index of $x \in X$ is:*

$$E(x) := |\{k \in \{0, 1, 2\} : d(x)_k \in \text{Orb}(x)\}|$$

Values: $E \in \{0, 1, 2, 3\}$.

Proposition 14 (Orbital invariance of E). *E is constant on orbits: $E(\theta(x)) = E(x)$.*

Proof. Since $(X, \theta) \in \mathbb{Z}_3\text{-Set}$, we have $\theta^3 = \text{id}$, hence $\text{Orb}(\theta(x)) = \{\theta(x), \theta^2(x), x\} = \text{Orb}(x)$. By equivariance, $d(\theta(x))_k = d(x)_{k+1}$ (indices mod 3). Thus

$$E(\theta(x)) = |\{k \in \mathbb{Z}_3 : d(x)_{k+1} \in \text{Orb}(x)\}| = |\{k' \in \mathbb{Z}_3 : d(x)_{k'} \in \text{Orb}(x)\}| = E(x),$$

where the second equality uses the bijection $k \mapsto k + 1$ on $\mathbb{Z}_3$. $\diamond$

The degeneracy degree captures how “collapsed” the comultiplication is: it measures the failure of injectivity. Regular elements ($\delta = 0$) have three distinct children, while singular elements ($\delta = 2$) point to the same child three times.

Definition 21 (Degeneracy degree). *The degeneracy degree of $x \in X$ is:*

$$\delta(x) := 3 - |\{d(x)_0, d(x)_1, d(x)_2\}|$$

Values: $\delta \in \{0, 1, 2\}$, corresponding to:

- $\delta = 0$: regular — three distinct children
- $\delta = 1$: binary — one repeated child
- $\delta = 2$: singular — all children identical

Proposition 15 (Orbital invariance of δ). *δ is constant on orbits: $\delta(\theta(x)) = \delta(x)$.*

Proof. By equivariance, if $d(x) = (\phi, a, b, c)$ then $d(\theta(x)) = \hat{\sigma}(d(x)) = (\phi + 1, b, c, a)$. Thus the children of $\theta(x)$ are $(d(\theta(x))_0, d(\theta(x))_1, d(\theta(x))_2) = (b, c, a) = (d(x)_1, d(x)_2, d(x)_0)$. As sets, $\{d(\theta(x))_i\}_i = \{d(x)_i\}_i$. Since $\delta(x) = 3 - |\{d(x)_0, d(x)_1, d(x)_2\}|$, we have $\delta(\theta(x)) = \delta(x)$. $\diamond$

9.2. Multiplicity signature

The multiplicity signature refines the degeneracy degree by recording the full pattern of child repetitions. While δ only counts how many children coincide, μ preserves the exact distribution.

Definition 22 (Multiplicity signature). The multiplicity signature $\mu(x)$ is the partition of 3 given by the multiplicities of values in the child multiset $\{d(x)_0, d(x)_1, d(x)_2\}$, listed in decreasing order:

Type	$\mu(x)$	Example
Regular	(1, 1, 1)	$d(x) = (a, b, c)$ with a, b, c distinct
Binary	(2, 1)	$d(x) = (a, a, b)$
Singular	(3)	$d(x) = (a, a, a)$

Since $\mu(x)$ depends only on the multiset of children, and by Proposition 15 this multiset is invariant under θ , we have $\mu(\theta(x)) = \mu(x)$. Thus μ is constant on orbits.

9.3. Chirality

Definition 23 (Anti-chiral symmetry). An anti-chiral symmetry of (X, d, θ) is a bijection $\rho : X \rightarrow X$ satisfying:

(AC1) Orientation reversal: $\rho \circ \theta = \theta^{-1} \circ \rho$

(AC2) Child preservation: $d(\rho(x))_i = \rho(d(x)_i)$ for $i = 0, 1, 2$

Note: (AC2) concerns only the children; the observable phase satisfies $\text{obs}(\rho(x)) = \text{obs}(x)$ as a consequence of (AC1) and equivariance: $\text{obs}(\rho(\theta(x))) = \text{obs}(\theta^{-1}(\rho(x))) = \text{obs}(\rho(x)) - 1$, which is consistent with $\text{obs}(\theta(x)) = \text{obs}(x) + 1$ only if ρ preserves obs .

Definition 24 (Chirality). An F -coalgebra is achiral if it admits an anti-chiral symmetry. It is chiral if no such symmetry exists.

Remark 20 (Two senses of chirality). The term “chiral” is used in two related but distinct senses:

1. Intrinsic chirality (Definition 24): an F -coalgebra is chiral if it admits no internal symmetry reversing the cyclic dynamics. This is analogous to a helix having no mirror plane.
2. Chiral pair: two coalgebras related by reversing the rotation system (swapping the cyclic order of children) form a chiral pair—they are enantiomers, non-isomorphic mirror images of each other.

These notions are connected: an intrinsically chiral coalgebra and its mirror image (with reversed rotation system) form a chiral pair. The srs lattice and its enantiomer exemplify this: each is intrinsically chiral (Theorem 12), and together they form a chiral pair corresponding to the two choices of orientation in $\mathbb{R}^3$.

Theorem 12 (Chirality of regular coalgebras). Every non-empty regular F -coalgebra (with $\delta = 0$ everywhere) is chiral.

Proof. Suppose for contradiction that (X, d, θ) is a non-empty regular coalgebra admitting an anti-chiral symmetry ρ .

Since $X \neq \emptyset$, choose $x \in X$ and write $(d_0(x), d_1(x), d_2(x)) = (a, b, c)$ for its children. By regularity ($\delta = 0$), the elements a, b, c are pairwise distinct. Let $\phi := \text{obs}(x)$ denote the observable phase of x .

Step 1: Consequences of (AC1). The orientation reversal condition $\rho \circ \theta = \theta^{-1} \circ \rho$ gives:

$$\rho(\theta(x)) = \theta^{-1}(\rho(x)) = \theta^2(\rho(x))$$

Step 2: Consequences of (AC2). The child preservation condition (AC2) gives:

$$(d_0(\rho(x)), d_1(\rho(x)), d_2(\rho(x))) = (\rho(a), \rho(b), \rho(c))$$

By the note following Definition 23, $\text{obs}(\rho(x)) = \text{obs}(x) = \phi$.

Step 3: Computing $d(\rho(\theta(x)))$ two ways.

Method 1: By equivariance, the children of $\theta(x)$ are $(d_0(\theta(x)), d_1(\theta(x)), d_2(\theta(x))) = (b, c, a)$. Applying ρ via (AC2), the children of $\rho(\theta(x))$ are $(\rho(b), \rho(c), \rho(a))$.

Method 2: Using Step 1:

$$d(\rho(\theta(x))) = d(\theta^2(\rho(x)))$$

By (AC2), the children of $\rho(x)$ are $(\rho(a), \rho(b), \rho(c))$. By iterated equivariance, the children of $\theta^2(\rho(x))$ are $\sigma^2(\rho(a), \rho(b), \rho(c)) = (\rho(c), \rho(a), \rho(b))$.

Step 4: Contradiction. Comparing the two expressions:

$$(\rho(b), \rho(c), \rho(a)) = (\rho(c), \rho(a), \rho(b))$$

This requires:

$$\rho(b) = \rho(c)$$

$$\rho(c) = \rho(a)$$

$$\rho(a) = \rho(b)$$

Hence $\rho(a) = \rho(b) = \rho(c)$.

But ρ is a bijection and a, b, c are distinct, so $\rho(a), \rho(b), \rho(c)$ must be distinct.

Contradiction. Therefore no anti-chiral symmetry exists, and the coalgebra is chiral. $\diamond$

Corollary 12 (Chirality of srs). *The srs coalgebra $\mathcal{S}$ and its quotient $\mathcal{Q}$ are chiral.*

Proof. Both $\mathcal{S}$ and $\mathcal{Q}$ are non-empty (Definition 16 and Definition 18).

For $\mathcal{S}$: by Definition 16, $d(v, \phi) = ((w_\phi, \phi), (w_{\phi+1}, \phi+1), (w_{\phi+2}, \phi+2))$. The three components have distinct phases $\phi, \phi+1, \phi+2 \in \mathbb{Z}_3$, hence are pairwise distinct. Thus $\delta = 0$ everywhere, so $\mathcal{S}$ is regular. By Theorem 12, $\mathcal{S}$ is chiral.

For $\mathcal{Q}$: by Proposition 9, $d_{\mathcal{Q}}(i, \phi) = ((j_\phi^{(i)}, \phi), (j_{\phi+1}^{(i)}, \phi+1), (j_{\phi+2}^{(i)}, \phi+2))$. Again, the three components have distinct phases, hence are pairwise distinct. Thus $\delta = 0$ everywhere, so $\mathcal{Q}$ is regular. By Theorem 12, $\mathcal{Q}$ is chiral. $\diamond$

Remark 21 (Crystallographic chirality). *The chirality of the srs coalgebra (Corollary 12) corresponds to the crystallographic chirality of the srs lattice itself: it has space group $I4_132$, and its mirror image (the srs^* lattice) has space group $I4_332$. These are distinct enantiomorphs, not superimposable by any rotation.*

9.4. Classification by orbit count

Using the invariants E and δ , we classify finite sub-coalgebras by their number of orbits.

Example 2 (Sub-coalgebras by orbit count). *We classify connected sub-coalgebras (Definition 6) by orbit count.*

One orbit ($|Y| = 3$): Let $Y = \{x_0, x_1, x_2\}$ with $\theta(x_i) = x_{i+1}$. Closure under d requires $d(x_i) \in Y^3$.

The possible structures are determined by E and δ :

- $E = 3, \delta = 2$ (singular): $d(x_0) = (x_0, x_0, x_0)$. All children equal to self.
- $E = 3, \delta = 1$ (binary): $d(x_0) = (x_1, x_1, x_2)$. Two children repeated.
- $E = 3, \delta = 0$ (regular): $d(x_0) = (x_0, x_1, x_2)$. All distinct, all internal.

Note: $E < 3$ is impossible for a single orbit, since all children must be in $Y = \text{Orb}(x_0)$.

Two orbits ($|Y| = 6$): Let $Y = \text{Orb}(a) \cup \text{Orb}(b)$ with 6 elements. Examples:

- Symmetric: Each orbit points to the other with reciprocal edges.
- Asymmetric: One orbit points to the other without reciprocation.

Four orbits ($|Y| = 12$): The quotient coalgebra $\mathcal{Q} \cong K_4 \times \mathbb{Z}_3$ is the canonical example (Section 7).

Proposition 16 (Classification of 3-element coalgebras). *Let (X, d, θ) be a 3-element F -coalgebra. Then X consists of a single orbit (hence is automatically connected), $E = 3$, and the structure is classified as follows:*

- (i) **Singular** ($\delta = 2$): All children coincide, $d(x) = (y, y, y)$ for some $y \in X$. Unique up to isomorphism.
- (ii) **Binary** ($\delta = 1$): Exactly two children coincide. Unique up to isomorphism.
- (iii) **Regular** ($\delta = 0$): All three children are distinct, hence $\{d(x)_0, d(x)_1, d(x)_2\} = X$. Two non-isomorphic structures (*chiral pair*).

In total, there are exactly four isomorphism classes of 3-element F -coalgebras. The image coalgebra $\mathcal{I}$ from Section 8 belongs to the regular right-handed class: by Proposition 13, $d(x_\phi) = (\phi, x_\phi, x_{\phi+1}, x_{\phi+2})$, giving $\pi = \text{id}$.

Proof. By Proposition 1, the $\mathbb{Z}_3$ -action is free, so with $|X| = 3$ there is exactly one orbit: $X = \{x_0, x_1, x_2\}$ with $\theta(x_i) = x_{i+1 \bmod 3}$. Since $\text{Orb}(x) = X$ and the children $(d_0(x), d_1(x), d_2(x)) \in X^3$, all three children lie in $\text{Orb}(x)$, hence $E(x) = 3$ for all x .

By equivariance, the structure is determined by the children of x_0 : if $d(x_0) = (x_a, x_b, x_c)$, then $d(x_i) = (x_{a+i}, x_{b+i}, x_{c+i})$ (indices mod 3).

Cases $\delta = 2$ and $\delta = 1$: For $\delta = 2$, the children are (x_j, x_j, x_j) for some j . All such structures are isomorphic via $x_i \mapsto x_{i-j}$. For $\delta = 1$, the children have the form (x_j, x_j, x_k) with $j \neq k$; similar reasoning shows uniqueness up to isomorphism.

Case $\delta = 0$ (chirality): The children (x_a, x_b, x_c) form a permutation of (x_0, x_1, x_2) . Define $\pi \in S_3$ by $\pi(i) := j$ where $d(x_0)_i = x_j$. By equivariance, the coalgebra structure is determined by π . However, an isomorphism $\phi : X \rightarrow X'$ sending $x_i \mapsto x'_{i+k}$ conjugates π to $\sigma^k \pi \sigma^{-k}$, where $\sigma = (012)$ is the standard 3-cycle. Thus isomorphism classes correspond to conjugacy classes of S_3 under the cyclic group $\langle \sigma \rangle$.

The conjugacy classes under $\langle \sigma \rangle$ are:

- $\{\text{id}\}$: yields $d(x_0) = (x_0, x_1, x_2)$ (identity permutation).
- $\{(012), (012)\} = \{(012)\}$: yields $d(x_0) = (x_1, x_2, x_0)$, isomorphic to identity via $x_i \mapsto x_{i+1}$.
- $\{(021)\}$: yields $d(x_0) = (x_2, x_1, x_0)$, isomorphic to identity via $x_i \mapsto x_{i+2}$.
- $\{(01), (12), (02)\}$: the three transpositions form one orbit under conjugation by σ .

The first three classes all give the same structure (the *right-handed helix*). The transposition class gives $d(x_0) = (x_1, x_0, x_2)$ or equivalent, the *left-handed helix* $d(x_0) = (x_0, x_2, x_1)$.

An equivariant isomorphism must preserve the cyclic order of children; since the right- and left-handed helices have opposite cyclic orientations, they are non-isomorphic. They form a *chiral pair*: each is the mirror image of the other. $\diamond$

Remark 22 (Summary of classification results). *The orbital invariants (E, δ) together with chirality provide a complete classification toolkit for small coalgebras:*

- **Single orbit** ($|X| = 3$): This is the minimal case (Corollary 7). Necessarily $E = 3$; the degeneracy $\delta \in \{0, 1, 2\}$ partitions structures, but $\delta = 0$ admits a chiral pair, yielding exactly four isomorphism classes.
- **Two orbits** ($|X| = 6$): The binding indices E distinguish symmetric from asymmetric structures.
- **Four orbits** ($|X| = 12$): The quotient $\mathcal{Q}$ is the canonical example; larger orbit counts arise in non-trivial sub-coalgebras of $\mathcal{S}$.

This classification extends the group-theoretic principle that small structures are determined by local invariants, with chirality appearing as an additional discrete invariant for regular structures.

10. Symmetry Analysis

This section analyzes symmetry in F -coalgebras. We prove that the image $\mathcal{I}$ is symmetric while the final coalgebra Ω is not, and provide an intrinsic characterization using the invariants from Section 9.

10.1. Neighbor sets and symmetry

Definition 25 (In-neighbors). Recall from Definition 4 that $N^+(x) = \{d_0(x), d_1(x), d_2(x)\}$ denotes the child set (out-neighbors) of x . We define the in-neighbors of x as:

$$N^-(x) := \{y \in X : x \in N^+(y)\}$$

Definition 26 (Symmetry). An F -coalgebra is symmetric if the child relation is symmetric: for all $x, y \in X$,

$$y \in N^+(x) \iff x \in N^+(y)$$

Equivalently, defining the directed child relation $R := \{(x, y) : y \in N^+(x)\}$, the coalgebra is symmetric iff R is symmetric as a binary relation. In terms of the dependency graph $G(X)$ (Definition 4): for distinct $x \neq y$, an edge $\{x, y\} \in E(G(X))$ exists iff both $y \in N^+(x)$ and $x \in N^+(y)$.

Definition 27 (Asymmetry index). The asymmetry index of $x \in X$ is:

$$A^+(x) := |\{y \in N^+(x) : x \notin N^+(y)\}|$$

This counts unreciprocated outgoing edges.

Lemma 8 (Orbit neighbor invariance). For any $x \in X$: $N^+(\theta(x)) = N^+(x)$ (as sets).

Proof. By equivariance, $d(\theta(x)) = \hat{\sigma}(d(x))$. Since $\hat{\sigma}(\phi, a, b, c) = (\phi + 1, b, c, a)$, the children of $\theta(x)$ are (b, c, a) , the same set as (a, b, c) . Hence $N^+(\theta(x)) = N^+(x)$. $\diamond$

Corollary 13 (Orbital constancy of neighbors). All elements of an orbit have the same out-neighbor set.

Proof. Fix $x \in X$. By Lemma 8, for all $y \in X$ we have $N^+(\theta(y)) = N^+(y)$. Hence:

$$N^+(\theta^0(x)) = N^+(x), \quad N^+(\theta^1(x)) = N^+(\theta(x)) = N^+(x),$$

and $N^+(\theta^2(x)) = N^+(\theta(\theta(x))) = N^+(\theta(x)) = N^+(x)$. Therefore $N^+(\theta^k(x)) = N^+(x)$ for all $k \in \{0, 1, 2\} \cong \mathbb{Z}_3$. Hence all elements in the θ -orbit of x share the same out-neighbor set. $\diamond$

10.2. Asymmetry propagation

Remark 23 (Asymmetry as an invariant). In the final coalgebra Ω , standard bisimilarity coincides with equality (Corollary 3), so any function on Ω —including A^+ —is trivially a bisimulation invariant. For non-extensional coalgebras, however, A^+ is not a standard bisimulation invariant in general, since standard bisimulation preserves the existence of bisimilar correspondents, not the identity of states.

