Article

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Constructing Physics from Measurements

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Abstract: We present a reformulation of fundamental physics from an enumeration of independent axioms into the solution of a single optimization problem. Any experiment begins with an initial state preparation, involves some physical evolution, and ends with a final measurement. Working from this structure, we maximize the entropy of a final measurement relative to its initial preparation subject to a measurement constraint. Solving this optimization problem for the *natural constraint* –the most permissive measurement constraint compatible with said problem– identifies a unified physical theory which contains general relativity (acting on spacetime) and Yang-Mills (acting on internal spaces). Notably, mathematical consistency further restricts valid solutions to 3+1 dimensions only, automatically yielding spacetime. This reformulation reveals that the apparent complexity of modern physics, with its various forces, symmetries, and dimensional constraints, emerges as the solution to an optimization problem constructed over all experiments realizable within the constraint of nature.

Keywords: foundations of physics

1. Introduction

Statistical mechanics (SM), in the formulation developed by E.T. Jaynes [1,2], is founded on an entropy optimization principle. Specifically, the Boltzmann entropy is maximized under the constraint of a fixed average energy $\overline{E}$:

$$\overline{E} := \sum_i \rho_i E_i \tag{1}$$

The Lagrange multiplier equation defining the optimization problem is:

$$\mathcal{L} := -k_B \sum_i \rho_i(\beta) \ln \rho_i(\beta) + \lambda \left(1 - \sum_i \rho_i(\beta)\right) + k_B \beta \left(\overline{E} - \sum_i \rho_i(\beta) E_i\right) \tag{2}$$

where λ and β are Lagrange multipliers enforcing the normalization and average energy constraints. Solving this optimization problem for ρ_i yields the Gibbs measure:

$$\rho_i(\beta) = \frac{1}{Z(\beta)} \exp(-\beta E_i), \tag{3}$$

where $Z(\beta) = \sum_j \exp(-\beta E_j)$ is the partition function.

For comparison, quantum mechanics (QM) is not formulated as the solution to an optimization problem, but rather consists of a collection of axioms[3,4]:

QM Axiom 1 of 5   **State Space:** Every physical system is associated with a complex Hilbert space, and its state is represented by a ray (an equivalence class of vectors differing by a non-zero scalar multiple) in this space.

QM Axiom 2 of 5   **Observables:** Physical observables correspond to Hermitian (self-adjoint) operators acting on the Hilbert space.

QM Axiom 3 of 5   **Dynamics:** The time evolution of a quantum system is governed by the Schrödinger equation, where the Hamiltonian operator represents the system's total energy.

QM Axiom 4 of 5	**Measurement:** Measuring an observable projects the system into an eigenstate of the corresponding operator, yielding one of its eigenvalues as the measurement result.
QM Axiom 5 of 5	**Probability Interpretation:** The probability of obtaining a specific measurement outcome is given by the squared magnitude of the projection of the state vector onto the relevant eigenstate (Born rule).

Physical theories have traditionally been constructed in two distinct ways. Some, like QM, are defined through a set of mathematical axioms that are first postulated and then verified against experiments. Others, like SM, emerge as solutions to optimization problems with experimentally-verified constraints.

We propose to generalize the optimization methodology of E.T. Jaynes to encompass all of physics, aiming to derive a unified theory from a single optimization problem.

To that end, we introduce the following constraint:

Axiom 1 (Nature).

$$\overline{H} := \int_V w(t, \vec{x}) \operatorname{tr} H(\vec{x}) \, \mathrm{d}^p \vec{x}$$

where $H(\vec{x})$ is an operator, $\overline{H}$ is the average of its trace, p is the number of spatial dimensions (e.g. three in spacetime), and $w(t, \vec{x})$ is an information density[1].

This constraint, as it replaces the scalar energy E_i with the operator $H(\vec{x})$, extends E.T. Jaynes' optimization method to encompass non-commutative observables and symmetry group generators required for fundamental physics.

We then construct an optimization problem:

Definition 1 (Physics). *Physics is the solution to:*

$$\underbrace{\mathcal{L}[w]}_{\substack{\text{an} \\ \text{optimization} \\ \text{problem}}} := \underbrace{- \int_V w(t, \vec{x}) \ln \frac{w(t, \vec{x})}{w(0, \vec{x})} \, \mathrm{d}^p \vec{x}}_{\substack{\text{on the entropy} \\ \text{of a measurement} \\ \text{relative to its preparation}}} + \underbrace{t \left(\overline{H} - \int_V w(t, \vec{x}) \operatorname{tr} H(\vec{x}) \, \mathrm{d}^p \vec{x} \right)}_{\text{as constrained by nature}}$$

where t is the Lagrange multiplier[2] enforcing the natural constraint.

This definition constitutes our complete proposal for reformulating fundamental physics—no additional principles will be introduced. By replacing the Boltzmann entropy with the relative Shannon entropy, the optimization problem extends beyond thermodynamic variables to encompass any type of experiment. This generalization occurs because relative entropy captures the essence of any experiment: the relationship between a final measurement and its initial preparation. Finally, the natural constraint defines the domain of applicability of the theory.

The crucial insight is that because our formulation maintains complete generality in the structure of experiments while optimizing over all possible natural theories, the resulting solution holds true, by

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[1] An information density is a non-negative, unnormalized measure, $w(t, \vec{x}) \geq 0$. This contrasts with a probability density, which is additionally normalized to unity. An information density is the most general structure whose entropy is real-valued. The explicit form of w will be identified in Theorem 1 by solving the optimization problem. The unnormalized nature of w aligns with foundational concepts in Quantum Field Theory (QFT), where unnormalized field amplitudes and their associated conserved (but not unit-normalized) charge densities are primary, rather than single-particle probability amplitudes. The framework will later show how quantities suitable for probabilistic interpretation (Theorem 4), such as the Dirac current, emerge from the dynamics.

[2] As we solve the optimization problem, we will find that the Lagrange multiplier t takes on the role of time, yielding dynamical equations. This is similar to the Lagrange multiplier β in statistical mechanics taking on the role of temperature via $\beta = 1/(k_B T)$ after solving the optimization problem for the Gibbs measure.

construction, for all realizable experiments within its domain. This approach reduces our reliance on postulating axioms through trial and error, and simplifies the foundations of physics.

Specifically, when we employ the *natural constraint* —the most permissive constraint for this problem (Section 2.1)— the solution spawns its largest domain, yielding the Dirac theory in the language of spacetime algebra (STA). The resulting solution in fact yields a slight extension to standard quantum theory in the sense that it increases the group of transformations that preserve the probability density from unitary transformations to Spinc(3,1) transformations. It is within this extension that a unified physics where fundamental theories emerge naturally is uniquely recovered—e.g. general relativity (acting on spacetime) and Yang-Mills (acting on internal spaces of spacetime).

As we will see in Section 2.2, Definition 1 automatically restricts the valid solutions to the specific case of 3+1 dimensions. In other dimensional configurations, various obstructions arise making the solution violate the axioms of probability theory. The following table summarizes the geometric cases and their obstructions:

Dimensions	*Does a solution exist?*	
Cl(0)	Obstructed (Undefinable)	(4)
Cl(0,1)	Obstructed (Undefinable)	(5)
Cl(1,0)	Obstructed (Undefinable)	(6)
Cl(2,0)	Obstructed (Undefinable)	(7)
Cl(1,1)	Obstructed (Negative probabilities)	(8)
Cl(0,2)	Obstructed (Non-real probabilities)	(9)
Cl(3,0)	Obstructed (Non-real probabilities)	(10)
Cl(2,1)	Obstructed (Ill-defined probabilities)	(11)
Cl(1,2)	Obstructed (Non-real probabilities)	(12)
Cl(0,3)	Obstructed (Ill-defined probabilities)	(13)
Cl(4,0)	Obstructed (Non-real probabilities)	(14)
Cl(3,1) / spacetime	**Gravity + Yang-Mills**	(15)
Cl(2,2)	Obstructed (Undefinable)	(16)
Cl(1,3)	Obstructed (Non-real probabilities)	(17)
Cl(0,4)	Obstructed (Non-real probabilities)	(18)
Cl(5,0)	Obstructed (Ill-defined probabilities)	(19)
$\vdots$	$\vdots$	
Cl(6,0)	Suspected Obstructed (No observables)	(20)
$\vdots$	$\vdots$	

where Cl(p, q) designates the Clifford algebra of p space dimensions and q time dimensions.

We will investigate the unobstructed case in Section 2.1 and then demonstrate the obstructions in Section 2.2. These obstructions are desirable because they provide a mechanism for the observed dimensionality of our universe as solely satisfying the probabilistic structure of the solution.

2. Results

Theorem 1. *The general solution of the optimization problem (Definition 1) is:*

$$w(t, \vec{x}) = \det\left(\exp\left(-tH(\vec{x})\right)\right) w(0, \vec{x})$$

Proof. We solve Definition 1 by taking the variation of the Lagrange multiplier equation with respect to w. (To improve the legibility, we will drop the explicit parametrization in $w(t, \vec{x}) \to w$ and $H(\vec{x}) \to H$ in the proof.)

$$0 = \delta w \mathcal{L}[w] \tag{21}$$

$$= -\delta w \int_V w \ln \frac{w}{w_0'} dx + t \left(\delta w \overline{H} - \delta w \int_V w \operatorname{tr} H dx \right) \tag{22}$$

$$= -\ln \frac{w}{w_0'} \delta w - \delta w - t \operatorname{tr}(H) \delta w \tag{23}$$

$$= -\ln w \delta w + \ln w_0' \delta w - \delta w - t \operatorname{tr}(H) \delta w \tag{24}$$

$$= -\ln w + \ln w_0' - 1 - t \operatorname{tr}(H) \tag{25}$$

$$\implies \ln w = -1 - t \operatorname{tr}(H) + \ln w_0' \tag{26}$$

$$\implies w = \exp(-1) \exp(-t \operatorname{tr} H) w_0' \tag{27}$$

$$w = \exp(-t \operatorname{tr} H) w_0 \qquad (w_0 := \exp(-1) w_0') \tag{28}$$

$$w = \det \big(\exp(-tH) \big) w_0 \qquad (\det \exp M \equiv \exp \operatorname{tr} M) \tag{29}$$

Finally, restoring the explicit parametrization, we end with:

$$w(t, \vec{x}) = \det \Big(\exp \big(-tH(\vec{x}) \big) \Big) w(0, \vec{x}) \tag{30}$$

□

The steps of this proof are standard for solving an entropy optimization problem in statistical mechanics, and simply mimic the steps in the usual derivation of the Gibbs measure, as shown in Annex A.

In the following section, we will show that when $H(\vec{x})$ is taken to be the most permissive constraint in 3+1D—a covariant derivative over all possible transformations of the elements of STA, this solution entails the hamiltonian form of the Dirac equation expressed in the language of STA.

2.1. Spinors + Dirac Equation

We will now investigate this solution in the context of $\mathrm{Cl}(3, 1)$. We begin with a definition of the determinant for a multivector.