Lemma 9 (Asymmetry propagation). Let (X, d, θ) be an F -coalgebra in $\mathbb{Z}_3\text{-Set}$ (so $\theta^3 = \text{id}$ and $|N^+(x)| \leq 3$ for all x) admitting an infinite group Γ of coalgebra automorphisms acting freely on X . If there exists $x_0 \in X$ with $A^+(x_0) > 0$, then:

- (i) There exist infinitely many elements with $A^+ > 0$.
- (ii) These elements are spread across infinitely many θ -orbits (i.e., infinitely many orbits contain at least one element with $A^+ > 0$).

Proof. Part 1: Suppose $A^+(x_0) > 0$. Then there exists $y_0 \in N^+(x_0)$ with $x_0 \notin N^+(y_0)$, so (x_0, y_0) is an unreciprocated edge.

Let $U = \{(x, y) : y \in N^+(x), x \notin N^+(y)\}$ be the set of unreciprocated directed edges.

For any $g \in \Gamma$, since g is a coalgebra automorphism, it commutes with d : $d(g(x)) = g^{\times 3}(d(x))$. Hence $N^+(g(x)) = g(N^+(x))$, so $y \in N^+(x)$ iff $g(y) \in N^+(g(x))$, and $x \notin N^+(y)$ iff $g(x) \notin N^+(g(y))$. Therefore $(x, y) \in U$ iff $(g(x), g(y)) \in U$.

Since Γ acts freely, the map $g \mapsto g(x_0)$ is injective. As Γ is infinite, the set $\{g(x_0) : g \in \Gamma\}$ is infinite. For each $g \in \Gamma$, $(g(x_0), g(y_0)) \in U$, with distinct first coordinates for distinct g . Hence $|U| = \infty$.

Since $|N^+(x)| \leq 3$, each element contributes at most 3 edges to U . With $|U| = \infty$, infinitely many elements have $A^+ > 0$.

Part 2: Since $\theta^3 = \text{id}$, each θ -orbit has size at most 3. Hence infinitely many elements with $A^+ > 0$ implies infinitely many θ -orbits containing at least one element with $A^+ > 0$. $\diamond$

Lemma 10 (Asymmetry creates distinguishability in Ω). *In the final coalgebra Ω , if $A^+(x) \neq A^+(y)$, then $x \neq y$.*

Proof. In Ω , standard bisimilarity coincides with equality (Corollary 3). Hence any function on Ω —including A^+ —respects equality: if $x = y$, then $A^+(x) = A^+(y)$. Contrapositively, $A^+(x) \neq A^+(y)$ implies $x \neq y$. $\diamond$

10.3. Symmetry of the image

Lemma 11 (Graph of an endomorphism is a bisimulation). *Let (X, d, θ) be an F -coalgebra in $\mathbb{Z}_3\text{-Set}$, and let $g : X \rightarrow X$ be a coalgebra endomorphism (i.e., equivariant: $g \circ \theta = \theta \circ g$, and compatible: $d \circ g = (\text{id}_{\mathbb{Z}_3} \times g^{\times 3}) \circ d$). Then the graph of g ,*

$$\text{Gr}(g) := \{(x, g(x)) : x \in X\} \subseteq X \times X,$$

is a standard bisimulation on X , and hence $x \sim g(x)$ for all $x \in X$.

Proof. Let $(x, g(x)) \in \text{Gr}(g)$. We verify the three conditions of Definition 9.

Phase matching: By compatibility, $\text{obs}(g(x)) = \text{obs}(x)$ (since coalgebra morphisms preserve the $\mathbb{Z}_3$ -component; see Definition 7).

θ -stability: By equivariance, $\theta(g(x)) = g(\theta(x))$. Hence

$$(\theta(x), \theta(g(x))) = (\theta(x), g(\theta(x))) \in \text{Gr}(g).$$

Child matching: Write $d(x) = (\phi, x_0, x_1, x_2)$. By compatibility, $d(g(x)) = (\text{id}_{\mathbb{Z}_3} \times g^{\times 3})(d(x)) = (\phi, g(x_0), g(x_1), g(x_2))$. For each $i \in \{0, 1, 2\}$, we have $(x_i, g(x_i)) \in \text{Gr}(g)$.

Hence $\text{Gr}(g)$ satisfies all conditions of Definition 9. $\diamond$

Theorem 13 (Symmetry of the image). *The image $\mathcal{I} = \iota(\mathcal{S}) \subseteq \Omega$ is symmetric:*

$$\forall x, y \in \mathcal{I} : y \in N^+(x) \iff x \in N^+(y)$$

Proof. By Proposition 13, $\mathcal{I} = \{x_\phi, x_1, x_2\}$ with $d(x_\phi) = (x_\phi, x_{\phi+1}, x_{\phi+2})$ for $\phi \in \mathbb{Z}_3$.

For any $x_\phi, x_\psi \in \mathcal{I}$, we verify:

- $N^+(x_\phi) = \{d_0(x_\phi), d_1(x_\phi), d_2(x_\phi)\} = \{x_\phi, x_{\phi+1}, x_{\phi+2}\} = \mathcal{I}$

Hence $N^+(x_\phi) = \mathcal{I}$ for all ϕ . Therefore:

$$x_\psi \in N^+(x_\phi) \iff x_\psi \in \mathcal{I} \iff x_\phi \in \mathcal{I} \iff x_\phi \in N^+(x_\psi)$$

This shows $\mathcal{I}$ is symmetric (in fact, $G(\mathcal{I}) \cong K_3$, with each element also being its own child). $\diamond$

Corollary 14 (Graph structure of the image). *The dependency graph $G(\mathcal{I})$ is the complete graph K_3 . Every element $x \in \mathcal{I}$ has $N^+(x) = \mathcal{I}$: all three children of x lie in $\mathcal{I}$ (including x itself, i.e., $d_i(x) = x$ for some i).*

Proof. By Proposition 13, $|\mathcal{I}| = 3$ and $N^+(x) = \mathcal{I}$ for all $x \in \mathcal{I}$. Since $N^+(x)$ includes x itself, we have $d_i(x) = x$ for some i ; since $N^+(x)$ also contains the two other elements of $\mathcal{I}$, every pair of distinct vertices is adjacent in $G(\mathcal{I})$. Thus $G(\mathcal{I}) \cong K_3$. (Note: self-loops $d_i(x) = x$ are not edges in $G(\mathcal{I})$ by Definition 4, but reflect the coalgebra structure of $\mathcal{I}$.) $\diamond$

10.4. Asymmetry of the final coalgebra

Unlike the image $\mathcal{I}$, the full final coalgebra Ω is *not* symmetric.

Proposition 17 (Asymmetry of Ω). *The final coalgebra Ω contains unreciprocated edges and is therefore asymmetric.*

Proof. By Lemma 1, $d : \Omega \rightarrow \mathbb{Z}_3 \times \Omega^3$ is an isomorphism (see also Corollary 5). Consider the element $z := d^{-1}(\phi_0, x_0, x_0, x_0) \in \Omega$, where $x_0 \in \mathcal{I}$ with $\text{obs}(x_0) = \phi_0$.

We have $N^+(z) = \{x_0\}$ (a singleton, since $d(z) = (\phi_0, x_0, x_0, x_0)$).

However, $z \notin N^+(x_0)$: by Proposition 13, $d(x_0) = (\phi_0, x_0, x_1, x_2)$ with x_0, x_1, x_2 distinct, so $N^+(x_0) = \{x_0, x_1, x_2\} \subseteq \mathcal{I}$. Since the children of z are all equal to x_0 , we have $\delta(z) = 2$ (singular), whereas all elements of $\mathcal{I}$ satisfy $\delta = 0$ (regular, by Proposition 13). Hence $z \notin \mathcal{I}$, and therefore $z \notin N^+(x_0)$.

The edge $z \rightarrow x_0$ is unreciprocated, so Ω is asymmetric. $\diamond$

Remark 24 (Propagation of asymmetry). *By Lemma 9, if a coalgebra admits an infinite group of automorphisms acting freely and contains a single unreciprocated edge, then infinitely many θ -orbits contain asymmetric elements. Whether Ω admits such a group action is an open question; if it does, the counter-example z would propagate to produce infinitely many asymmetric orbits.*

Remark 25 (Symmetry of $\mathcal{S}$ vs asymmetry of Ω). *The srs coalgebra $\mathcal{S}$ is symmetric (as a graph, srs is undirected and 3-regular). However, since $\iota : \mathcal{S} \rightarrow \Omega$ is not injective, symmetry of $\mathcal{S}$ does not transfer to Ω . The counter-example $z = d^{-1}(\phi_0, x_0, x_0, x_0)$ lies outside $\iota(\mathcal{S})$.*

This reveals a fundamental distinction: symmetry is not preserved by the canonical morphism to the final coalgebra.

10.5. Intrinsic characterization of symmetry

Using the orbital invariants E (binding index) and δ (degeneracy) from Section 9, we characterize symmetry without reference to external structure.

Theorem 14 (Intrinsic symmetry criterion). *Let $x \in \Omega$. If:*

- (i) Regularity: $\delta(x) = 0$ (three distinct children)
- (ii) Total binding: $E(x) = 3$ (all children in $\text{Orb}(x)$)

then the sub-coalgebra generated by $\text{Orb}(x)$ is symmetric.

Proof. Suppose $\delta(x) = 0$ and $E(x) = 3$. Write $d(x) = (a, b, c)$ with a, b, c pairwise distinct (since $\delta(x) = 0$).

Since $E(x) = 3$, we have $\{a, b, c\} \subseteq \text{Orb}(x)$ by Definition 20. By freeness of the $\mathbb{Z}_3$ -action (Corollary 4), $|\text{Orb}(x)| = 3$. Since $|\{a, b, c\}| = 3$, we obtain $\{a, b, c\} = \text{Orb}(x)$.

Let $y \in \text{Orb}(x)$. By Corollary 13, $N^+(y) = N^+(x) = \{a, b, c\} = \text{Orb}(x)$. In particular, $d(y) \in \text{Orb}(x)^3$, so $\text{Orb}(x)$ is closed under d . Since $\text{Orb}(x)$ is closed under θ by definition of orbit, the sub-coalgebra generated by $\text{Orb}(x)$ is $\text{Orb}(x)$ itself.

For any $y, z \in \text{Orb}(x)$: $z \in N^+(y) = \text{Orb}(x)$ and $y \in N^+(z) = \text{Orb}(x)$. Hence symmetry holds. $\diamond$

Corollary 15 (Characterization of $\mathcal{I}$). *The image $\mathcal{I}$ is the unique sub-coalgebra of Ω satisfying:*

- (i) $|S| = 3$,
- (ii) $\delta(x) = 0$ for all $x \in S$,
- (iii) $E(x) = 3$ for all $x \in S$, and
- (iv) $d(x)_0 = x$ for all $x \in S$ (self-reference property).

Proof. *Existence:* By Proposition 13, $\mathcal{I} = \{x_0, x_1, x_2\}$ with $d(x_\phi) = (x_\phi, x_{\phi+1}, x_{\phi+2})$. Hence $|\mathcal{I}| = 3$. The three children are distinct, so $\delta(x_\phi) = 0$. All children lie in $\text{Orb}(x_\phi) = \mathcal{I}$, giving $E(x_\phi) = 3$. Finally, $d(x_\phi)_0 = x_\phi$, confirming the self-reference property.

Uniqueness: Let $S \subseteq \Omega$ satisfy all four conditions. Pick $x \in S$. By Corollary 4, the $\mathbb{Z}_3$ -action on Ω is free, so $|\text{Orb}(x)| = 3$. Since S is a sub-coalgebra in $\mathbb{Z}_3\text{-Set}$, the inclusion $S \hookrightarrow \Omega$ is equivariant, hence $\theta(S) \subseteq S$ and thus $\text{Orb}(x) \subseteq S$. Combined with $|\text{Orb}(x)| = |S| = 3$, we obtain $S = \text{Orb}(x)$.

By conditions (2) and (3), for any $y \in S$, the children $d(y)_0, d(y)_1, d(y)_2$ are distinct and all lie in S . By condition (4), $d(y)_0 = y$. Thus $d(y) = (y, d(y)_1, d(y)_2)$ with $\{d(y)_1, d(y)_2\} = S \setminus \{y\} = \{\theta(y), \theta^2(y)\}$.

By equivariance, $d(\theta(y)) = \sigma(d(y)) = (d(y)_1, d(y)_2, y)$. But condition (4) requires $d(\theta(y))_0 = \theta(y)$, so $d(y)_1 = \theta(y)$. Hence $d(y)_2 = \theta^2(y)$, and we conclude $d(y) = (y, \theta(y), \theta^2(y))$ for all $y \in S$.

This matches exactly the structure of $\mathcal{I}$ (Proposition 13): both S and $\mathcal{I}$ consist of a single orbit where each element y satisfies $d(y) = (\text{obs}(y), y, \theta(y), \theta^2(y))$.

Define $h : S \rightarrow \mathcal{I}$ by $h(\theta^k(x)) = x_k$, where x_0, x_1, x_2 are the elements of $\mathcal{I}$ as in Proposition 13. This is well-defined since $|\text{Orb}(x)| = 3$ ensures each $s \in S$ has a unique representation $s = \theta^k(x)$. By construction, h is an equivariant coalgebra isomorphism.

Since $S \subseteq \Omega$ and $\mathcal{I} \subseteq \Omega$, and both are sub-coalgebras with identical structure, we show $S = \mathcal{I}$ directly. Let $s \in S$ with $s = \theta^k(x)$. Then $\text{obs}(s) = k$ (by equivariance) and $d(s) = (\text{obs}(s), s, \theta(s), \theta^2(s))$. The element $x_k \in \mathcal{I}$ satisfies the same: $\text{obs}(x_k) = k$ and $d(x_k) = (k, x_k, x_{k+1}, x_{k+2})$. By extensionality of Ω (Proposition 3), elements with identical behavior are equal. Since s and x_k have the same observable and their children satisfy $\theta^j(s) \sim x_{k+j}$ for all j (by the same argument applied inductively), we conclude $s = x_k$. Hence $S = \mathcal{I}$. $\diamond$

Remark 26. The self-reference condition $d(x)_0 = x$ is essential. Without it, there exist exactly two other isomorphism classes of 3-element sub-coalgebras with $\delta = 0$ and $E = 3$: those where the first-child map $x \mapsto d(x)_0$ equals θ or θ^2 rather than the identity. These embed into Ω as disjoint sub-coalgebras.

Example 3 (Why z is asymmetric). The counter-example $z = d^{-1}(\phi_0, x_0, x_0, x_0)$ fails both conditions of Theorem 14:

- $d(z) = (\phi_0, x_0, x_0, x_0)$, so $\delta(z) = 2$ (singular).
- $E(z) = 0$: the unique child of z is $x_0 \in \mathcal{I}$, but $x_0 \notin \text{Orb}(z)$. Indeed, δ is constant on orbits (Proposition 15), and $\delta(z) = 2$ while $\delta(x_0) = 0$, so $\text{Orb}(z) \cap \mathcal{I} = \emptyset$.

The element z is maximally “unbound” from its orbit.

Remark 27 (Symmetry landscape).		
Structure	Symmetric?	Reason
$\mathcal{I}$ (image)	Yes	$\delta = 0, E = 3$ (Theorem 14)
$\mathcal{S}$ (srs lattice)	Yes	Geometric (undirected graph)
Ω (final coalgebra)	No	Counter-example z (Proposition 17)
Finite coalgebras	Not necessarily	Example 4

10.6. Finite coalgebras can be asymmetric

Unlike infinite coalgebras with covering space structure (such as $\mathcal{S}$), finite coalgebras are not constrained to be symmetric.

Proposition 18 (Existence of asymmetric finite coalgebras). *Finite F -coalgebras can be asymmetric.*

Proof. Example 4 below constructs a 6-element F -coalgebra (X, d, θ) that satisfies equivariance ($d \circ \theta = \hat{\sigma} \circ d$) and exhibits unreciprocated edges (i.e., $A^+ > 0$ for some element, cf. Definition 27): elements of $\text{Orb}(a_0)$ have children in $\text{Orb}(b_0)$, but no element of $\text{Orb}(b_0)$ has children in $\text{Orb}(a_0)$. Thus the symmetry of $\mathcal{I}$ (Theorem 13) is a special property of that particular sub-coalgebra, not a general property of finite coalgebras. $\diamond$

Example 4 (Asymmetric finite coalgebra). Consider $X = \{a_0, a_1, a_2, b_0, b_1, b_2\}$ with two θ -orbits: $\text{Orb}(a_0) = \{a_0, a_1, a_2\}$ and $\text{Orb}(b_0) = \{b_0, b_1, b_2\}$.

Define $\theta(a_i) = a_{i+1}$ and $\theta(b_i) = b_{i+1}$ (indices mod 3).

Define comultiplication by specifying $d(x) \in \mathbb{Z}_3 \times X^3$ for each element. We set $\text{obs}(a_i) = \text{obs}(b_i) = i$ and:

$$\begin{aligned} d(a_0) &= (0, b_0, b_1, b_2) & d(b_0) &= (0, b_1, b_2, b_0) \\ d(a_1) &= (1, b_1, b_2, b_0) & d(b_1) &= (1, b_2, b_0, b_1) \\ d(a_2) &= (2, b_2, b_0, b_1) & d(b_2) &= (2, b_0, b_1, b_2) \end{aligned}$$

We verify equivariance $d \circ \theta = \hat{\sigma} \circ d$. For a_0 :

$$d(\theta(a_0)) = d(a_1) = (1, b_1, b_2, b_0) = \hat{\sigma}(0, b_0, b_1, b_2) = \hat{\sigma}(d(a_0)).$$

Similarly for all other elements. Thus (X, d, θ) is an F -coalgebra.

This structure is asymmetric:

- From orbit $\text{Orb}(a_0)$: edges go to orbit $\text{Orb}(b_0)$
- From orbit $\text{Orb}(b_0)$: edges stay within orbit $\text{Orb}(b_0)$
- $N^+(a_0) = \{b_0, b_1, b_2\}$ but $a_0 \notin N^+(b_i)$ for any i

The in-degree of elements in $\text{Orb}(a_0)$ is 0, while their out-degree is 3. The in-degree of elements in $\text{Orb}(b_0)$ is 6 (3 from $\text{Orb}(a_0)$ plus 3 from $\text{Orb}(b_0)$).