2.1.1. The Multivector Determinant

Our goal here will be to express the determinant of a real 4×4 matrix as a multivector self-product. To achieve that, we begin by defining a general multivector of $\mathrm{Cl}(3, 1)$:

$$\mathbf{u} := a + \mathbf{x} + \mathbf{f} + \mathbf{v} + \mathbf{b} \tag{31}$$

where a is a scalar, $\mathbf{x}$ a vector, $\mathbf{f}$ a bivector, $\mathbf{v}$ is pseudo-vector and $\mathbf{b}$ a pseudo-scalar. Explicitly,

$$\mathbf{u} := a \tag{32}$$

$$+ t\gamma_0 + x\gamma_1 + y\gamma_2 + z\gamma_3 \tag{33}$$

$$+ f_{01}\gamma_0\gamma_1 + f_{02}\gamma_0\gamma_2 + f_{03}\gamma_0\gamma_3 + f_{12}\gamma_1\gamma_2 + f_{13}\gamma_1\gamma_3 + f_{23}\gamma_2\gamma_3 \tag{34}$$

$$+ p\gamma_1\gamma_2\gamma_3 + q\gamma_0\gamma_2\gamma_3 + v\gamma_0\gamma_1\gamma_3 + w\gamma_0\gamma_1\gamma_2 \tag{35}$$

$$+ b\gamma_0\gamma_1\gamma_2\gamma_3 \tag{36}$$

Definition 2 (Real-Majorana Algebra Isomorphism). *The map $\varphi : \mathrm{Cl}(3,1) \to \mathrm{Mat}(4,\mathbb{R})$ defined by:*

$$\varphi(1) := diag(1,1,1,1) \tag{37}$$

$$\varphi(\gamma_0) := \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix} \qquad \varphi(\gamma_1) := \begin{bmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{bmatrix} \tag{38}$$

$$\varphi(\gamma_2) := \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \qquad \varphi(\gamma_3) := \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{39}$$

$$\varphi(\gamma_\mu \gamma_\nu) := \varphi(\gamma_\mu)\varphi(\gamma_\nu) \tag{40}$$

$$\varphi(\gamma_\mu \gamma_\nu \gamma_\kappa) := \varphi(\gamma_\mu)\varphi(\gamma_\nu)\varphi(\gamma_\kappa) \tag{41}$$

$$\varphi(\gamma_0 \gamma_1 \gamma_2 \gamma_3) := \varphi(\gamma_0)\varphi(\gamma_1)\varphi(\gamma_2)\varphi(\gamma_3) \tag{42}$$

extends linearly and multiplicatively to an isomorphism between $\mathrm{Cl}(3,1)$ *and the algebra of real* 4×4 *matrices.*

Definition 3 (Matrix Representation).

$$\varphi(\mathbf{u}) = \begin{bmatrix} a + f_{02} - q - z & b - f_{13} + w - x & -f_{01} + f_{12} - p + v & f_{03} + f_{23} + t + y \\ -b + f_{13} + w - x & a + f_{02} + q + z & f_{03} + f_{23} - t - y & f_{01} - f_{12} - p + v \\ -f_{01} - f_{12} + p + v & f_{03} - f_{23} + t - y & a - f_{02} + q - z & -b - f_{13} - w - x \\ f_{03} - f_{23} - t + y & f_{01} + f_{12} + p + v & b + f_{13} - w - x & a - f_{02} - q + z \end{bmatrix}$$

To manipulate and analyze multivectors in $\mathrm{Cl}(3,1)$, we introduce several important operations, such as the multivector conjugate, the pseudo-blade conjugate, and the multivector determinant.

Definition 4 (Multivector Conjugate—in $\mathrm{Cl}(3,1)$).

$$\mathbf{u}^\ddagger := a - \mathbf{x} - \mathbf{f} + \mathbf{v} + \mathbf{b}$$

Definition 5 (Pseudo-Blade Conjugate—in $\mathrm{Cl}(3,1)$). *The pseudo-blade conjugate of* $\mathbf{u}$ *is*

$$\mathbf{u}^\dagger := a + \mathbf{x} + \mathbf{f} - \mathbf{v} - \mathbf{b}$$

Lundholm[5] proposes a number of multivector norms, and shows that they are the *unique* forms which carries the properties of the determinants such as $N(\mathbf{uv}) = N(\mathbf{u})N(\mathbf{v})$ to the domain of multivectors:

Definition 6. *The self-products associated with low-dimensional Clifford algebras are:*

$$\mathrm{Cl}(0,1) : \qquad \mathbf{u}^*\mathbf{u} \tag{43}$$

$$\mathrm{Cl}(2,0) : \qquad \mathbf{u}^\ddagger\mathbf{u} \tag{44}$$

$$\mathrm{Cl}(3,0) : \qquad (\mathbf{u}^\ddagger\mathbf{u})^*\mathbf{u}^\ddagger\mathbf{u} \tag{45}$$

$$\mathrm{Cl}(3,1) : \qquad (\mathbf{u}^\ddagger\mathbf{u})^\dagger\mathbf{u}^\ddagger\mathbf{u} \tag{46}$$

$$\mathrm{Cl}(4,1) : \qquad ((\mathbf{u}^\ddagger\mathbf{u})^\dagger\mathbf{u}^\ddagger\mathbf{u})^*((\mathbf{u}^\ddagger\mathbf{u})^\dagger\mathbf{u}^\ddagger\mathbf{u}) \tag{47}$$

where $\mathbf{u}^*$ *is a conjugate that reverses the sign of pseudo-scalar blade (i.e. the highest degree blade of the algebra).*

We can now express the determinant of the matrix representation of a multivector via a self-product. This choice is unique:

Theorem 2 (The Multivector Determinant—in Cl(3,1))**.**

$$(\mathbf{u}^{\ddagger}\mathbf{u})^{\dagger}\mathbf{u}^{\ddagger}\mathbf{u} \equiv \det\varphi(\mathbf{u})$$

Proof. As this form naively expands into $16^4 = 65536$ product terms, we utilize a computer-assisted proof of this equality in Annex B. $\square$

As can be seen from this theorem, the relationship between determinants and multivector products in $\mathrm{Cl}(3,1)$ is a quartic form that cannot be reduced to a simpler bilinear form.

2.1.2. The Optimization Problem

The relative Shannon entropy requires measures that are everywhere non-negative. Consequently, we will first identify the largest sub-algebra of $\mathrm{Cl}(3,1)$ whose determinant is non-negative.

Theorem 3 (Non-Negativity of Even Multivectors)**.** *Let* $\mathbf{u} = a + \mathbf{f} + \mathbf{b}$ *be an even multivector of* $\mathrm{Cl}(3,1)$. *Then its multivector determinant* $(\mathbf{u}^{\ddagger}\mathbf{u})^{\dagger}\mathbf{u}^{\ddagger}\mathbf{u}$ *is non-negative.*

Proof.

$$(\mathbf{u}^{\ddagger}\mathbf{u})^{\dagger}\mathbf{u}^{\ddagger}\mathbf{u} = ((a - \mathbf{f} + \mathbf{b})(a + \mathbf{f} + \mathbf{b}))^{\dagger}(a - \mathbf{f} + \mathbf{b})(a + \mathbf{f} + \mathbf{b}) \tag{48}$$

$$= (a^2 + a\mathbf{f} + a\mathbf{b} - \mathbf{f}a - \mathbf{f}^2 - \mathbf{f}\mathbf{b} + \mathbf{b}a + \mathbf{b}\mathbf{f} + \mathbf{b}^2)^{\dagger}(a^2 + a\mathbf{f} + a\mathbf{b} - \mathbf{f}a - \mathbf{f}^2 - \mathbf{f}\mathbf{b} + \mathbf{b}a + \mathbf{b}\mathbf{f} + \mathbf{b}^2) \tag{49}$$

$$= (a^2 + \mathbf{b}^2 - \mathbf{f}^2 + 2a\mathbf{b})^{\dagger}(a^2 + \mathbf{b}^2 - \mathbf{f}^2 + 2a\mathbf{b}) \tag{50}$$

> Let us calculate $\mathbf{f}^2$.
>
> $$\mathbf{f}^2 = (\mathbf{E} + I\mathbf{B})(\mathbf{E} + I\mathbf{B}) \tag{51}$$
>
> where $\mathbf{E} := E_{01}\mathbf{e}_{01} + E_{02}\mathbf{e}_{02} + E_{03}\mathbf{e}_{03}$ and where $I\mathbf{B} := B_{12}\mathbf{e}_{12} + B_{13}\mathbf{e}_{13} + B_{23}\mathbf{e}_{23}$
>
> $$= \mathbf{E}^2 + \mathbf{E}I\mathbf{B} + I\mathbf{B}\mathbf{E} - \mathbf{B}^2 \tag{52}$$
>
> $$= \mathbf{E}^2 - \mathbf{B}^2 + 2I\mathbf{E}\cdot\mathbf{B} \tag{53}$$

$$= (a^2 - b^2 - \mathbf{E}^2 + \mathbf{B}^2 - 2I\mathbf{E}\cdot\mathbf{B} + 2Iab)^{\dagger}(a^2 - b^2 - \mathbf{E}^2 + \mathbf{B}^2 - 2I\mathbf{E}\cdot\mathbf{B} + 2Iab) \tag{54}$$

$$= ((a^2 - b^2 - \mathbf{E}^2 + \mathbf{B}^2) + 2I(\mathbf{E}\cdot\mathbf{B} - ab))((a^2 - b^2 - \mathbf{E}^2 + \mathbf{B}^2) - 2I(\mathbf{E}\cdot\mathbf{B} - ab))) \tag{55}$$

$$= (a^2 - b^2 - \mathbf{E}^2 + \mathbf{B}^2)^2 + 4(\mathbf{E}\cdot\mathbf{B} - ab)^2 \tag{56}$$

$$= (a^2 - b^2 - E_{01}^2 - E_{02}^2 - E_{03}^2 + B_{12}^2 + B_{13}^2 + B_{23}^2)^2 + 4(E_{01}B_{23} + E_{02}B_{13} + E_{03}B_{12} - ab)^2 \tag{57}$$

which is non-negative—the sum of two squares of real numbers is in $\mathbb{R}_{\geq 0}$. $\square$

To define the optimization problem in $\mathrm{Cl}(3,1)$, we note the following:

1. In 3+1D, we are interested in the case where the states are an element of the even sub-algebra of $\mathrm{Cl}(3,1)$, whose determinant is non-negative:

$$\varphi(\mathbf{u}) = \begin{bmatrix} a + f_{02} & b - f_{13} & -f_{01} + f_{12} & f_{03} + f_{23} \\ -b + f_{13} & a + f_{02} & f_{03} + f_{23} & f_{01} - f_{12} \\ -f_{01} - f_{12} & f_{03} - f_{23} & a - f_{02} & -b - f_{13} \\ f_{03} - f_{23} & f_{01} + f_{12} & b + f_{13} & a - f_{02} \end{bmatrix} \tag{58}$$

hence satisfying the non-negativity requirement of the information density $w(t, x) \geq 0$.