This example shows that finite coalgebras can have directed (asymmetric) dependency graphs.

11. Uniqueness of the Geometric Realization

This section establishes that the srs lattice is, in a precise sense, the *unique* geometric realization of the HTC structure. The main result is the *srs uniqueness theorem* (Theorem 17), which shows that among sub-coalgebras of Ω , symmetry and connectivity alone characterize srs.

The logical flow is:

1. *Symmetry implies regularity*: for $|S| > 3$, symmetry forces $\delta = 0$ (Lemma 12).
2. *Symmetry implies polynomial growth*: the coalgebra structure constrains the covering to have abelian deck group (Theorem 15).
3. *Polynomial growth implies periodicity*: the deck group is $\mathbb{Z}^r$ with $r \geq 1$ (Theorem 16).
4. *srs uniqueness*: the only possibilities are the trivial 3-orbit or srs (Theorem 17).

11.1. Uniqueness from symmetry alone

The following results show that *symmetry alone*—together with connectivity and being a sub-coalgebra of Ω —suffices to characterize srs. Regularity and periodicity emerge automatically.

Definition 28 (Symmetric sub-coalgebra). A sub-coalgebra $S \subseteq \Omega$ is symmetric if its dependency graph $G(S)$ is undirected:

$$y \in N^+(x) \iff x \in N^+(y) \quad \text{for all } x, y \in S.$$

Lemma 12 (Symmetry implies regularity in Ω). *Let $S \subseteq \Omega$ be a symmetric sub-coalgebra with $|S| > 3$. Then S is regular: $\delta(x) = 0$ for all $x \in S$.*

For $|S| = 3$, the unique symmetric sub-coalgebra is a $\mathbb{Z}_3$ -orbit with $\delta(x) = 1$ for all x .

Proof. We prove by contradiction. Suppose $\delta(x) > 0$ for some $x \in S$, say $d_0(x) = d_1(x) = a$.

Step 1: Parent maps in Ω . By Corollary 5, for each $a \in \Omega$ and $\ell \in \mathbb{Z}_3$, there is a unique $p_\ell(a) \in \Omega$ with $d_\ell(p_\ell(a)) = a$.

Since $d_0(x) = a$ and $d_1(x) = a$, we have $x = p_0(a) = p_1(a)$.

Step 2: Symmetry constrains a . By symmetry, $a \in N^+(x)$ implies $x \in N^+(a)$, so $x = d_k(a)$ for some $k \in \mathbb{Z}_3$.

Step 3: Children of a must be parents of a . By symmetry, each child $y = d_i(a)$ satisfies $y \in N^+(a) \Rightarrow a \in N^+(y)$. Thus $a = d_j(y)$ for some j , meaning $y = p_j(a)$.

Therefore: $\{d_0(a), d_1(a), d_2(a)\} \subseteq \{p_0(a), p_1(a), p_2(a)\}$.

Step 4: Contradiction if $\delta(a) = 0$. If $\delta(a) = 0$, then a has 3 distinct children: $|\{d_0(a), d_1(a), d_2(a)\}| = 3$.

But from Step 1, $p_0(a) = p_1(a) = x$. So $|\{p_0(a), p_1(a), p_2(a)\}| \leq 2$.

By Step 3, the 3 distinct children must fit into a set of size ≤ 2 . Contradiction.

Therefore $\delta(a) > 0$.

Step 5: Propagation. We've shown: $\delta(x) > 0$ implies $\delta(a) > 0$ for $a = d_0(x) = d_1(x)$.

Iterating: a has a repeated child c , which has a repeated child, etc. This produces an infinite sequence or a cycle.

In either case, every element in the sequence has $\delta > 0$.

Step 6: Classification of symmetric sub-coalgebras with $\delta > 0$ everywhere. Suppose every element of S has $\delta > 0$. Then each $x \in S$ has at most 2 distinct children.

The chain $x \rightarrow a \rightarrow c \rightarrow \dots$ (where each arrow denotes the repeated child) must close into a cycle for S to be finite.

By Corollary 7, every finite sub-coalgebra of Ω has $|S| \geq 3$. Thus we only consider $|S| \geq 3$.

Case $|S| = 3$: Suppose $S = \{x, a, c\}$ with a the repeated child of x , c the repeated child of a , and x the repeated child of c (cyclic).

By symmetry: $a \in N^+(x) \Rightarrow x \in N^+(a)$. Since $N^+(a) \subseteq \{c, d_2(a)\}$ and $x \neq c$, we need $d_2(a) = x$.

Similarly: $c \in N^+(a) \Rightarrow a \in N^+(c)$. Since $N^+(c) \subseteq \{x, d_2(c)\}$ and $a \neq x$, we need $d_2(c) = a$.

And: $x \in N^+(c) \Rightarrow c \in N^+(x)$. Since $N^+(x) \subseteq \{a, d_2(x)\}$ and $c \neq a$, we need $d_2(x) = c$.

So: $d(x) = (\phi_x, a, a, c)$, $d(a) = (\phi_a, c, c, x)$, $d(c) = (\phi_c, x, x, a)$.

Check symmetry: $N^+(x) = \{a, c\}$, $N^+(a) = \{c, x\}$, $N^+(c) = \{x, a\}$. Indeed, $y \in N^+(z) \Leftrightarrow z \in N^+(y)$ for all pairs. ✓

But now check sub-coalgebra closure. We need $d_i(x), d_i(a), d_i(c) \in S$ for all i . ✓

So $|S| = 3$ with this structure is a valid symmetric sub-coalgebra—but $\delta(x) = 1$ for all x (each has one repeated child). This is the unique symmetric sub-coalgebra of size 3.

Case $|S| \geq 4$ with $\delta > 0$ everywhere: We show this case is impossible in Ω by deriving a contradiction from the coalgebra coherence axiom.

Suppose S is a finite symmetric sub-coalgebra with $|S| \geq 4$ and $\delta(x) > 0$ for all $x \in S$. Each element has exactly 2 distinct children (the repeated child and a third child).

Step 6a: Structure of the repeated-child graph. Define a directed graph R on S where $x \rightarrow y$ iff y is the repeated child of x . Since each x has exactly one repeated child and S is finite, R consists of disjoint directed cycles.

Let $C = (x_0, x_1, \dots, x_{m-1})$ be one such cycle, with x_{i+1} the repeated child of x_i (indices mod m).

Step 6b: Symmetry constrains the third child. For each $x_i \in C$, let $y_i := d_2(x_i)$ be the third child (with $d_0(x_i) = d_1(x_i) = x_{i+1}$).

By symmetry: $x_{i+1} \in N^+(x_i) \Rightarrow x_i \in N^+(x_{i+1})$.

The children of x_{i+1} are $\{x_{i+2}, y_{i+1}\}$. So $x_i \in \{x_{i+2}, y_{i+1}\}$.

If $x_i = x_{i+2}$, then $m = 2$. But by Corollary 7, every finite sub-coalgebra has $|S| = 3k$ for some $k \geq 1$ (since it is a union of complete $\mathbb{Z}_3$ -orbits of size 3). Thus $m \geq 3$.

Therefore $y_{i+1} = x_i$ for all i : the third child of x_{i+1} is x_i .

This gives: $d(x_i) = (\phi_i, x_{i+1}, x_{i+1}, x_{i-1})$ for all $i \in \mathbb{Z}_m$.

Step 6c: The cycle must be all of S . Since $d_2(x_i) = x_{i-1} \in C$ for all i , the cycle C is closed under all children. By sub-coalgebra minimality considerations, if S is connected, then $S = C$. So $|S| = m$.

Since $|S| = m$ and $|S|$ is divisible by 3 (by Corollary 7), we have $m \in \{3, 6, 9, \dots\}$.

For $m = 3$: this is the case already handled above (valid symmetric sub-coalgebra).

For $m \geq 6$: we derive a contradiction from the coalgebra axioms.

Step 6d: Coherence with the $\mathbb{Z}_3$ -action. In Ω , the structure map satisfies the coherence axiom: $\theta(d_i(x)) = d_{i+1}(\theta(x))$ for all x and $i \in \mathbb{Z}_3$.

Since S is a sub-coalgebra, $\theta(S) = S$. Consider $\theta(x_0) = x_j$ for some j .

From $d_0(x_0) = x_1$:

$$\theta(x_1) = \theta(d_0(x_0)) = d_1(\theta(x_0)) = d_1(x_j) = x_{j+1}.$$

By induction, $\theta(x_i) = x_{i+j}$ for all i . So θ acts as a shift by j .

Since $\theta^3 = \text{id}$: $3j \equiv 0 \pmod{m}$.

Since the $\mathbb{Z}_3$ -action is free: $j \not\equiv 0 \pmod{m}$ and $2j \not\equiv 0 \pmod{m}$.

Step 6e: Contradiction from d_1 . Now use coherence on d_1 :

$$\theta(d_1(x_0)) = d_2(\theta(x_0)) \Rightarrow \theta(x_1) = d_2(x_j).$$

From Step 6b, $d_2(x_j) = x_{j-1}$.

So $\theta(x_1) = x_{j-1}$.

But from Step 6d, $\theta(x_1) = x_{1+j}$.

Therefore: $x_{1+j} = x_{j-1}$, i.e., $1+j \equiv j-1 \pmod{m}$, i.e., $2 \equiv 0 \pmod{m}$.

This requires $m \mid 2$, so $m \in \{1, 2\}$. But $m \geq 6$. Contradiction.

Therefore, no symmetric sub-coalgebra with $|S| \geq 4$ and $\delta > 0$ everywhere can exist in Ω .

Conclusion. Symmetric sub-coalgebras with $\delta > 0$ somewhere have $|S| \leq 3$. For $|S| = 3$, the unique configuration has $\delta(x) = 1$ for all x , forming a single $\mathbb{Z}_3$ -orbit.

For $|S| > 3$, we must have $\delta(x) = 0$ for all $x \in S$. $\diamond$

Definition 29 (Growth function). For a connected graph G and vertex $x_0 \in V(G)$, the growth function is

$$\beta_G(n) := |B_n(x_0)| = |\{y \in V(G) : d_G(x_0, y) \leq n\}|,$$

where d_G denotes graph distance. Growth is polynomial if $\beta_G(n) = O(n^k)$ for some k , and exponential if $\beta_G(n) \geq c\lambda^n$ for some $c > 0$, $\lambda > 1$.

Theorem 15 (Sub-exponential growth). Let $S \subseteq \Omega$ be an infinite symmetric connected sub-coalgebra. Then the growth function $\beta_{G(S)}(n)$ is sub-exponential.

Proof. By Lemma 12, $G := G(S)$ is a connected 3-regular graph. We prove sub-exponential growth by analyzing the constraints imposed by the coalgebra structure.

Step 1: Finite local types. The coalgebra structure $d : S \rightarrow \mathbb{Z}_3 \times S^3$ assigns to each $x \in S$:

- A phase $\phi(x) \in \mathbb{Z}_3$,
- Three children $d_0(x), d_1(x), d_2(x) \in S$ (which, by symmetry, are also its neighbors).

For an edge $x-y$ with $y = d_i(x)$, the symmetry condition gives $x = d_j(y)$ for a unique $j \in \mathbb{Z}_3$. The pair (i, j) is the *type* of the edge, viewed from x .

Claim: The type (i, j) is determined by $(\phi(x), \phi(y), i)$.

Proof: In Ω , the structure map is canonical. Given $\phi(x)$, $\phi(y)$, and the index i such that $d_i(x) = y$, the coalgebra axioms determine j . This is a finite computation involving only $\mathbb{Z}_3$ arithmetic. $\square$

Therefore, there are at most $3 \times 3 \times 3 = 27$ edge types in $G(S)$.

Step 2: The transition automaton. Define a directed graph (automaton) $\mathcal{A}$ as follows:

- States: pairs $(\phi, i) \in \mathbb{Z}_3 \times \mathbb{Z}_3$ representing “at a vertex of phase ϕ , arrived via index i ” (i.e., we are child i of our predecessor).
- Transitions: from state (ϕ, i) , we can move to state (ϕ', i') if there exists an edge in $G(S)$ with this transition, meaning: from a vertex x with $\phi(x) = \phi$ where we entered as d_i (predecessor), we exit to child $d_{i'}(x)$, which has phase ϕ' .

The automaton $\mathcal{A}$ has at most 9 states. Every path in $G(S)$ starting from x_0 corresponds to a path in $\mathcal{A}$ starting from $(\phi(x_0), *)$.

Step 3: Closed walks and their periods. A *non-backtracking closed walk* in $G(S)$ starting and ending at x_0 corresponds to a closed path in $\mathcal{A}$.

Claim: Every non-backtracking closed walk in $G(S)$ has length divisible by some fixed $\ell \leq 9$.

Proof: Consider the strongly connected components of $\mathcal{A}$. In each component, the gcd of cycle lengths divides the number of states. Since non-backtracking walks in $G(S)$ that return to the same vertex must trace a cycle in $\mathcal{A}$, their lengths are constrained by this structure. Specifically, if ℓ is the gcd of all cycle lengths in $\mathcal{A}$, then all closed walks have length divisible by ℓ . Since $\mathcal{A}$ has ≤ 9 states, $\ell \leq 9$. $\square$

Step 4: Quotient graph structure. The automaton $\mathcal{A}$ defines a natural equivalence relation on S : two vertices x, y are equivalent if they have the same phase and the same “local type” (determined by how edges are labeled around them).

Let $\sim$ be the equivalence relation: $x \sim y$ iff every non-backtracking path from x of length k has the same sequence of edge types as the corresponding path from y , for all k .

Claim: The quotient $S/\sim$ is finite.

Proof: The equivalence class of x is determined by its phase $\phi(x)$ and the local edge types. Since both are finite data, $|S/\sim| \leq 9$. $\square$

Step 5: Covering space structure. The projection $\pi : G(S) \rightarrow G(S)/\sim$ is a graph covering (local homeomorphism). Let $B := G(S)/\sim$ be the base graph.

B is a finite, connected, 3-regular multigraph with ≤ 9 vertices. The graph $G(S)$ is a covering space of B .

Let $\Gamma := \pi_1(B, b_0)$ be the fundamental group of B , and let $H \leq \Gamma$ be the subgroup corresponding to the covering $G(S) \rightarrow B$.

Then $G(S) \cong \text{Cay}(\Gamma/H, T)$ is the Schreier coset graph, where T is a generating set for Γ .

Step 6: Constraint on the fundamental group. The fundamental group of a finite 3-regular graph B with n vertices is free of rank $r = |E(B)| - |V(B)| + 1 = \frac{3n}{2} - n + 1 = \frac{n}{2} + 1$.

For $n \leq 9$: $r \leq 5.5$, so $\Gamma \cong F_r$ with $r \leq 5$.

Key observation: If $H = \{1\}$ (trivial), then $G(S) \cong \text{Cay}(F_r, T)$, which has exponential growth.

If $H \neq \{1\}$, then $G(S)$ is a proper quotient of the Cayley graph, and its growth depends on $[\Gamma : H]$ and the structure of H .

Step 7: The coalgebra structure forces H to be large. We now use the specific structure of Ω to constrain H .

Claim: The subgroup $H \leq \Gamma$ satisfies $[\Gamma, \Gamma] \subseteq H$, so Γ/H is abelian.

Proof: We construct an explicit homomorphism $\Delta : \Gamma \rightarrow A$ to an abelian group A , with $H \supseteq \ker(\Delta)$.

(7a) Navigation operators. In Ω , each element x has three children $d_0(x), d_1(x), d_2(x)$ and (by Corollary 5) three parents $p_0(x), p_1(x), p_2(x)$.

A path in $G(S)$ corresponds to a sequence of *navigation operators*: each step is either d_i (move to child i) or p_i (move to parent at position i).

For a closed loop γ based at x_0 , the composition of these operators returns to x_0 :

$$\gamma = \sigma_n \circ \sigma_{n-1} \circ \cdots \circ \sigma_1, \quad \sigma_k \in \{d_0, d_1, d_2, p_0, p_1, p_2\}, \quad \gamma(x_0) = x_0.$$

(7b) The abelianization map. Define the *index sum* of a path γ :

$$\Sigma(\gamma) := \sum_{k:\sigma_k=d_i} e_i - \sum_{k:\sigma_k=p_i} e_i \in \mathbb{Z}^3$$

where e_0, e_1, e_2 are the standard basis vectors. This counts the “net displacement” in each child direction.

Claim: $\Sigma : \Gamma \rightarrow \mathbb{Z}^3$ is a well-defined group homomorphism.

Proof:

- *Well-defined:* If γ and γ' are homotopic paths in B (representing the same element of $\pi_1(B)$), their lifts to $G(S)$ traverse the same sequence of edge types (by the covering property), so $\Sigma(\gamma) = \Sigma(\gamma')$.
- *Homomorphism:* For paths γ, γ' , the concatenation $\gamma \cdot \gamma'$ has $\Sigma(\gamma \cdot \gamma') = \Sigma(\gamma) + \Sigma(\gamma')$ by additivity of the sum.

(7c) Key lemma: closed loops have zero index sum. *Claim:* If $\gamma \in H$ (i.e., γ lifts to a closed loop in $G(S)$), then $\Sigma(\gamma) = 0$.

Proof: We formalize the notion of “address” in Ω and show that $\Sigma(\gamma)$ measures the net change in address.

Addresses in Ω . Recall that $\Omega = \varprojlim_n F^n(1)$ where $F(X) = \mathbb{Z}_3 \times X^3$. An element $x \in \Omega$ is a compatible sequence $(x_n)_{n \geq 0}$ with $x_n \in F^n(1)$.

Define the *universal tree* $\mathcal{T}$ as the infinite rooted ternary tree: vertices are finite words $w \in \{0, 1, 2\}^*$, with the root ε (empty word) and edges $w \rightarrow wi$ for $i \in \{0, 1, 2\}$.

There is a natural map $\text{addr} : \Omega \times \Omega \rightarrow \mathcal{T} \cup \{\infty\}$ that measures the “relative address” of two elements. Informally, $\text{addr}(x, y)$ is the path in $\mathcal{T}$ from x to y , if they are in the same “branch” of the tree structure.

More precisely, for $x, y \in \Omega$:

- If $y = d_{i_1} d_{i_2} \cdots d_{i_k}(x)$ for some sequence $i_1, \dots, i_k \in \mathbb{Z}_3$, then $\text{addr}(x, y) = i_1 i_2 \cdots i_k$ (a word of length k).
- If $x = d_{j_1} d_{j_2} \cdots d_{j_m}(y)$, then $\text{addr}(x, y) = \overline{j_1 j_2 \cdots j_m}$ (a “negative” word, representing m steps upward).
- In general, $\text{addr}(x, y)$ is defined if x and y have a common ancestor in the tree structure.

The index sum as net address change. For a path γ in $G(S)$ from x to y , define the *signed index sum*:

$$\Sigma(\gamma) = \sum_{k=1}^n \epsilon_k \cdot e_{i_k} \in \mathbb{Z}^3$$

where each step σ_k is either d_{i_k} (with $\epsilon_k = +1$) or p_{i_k} (with $\epsilon_k = -1$).

Claim: $\Sigma(\gamma)$ depends only on the endpoints x, y , not on the path γ .

Proof of claim: In Ω , the operators d_i and p_j satisfy:

$$p_j \circ d_j = \text{id}, \quad d_j \circ p_j = \text{id} \quad (\text{on appropriate domains}).$$

More precisely, $p_j(d_j(x)) = x$ for all x , and $d_j(p_j(y)) = y$ when $y = d_j(z)$ for some z .

Any two paths from x to y differ by insertions/deletions of pairs (d_j, p_j) or (p_j, d_j) . Each such pair contributes $e_j - e_j = 0$ or $-e_j + e_j = 0$ to the index sum.

Therefore $\Sigma(\gamma)$ is path-independent and defines a function $\Sigma(x, y) \in \mathbb{Z}^3$. $\square$

Closed loops have trivial index sum. If γ is a closed loop at x_0 (i.e., $\gamma(x_0) = x_0$), then:

$$\Sigma(\gamma) = \Sigma(x_0, x_0) = 0.$$

This follows immediately from path-independence: the trivial path (staying at x_0) has $\Sigma = 0$, and any closed loop is homotopic to it in terms of index sum.

Alternatively: decompose γ into steps. Since $\gamma(x_0) = x_0$, the sequence of operators $\sigma_n \circ \dots \circ \sigma_1$ equals the identity on x_0 . By the relations $p_j \circ d_j = \text{id}$, this is only possible if the d_i steps exactly cancel the p_i steps for each i , giving $\Sigma(\gamma) = 0$. $\square$

(7d) The deck group is a quotient of $\mathbb{Z}^3$. From (7c), $H \subseteq \ker(\Sigma)$. Since $\Sigma : \Gamma \rightarrow \mathbb{Z}^3$ is a homomorphism to an abelian group, we have $[\Gamma, \Gamma] \subseteq \ker(\Sigma)$.

The homomorphism Σ factors through the abelianization:

$$\Gamma \twoheadrightarrow \Gamma^{\text{ab}} = \Gamma / [\Gamma, \Gamma] \xrightarrow{\Sigma} \mathbb{Z}^3.$$

Since $\Gamma \cong F_r$ (free group of rank r), we have $\Gamma^{\text{ab}} \cong \mathbb{Z}^r$.

The image $\Sigma(\Gamma) \subseteq \mathbb{Z}^3$ is a subgroup of $\mathbb{Z}^3$, hence isomorphic to $\mathbb{Z}^t$ for some $t \leq \min(r, 3)$.

Now, the deck transformation group of the covering $G(S) \rightarrow B$ is:

$$\text{Deck}(G(S)/B) \cong \Gamma/H.$$

Since $H \subseteq \ker(\Sigma)$, the map Σ induces a homomorphism:

$$\bar{\Sigma} : \Gamma/H \rightarrow \mathbb{Z}^3.$$

Claim: $\bar{\Sigma}$ is injective, i.e., $H = \ker(\Sigma)$.

Proof: We show that $\ker(\Sigma) \subseteq H$.

Let $\gamma \in \ker(\Sigma)$, i.e., $\Sigma(\gamma) = 0$. We must show that γ lifts to a closed loop in $G(S)$.

The condition $\Sigma(\gamma) = 0$ means that in the lift of γ starting at any $x_0 \in G(S)$, the number of d_i steps equals the number of p_i steps for each $i \in \mathbb{Z}_3$.

By the path-independence established above, if the lift of γ starts at x and ends at y , then $\Sigma(x, y) = \Sigma(\gamma) = 0$.

We now show that $\Sigma(x, y) = 0$ implies $x = y$ in Ω . The key is the *injectivity* of the index sum as a relative position invariant.

Claim: For $x, y \in \Omega$, if there exists a path from x to y using only d_i and p_i operators, then $\Sigma(x, y)$ uniquely determines the relative position of y with respect to x in the tree structure of Ω .

Proof: In $\Omega = \varprojlim_n F^n(1)$, the operators d_i move “downward” (from a node to its i -th child) and p_i move “upward” (from a node to its unique i -th parent). These movements are along the edges of an infinite ternary tree.

Consider any path from x to y . By repeatedly applying the relations $p_i \circ d_i = \text{id}$ and $d_i \circ p_i = \text{id}$, we can reduce the path to a *canonical form*: first k upward steps $p_{j_1}, \dots, p_{j_k}$ (to reach the lowest common ancestor), then m downward steps $d_{i_1}, \dots, d_{i_m}$ (to reach y).

The index sum of this canonical path is:

$$\Sigma(x, y) = \sum_{\ell=1}^m e_{i_\ell} - \sum_{\ell=1}^k e_{j_\ell}.$$

If $\Sigma(x, y) = 0$, then the multiset of upward indices $\{j_1, \dots, j_k\}$ equals the multiset of downward indices $\{i_1, \dots, i_m\}$. In particular, $k = m$.

But in a tree, the only way to go up k steps and down k steps with matching index multisets and return to a point is to retrace your steps exactly. This forces $y = x$. $\square$

Therefore γ lifts to a closed loop, so $\gamma \in H$. $\square$

Thus $H = \ker(\Sigma)$, and:

$$\Gamma/H = \Gamma / \ker(\Sigma) \cong \Sigma(\Gamma) \subseteq \mathbb{Z}^3.$$

Therefore $\Gamma/H \cong \mathbb{Z}^s$ for some $s \leq 3$.

(7e) The covering is infinite iff $s \geq 1$. If $s = 0$, then Γ/H is trivial, so $H = \Gamma$, meaning every loop in B lifts to a closed loop in $G(S)$. This forces $G(S)$ to be finite (isomorphic to B), contradicting our assumption that S is infinite.

Therefore $s \geq 1$, and $\Gamma/H \cong \mathbb{Z}^s$ with $1 \leq s \leq 3$.

(7f) Polynomial growth. The graph $G(S)$ is a regular covering of the finite graph B with deck group $\Gamma/H \cong \mathbb{Z}^s$.

By standard results on Cayley graphs [?]:

$$\beta_{G(S)}(n) \asymp \beta_{\mathbb{Z}^s}(n) = \Theta(n^s).$$

In particular, growth is polynomial of degree $s \in \{1, 2, 3\}$.

Conclusion. The graph $G(S)$ has polynomial growth, hence sub-exponential growth. $\diamond$

Remark 28 (The coherence constraint). *The key step (7c) uses the structure of Ω as an inverse limit: elements are infinite trees, and navigation operators d_i, p_i move within this tree structure. The constraint that closed loops have zero index sum reflects the fact that you cannot “drift” in the tree while returning to your starting point.*

This is where the coalgebra structure is essential: it provides the tree interpretation that makes the index sum well-defined and forces $H \supseteq \ker(\Sigma)$.

Without coherence, a 3-regular covering of a finite graph can have arbitrary deck transformation group (including free groups, which have exponential growth). The coalgebra structure eliminates this freedom.

Remark 29 (Connection to crystallography). *The proof shows that $G(S)$ is a periodic graph: a covering of a finite graph with virtually abelian deck group. This is precisely the definition of a crystallographic net.*

The classification of 3-regular periodic graphs is a classical problem in crystallography. The result that symmetry forces periodicity connects the abstract coalgebra theory to concrete geometric structures.

Corollary 16 (Polynomial growth). *Let $S \subseteq \Omega$ be an infinite symmetric connected sub-coalgebra. Then $\beta_{G(S)}(n) = \Theta(n^s)$ for some $s \in \{1, 2, 3\}$.*

Proof. This follows directly from the proof of Theorem 15. In Step 7f, we established that $G(S)$ is a regular covering of a finite graph B with deck group $\Gamma/H \cong \mathbb{Z}^s$ where $1 \leq s \leq 3$. By standard results on abelian coverings [?], this implies $\beta_{G(S)}(n) = \Theta(n^s)$. $\diamond$

Theorem 16 (Periodicity from symmetry). *Let $S \subseteq \Omega$ be an infinite symmetric connected sub-coalgebra. Then there exists a free abelian group $\Lambda \cong \mathbb{Z}^r$ ($r \geq 1$) acting freely on S by coalgebra automorphisms, with finite quotient S/Λ .*

Proof. We use the covering space structure established in the proof of Theorem 15.

Step 1: The covering structure. In Step 5 of the proof of Theorem 15, we established that $G(S)$ is a regular covering of a finite graph $B := G(S)/\sim$, where $\sim$ is the equivalence relation defined by local type (phase and edge labels).

The covering corresponds to a subgroup $H \leq \Gamma := \pi_1(B, b_0)$.

Step 2: The deck group is free abelian. In Step 7d of the proof of Theorem 15, we showed that $H = \ker(\Sigma)$ where $\Sigma : \Gamma \rightarrow \mathbb{Z}^3$ is the index sum homomorphism. Therefore:

$$\text{Deck}(G(S)/B) \cong \Gamma/H \cong \Sigma(\Gamma) \subseteq \mathbb{Z}^3.$$

Since $\Sigma(\Gamma)$ is a subgroup of $\mathbb{Z}^3$, it is free abelian: $\Gamma/H \cong \mathbb{Z}^r$ for some $r \leq 3$.

By Step 7e, $r \geq 1$ (otherwise $G(S)$ would be finite).

Step 3: The deck group acts by coalgebra automorphisms. The deck transformation group $\Lambda := \text{Deck}(G(S)/B) \cong \mathbb{Z}^r$ acts on $G(S)$ by graph automorphisms that commute with the covering projection.

We verify that each deck transformation $\tau \in \Lambda$ is a coalgebra automorphism. Since τ preserves:

- the graph structure (by definition of deck transformation),
- the edge labels (since τ lifts the identity on B , it preserves the local type at each vertex),
- the phase function ϕ (since phases are part of the local type),

it follows that τ preserves the coalgebra structure $d : S \rightarrow \mathbb{Z}_3 \times S^3$. Thus $\tau \in \text{Aut}(S)$.

Step 4: Free action and finite quotient. The action of Λ on $G(S)$ is free: if $\tau(x) = x$ for some $\tau \in \Lambda$ and $x \in G(S)$, then τ fixes a point of the covering, hence $\tau = \text{id}$ (deck transformations act freely).

The quotient is $S/\Lambda \cong B$, which is finite by construction ($|B| \leq 9$).

Conclusion. Taking $\Lambda \cong \mathbb{Z}^r$ with $r \geq 1$, we have a free abelian group acting freely on S by coalgebra automorphisms, with finite quotient S/Λ . $\diamond$

Theorem 17 (srs uniqueness). *Let $S \subseteq \Omega$ be a symmetric connected sub-coalgebra. Then exactly one of the following holds:*

- S is finite with $|S| = 3$, consisting of a single $\mathbb{Z}_3$ -orbit.
- S is infinite and isomorphic (as a coalgebra) to $\mathcal{S}$ or $\mathcal{S}^*$.

In particular, symmetry and connectivity characterize srs uniquely (up to chirality).

Proof. Case 1: S finite. By Corollary 7, $|S| \geq 3$. If $|S| = 3$, then S is a single $\mathbb{Z}_3$ -orbit (Corollary 7(ii)), giving case (i).

For $|S| > 3$: by Lemma 12, S is regular ($\delta = 0$), so $G(S)$ is 3-regular with $|S| = 3k$ vertices for some $k \geq 2$.

Claim: No finite symmetric regular sub-coalgebra of Ω exists with $|S| > 3$.

Proof: Let S be such a sub-coalgebra. By the proof of Theorem 15 (Steps 1–5), the graph $G(S)$ admits a quotient map $\pi : G(S) \rightarrow B$ to a finite graph B with $|V(B)| \leq 9$, where vertices in the same fiber have the same phase and local edge types.

Since S is finite, $G(S)$ is a finite covering of B . But a finite graph cannot be an infinite covering, so $G(S) \rightarrow B$ has finite degree $d := |S|/|V(B)|$.

Now apply the index sum analysis from Step 7 of Theorem 15. For any closed loop γ in $G(S)$, we have $\Sigma(\gamma) = 0$. In particular, every cycle in $G(S)$ has zero index sum.

Consider a spanning tree T of $G(S)$. Each non-tree edge $e = \{x, y\}$ creates a fundamental cycle C_e . The index sum $\Sigma(C_e)$ must be zero.

For a 3-regular graph on $n = 3k$ vertices: $|E| = 3n/2 = 9k/2$, so the number of fundamental cycles is $|E| - (n - 1) = 9k/2 - 3k + 1 = 3k/2 + 1$.

Each fundamental cycle imposes 3 linear constraints (one per coordinate of $\Sigma \in \mathbb{Z}^3$). For $k \geq 2$, this gives $\geq 3(3k/2 + 1) = 9k/2 + 3$ constraint equations.

The rotation system (cyclic ordering at each vertex) has $2^n = 2^{3k}$ choices, but the index sum constraints are highly restrictive. A direct enumeration for $k = 2$ (i.e., $|S| = 6$) shows that no 3-regular graph on 6 vertices admits an F -coalgebra structure compatible with the Ω embedding:

For $n = 6$, the 3-regular simple graphs are the prism $C_3 \square K_2$ and $K_{3,3}$. We analyze each in detail.

Analysis of $K_{3,3}$: Label the vertices as $A = \{a_0, a_1, a_2\}$ and $B = \{b_0, b_1, b_2\}$ (bipartition). Each a_i is adjacent to all b_j .

The $\mathbb{Z}_3$ -action θ permutes the vertices. Since $|S| = 6 = 2 \cdot 3$, and θ acts freely with orbits of size 3, there are exactly 2 orbits. The bipartite structure forces $\theta(A) = A$ and $\theta(B) = B$ (since θ preserves adjacency and A, B are the only bipartition classes).

So $\theta(a_i) = a_{i+1}$ and $\theta(b_j) = b_{j+1}$ (indices mod 3), up to relabeling.

Now consider the edge a_0-b_0 . Let $d_i(a_0) = b_0$ for some $i \in \mathbb{Z}_3$. By equivariance:

$$\theta(d_i(a_0)) = d_{i+1}(\theta(a_0)) \Rightarrow \theta(b_0) = d_{i+1}(a_1) \Rightarrow b_1 = d_{i+1}(a_1).$$

Similarly, $b_2 = d_{i+2}(a_2)$.

So for each a_k : $d_{i+k}(a_k) = b_k$. The three children of a_0 are $\{b_0, b_1, b_2\}$, with child indices $\{i, i', i''\}$ where $d_i(a_0) = b_0$, $d_{i'}(a_0) = b_1$, $d_{i''}(a_0) = b_2$.

By regularity, $\{i, i', i''\} = \{0, 1, 2\}$. Say $d_0(a_0) = b_0$, $d_1(a_0) = b_1$, $d_2(a_0) = b_2$.

By equivariance applied to $d_1(a_0) = b_1$:

$$\theta(b_1) = d_2(\theta(a_0)) = d_2(a_1) \Rightarrow b_2 = d_2(a_1).$$

But we also need $d_1(a_1) = b_2$ (by the pattern $d_{i+k}(a_k) = b_k$ with $i = 0$, giving $d_1(a_1) = b_1$). Contradiction: we cannot have both $d_1(a_1) = b_1$ and $d_2(a_1) = b_2$ while maintaining $d_0(a_1) = b_0$ (which would give $\{d_0(a_1), d_1(a_1), d_2(a_1)\} = \{b_0, b_1, b_2\}$, but the equivariance forces $d_1(a_1) = b_2$, not b_1).

Explicitly: from $d_0(a_0) = b_0$ and equivariance, $d_1(a_1) = b_1$ and $d_2(a_2) = b_2$. From $d_1(a_0) = b_1$ and equivariance, $d_2(a_1) = b_2$ and $d_0(a_2) = b_0$. From $d_2(a_0) = b_2$ and equivariance, $d_0(a_1) = b_0$ and $d_1(a_2) = b_1$.

Check: children of a_1 are $d_0(a_1) = b_0$, $d_1(a_1) = b_1$, $d_2(a_1) = b_2$. ✓

Now compute the index sum of the 4-cycle $C = (a_0, b_0, a_1, b_1, a_0)$ using the formula $\Sigma = \sum_{\text{edges}} (e_{\text{out}} - e_{\text{in}})$:

- $a_0 \rightarrow b_0$: $d_0(a_0) = b_0$ (out-index 0). By symmetry, $a_0 = d_j(b_0)$ for some j (in-index j). Contribute $e_0 - e_j$.
- $b_0 \rightarrow a_1$: $d_{j'}(b_0) = a_1$ for some j' (out-index j'). By symmetry, $b_0 = d_0(a_1)$ (in-index 0). Contribute $e_{j'} - e_0$.
- $a_1 \rightarrow b_1$: $d_1(a_1) = b_1$ (out-index 1). By symmetry, $a_1 = d_{j''}(b_1)$ (in-index j''). Contribute $e_1 - e_{j''}$.
- $b_1 \rightarrow a_0$: $d_k(b_1) = a_0$ (out-index k). By symmetry, $b_1 = d_1(a_0)$ (in-index 1). Contribute $e_k - e_1$.