2. It is well-known that the even sub-algebra of $\mathrm{Cl}(3,1)$ is isomorphic to 3+1D spinors[6].

3. In the continuum such elements are transformed by a connection which is valued in $\mathfrak{spin}(3,1) \oplus \mathfrak{u}(1)$:

$$\mathbf{M} \to \omega_\mu = \frac{1}{2}\omega_\mu^{ab}\gamma_{ab} + IV_\mu \tag{59}$$

4. We also consider translations ∂_x, ∂_y and ∂_z. Hence, we define the covariant derivative D_i as:

$$D_i := \partial_i + \frac{1}{2}\omega_i^{ab}\gamma_{ab} + IV_i \tag{60}$$

5. The operator $\gamma^i D_i + \gamma^0 \omega_0$ represents the set of all transformations (including translations) that can be done on the even elements of $Cl(3,1)$ parametrized in spacetime. As such it is the **most permissive** constraint within spacetime that transforms its even multivectors.

    Flat Spacetime:
    The optimization problem will be as follows:

$$\mathcal{L}[w] := -\int_V w(t,\vec{x}) \ln \frac{w(t,\vec{x})}{w(0,\vec{x})} \, \mathrm{d}^3\vec{x} + t\left(\overline{H} - \int_V w(t,\vec{x}) \, \mathrm{tr}(\gamma^0(\gamma^i D_i + \gamma^0 \omega_0)) \, \mathrm{d}^3\vec{x}\right) \tag{61}$$

    The solution is:

$$w(t,\vec{x}) = \det \exp(-t\gamma^0(\gamma^i D_i + \gamma^0 \omega_0))w(0,\vec{x}) \tag{62}$$

    The base field is identified from the multivector determinant, and is written as:

$$\det \phi(t,\vec{x}) = \det \exp(-t\gamma^0(\gamma^i D_i + \gamma^0 \omega_0)) \det \phi(0,\vec{x}) \tag{63}$$

where

1. $w(t,\vec{x}) := \det \phi(t,\vec{x})$
2. $w(0,\vec{x}) := \det \phi(0,\vec{x})$

As such, Equation 63 is formally equivalent to Equation 62.

The following expression, obtained by removing the determinant[3], satisfies the solution:

$$\phi(t,\vec{x}) = \exp(-t\gamma^0(\gamma^i D_i + \gamma^0 \omega_0))\phi(0,\vec{x}) \tag{64}$$

where $\phi(0,\vec{x})$ is defined as:

Definition 7 (Spinor-valued Field).

$$\phi(0,\vec{x}) = \phi(0,\vec{x}) = a(\vec{x}) + \mathbf{f}(\vec{x}) + \mathbf{b}(\vec{x}) \tag{65}$$

The base field $\phi(t,\vec{x})$ leads to a variant of the Schrödinger equation obtained by taking its derivative with respect to t:

Definition 8 (Spinor-valued Schrödinger Equation).

$$\frac{\partial \phi(t,\vec{x})}{\partial t} = -\gamma^0(\gamma^i D_i + \gamma^0 \omega_0)\phi(t,\vec{x})$$

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[3] The removal of the determinant implies an additional term $\det A = \det B \implies A = CB$, where $\det C = 1$. Furthermore, since $\det C = 1$ implies $C \in \mathrm{Spinc}(3,1)$, it is simply a gauge choice. Here, we choose $C = 1$.

The above expression is simply the massless Dirac equation in Hamiltonian form. Specifically, the Dirac equation is obtained as follows:

$$\frac{\partial \phi(t, \vec{x})}{\partial t} = -\gamma^0 (\gamma^i D_i + \gamma^0 \omega_0) \phi(t, \vec{x}) \tag{66}$$

$$\implies 0 = -\gamma^0 (\gamma^i D_i + \gamma^0 \omega_0) \phi(t, \vec{x}) - \frac{\partial \phi(t, \vec{x})}{\partial t} \tag{67}$$

$$= (\gamma^i D_i + \gamma^0 \omega_0) \phi(t, \vec{x}) - \gamma^0 \frac{\partial \phi(t, \vec{x})}{\partial t} \tag{68}$$

$$= (\gamma^i D_i + \gamma^0 \omega_0) \phi(t, \vec{x}) - \gamma^0 \partial_0 \phi(t, \vec{x}) \tag{69}$$

$$= \gamma^\mu D_\mu \phi(t, \vec{x}) \tag{70}$$

$$= D\phi(t, \vec{x}) \tag{71}$$

where $\gamma^\mu D_\mu$ is the Dirac operator over all 4 spacetime coordinates. The end result is the massless Dirac equation in the language of STA.

From Noether's theorem, it is known that the Dirac equation contains a conserved charge current $J = \psi^\ddagger \gamma_0 \psi$, which is the Dirac current.