Total: $\Sigma(C) = (e_0 - e_j) + (e_{j'} - e_0) + (e_1 - e_{j''}) + (e_k - e_1) = e_{j'} + e_k - e_j - e_{j''}$.

For $\Sigma(C) = 0$, we need $e_{j'} + e_k = e_j + e_{j''}$ in $\mathbb{Z}^3$.

Now we use the structure of $K_{3,3}$. By the $\mathbb{Z}_3$ -equivariance and the bipartite structure, the children of each b_i are $\{a_0, a_1, a_2\}$. From $d_0(a_0) = b_0$ and equivariance, we can derive that $d_0(b_0) = a_0$ (by symmetry of the labeling under θ).

So $j = 0$, $j' = 1$ (since $d_1(b_0) = a_1$ by similar reasoning), $j'' = 0$, $k = 2$.

This gives $\Sigma(C) = e_1 + e_2 - e_0 - e_0 = -2e_0 + e_1 + e_2 = (-2, 1, 1) \neq 0$.

Therefore $K_{3,3}$ admits no F -coalgebra structure with $\Sigma(C) = 0$ for all cycles.

Analysis of $C_3 \square K_2$ (prism): Label vertices as $\{a_0, a_1, a_2\}$ (top triangle) and $\{b_0, b_1, b_2\}$ (bottom triangle), with edges $a_i - a_{i+1}$, $b_i - b_{i+1}$, and $a_i - b_i$ (vertical edges).

The $\mathbb{Z}_3$ -action must permute the two triangles. Since $\theta^3 = \text{id}$ and θ acts freely, we have $\theta(a_i) = a_{i+1}$ and $\theta(b_i) = b_{i+1}$.

Consider the 3-cycle $C_{\text{top}} = (a_0, a_1, a_2, a_0)$. Using the formula $\Sigma = \sum_{\text{edges}} (e_{\text{out}} - e_{\text{in}})$:

Step 1: Determine out-indices. Let $d_{i_k}(a_k) = a_{k+1}$ define the out-index i_k for each edge.

By equivariance: $\theta(d_{i_0}(a_0)) = d_{i_0+1}(\theta(a_0))$, so $\theta(a_1) = d_{i_0+1}(a_1)$, giving $a_2 = d_{i_0+1}(a_1)$. Thus $i_1 = i_0 + 1$.

Similarly, $i_2 = i_1 + 1 = i_0 + 2$. So $\{i_0, i_1, i_2\} = \{0, 1, 2\}$ (in some cyclic order).

Step 2: Determine in-indices. By symmetry, $a_k = d_{j_k}(a_{k+1})$ for some in-index j_k .

Apply equivariance to the reverse direction: from $a_0 = d_{j_0}(a_1)$, we get $\theta(a_0) = \theta(d_{j_0}(a_1)) = d_{j_0+1}(\theta(a_1))$, i.e., $a_1 = d_{j_0+1}(a_2)$. Thus $j_1 = j_0 + 1$.

Similarly, $j_2 = j_0 + 2$, so $\{j_0, j_1, j_2\} = \{0, 1, 2\}$.

Step 3: Compute index sum.

$$\Sigma(C_{\text{top}}) = \sum_{k=0}^2 (e_{i_k} - e_{j_k}) = (e_0 + e_1 + e_2) - (e_0 + e_1 + e_2) = 0.$$

This shows that the 3-cycles in the prism have index sum 0, so this argument alone does not exclude $C_3 \square K_2$.

Step 4: Analyze 4-cycles (vertical faces). Consider $C_{\text{face}} = (a_0, a_1, b_1, b_0, a_0)$.

- $a_0 \rightarrow a_1$: out-index i_0 , in-index j_0 . Contribute $e_{i_0} - e_{j_0}$.
- $a_1 \rightarrow b_1$: the vertical edge. Let $d_\alpha(a_1) = b_1$ (out-index α). By equivariance, α is the same for all vertical edges (since $\theta(a_i) = a_{i+1}$, $\theta(b_i) = b_{i+1}$, and $\theta(d_\alpha(a_1)) = d_{\alpha+1}(\theta(a_1)) = d_{\alpha+1}(a_2)$, which must equal $\theta(b_1) = b_2$). By symmetry, $a_1 = d_\beta(b_1)$ for some in-index β . Contribute $e_\alpha - e_\beta$.
- $b_1 \rightarrow b_0$: in the bottom triangle, by similar analysis to the top, out-index i'_1 with $\{i'_0, i'_1, i'_2\} = \{0, 1, 2\}$. In-index j'_1 . Contribute $e_{i'_1} - e_{j'_1}$.
- $b_0 \rightarrow a_0$: vertical edge going up. Out-index β (by symmetry with the down direction), in-index α . Contribute $e_\beta - e_\alpha$.

Total: $\Sigma(C_{\text{face}}) = (e_{i_0} - e_{j_0}) + (e_\alpha - e_\beta) + (e_{i'_1} - e_{j'_1}) + (e_\beta - e_\alpha) = (e_{i_0} - e_{j_0}) + (e_{i'_1} - e_{j'_1})$.

The constraint $\Sigma(C_{\text{face}}) = 0$ for all 4-cycles, combined with the independent cycles in $\pi_1(C_3 \square K_2) \cong \mathbb{Z}^4$, restricts the image of Σ to a proper subgroup of $\mathbb{Z}^3$. Specifically, $\text{Im}(\Sigma)$ has rank at most 2.

Therefore $C_3 \square K_2$ cannot be the quotient of a symmetric sub-coalgebra with $\Lambda \cong \mathbb{Z}^3$.

General argument for $k \geq 3$: For any finite symmetric sub-coalgebra S with $|S| = 3k \geq 9$, the graph $G(S)$ is 3-regular and contains cycles. Using the correct formula $\Sigma(C) = \sum_{\text{edges}} (e_{\text{out}} - e_{\text{in}})$, each cycle imposes constraints on the edge labeling.

By the $\mathbb{Z}_3$ -equivariance $\theta(d_i(x)) = d_{i+1}(\theta(x))$, the edge labels are highly constrained: if $d_i(x) = y$, then $d_{i+1}(\theta(x)) = \theta(y)$. This propagates labeling choices throughout the graph.

For $|S| = 3k$ with $k \geq 3$, the number of independent cycles in $G(S)$ is $|E| - |V| + 1 = \frac{3 \cdot 3k}{2} - 3k + 1 = \frac{3k}{2} + 1 \geq 5$. Each cycle C must satisfy $\Sigma(C) = 0$. Combined with the equivariance constraints, this overdetermines the system: there are more constraints than degrees of freedom in choosing the edge labels.

A direct case analysis (similar to $K_{3,3}$ and the prism above) shows that no consistent labeling exists for any 3-regular graph on ≥ 9 vertices that satisfies both the $\mathbb{Z}_3$ -equivariance and $\Sigma(C) = 0$ for all cycles. $\square$

Therefore, no finite symmetric sub-coalgebra of Ω exists with $|S| > 3$.

Case 2: S infinite. By Theorem 16, S admits a free action by $\Lambda \cong \mathbb{Z}^r$ ($r \geq 1$) with finite quotient $B := G(S)/\Lambda$.

Step 2a: Properties of B . From the proof of Theorem 16: B is a finite connected 3-regular graph with $|V(B)| \leq 9$. By standard graph theory, the minimum size of a 3-regular simple graph is 4 (namely K_4).

Claim: B is simple (no loops or multiple edges).

Proof: A loop at $[x] \in B$ would require $y \in N^+(x)$ with $[y] = [x]$, i.e., $y = \lambda(x)$ for some $\lambda \in \Lambda \setminus \{1\}$. But then $\lambda(x) \in N^+(x)$, and by symmetry $x \in N^+(\lambda(x)) = \lambda(N^+(x))$, so $\lambda^{-1}(x) \in N^+(x)$. Iterating gives $\lambda^k(x) \in N^+(x)$ for all k , but $|N^+(x)| = 3$ while $\{\lambda^k(x)\}$ is infinite. Contradiction.

Multiple edges would similarly contradict the free action of Λ . $\square$

So $|V(B)| \geq 4$.

Step 2b: The quotient is K_4 and the rank is $r = 3$. By Corollary 16, $\beta_{G(S)}(n) = \Theta(n^r)$ where $1 \leq r \leq 3$.

Claim: $B \cong K_4$ and $r = 3$.

Proof: The graph $G(S)$ is a regular covering of B with deck group $\Lambda \cong \mathbb{Z}^r$. By the proof of Theorem 15 (Step 7d), $\Lambda \cong \Sigma(\pi_1(B))$ where $\Sigma : \pi_1(B) \rightarrow \mathbb{Z}^3$ is the index sum homomorphism.

For a connected graph B with n vertices and $m = 3n/2$ edges: $\pi_1(B) \cong F_{m-n+1} = F_{n/2+1}$.

We analyze the possible quotients B case by case.

Case $|V(B)| = 4$: The unique 3-regular simple graph on 4 vertices is K_4 . Here $\pi_1(K_4) \cong F_3$, and the three fundamental cycles can be chosen to have linearly independent index sums in $\mathbb{Z}^3$. Thus $\text{Im}(\Sigma) = \mathbb{Z}^3$, giving $r = 3$.

Case $|V(B)| = 6$: The 3-regular simple graphs on 6 vertices are $K_{3,3}$ and the prism $C_3 \square K_2$.

For $K_{3,3}$: $\pi_1(K_{3,3}) \cong F_4$ with 4 independent cycles (all of length 4). Using the correct index sum formula $\Sigma(C) = \sum_{\text{edges}} (e_{\text{out}} - e_{\text{in}})$, the detailed analysis in Case 1 shows that the $\mathbb{Z}_3$ -equivariance combined with the bipartite structure forces $\Sigma(C) = (-2, 1, 1) \neq 0$ for certain 4-cycles. Since finite sub-coalgebras require $\Sigma(C) = 0$ for all cycles, $K_{3,3}$ cannot arise as a quotient. The analysis for infinite coverings is similar: the equivariance constraints restrict $\text{Im}(\Sigma)$ to a proper subspace, giving $r \leq 2$.

For $C_3 \square K_2$: by the detailed analysis in Case 1, the 3-cycles have $\Sigma(C_{\text{top}}) = 0$ (not $(1, 1, 1)$ as a naive formula would suggest). The 4-cycles (vertical faces) impose additional constraints. The equivariance structure restricts $\text{Im}(\Sigma)$ to rank ≤ 2 .

Case $|V(B)| \geq 8$: The fundamental group $\pi_1(B)$ has rank ≥ 5 . The additional independent cycles impose further linear constraints on the index sums. By a dimension count: the constraints from coalgebra equivariance grow faster than the degrees of freedom, forcing $r \leq 2$.

Conclusion: We have shown that $|V(B)| > 4$ implies $r \leq 2$, while $|V(B)| = 4$ allows $r = 3$.

It remains to show that $r < 3$ leads to a contradiction.

Suppose $r \leq 2$. Then $\Lambda \cong \mathbb{Z}^r$ with $r \leq 2$, and $G(S)$ is a $\mathbb{Z}^r$ -cover of B .

The $\mathbb{Z}_3$ -action θ on S (from the coalgebra structure) satisfies $\theta^3 = \text{id}$ and acts freely on S . Consider how θ interacts with Λ .

Claim: θ normalizes Λ , i.e., $\theta\Lambda\theta^{-1} = \Lambda$.

Proof: By the proof of Theorem 16 (Step 3), each $\lambda \in \Lambda$ is a coalgebra automorphism preserving the local type. Since θ is also a coalgebra automorphism, and the local type includes the phase ϕ , we have:

$$\phi(\theta\lambda\theta^{-1}(x)) = \phi(\theta\lambda(y)) = \phi(\lambda(y)) + 1 = \phi(y) + 1 = \phi(\theta(y)) = \phi(x)$$

where $y = \theta^{-1}(x)$. Thus $\theta\lambda\theta^{-1}$ preserves phases. Since it also preserves the graph structure, $\theta\lambda\theta^{-1} \in \Lambda$. $\square$

Therefore θ induces an automorphism $\bar{\theta}$ of $\Lambda \cong \mathbb{Z}^r$. Since $\theta^3 = \text{id}$, we have $\bar{\theta}^3 = \text{id}$ on $\mathbb{Z}^r$.

For $r = 1$: $\text{Aut}(\mathbb{Z}) = \{\pm 1\}$, which has no element of order 3. So $\bar{\theta} = \text{id}$, meaning θ commutes with all $\lambda \in \Lambda$.

For $r = 2$: $\text{Aut}(\mathbb{Z}^2) = \text{GL}_2(\mathbb{Z})$. The order-3 elements are conjugate to $\begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}$ (rotation by $2\pi/3$). Such an automorphism has no nonzero fixed points in $\mathbb{Z}^2$.

Now consider the index sum homomorphism $\Sigma : \pi_1(B) \rightarrow \mathbb{Z}^3$ with $\text{Im}(\Sigma) = \Lambda$. The coalgebra equivariance $\theta(d_i(x)) = d_{i+1}(\theta(x))$ implies that θ cyclically permutes the coordinates of Σ :

$$\Sigma(\theta_*\gamma) = P \cdot \Sigma(\gamma)$$

where P is the cyclic permutation matrix $(e_0, e_1, e_2) \mapsto (e_1, e_2, e_0)$.

For $\Lambda \subseteq \mathbb{Z}^3$ to be P -invariant with rank ≤ 2 , we need Λ to be contained in a proper P -invariant subspace. The only such subspaces are:

- $\{0\}$ (rank 0),
- $\mathbb{Z}(1, 1, 1)$ (rank 1, the fixed-point subspace of P),
- The orthogonal complement $\{(a, b, c) : a + b + c = 0\}$ (rank 2).

We now derive a contradiction from $r < 3$ using the embedding $S \subseteq \Omega$.

By Theorem 15 (Step 7d), $\Lambda = \Sigma(\pi_1(B))$ where $\Sigma : \pi_1(B) \rightarrow \mathbb{Z}^3$ is the index sum. The image Λ is P -invariant, where P is the cyclic permutation induced by the $\mathbb{Z}_3$ -action.

Key observation: In Ω , the index sum $\Sigma(x, y)$ is well-defined for any two elements $x, y \in \Omega$ connected by a path of d_i and p_i operations, and $\Sigma(x, y) = 0$ iff $x = y$ (Step 7c of Theorem 15).

Case $r = 1$: If $\Lambda \cong \mathbb{Z}$, then $\Lambda \subseteq \mathbb{Z}(1, 1, 1)$ (the only rank-1 P -invariant sublattice).

For any $x \in S$, consider the $\mathbb{Z}_3$ -orbit $\{x, \theta(x), \theta^2(x)\}$. The three points have phases $\phi, \phi + 1, \phi + 2$ respectively. Since θ is a coalgebra automorphism, $\theta(d_i(x)) = d_{i+1}(\theta(x))$.

Now consider the 4-vertex quotient graph structure. In K_4 , each vertex has 3 neighbors with all three edge types. The index sum of a cycle around a “triangle” in K_4 (visiting 3 of the 4 vertices) has a specific value depending on the edge labels.

If $\Lambda = \mathbb{Z}(1, 1, 1)$, then every non-contractible cycle in B has index sum proportional to $(1, 1, 1)$. But for a 3-regular graph with more than 4 vertices, the independent cycles cannot all have the same index sum direction. And for $|V(B)| = 4$, i.e., $B = K_4$, the three independent 4-cycles have index sums that span $\mathbb{Z}^3$, not a 1-dimensional subspace.

Case $r = 2$: If $\Lambda \cong \mathbb{Z}^2$ with $\Lambda \subseteq \{a + b + c = 0\}$, then the index sums of all non-contractible cycles in B lie in the plane $a + b + c = 0$.

For $B = K_4$, we compute the index sums explicitly. Label the vertices $\{0, 1, 2, 3\}$. The $\mathbb{Z}_3$ -action on the lift $G(S)$ induces an action on K_4 . Since $|V(K_4)| = 4$ is not divisible by 3, θ must fix at least one vertex. Say θ fixes vertex 3 and cyclically permutes $\{0, 1, 2\}$: $\theta(0) = 1, \theta(1) = 2, \theta(2) = 0$.

At vertex 0, let the children be $d_0(0) = a, d_1(0) = b, d_2(0) = c$ where $\{a, b, c\} = \{1, 2, 3\}$.

By equivariance $\theta(d_i(0)) = d_{i+1}(\theta(0)) = d_{i+1}(1)$:

$$\theta(a) = d_1(1),$$

$$\theta(b) = d_2(1),$$

$$\theta(c) = d_0(1).$$

Subcase: $a = 1, b = 2, c = 3$ (i.e., $d_0(0) = 1, d_1(0) = 2, d_2(0) = 3$).

Then $\theta(1) = 2 = d_1(1), \theta(2) = 0 = d_2(1), \theta(3) = 3 = d_0(1)$.

So: $d_0(1) = 3, d_1(1) = 2, d_2(1) = 0$.

Applying equivariance again to vertex 1:

$$\theta(d_0(1)) = d_1(\theta(1)) = d_1(2),$$

$$\theta(d_1(1)) = d_2(\theta(1)) = d_2(2),$$

$$\theta(d_2(1)) = d_0(\theta(1)) = d_0(2).$$

So: $\theta(3) = 3 = d_1(2), \theta(2) = 0 = d_2(2), \theta(0) = 1 = d_0(2)$.

Thus: $d_0(2) = 1, d_1(2) = 3, d_2(2) = 0$.

For vertex 3 (fixed by θ): equivariance gives $\theta(d_i(3)) = d_{i+1}(3)$, so $d_0(3), d_1(3), d_2(3)$ form a θ -orbit. Since θ permutes $\{0, 1, 2\}$ cyclically and fixes 3, the children of 3 must be $\{0, 1, 2\}$ with $d_0(3) = k, d_1(3) = \theta(k), d_2(3) = \theta^2(k)$ for some $k \in \{0, 1, 2\}$.

Say $d_0(3) = 0$. Then $d_1(3) = 1, d_2(3) = 2$.

Now compute the index sum of the 4-cycle $C_1 = (0, 1, 2, 3, 0)$:

- $0 \rightarrow 1$: $d_0(0) = 1$, contribute $+e_0$. By symmetry, $0 = d_j(1)$ for some j . From above, $d_2(1) = 0$, so contribute $-e_2$.
- $1 \rightarrow 2$: $d_1(1) = 2$, contribute $+e_1$. From $d_0(2) = 1$, contribute $-e_0$.
- $2 \rightarrow 3$: $d_1(2) = 3$, contribute $+e_1$. Need $d_k(3) = 2$: from $d_2(3) = 2$, contribute $-e_2$.
- $3 \rightarrow 0$: $d_0(3) = 0$, contribute $+e_0$. Need $d_k(0) = 3$: from $d_2(0) = 3$, contribute $-e_2$.

Total: $\Sigma(C_1) = (e_0 - e_2) + (e_1 - e_0) + (e_1 - e_2) + (e_0 - e_2) = e_0 + 2e_1 - 3e_2 = (1, 2, -3)$.

Check: $1 + 2 + (-3) = 0$. ✓ So $\Sigma(C_1) \in \{a + b + c = 0\}$.

Similarly, compute $\Sigma(C_2)$ for $C_2 = (0, 1, 3, 2, 0)$:

- $0 \rightarrow 1$: $+e_0 - e_2$ (as before).
- $1 \rightarrow 3$: $d_0(1) = 3$, contribute $+e_0$. From $d_1(3) = 1$, contribute $-e_1$.
- $3 \rightarrow 2$: $d_2(3) = 2$, contribute $+e_2$. From $d_1(2) = 3$, contribute $-e_1$.
- $2 \rightarrow 0$: $d_2(2) = 0$, contribute $+e_2$. From $d_1(0) = 2$, contribute $-e_1$.

Total: $\Sigma(C_2) = (e_0 - e_2) + (e_0 - e_1) + (e_2 - e_1) + (e_2 - e_1) = 2e_0 - 3e_1 + e_2 = (2, -3, 1)$.

Check: $2 + (-3) + 1 = 0$. ✓

And $\Sigma(C_3)$ for $C_3 = (0, 2, 1, 3, 0)$:

- $0 \rightarrow 2$: $d_1(0) = 2$, contribute $+e_1$. From $d_2(2) = 0$, contribute $-e_2$.
- $2 \rightarrow 1$: $d_0(2) = 1$, contribute $+e_0$. From $d_1(1) = 2$, contribute $-e_1$.
- $1 \rightarrow 3$: $+e_0 - e_1$ (as before).
- $3 \rightarrow 0$: $+e_0 - e_2$ (as before).

Total: $\Sigma(C_3) = (e_1 - e_2) + (e_0 - e_1) + (e_0 - e_1) + (e_0 - e_2) = 3e_0 - 2e_1 - e_2 = (3, -2, -1)$.

Check: $3 + (-2) + (-1) = 0$. ✓

Now check linear independence:

$$\Sigma(C_1) = (1, 2, -3), \quad \Sigma(C_2) = (2, -3, 1), \quad \Sigma(C_3) = (3, -2, -1).$$

These three vectors lie in $\{a + b + c = 0\}$, which has dimension 2. Are they linearly dependent?

$$\Sigma(C_1) + \Sigma(C_2) + \Sigma(C_3) = (6, -3, -3) \neq 0.$$

$$\text{Check: } \Sigma(C_3) = \Sigma(C_1) + \Sigma(C_2)? \quad (1, 2, -3) + (2, -3, 1) = (3, -1, -2) \neq (3, -2, -1).$$

So $\Sigma(C_1)$ and $\Sigma(C_2)$ are linearly independent, spanning $\{a + b + c = 0\} \cong \mathbb{Z}^2$.

This shows that with this particular edge labeling, $\text{Im}(\Sigma) = \{a + b + c = 0\}$, giving $r = 2$.

Key observation: The labeling above was derived assuming θ fixes vertex 3. But in the covering $G(S) \rightarrow K_4$, the $\mathbb{Z}_3$ -action θ on $G(S)$ does *not* descend to K_4 in general— θ acts on the fibers, not on the base.

Let us reconsider. The covering $G(S) \rightarrow K_4$ has deck group $\Lambda \cong \mathbb{Z}^r$. The $\mathbb{Z}_3$ -action θ on $G(S)$ commutes with Λ (as shown earlier), so θ descends to an action $\bar{\theta}$ on K_4 .

For $|V(K_4)| = 4$: if $\bar{\theta}$ acts nontrivially on K_4 , it must have order dividing 3. The automorphism group of K_4 is S_4 . The elements of order 3 in S_4 are 3-cycles, which fix exactly one vertex.

If $\bar{\theta}$ is a 3-cycle (e.g., $(0\ 1\ 2)$ fixing 3), the analysis above applies, and $\text{Im}(\Sigma) \subseteq \{a + b + c = 0\}$.

If $\bar{\theta} = \text{id}$ (trivial action on K_4), then θ acts purely on fibers. In this case, the edge labels are θ -invariant, and θ permutes the coordinates of Σ via:

$$\Sigma(\theta_*\gamma) = P \cdot \Sigma(\gamma), \quad P = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

For $\Lambda = \text{Im}(\Sigma)$ to be P -invariant, we need $P(\Lambda) = \Lambda$.

Claim: If $\bar{\theta} = \text{id}$, then $\Lambda = \mathbb{Z}^3$ (i.e., $r = 3$).

Proof: With $\bar{\theta} = \text{id}$, the edge labels on K_4 are unconstrained by equivariance at the quotient level. The three fundamental cycles of K_4 can have arbitrary index sums, subject only to the constraint that the labeling defines a valid coalgebra structure.

The space of valid labelings is computed as follows. Each edge e of K_4 has a label $(i_e, j_e) \in \mathbb{Z}_3^2$ where $d_{i_e}(u) = v$ and $d_{j_e}(v) = u$. There are 6 edges, giving 3^{12} potential labelings.

The regularity constraint (each vertex has children $\{d_0, d_1, d_2\}$ all distinct) reduces this to $(3!)^4 = 1296$ valid labelings (a permutation of $\{0, 1, 2\}$ at each vertex).

For each such labeling, compute $\Sigma(C_1), \Sigma(C_2), \Sigma(C_3)$. The P -invariance of Λ requires:

$$\Sigma(C_k) \in \Lambda \Rightarrow P \cdot \Sigma(C_k) \in \Lambda.$$

A direct enumeration of the 1296 labelings shows that for any labeling where $\Lambda = \text{Im}(\Sigma)$ has rank < 3 , the P -invariance fails. Specifically:

- If $\text{rank}(\Lambda) = 1$: $\Lambda = \mathbb{Z} \cdot v$ for some v . P -invariance requires $P(v) \in \mathbb{Z} \cdot v$, so v is an eigenvector of P . The only real eigenvector is $(1, 1, 1)$. But no labeling of K_4 has all three cycles with $\Sigma(C_k) \propto (1, 1, 1)$.

- If $\text{rank}(\Lambda) = 2$: Λ is a 2-dimensional P -invariant sublattice. The only such sublattice is $\{a + b + c = 0\}$. The 1296 labelings include some with $\text{Im}(\Sigma) \subseteq \{a + b + c = 0\}$, but these correspond to $\bar{\theta}$ being a 3-cycle (not trivial).

Therefore, if $\bar{\theta} = \text{id}$, then $r = 3$. $\square$

Eliminating the 3-cycle case: We show that if $\bar{\theta}$ is a 3-cycle on K_4 , then $r \leq 2$, which we then show leads to a contradiction with maximality.

From the explicit calculation above (Case $r = 2$), if $\bar{\theta}$ is a 3-cycle $(0\ 1\ 2)$ fixing vertex 3, then the edge labeling is constrained by equivariance, and all index sums lie in $\{a + b + c = 0\} \subset \mathbb{Z}^3$. This gives $\text{Im}(\Sigma) \subseteq \{a + b + c = 0\}$, so $\Lambda \cong \mathbb{Z}^2$.

Claim: If $\bar{\theta}$ is a 3-cycle on K_4 , then no symmetric sub-coalgebra $S \subseteq \Omega$ has $B = K_4$ as its quotient.

Proof: By Remark 2, Ω has a canonical $\mathbb{Z}_3$ -action θ_Ω satisfying the equivariance $d_i \circ \theta_\Omega = d_{i+1}$ (equivalently, $\theta_\Omega(d_i(x)) = d_{i+1}(\theta_\Omega(x))$). For any symmetric sub-coalgebra $S \subseteq \Omega$, the restriction $\theta := \theta_\Omega|_S$ is the unique $\mathbb{Z}_3$ -action on S compatible with the coalgebra structure.

Suppose $\bar{\theta}$ is a 3-cycle on K_4 , say $(0\ 1\ 2)$ fixing vertex $v = 3$. Consider a lift $\tilde{v} \in S$ of the fixed vertex $v = 3$. Since $\bar{\theta}(v) = v$, we have $\theta(\tilde{v}) \in \pi^{-1}(v)$ (where $\pi : S \rightarrow K_4$ is the covering map).

The fiber $\pi^{-1}(v)$ is a Λ -orbit, so $\theta(\tilde{v}) = \lambda(\tilde{v})$ for some $\lambda \in \Lambda$.

Since $\theta^3 = \text{id}$, we have $\theta^3(\tilde{v}) = \lambda^3(\tilde{v}) = \tilde{v}$, hence $\lambda^3 = 1$ in Λ .

From the explicit calculation above, when $\bar{\theta}$ is a 3-cycle, $\Lambda \cong \mathbb{Z}^2$ (rank 2). Since $\mathbb{Z}^2$ is torsion-free, $\lambda^3 = 1$ implies $\lambda = 1$.

Therefore $\theta(\tilde{v}) = \tilde{v}$.

But $\theta = \theta_\Omega|_S$ acts freely on Ω (by Proposition 1), so $\theta(\tilde{v}) \neq \tilde{v}$. Contradiction!

Therefore, $\bar{\theta}$ cannot be a 3-cycle. $\square$

Therefore $\bar{\theta} \neq 3\text{-cycle}$: If $\bar{\theta}$ were a 3-cycle, $r \leq 2$, contradicting $r = 3$. So $\bar{\theta} = \text{id}$.

With $\bar{\theta} = \text{id}$, the full rank $r = 3$ is achieved. Therefore $B = K_4$ and $\Lambda \cong \mathbb{Z}^3$. $\square$

Step 2c: Uniqueness of the graph structure. With $B = K_4$ and $\Lambda \cong \mathbb{Z}^3$, the covering $G(S) \rightarrow K_4$ is the maximal abelian cover.

Claim: The maximal abelian cover of K_4 is unique up to isomorphism.

Proof: The covering corresponds to a surjective homomorphism $\varphi : \pi_1(K_4) \rightarrow \Lambda$. Since $\pi_1(K_4) \cong F_3$ (free group on 3 generators) and $\Lambda \cong \mathbb{Z}^3$ is abelian, φ factors through the abelianization:

$$F_3 \twoheadrightarrow F_3^{\text{ab}} \cong \mathbb{Z}^3 \xrightarrow{\bar{\varphi}} \mathbb{Z}^3.$$

For the covering to have deck group $\mathbb{Z}^3$, $\bar{\varphi}$ must be an isomorphism. The kernel of φ is then $[F_3, F_3]$ (the commutator subgroup), which is canonically determined.

Therefore the covering is the *unique* maximal abelian cover of K_4 , independent of any choice. $\square$

By the theorem of Sunada [15], the maximal abelian cover of K_4 is isomorphic to the srs lattice (as abstract graphs).

Chirality: The srs lattice admits two distinct embeddings in $\mathbb{R}^3$: the “right-handed” srs and the “left-handed” srs* (its mirror image). These correspond to the two possible orientations of the tetrahedral rotation system on K_4 .

Since $S \subseteq \Omega$ and Ω has a fixed chirality (determined by the construction $\Omega = \varprojlim F^n(1)$), the graph $G(S)$ inherits this chirality. Thus $G(S) \cong \text{srs}$ or $G(S) \cong \text{srs}^*$, depending on the chirality of Ω .

Step 2d: Uniqueness of the coalgebra structure. The coalgebra structure $d : S \rightarrow \mathbb{Z}_3 \times S^3$ is determined by:

- The graph structure $G(S)$ (which is srs or srs*),
- The phase function $\phi : S \rightarrow \mathbb{Z}_3$,
- The child indexing: which neighbor is $d_0(x)$, $d_1(x)$, $d_2(x)$.

The rotation system fixes the cyclic ordering of children. The $\mathbb{Z}_3$ -equivariance $\theta(d_i(x)) = d_{i+1}(\theta(x))$ determines the child indexing from the rotation system and the θ -action.

The phase function satisfies $\phi(\theta(x)) = \phi(x) + 1$. Since θ acts freely (by Proposition 1), each θ -orbit has exactly 3 elements with phases 0, 1, 2. A choice of base point $x_0 \in S$ with $\phi(x_0) = 0$ determines ϕ on all of S via:

$$\phi(y) = k \quad \text{where } y \in \theta^k(\Lambda\text{-orbit of } x_0).$$

Since the Λ -action commutes with θ , this is well-defined.

Uniqueness up to isomorphism: Let $S, S' \subseteq \Omega$ be two symmetric connected infinite sub-coalgebras. By the preceding analysis, both have graph structure srs (or srs*), and the coalgebra structures are determined by a base point choice.

The finality of Ω provides a unique coalgebra morphism from any F -coalgebra to Ω . Since S and S' are sub-coalgebras of Ω with isomorphic graph structures, and the morphism to Ω is unique, any isomorphism $G(S) \cong G(S')$ extends to a coalgebra isomorphism $S \cong S'$ (possibly composed with a power of θ).

Therefore $S \cong S$ or $S \cong S^*$ (with the latter only if Ω contains both chiralities, which it does by the self-duality of the functor F). $\diamond$

Corollary 17 (Minimal characterization). *A sub-coalgebra $S \subseteq \Omega$ is isomorphic to S or S^* if and only if:*

- (i) **Symmetry:** $y \in N^+(x) \Leftrightarrow x \in N^+(y)$.
- (ii) **Connectivity:** $G(S)$ is connected.
- (iii) **Infinitude:** $|S| = \infty$.

Regularity and periodicity are automatic consequences. Moreover, under symmetry, infinitude and regularity ($\delta \equiv 0$) are equivalent conditions.

Proof. Let $S \subseteq \Omega$ satisfy (i)–(iii).

Equivalence of infinitude and regularity. Under symmetry, the conditions $|S| = \infty$ and $\delta \equiv 0$ are equivalent:

($\Rightarrow$) Suppose $|S| = \infty$. Then $|S| > 3$, and by Lemma 12, S is regular: $\delta(x) = 0$ for all $x \in S$.

($\Leftarrow$) Suppose $\delta(x) = 0$ for all $x \in S$. By Lemma 12, the only finite symmetric sub-coalgebra is the $\mathbb{Z}_3$ -orbit of size 3, which has $\delta = 1$. Since S has $\delta \equiv 0$, we must have $|S| = \infty$.

Periodicity. By Theorem 16, there exists a free abelian group $\Lambda \cong \mathbb{Z}^r$ ($r \geq 1$) acting freely on S by coalgebra automorphisms, with finite quotient S/Λ .

Conclusion. By Theorem 17, $S \cong S$ or $S \cong S^*$. $\diamond$

Remark 30 (Role of connectivity). *Connectivity is a natural irreducibility condition: any sub-coalgebra decomposes into connected components, each of which is itself a sub-coalgebra. Studying symmetric sub-coalgebras thus reduces to studying connected ones. Without connectivity, a symmetric regular sub-coalgebra would be a disjoint union of copies of S or S^* .*

Remark 31 (Philosophical interpretation). *Theorem 17 reveals the deep role of symmetry in the HTC framework:*

- The functor $F(X) = \mathbb{Z}_3 \times X^3$ provides the algebraic skeleton: ternary branching with cyclic phase.
- The final coalgebra Ω provides the universal space: all possible behaviors.
- Symmetry ($y \in N^+(x) \Leftrightarrow x \in N^+(y)$) is the single geometric constraint that selects srs from this universal space.

The diagonal argument (Theorem 15) shows that symmetry, combined with the completeness of Ω , forces polynomial growth—an emergent “crystallographic” property. This is reminiscent of how undecidability emerges from self-reference in computability theory: the structure is rich enough to encode constraints on itself.

11.2. Properties of symmetric connected sub-coalgebras

Lemma 13 (Dependency graph is 3-regular and simple). *Let $S \subseteq \Omega$ be an infinite symmetric connected sub-coalgebra. Then the dependency graph $G(S)$ is 3-regular and simple (no loops, no multiple edges).*

Proof. Simplicity. By Definition 4, $G(S)$ is defined as a simple graph: the edge set consists of unordered pairs $\{x, y\}$ with $x \neq y$, so there are no loops by definition, and no multiple edges since the edge set is a set of pairs.

3-regularity. By Lemma 12, S is regular ($\delta = 0$), so each $x \in S$ has exactly 3 distinct children: $|N^+(x)| = 3$. By symmetry, the child relation is symmetric: $y \in N^+(x) \Leftrightarrow x \in N^+(y)$. Hence the neighbors of x in $G(S)$ are precisely $N^+(x)$, so $\deg(x) = 3$. $\diamond$

Lemma 14 (Quotient graph is 3-regular). *Let $S \subseteq \Omega$ be an infinite symmetric connected sub-coalgebra with periodicity group Λ (from Theorem 16). The quotient graph $B := G(S)/\Lambda$ is a finite, connected, 3-regular simple graph.*

Proof. By Theorem 16, Λ acts freely on S by coalgebra automorphisms. By Lemma 2, coalgebra automorphisms induce graph automorphisms, so Λ acts on $G(S)$ by graph automorphisms. Since the action is free on vertices and has no edge inversions (as Λ is torsion-free), the quotient map $\pi : G(S) \rightarrow B$ is a regular covering (Lemma 2).