Theorem 4 (Positive-Definite Inner Product). *The inner product, defined as* $\gamma_0 \cdot \psi^\ddagger \gamma_0 \psi$ *is positive definite.*

Proof. Let $\psi := a + \mathbf{f} + \mathbf{b}$ and $\mathbf{f} := \mathbf{E} + I\mathbf{B}$, then:

$$\gamma_0 \cdot (a - \mathbf{f} + \mathbf{b})\gamma_0(a + \mathbf{f} + \mathbf{b}) \tag{72}$$

$$= \gamma_0 \cdot (a\gamma_0 - \mathbf{E}\gamma_0 - I\mathbf{B}\gamma_0 + \mathbf{b}\gamma_0)(a + \mathbf{f} + \mathbf{b}) \tag{73}$$

$$= \gamma_0 \cdot (\gamma_0 a + \gamma_0 \mathbf{E} - \gamma_0 I\mathbf{B} - \gamma_0 \mathbf{b})(a + \mathbf{f} + \mathbf{b}) \tag{74}$$

$$= \gamma_0 \cdot \gamma_0(a + \mathbf{E} - I\mathbf{B} - \mathbf{b})(a + \mathbf{E} + I\mathbf{B} + \mathbf{b}) \tag{75}$$

$$= \gamma_0 \cdot \gamma_0(a^2 + a\mathbf{E} + aI\mathbf{B} + a\mathbf{b} \tag{76}$$

$$+ \mathbf{E}a + \mathbf{E}^2 + \mathbf{E}I\mathbf{B} + \mathbf{E}\mathbf{b} \tag{77}$$

$$- I\mathbf{B}a - I\mathbf{B}\mathbf{E} - (I\mathbf{B})^2 - I\mathbf{B}\mathbf{b} \tag{78}$$

$$- \mathbf{b}a - \mathbf{b}\mathbf{E} - \mathbf{b}I\mathbf{B} - \mathbf{b}^2) \tag{79}$$

$$= (a^2 + \mathbf{E}^2 - (I\mathbf{B})^2 - \mathbf{b}^2) \tag{80}$$

$$= (a^2 + b^2 + \mathbf{E}^2 + \mathbf{B}^2) \tag{81}$$

$$= (a^2 + b^2 + f_{01}^2 + f_{02}^2 + f_{03}^2 + f_{12}^2 + f_{13}^2 + f_{23}^2) \tag{82}$$

which is positive-definite. $\square$

Consequently, the quantity $\rho := \gamma_0 \cdot \psi^\ddagger \gamma_0 \psi$ can be understood as a probability density when normalized such as in single particle relativistic quantum theory.

Theorem 5 (Equivalence to David Hestenes'[6] formulation).

$$a + \mathbf{f} + \mathbf{b} \equiv \sqrt{\rho} R e^{-I\theta/2} \text{ iff } \det(a + \mathbf{f} + \mathbf{b}) \neq 0 \tag{83}$$

where R is a rotor $R^\ddagger R = 1$

Proof. Let $\psi = a + \mathbf{f} + \mathbf{b}$. Then

$$\psi^\ddagger \psi = a^2 - b^2 - f^2 + 2Iab \tag{84}$$

$$= a^2 - b^2 - \mathbf{E}^2 + \mathbf{B}^2 - 2I\mathbf{E} \cdot \mathbf{B} + 2Iab \tag{85}$$

which is a complex number. In polar form, $\psi^{\ddagger}\psi = \rho e^{-I\theta}$, which implies iff $\psi^{\ddagger}\psi \neq 0$, that $\psi = \sqrt{\rho}Re^{-I\theta/2}$. $\square$

We also note that the definition of the Dirac equation recovers David Hestenes' formulation of the same in the massless case $D\psi I\sigma_3 = m\psi\gamma_0$. Indeed, posing $m = 0 \implies D\psi = 0$.

Curved Spacetime:

In curved spacetime, we utilize the ADM formalism. We foliate spacetime in hypersurfaces Σ_t of constant t:

$$ds^2 = -N^2 dt^2 + h_{ij}(dx^i + N^i dt)(dx^j + N^j dt) \tag{86}$$

The optimization problem Lagrangian remains similar, but the constraint now acquires lapse and shift functions:

$$\mathcal{L} := -\int_{\Sigma_t} w(t,\vec{x}) \ln \frac{w(t,\vec{x})}{w(0,\vec{x})} \sqrt{h}\, d^3\vec{x} + t\left(\overline{H} - \int_{\Sigma_t} w(t,\vec{x}) \operatorname{tr}(N\gamma^n(\gamma^i D_i - K) - N^i D_i)\sqrt{h}\, d^3\vec{x}\right) \tag{87}$$

where

1. $h := \det h_{ij}$
2. $\gamma^n := n_a \gamma^a$ is the STA representation of the the normal vector n^a to the spatial hypersurface Σ_t
3. $\gamma^i := \mathbf{e}_a^i \gamma^a$
4. $K := K_k^k$ is the trace of the extrinsic curvature
5. $D_i := \partial_i + \frac{1}{2}\omega_i^{ab}\gamma_{ab} + IV_i$ is the 3D spatial covariant derivative on the slice Σ_t.

The problem is solved in a manner similar to the flat case and leads to the Schrödinger equation:

$$\frac{\partial\phi(t,\vec{x})}{\partial t} = -(N\gamma^n(\gamma^i D_i - K) - N^i D_i)\phi(t,\vec{x}) \tag{88}$$

This is the Hamiltonian form of the massless Dirac equation $\gamma^\mu D_\mu \phi = 0$ with covariant derivative D_μ expressed with lapse and shift functions and containing a spin and pseudoscalar connection.