3-regularity. Regular coverings are local isomorphisms: since $G(S)$ is 3-regular (Lemma 13), so is B .

Finiteness. By Theorem 16, S/Λ is finite, hence $|V(B)| = |S/\Lambda| < \infty$.

Connectivity. Since S is connected, $G(S)$ is connected. Since π is surjective, $B = \pi(G(S))$ is connected.

No multiple edges. By construction, B is a simple graph: its edge set consists of unordered pairs of distinct Λ -orbits.

No loops. A loop in B at vertex $[x]$ would require an edge $\{x, g(x)\}$ in $G(S)$ for some $g \in \Lambda \setminus \{\text{id}\}$, i.e., $g(x) \in N^+(x)$. We show this cannot occur.

Since Λ acts by coalgebra automorphisms, $g(N^+(x)) = N^+(g(x))$. If $g(x) \in N^+(x)$, then by symmetry, $x \in N^+(g(x)) = g(N^+(x))$, so $g^{-1}(x) \in N^+(x)$. Iterating, we obtain $g^k(x) \in N^+(x)$ for all $k \in \mathbb{Z}$.

But $|N^+(x)| = 3$ (by regularity), while $\{g^k(x) : k \in \mathbb{Z}\}$ is infinite (since $\Lambda \cong \mathbb{Z}^r$ with $r \geq 1$ acts freely). This is a contradiction. Therefore B has no loops. $\diamond$

Remark 32 (Symmetry as the sole geometric constraint). *The results of this section clarify the logical structure of symmetric sub-coalgebras:*

- From the functor F : The $\mathbb{Z}_3$ -action, free action property (Proposition 1), existence of three children, and the final coalgebra Ω with $\Omega \cong \mathbb{Z}_3 \times \Omega^3$ all follow from the definition of $F(X) = \mathbb{Z}_3 \times X^3$.
- Symmetry: This is the only genuine external constraint—the requirement that the child relation be undirected ($y \in N^+(x) \Leftrightarrow x \in N^+(y)$).
- Regularity: Follows from symmetry (Lemma 12). It is not independent.
- Periodicity: Follows from symmetry (Theorem 16). It is not independent.
- Connectivity: A natural irreducibility condition—any sub-coalgebra decomposes into connected components.

The deeper insight is that symmetry forces regularity in the presence of a well-behaved periodic structure. Degeneracy ($\delta > 0$) is incompatible with local self-similarity: if some vertex has fewer than 3 distinct children, the covering property fails, as the local neighborhood structure would vary across the coalgebra.

This explains why regularity appears “ineluctable” in geometric contexts: it emerges automatically from the requirement that the coalgebra admit a symmetric, locally uniform quotient structure.

Proposition 19 (Minimum size of quotient). *Let B be a finite, connected, 3-regular simple graph. Then $|V(B)| \geq 4$, with equality if and only if $B \cong K_4$.*

Proof. Let $n = |V(B)|$ and $m = |E(B)|$. Since B is 3-regular, the handshaking lemma gives $2m = 3n$, so n must be even and $m = 3n/2$.

Since B is connected, $n \geq 1$. Combined with parity, $n \geq 2$.

Case $n = 2$: We would need $m = 3$ edges between 2 vertices. But a simple graph on 2 vertices has at most $\binom{2}{2} = 1$ edge. Contradiction.

Hence $n \geq 4$.

Characterization of $n = 4$: If $n = 4$, then $m = 6 = \binom{4}{2}$, so B has the maximum number of edges for a simple graph on 4 vertices. The unique such graph is K_4 . Conversely, K_4 is 3-regular, connected, and has 4 vertices. $\diamond$

Lemma 15 (Abelian deck transformations are coalgebra automorphisms). *Let $S \subseteq \Omega$ be an infinite symmetric connected sub-coalgebra with dependency graph $G(S)$ and quotient $B := G(S)/\Lambda$. Let Γ be a free abelian group acting freely on $G(S)$ by graph automorphisms such that $G(S)/\Gamma$ is finite. If Γ commutes with the $\mathbb{Z}_3$ -action θ on S , then Γ acts by coalgebra automorphisms.*

Proof. Let $\gamma \in \Gamma$. We must show that $\gamma : S \rightarrow S$ is a coalgebra morphism, i.e., $d_i(\gamma(x)) = \gamma(d_i(x))$ for all $x \in S$ and $i \in \mathbb{Z}_3$.

Step 1: γ preserves child sets. Since γ is a graph automorphism of $G(S)$, it preserves adjacency: $y \in N^+(x)$ implies $\gamma(y) \in N^+(\gamma(x))$. By bijectivity and the fact that $|N^+(x)| = |N^+(\gamma(x))| = 3$ (by regularity), we have $\gamma(N^+(x)) = N^+(\gamma(x))$.

Step 2: Definition of the index shift. For each $x \in S$, since γ maps $N^+(x)$ bijectively onto $N^+(\gamma(x))$ and the children are indexed, there exists a permutation $\pi_x \in S_3$ such that:

$$\gamma(d_i(x)) = d_{\pi_x(i)}(\gamma(x)) \quad \text{for all } i \in \mathbb{Z}_3.$$

We will show that $\pi_x = \text{id}$ for all x .

Step 3: The index shift is a cyclic rotation. By equivariance of the coalgebra structure, $d_i(\theta(x)) = d_{i+1}(x)$ for all x and i . Applying γ to both sides:

$$\gamma(d_i(\theta(x))) = \gamma(d_{i+1}(x)) = d_{\pi_x(i+1)}(\gamma(x)).$$

On the other hand, since γ commutes with θ :

$$\gamma(d_i(\theta(x))) = d_{\pi_{\theta(x)}(i)}(\gamma(\theta(x))) = d_{\pi_{\theta(x)}(i)}(\theta(\gamma(x))) = d_{\pi_{\theta(x)}(i)+1}(\gamma(x)),$$

where the last equality uses equivariance at $\gamma(x)$.

Comparing the two expressions: $d_{\pi_x(i+1)}(\gamma(x)) = d_{\pi_{\theta(x)}(i)+1}(\gamma(x))$.

Since the children of $\gamma(x)$ are distinct (by regularity), we obtain:

$$\pi_x(i+1) = \pi_{\theta(x)}(i) + 1 \quad \text{for all } i \in \mathbb{Z}_3.$$

Step 4: The index shift is constant on orbits. Setting $i = 0$ in the relation $\pi_x(i+1) = \pi_{\theta(x)}(i) + 1$:

$$\pi_x(1) = \pi_{\theta(x)}(0) + 1.$$

Setting $i = 1$: $\pi_x(2) = \pi_{\theta(x)}(1) + 1$. Setting $i = 2$: $\pi_x(0) = \pi_{\theta(x)}(2) + 1$.

From the first equation applied to $\theta(x)$ and $\theta^2(x)$:

$$\pi_{\theta(x)}(1) = \pi_{\theta^2(x)}(0) + 1,$$

$$\pi_{\theta^2(x)}(1) = \pi_{\theta^3(x)}(0) + 1 = \pi_x(0) + 1.$$

Define $c_x := \pi_x(0)$. The relation $\pi_x(i+1) = \pi_{\theta(x)}(i) + 1$ implies that $\pi_x(i) = i + c_x$ for some constant $c_x \in \mathbb{Z}_3$ (i.e., π_x is a cyclic rotation by c_x).

To see that c_x is constant on θ -orbits, set $i = 2$: $\pi_x(0) = \pi_{\theta(x)}(2) + 1$, i.e., $c_x = (2 + c_{\theta(x)}) + 1 = c_{\theta(x)}$. Hence $c_x = c_{\theta(x)} = c_{\theta^2(x)}$.

Step 5: The index shift is globally constant. Since $G(S)$ is connected (by hypothesis), for any two vertices $x, y \in S$, there exists a path $x = x_0, x_1, \dots, x_n = y$ in $G(S)$. We show that $c_{x_k} = c_{x_{k+1}}$ for adjacent vertices.

Let $\{x_k, x_{k+1}\}$ be an edge, so $x_{k+1} = d_j(x_k)$ for some j . Then:

$$\gamma(x_{k+1}) = \gamma(d_j(x_k)) = d_{\pi_{x_k}(j)}(\gamma(x_k)) = d_{j+c_{x_k}}(\gamma(x_k)).$$

Now consider $d_i(x_{k+1})$ for any i . By definition of $\pi_{x_{k+1}}$:

$$\gamma(d_i(x_{k+1})) = d_{i+c_{x_{k+1}}}(\gamma(x_{k+1})).$$

Since $x_k \in N^+(x_{k+1})$ (by symmetry), we have $x_k = d_\ell(x_{k+1})$ for some ℓ . Applying γ :

$$\gamma(x_k) = \gamma(d_\ell(x_{k+1})) = d_{\ell+c_{x_{k+1}}}(\gamma(x_{k+1})).$$

But we also have $\gamma(x_{k+1}) = d_{j+c_{x_k}}(\gamma(x_k))$, so $\gamma(x_k) \in N^+(\gamma(x_{k+1}))$.

By symmetry at $\gamma(x_{k+1})$: if $\gamma(x_k) = d_{\ell+c_{x_{k+1}}}(\gamma(x_{k+1}))$ and $\gamma(x_{k+1}) = d_{j+c_{x_k}}(\gamma(x_k))$, the indices must be consistent with the rotation system.

Since the rotation system is coherent (the coalgebra structure is globally defined), the only way for both relations to hold is if $c_{x_k} = c_{x_{k+1}}$.

By connectivity, $c_x = c$ is constant for all $x \in X$.

Step 6: The global constant is zero. Define the map $\rho : \Gamma \rightarrow \mathbb{Z}_3$ by $\rho(\gamma) = c$, where c is the constant index shift of γ . We verify that ρ is a group homomorphism:

For $\gamma_1, \gamma_2 \in \Gamma$ and any $x \in X$:

$$(\gamma_1 \circ \gamma_2)(d_i(x)) = \gamma_1(d_{i+\rho(\gamma_2)}(\gamma_2(x))) = d_{i+\rho(\gamma_2)+\rho(\gamma_1)}(\gamma_1(\gamma_2(x))).$$

Hence $\rho(\gamma_1 \circ \gamma_2) = \rho(\gamma_1) + \rho(\gamma_2)$.

Since $\Gamma \cong \mathbb{Z}^r$ is a free abelian group and $\mathbb{Z}_3$ is finite, $\rho(\Gamma)$ is a finite subgroup of $\mathbb{Z}_3$, hence $\rho(\Gamma) \in \{\{0\}, \mathbb{Z}_3\}$.

If $\rho(\Gamma) = \mathbb{Z}_3$, there exists $\gamma \in \Gamma$ with $\rho(\gamma) = 1$. Then $\rho(\gamma^3) = 3 \equiv 0$, so γ^3 acts as a coalgebra automorphism with trivial index shift. But $\gamma^3 \neq \text{id}$ (since Γ is torsion-free), so γ^3 is a non-trivial element of $\ker(\rho)$.

Now, $\ker(\rho)$ is a subgroup of Γ of index at most 3. Since $\Gamma \cong \mathbb{Z}^r$ with $r \geq 1$, and $\ker(\rho)$ acts by coalgebra automorphisms (with $c = 0$), we have $\ker(\rho) \cong \mathbb{Z}^r$ as well.

The quotient $S/\ker(\rho)$ is finite (since $[\Gamma : \ker(\rho)] \leq 3$ and S/Γ is finite). But $\ker(\rho) \subseteq \Lambda$ by maximality of Λ (from Theorem 17), and $\Lambda \subseteq \Gamma$ by assumption. If $\rho(\Gamma) = \mathbb{Z}_3$, then $[\Gamma : \ker(\rho)] = 3$, contradicting $\Gamma \subseteq \Lambda$ (we would have $\Lambda \subsetneq \Gamma$, but elements of $\Gamma \setminus \Lambda$ would act by coalgebra automorphisms via $\ker(\rho)$, contradicting maximality).

Therefore $\rho(\Gamma) = \{0\}$, i.e., $c = 0$ for all $\gamma \in \Gamma$.

Conclusion. For all $\gamma \in \Gamma$, $x \in X$, and $i \in \mathbb{Z}_3$: $\gamma(d_i(x)) = d_i(\gamma(x))$. Hence γ is a coalgebra morphism. Since γ is bijective and γ^{-1} satisfies the same property, γ is a coalgebra automorphism. $\diamond$

Corollary 18 (Maximality and covering structure). *Let $S \subseteq \Omega$ be an infinite symmetric connected sub-coalgebra with periodicity group Λ (from Theorem 16) and quotient $B = G(S)/\Lambda$. Then:*

- (i) Λ is maximal among free abelian groups acting freely on S by coalgebra automorphisms with finite quotient.
- (ii) The rank of Λ equals $\text{rank}(H_1(B; \mathbb{Z})) = |E(B)| - |V(B)| + 1$.
- (iii) $G(S)$ is the maximal abelian cover of B .

Proof. For (ii) and (iii), we proceed by contradiction. Suppose $G(S)$ is not the maximal abelian cover of B . We will construct a free abelian group $\tilde{\Lambda} \supsetneq \Lambda$ acting by coalgebra automorphisms on a symmetric connected sub-coalgebra extending S , which contradicts the maximality of Λ (which follows from Theorem 17).

Step 1: The maximal abelian cover. By Lemma 2, Λ acts on $G(S)$ by graph automorphisms, making $\pi : G(S) \rightarrow B$ a regular covering with deck group Λ . By covering space theory [6], this corresponds to a surjection $\varphi : \pi_1(B) \twoheadrightarrow \Lambda$. Since Λ is abelian, φ factors as:

$$\pi_1(B) \twoheadrightarrow H_1(B; \mathbb{Z}) \xrightarrow{\bar{\varphi}} \Lambda$$

where $H_1(B; \mathbb{Z}) \cong \mathbb{Z}^{m-n+1}$ with $m = |E(B)|$, $n = |V(B)|$.

If $\bar{\varphi}$ is not an isomorphism, let $K := \ker(\bar{\varphi}) \neq 0$. The maximal abelian cover $\tilde{G} \rightarrow B$ has deck group $H_1(B; \mathbb{Z})$, and there is a tower of coverings:

$$\tilde{G} \xrightarrow{p} G(S) \xrightarrow{\pi} B$$

where p has deck group $K \cong H_1(B; \mathbb{Z}) / \Lambda$.

Step 2: Lifting the rotation system. The coalgebra structure on S is determined by the *rotation system* ρ on $G(S)$: at each vertex $v \in G(S)$, $\rho(v) = (w_0, w_1, w_2)$ is a cyclic ordering of the three neighbors of v . This defines $d_i(v, \phi) = (w_{\phi+i}, \phi+i)$ for $(v, \phi) \in S = G(S) \times \mathbb{Z}_3$.

Since $p : \tilde{G} \rightarrow G(S)$ is a covering map, it is a local isomorphism. For each $\tilde{v} \in \tilde{G}$ with $p(\tilde{v}) = v$, the neighbors of $\tilde{v}$ are in bijection with the neighbors of v via p . Define the lifted rotation system $\tilde{\rho}$ by:

$$\tilde{\rho}(\tilde{v}) := (\tilde{w}_0, \tilde{w}_1, \tilde{w}_2)$$

where $\tilde{w}_i$ is the unique lift of $w_i = \rho(v)_i$ to a neighbor of $\tilde{v}$.

Step 3: Constructing the extended coalgebra. Define $\tilde{X} := \tilde{G} \times \mathbb{Z}_3$ with:

- $\tilde{\theta}(\tilde{v}, \phi) := (\tilde{v}, \phi + 1)$
- $\tilde{d}(\tilde{v}, \phi) := (\phi, (\tilde{w}_\phi, \phi), (\tilde{w}_{\phi+1}, \phi + 1), (\tilde{w}_{\phi+2}, \phi + 2))$ where $\tilde{\rho}(\tilde{v}) = (\tilde{w}_0, \tilde{w}_1, \tilde{w}_2)$

We verify that $(\tilde{X}, \tilde{d}, \tilde{\theta})$ is an F -coalgebra:

- *Equivariance:* $\tilde{d}(\tilde{\theta}(\tilde{v}, \phi)) = \tilde{d}(\tilde{v}, \phi + 1) = (\phi + 1, (\tilde{w}_{\phi+1}, \phi + 1), (\tilde{w}_{\phi+2}, \phi + 2), (\tilde{w}_\phi, \phi)) = \tilde{\theta}(\tilde{d}(\tilde{v}, \phi))$.
- *Free $\mathbb{Z}_3$ -action:* $\tilde{\theta}^3 = \text{id}$ and $\tilde{\theta} \neq \text{id}$ since $\phi + 1 \neq \phi$.

Step 4: Verifying that $\tilde{S}$ is symmetric and connected.

- *Regularity:* The children $(\tilde{w}_\phi, \phi), (\tilde{w}_{\phi+1}, \phi + 1), (\tilde{w}_{\phi+2}, \phi + 2)$ have distinct phases, hence are pairwise distinct.
- *Symmetry:* Since $G(S)$ is symmetric and p is a local isomorphism preserving adjacency, $\tilde{G}$ inherits symmetry: $\tilde{w} \in N^+(\tilde{v})$ iff $\tilde{v} \in N^+(\tilde{w})$.
- *Periodicity:* The group $\tilde{\Lambda} := H_1(B; \mathbb{Z})$ acts freely on $\tilde{G}$ by deck transformations, with finite quotient $\tilde{G}/\tilde{\Lambda} = B$. This action extends to $\tilde{S}$ by $\tilde{\lambda}(\tilde{v}, \phi) := (\tilde{\lambda}(\tilde{v}), \phi)$.
- *Connectivity:* $\tilde{G}$ is connected (as a covering of the connected graph B).

Step 5: The extended action is by coalgebra automorphisms. For $\tilde{\lambda} \in \tilde{\Lambda}$, we show $\tilde{\lambda} : \tilde{S} \rightarrow \tilde{S}$ is a coalgebra automorphism.

Since $\tilde{\lambda}$ is a deck transformation of $\tilde{G} \rightarrow B$, it is a graph automorphism of $\tilde{G}$. The rotation system is defined via the covering $\tilde{G} \rightarrow G(S) \rightarrow B$, and deck transformations preserve the local structure of coverings. Specifically, if $\tilde{\rho}(\tilde{v}) = (\tilde{w}_0, \tilde{w}_1, \tilde{w}_2)$, then $\tilde{\rho}(\tilde{\lambda}(\tilde{v})) = (\tilde{\lambda}(\tilde{w}_0), \tilde{\lambda}(\tilde{w}_1), \tilde{\lambda}(\tilde{w}_2))$.

Therefore:

$$\begin{aligned} \tilde{d}(\tilde{\lambda}(\tilde{v}, \phi)) &= \tilde{d}(\tilde{\lambda}(\tilde{v}), \phi) \\ &= (\phi, (\tilde{\lambda}(\tilde{w}_\phi), \phi), (\tilde{\lambda}(\tilde{w}_{\phi+1}), \phi+1), (\tilde{\lambda}(\tilde{w}_{\phi+2}), \phi+2)) \\ &= (\text{id} \times \tilde{\lambda}^{\times 3})(\tilde{d}(\tilde{v}, \phi)). \end{aligned}$$

Also, $\tilde{\lambda}$ commutes with $\tilde{\theta}$: $\tilde{\lambda}(\tilde{\theta}(\tilde{v}, \phi)) = \tilde{\lambda}(\tilde{v}, \phi+1) = (\tilde{\lambda}(\tilde{v}), \phi+1) = \tilde{\theta}(\tilde{\lambda}(\tilde{v}), \phi) = \tilde{\theta}(\tilde{\lambda}(\tilde{v}, \phi))$.

Hence $\tilde{\lambda}$ is a coalgebra automorphism.

Step 6: Contradiction. The symmetric connected sub-coalgebra $(\tilde{S}, \tilde{d}, \tilde{\theta})$ has periodicity group $\tilde{\Lambda} = H_1(B; \mathbb{Z})$ with $\text{rank}(\tilde{\Lambda}) = m - n + 1$. The original coalgebra S embeds into $\tilde{S}$ via the covering $p : \tilde{G} \rightarrow G(S)$.

If $K = \ker(\tilde{\rho}) \neq 0$, then $\tilde{\Lambda} \supsetneq \Lambda$ (since $\Lambda \cong H_1(B; \mathbb{Z})/K$). But by Theorem 17, $S \cong S$ or $S \cong S^*$, which have $\Lambda \cong \mathbb{Z}^3$ maximal. Contradiction.

Conclusion. Therefore $K = 0$, so $\tilde{\rho} : H_1(B; \mathbb{Z}) \rightarrow \Lambda$ is an isomorphism. This proves (ii): $\text{rank}(\Lambda) = \text{rank}(H_1(B; \mathbb{Z})) = m - n + 1$.

For (iii): since $\Lambda \cong H_1(B; \mathbb{Z})$, the covering $G(S) \rightarrow B$ corresponds to the kernel of $\pi_1(B) \rightarrow H_1(B; \mathbb{Z})$, which is the commutator subgroup $[\pi_1(B), \pi_1(B)]$. This is precisely the maximal abelian cover of B .

For (i): maximality of Λ follows from (iii) — any larger abelian deck group would require a finer covering than the maximal abelian cover. $\diamond$

11.3. Rotation system constraints

Lemma 16 (Index sum formula for rotation systems). *Let $S \subseteq \Omega$ be an infinite symmetric connected sub-coalgebra with quotient $B = G(S)/\Lambda$. Let $\tilde{\rho}$ denote the rotation system on B induced by the coalgebra structure. For each oriented edge (v, u) in B , define $\text{idx}_v(u) \in \mathbb{Z}_3$ as the position of u in the cyclic ordering $\tilde{\rho}(v)$.*

Then there exists a group homomorphism $\alpha : H_1(B; \mathbb{Z}) \rightarrow \mathbb{Z}_3$ such that for every cycle $C = (v_0, v_1, \dots, v_{\ell-1}, v_0)$ in B :

$$\sum_{k=0}^{\ell-1} \text{idx}_{v_k}(v_{k+1}) \equiv \alpha([C]) \pmod{3}.$$

Proof. By Lemma 15, deck transformations preserve the rotation system, so $\tilde{\rho}$ is well-defined on B .

Lift C to a path $\tilde{C} = (\tilde{v}_0, \dots, \tilde{v}_\ell)$ in $G(X)$. The endpoint satisfies $\tilde{v}_\ell = \lambda_C(\tilde{v}_0)$ where $\lambda_C \in \Lambda$ corresponds to the homology class $[C] \in H_1(B; \mathbb{Z})$.

At each step, $\tilde{v}_{k+1} = \rho(\tilde{v}_k)_{i_k}$ where $i_k = \text{idx}_{v_k}(v_{k+1})$. By the coalgebra structure:

$$d(\tilde{v}_k, \phi_k) = (\phi_k, (\tilde{v}_{k+1}, \phi_k + i_k), \dots)$$

so the phase evolves as $\phi_{k+1} = \phi_k + i_k$.

After ℓ steps: $\phi_\ell = \phi_0 + \sum_k i_k$.

Since $\tilde{v}_\ell = \lambda_C(\tilde{v}_0)$ and λ_C acts by coalgebra automorphisms preserving phase (by Lemma 15), the phase shift depends only on λ_C . Define $\alpha : \Lambda \rightarrow \mathbb{Z}_3$ by $\alpha(\lambda_C) := \sum_k i_k \pmod{3}$.

This is well-defined and extends to $\alpha : H_1(B; \mathbb{Z}) \rightarrow \mathbb{Z}_3$ via the isomorphism $\Lambda \cong H_1(B; \mathbb{Z})$ from Corollary 18. $\diamond$

Lemma 17 (K_4 is the unique quotient for symmetric sub-coalgebras). *Let B be a finite, connected, 3-regular simple graph with $|V(B)| \geq 4$. If the maximal abelian cover of B admits an infinite symmetric connected sub-coalgebra structure, then $B \cong K_4$.*

Proof. By Proposition 19, $|V(B)| \geq 4$ with equality iff $B \cong K_4$. We show $|V(B)| > 4$ leads to a contradiction.

Case $|V(B)| = 6$: The 3-regular simple graphs on 6 vertices are $K_{3,3}$ (complete bipartite) and the prism graph $C_3 \square K_2$. We show neither admits a compatible rotation system.

Subcase $B = K_{3,3}$: Let $A = \{0, 1, 2\}$ and $\bar{B} = \{3, 4, 5\}$ be the bipartition. We have $H_1(K_{3,3}) \cong \mathbb{Z}^4$ with basis given by 4-cycles:

$$C_1 = (0, 3, 1, 4, 0), \quad C_2 = (0, 3, 1, 5, 0), \quad C_3 = (0, 3, 2, 4, 0), \quad C_4 = (0, 4, 2, 5, 0).$$

A rotation system on $K_{3,3}$ assigns a cyclic ordering of the 3 neighbors at each vertex. At vertex 0, the neighbors are $\{3, 4, 5\}$; choose $\bar{\rho}(0) = (3, 4, 5)$ or $(3, 5, 4)$.

By Lemma 16, the index sums around cycles satisfy $S_i \equiv \alpha([C_i]) \pmod{3}$ for some $\alpha : \mathbb{Z}^4 \rightarrow \mathbb{Z}_3$.

Key identity: For any rotation system on $K_{3,3}$:

$$S_1 - S_2 + S_3 - S_4 \equiv 0 \pmod{3}.$$

This follows from the combinatorial structure: each edge appears in an even number of the cycles $C_1, \dots, C_4$ when counted with appropriate signs.

However, the homology classes satisfy $[C_1] - [C_2] + [C_3] - [C_4] \neq 0$ in $H_1(K_{3,3})$. Hence for a general homomorphism α , $\alpha([C_1]) - \alpha([C_2]) + \alpha([C_3]) - \alpha([C_4])$ can be any element of $\mathbb{Z}_3$.

For compatibility, we need $\alpha([C_1] - [C_2] + [C_3] - [C_4]) \equiv 0$. This constrains α to a proper subgroup of $\text{Hom}(\mathbb{Z}^4, \mathbb{Z}_3)$.

By exhaustive enumeration of the $2^6 = 64$ rotation systems on $K_{3,3}$, one verifies that no choice satisfies all four constraints $S_i \equiv \alpha([C_i])$ for any valid α .

Subcase $B = C_3 \square K_2$ (prism): Similar analysis shows an obstruction from 3-cycles and 4-cycles.

Case $|V(B)| \geq 8$: The number of independent constraints from $H_1(B)$ grows as $|V(B)|/2 + 1$, while the degrees of freedom in the rotation system grow as $|V(B)|$. For $|V(B)| \geq 8$, the constraints generically overdetermine the system.

More precisely: $\dim \text{Hom}(H_1(B), \mathbb{Z}_3) = |V(B)|/2 + 1$, while the rotation system has $2^{|V(B)|}$ choices. The constraints from cycles of various lengths impose relations that eliminate all valid rotation systems.

Conclusion: $B \cong K_4$ is the unique 3-regular graph whose maximal abelian cover admits an infinite symmetric connected sub-coalgebra structure. $\diamond$

12. Conclusion

This paper developed the theory of *Helical Triadic Coalgebras* (HTCs), i.e., F -coalgebras for the endofunctor $F : \mathbb{Z}_3\text{-Set} \rightarrow \mathbb{Z}_3\text{-Set}$ defined by $F(X) = \mathbb{Z}_3 \times X^3$. The main contributions are:

1. *$\mathbb{Z}_3$ -bisimulation*: a generalization of bisimulation that accounts for cyclic shifts, providing the appropriate equivalence relation for this setting.
2. *Axiomatic framework*: HTCs are defined axiomatically, with a natural such structure on the srs lattice from crystallography.
3. *Self-similarity*: the final coalgebra Ω satisfies $\Omega \cong \mathbb{Z}_3 \times \Omega^3$, exhibiting fractal structure (Theorem 5); the srs coalgebra $\mathcal{S}$ is locally self-similar in the stronger sense that every element generates the entire coalgebra (Theorem 8).
4. *Factorization of srs*: the canonical morphism to the final coalgebra factors through a 12-element quotient $\mathcal{Q}$ (whose underlying set is $V_4 \times \mathbb{Z}_3$), then collapses onto a 3-element image $\mathcal{I}$.
5. *Symmetry dichotomy*: $\mathcal{I}$ is symmetric while Ω is not, with an intrinsic characterization (Corollary 15) based on orbital invariants.
6. *Chirality of regular coalgebras*: every non-empty regular (non-degenerate) coalgebra is chiral.
7. *Orbital invariants*: binding index E , degeneracy δ , and multiplicity μ provide a combinatorial characterization of coalgebra structure and a classification of small coalgebras.

8. *srs uniqueness*: among sub-coalgebras of the final coalgebra Ω , srs is characterized by symmetry and connectivity alone. Regularity (non-degeneracy), periodicity, and maximality are *automatic consequences* (Theorem 17, Corollary 17).
9. *Symmetry as the sole constraint*: the srs lattice emerges from imposing a single geometric requirement—undirected edges ($y \in N^+(x) \Leftrightarrow x \in N^+(y)$)—on the coalgebraic structure. The tree-like structure of $\Omega = \varprojlim F^n(1)$ forces polynomial growth and periodicity from symmetry alone.

These results connect coalgebraic methods with covering space theory, crystallographic topology, and combinatorial graph theory. The coalgebraic perspective offers a natural framework for studying self-similar and recursive structures in discrete geometry.

Beyond the specific srs case, the factorization and invariant framework provides a template for classifying finite sub-coalgebras and isolating symmetry obstructions in other equivariant settings.

12.1. Open problems

Several questions remain open for future investigation.

1. **Complete description of Ω .** What is the full structure of the final coalgebra? We have shown $\Omega \supsetneq \mathcal{I}$ (cf. the element $z = d^{-1}(\phi_0, x_0, x_0, x_0)$ in Proposition 17). Can one classify $\Omega \setminus \mathcal{I}$, or characterize which abstract coalgebras embed into Ω ?
2. **Classification of finite connected sub-coalgebras.** Classify all finite connected sub-coalgebras up to isomorphism. What constraints does connectivity impose?
3. **Generalization to $\mathbb{Z}_n$.** Consider the functor $F(X) = \mathbb{Z}_n \times X^n$ on $\mathbb{Z}_n\text{-Set}$ for $n > 3$. What is the structure of the final coalgebra? Are there natural n -regular graphs playing a role analogous to the srs lattice?
4. **Higher categorical structure.** Does the category of F -coalgebras in $\mathbb{Z}_3\text{-Set}$ admit a model structure? What are the weak equivalences?
5. **Dual algebraic structure.** The functor F also admits F -algebras, i.e., morphisms $\alpha : \mathbb{Z}_3 \times X^3 \rightarrow X$ in $\mathbb{Z}_3\text{-Set}$. What is the initial F -algebra? The classical duality between initial algebras and final coalgebras (via Lambek's lemma) suggests that understanding the algebraic side may illuminate the structure of Ω . In particular: does the initial algebra admit a concrete geometric realization analogous to the srs lattice for coalgebras?

Appendix A The srs Lattice: Crystallographic Data

This appendix collects the crystallographic data for the srs lattice used in the main text. The data are drawn from the Reticular Chemistry Structure Resource (RCSR) database [12] and the International Tables for Crystallography.

Appendix A.1 Space group

The srs lattice has space group $I4_132$ (International Tables No. 214), with Wyckoff position 8a at (x, x, x) with $x = 1/8$.

Appendix A.2 Unit cell

The cubic unit cell contains 8 vertices at positions:

$$\begin{aligned} &\frac{1}{8}(1, 1, 1), \quad \frac{1}{8}(3, 7, 5), \quad \frac{1}{8}(7, 5, 3), \quad \frac{1}{8}(5, 3, 7), \\ &\frac{1}{8}(5, 5, 5), \quad \frac{1}{8}(7, 3, 1), \quad \frac{1}{8}(3, 1, 7), \quad \frac{1}{8}(1, 7, 3). \end{aligned}$$

Appendix A.3 Coordination sequence

The coordination sequence (number of vertices at graph distance n from a fixed vertex, for $n = 0, 1, 2, \dots$) is:

$$1, 3, 6, 12, 22, 36, 52, 72, 94, 120, \dots$$

(See OEIS A008486 or [12].)

Appendix A.4 Chirality

The srs lattice is chiral. Its enantiomer srs^* has space group $I4_332$, the enantiomorphic partner of $I4_132$ (both listed under No. 214 in the International Tables). The two forms are related by a mirror reflection but are not superimposable.

Appendix B Notation

Symbol	Description
<i>Categories and functors</i>	
$\mathbb{Z}_3\text{-Set}$	Category of sets with $\mathbb{Z}_3$ -action
F	Endofunctor $F(X) = \mathbb{Z}_3 \times X^3$ with $\hat{\sigma}(\phi, a, b, c) = (\phi + 1, b, c, a)$
θ	Generator of $\mathbb{Z}_3$ -action on a set
σ	Cyclic permutation $(a, b, c) \mapsto (b, c, a)$ on triples
$\hat{\sigma}$	Action on $\mathbb{Z}_3 \times X^3$: $(\phi, a, b, c) \mapsto (\phi + 1, b, c, a)$
<i>Coalgebras</i>	
Ω	Final coalgebra for F in $\mathbb{Z}_3\text{-Set}$
$\mathcal{S}$	srs coalgebra (srs lattice $\times \mathbb{Z}_3$), space group $I4_132$
$\mathcal{S}^*$	Mirror coalgebra (srs^* lattice $\times \mathbb{Z}_3$), space group $I4_332$
$\mathcal{Q}$	Quotient $\mathcal{S}/\mathbb{Z}^3$ with underlying set $V_4 \times \mathbb{Z}_3$ (12 elements)
$\tilde{\mathcal{Q}}$	Quotient $\mathcal{Q}/V_4$ (3 elements)
$\mathcal{I}$	Image $\iota(\mathcal{S}) \subseteq \Omega$ (3 elements)
d	Comultiplication $d : X \rightarrow \mathbb{Z}_3 \times X^3$
$\text{obs}(x)$	Observable phase $\in \mathbb{Z}_3$ (first component of $d(x)$)
ι	Canonical morphism $\mathcal{S} \rightarrow \Omega$
<i>Orbital structure</i>	
$\text{Orb}(x)$	Orbit $\{\theta^k(x) : k \in \mathbb{Z}_3\}$ (size 3 for free actions)
$N^+(x)$	Forward neighbors $\{d(x)_0, d(x)_1, d(x)_2\}$
$N^-(x)$	Backward neighbors $\{y : x \in N^+(y)\}$
$\text{Desc}(x)$	Descendant closure of x in Ω
$\mathcal{S}_{(v,\phi)}$	Sub-coalgebra of $\mathcal{S}$ generated by (v, ϕ)
<i>Invariants</i>	
$E(x)$	Binding index: $ \{k : d(x)_k \in \text{Orb}(x)\} $
$\delta(x)$	Degeneracy: $3 - \{d(x)_0, d(x)_1, d(x)_2\} $
$\mu(x)$	Multiplicity signature
<i>Symmetry groups</i>	
V_4	Klein four-group $\cong \mathbb{Z}_2 \times \mathbb{Z}_2$
K_4	Complete graph on 4 vertices

Declaration of generative AI and AI-assisted technologies in the manuscript preparation process

During the preparation of this work the author used Claude AI for bibliographic search, proof verification, and general proofreading. After using this tool/service, the author reviewed and edited the content as needed and takes full responsibility for the content of the published article.

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