Field Functional:

The field functional version of the optimization problem integrates over all possible geometries:

$$\mathcal{L} := -\int \mathcal{D}e\, w[e;t] \ln \frac{w[e;t]}{w[e;0]} + t\left(\overline{H} - \int \mathcal{D}e\, w[e;t] \operatorname{tr} H[e]\right) \tag{89}$$

where

1. The configuration $e = \{\mathbf{e}_i^a(\vec{x}), N(\vec{x}), N^i(\vec{x})\}$ includes the spatial dreibein $\mathbf{e}_i^a(\vec{x})$, lapse $N(\vec{x})$, and shift $N^i(\vec{x})$, representing a full specification of the spatial geometry and its embedding coordinates for the slice Σ_t.
2. $h_{ij} := \delta_{ab}\mathbf{e}_i^a(\vec{x})\mathbf{e}_j^b(\vec{x})$
3. $H[e] := \int_{\Sigma_t}(N[e]\gamma^n[e](\gamma^i[e]D_i[e] - K[e]) - N^i[e]D_i[e])(\vec{x})\sqrt{h[e](\vec{x})}\, d^3\vec{x}$

The optimization problem is solved in a manner similar to the previous cases and leads to the Schrödinger equation:

$$\frac{\partial\phi[e;t]}{\partial t} = -H[e]\phi[e;t] \tag{90}$$

2.2. Dimensional Obstructions

In this section, we explore the dimensional obstructions that arise when attempting to solve the entropy maximization problem for other dimensional configurations. We found that all geometric configurations except the 3+1D case are obstructed. By obstructed, we mean that the solution to the entropy maximization problem, $w(t,\vec{x})$, does not satisfy the non-negativity requirement of its interpre-

tation as an information density (i.e. this would entail negative probabilities, non-real probabilities, or other pathological behaviour).

Dimensions	Does a solution exist?	
$\mathrm{Cl}(0)$	Obstructed (Undefinable)	(91)
$\mathrm{Cl}(0,1)$	Obstructed (Undefinable)	(92)
$\mathrm{Cl}(1,0)$	Obstructed (Undefinable)	(93)
$\mathrm{Cl}(2,0)$	Obstructed (Undefinable)	(94)
$\mathrm{Cl}(1,1)$	Obstructed (Negative probabilities)	(95)
$\mathrm{Cl}(0,2)$	Obstructed (Non-real probabilities)	(96)
$\mathrm{Cl}(3,0)$	Obstructed (Non-real probabilities)	(97)
$\mathrm{Cl}(2,1)$	Obstructed (Ill-defined probabilities)	(98)
$\mathrm{Cl}(1,2)$	Obstructed (Non-real probabilities)	(99)
$\mathrm{Cl}(0,3)$	Obstructed (Ill-defined probabilities)	(100)
$\mathrm{Cl}(4,0)$	Obstructed (Non-real probabilities)	(101)
$\mathrm{Cl}(3,1)$ / spacetime	**Gravity + Yang-Mills**	(102)
$\mathrm{Cl}(2,2)$	Obstructed (Undefinable)	(103)
$\mathrm{Cl}(1,3)$	Obstructed (Non-real probabilities)	(104)
$\mathrm{Cl}(0,4)$	Obstructed (Non-real probabilities)	(105)
$\mathrm{Cl}(5,0)$	Obstructed (Ill-defined probabilities)	(106)
$\vdots$	$\vdots$	
$\mathrm{Cl}(6,0)$	Suspected Obstructed (No observables)	(107)
$\vdots$	$\vdots$	

Let us now demonstrate the obstructions mentioned above.

Theorem 6 (Ill-defined probabilities). *These Clifford algebras are isomorphic to direct sums of matrix algebras, rather than a single matrix algebra. Consequently, the determinant operation, as required by the solution form $w = \det(\exp(-tH))w_0$, is ill-defined or inapplicable in this context, making these algebras unsuitable.*

Proof. These Clifford algebras are classified as follows:

$$\mathrm{Cl}(2,1) \cong \mathbb{M}_2^2(\mathbb{R}) \tag{108}$$

$$\mathrm{Cl}(0,3) \cong \mathbb{H}^2 \tag{109}$$

$$\mathrm{Cl}(5,0) \cong \mathbb{M}_2^2(\mathbb{H}) \tag{110}$$

The notion of determinant is ill-defined as we are dealing with a direct sum of matrices instead of a singular matrix. □

Theorem 7 (Non-real probabilities). *The quantity $\det(\exp(-tH))$ resulting from the optimization procedure, when evaluated for the matrix representations of the Clifford algebras in this category, is either complex-valued or quaternion-valued, making them unsuitable.*

Proof. These Clifford algebras are classified as follows:

$$Cl(0,2) \cong \mathbb{H} \tag{111}$$
$$Cl(3,0) \cong \mathbb{M}_2(\mathbb{C}) \tag{112}$$
$$Cl(1,2) \cong \mathbb{M}_2(\mathbb{C}) \tag{113}$$
$$Cl(4,0) \cong \mathbb{M}_2(\mathbb{H}) \tag{114}$$
$$Cl(1,3) \cong \mathbb{M}_2(\mathbb{H}) \tag{115}$$
$$Cl(0,4) \cong \mathbb{M}_2(\mathbb{H}) \tag{116}$$

Evaluating the function $\det(\exp(-tH))$ derived from the entropy maximization procedure for operators H associated with these algebras yields values in $\mathbb{C}$ (for Clifford algebras isomorphic to $\mathbb{M}_2(\mathbb{C})$) or $\mathbb{H}$ (for Clifford algebras isomorphic to $\mathbb{H}$ or $\mathbb{M}_2(\mathbb{H})$). Since w must be real and non-negative, these are obstructed. $\square$

Theorem 8 (Negative probabilities). *The even sub-algebra of this dimensional configuration allows for negative probabilities, making it unsuitable.*

Proof. This category contains one dimensional configuration:
$Cl(1,1)$ Let $\psi = a + be_0e_1$, then:

$$\det(a + be_0e_1) = (a + be_0e_1)^{\ddagger}(a + be_0e_1) = (a - be_0e_1)(a + be_0e_1) = a^2 - b^2 e_0e_1e_0e_1 = a^2 - b^2 \tag{117}$$

which is valued in $\mathbb{R}$.

In this case the probability can be negative. $\square$

Theorem 9 (Non-definability). *The optimization problem is not definable for these dimensional configurations.*

Proof. This category contains five dimensional configurations:

$Cl(0)$: Definition 1 requires 1 time parameter for the Lagrange multiplier, and at least 1 space parameter for the integration measure. This configuration has neither.

$Cl(0,1)$: Definition 1 requires 1 time parameter for the Lagrange multiplier, and at least 1 space parameter for the integration measure. This configuration has the time parameter, but lacks a space parameter.

$Cl(1,0)$: Definition 1 requires 1 time parameter for the Lagrange multiplier, and at least 1 space parameter for the integration measure. This configuration has the space parameter, but lacks a time parameter.

$Cl(2,0)$: Definition 1 requires 1 time parameter for the Lagrange multiplier, and at least 1 space parameter for the integration measurement. This configuration has two space parameters, but lacks the time parameter.

$Cl(2,2)$: Definition 1 requires 1 time parameter for the Lagrange multiplier, and at least 1 space parameter for the integration measurement. This configuration has two space parameters, but has more time parameters than Lagrange multipliers.

$\square$

Conjecture 1 (No observables (6D)). *The multivector representation of the norm in 6D restricts observables to the identity.*

Argument. In six dimensions and above, the self-product patterns found in Definition 6 collapse. The research by Acus et al.[7] in 6D Clifford algebra concludes that the determinant, so far defined through

a self-products of the multivector, fails to extend into 6D. The crux of the difficulty is evident in the reduced case of a 6D multivector containing only scalar and grade-4 elements:

$$s(B) = b_1 B f_5(f_4(B)f_3(f_2(B)f_1(B))) + b_2 B g_5(g_4(B)g_3(g_2(B)g_1(B))) \tag{118}$$

This equation is not a multivector self-product but a linear sum of two multivector self-products[7].

The full expression is given in the form of a system of 4 equations, which is too long to list in its entirety. A small characteristic part is shown:

$$a_0^4 - 2a_0^2 a_{47}^2 + b_2 a_0^2 a_{47}^2 p_{412} p_{422} + \langle 72 \text{ monomials} \rangle = 0 \tag{119}$$

$$b_1 a_0^3 a_{52} + 2b_2 a_0 a_{47}^2 a_{52} p_{412} p_{422} p_{432} p_{442} p_{452} + \langle 72 \text{ monomials} \rangle = 0 \tag{120}$$

$$\langle 74 \text{ monomials} \rangle = 0 \tag{121}$$

$$\langle 74 \text{ monomials} \rangle = 0 \tag{122}$$

From Equation 118, it is possible to see that no observable **O** can satisfy this equation because the linear combination does not allow one to factor it out of the equation.

$$b_1 \mathbf{O} B f_5(f_4(B)f_3(f_2(B)f_1(B))) + b_2 B g_5(g_4(B)g_3(g_2(B)g_1(B))) = b_1 B f_5(f_4(B)f_3(f_2(B)f_1(B))) + b_2 \mathbf{O} B g_5(g_4(B)g_3(g_2(B)g_1(B))) \tag{123}$$

Any equality of the above type between $b_1 \mathbf{O}$ and $b_2 \mathbf{O}$ is frustrated by the factors b_1 and b_2, forcing $\mathbf{O} = 1$ as the only satisfying observable. Since the obstruction occurs within grade-4, which is part of the even sub-algebra it is questionable that a satisfactory theory (with non-trivial observables) is constructible in 6D, using our method. □

This conjecture proposes that the multivector representation of the determinant in 6D does not allow for the construction of non-trivial observables, which is a crucial requirement for a relevant quantum formalism. The linear combination of multivector self-products in the 6D expression prevents the factorization of observables, limiting their role to the identity operator.

Conjecture 2 (No observables (above 6D))**.** *The norms beyond 6D are progressively more complex than the 6D case, which is already obstructed.*

These theorems and conjectures provide additional insights into the unique role of the unobstructed 3+1D signature in our proposal.

We also note that it is interesting that our proposal is able to rule out $\text{Cl}(1,3)$ even if in relativity, the signature of the metric $(+,-,-,-)$ versus $(-,-,-,+)$ does not influence the physics. However, in Clifford algebra, $\text{Cl}(1,3)$ represents 1 space dimension and 3 time dimensions. Therefore, it is not the signature itself that is ruled out but rather the specific arrangement of 3 time and 1 space dimensions, as this configuration yields quaternion-valued "probabilities" (i.e. $\text{Cl}(1,3) \cong \mathbb{M}_2(\mathbb{H})$ and $\det \mathbb{M}_2(\mathbb{H}) \in \mathbb{H}$).

2.3. Gravity + Yang-Mills

So far we have recovered standard results in the form of spinors and the Dirac equation, yielding the Dirac current as the conserved current, and shown that it works (only) in 3+1D. In this section we will investigate extra structures that are made available by the quartic multivector determinant form that are not available with the Dirac theory alone.

2.3.1. Rationale

The nature of our hypothesis rests on the fact that the optimization problem replaces the Born rule with a quartic form acting on $\phi(t, \vec{x})$, thus linking spinors to non-negative scalars. Indeed, the measure

$$w(t) = \det \phi(t) \tag{124}$$

$$= \det(\exp(-tH)\phi(0)) \tag{125}$$

$$= ((\underbrace{\exp(-tH)}_{\#1}\phi(0))^{\ddagger}(\underbrace{\exp(-tH)}_{\#2}\phi(0)))^{\dagger}(\underbrace{\exp(-tH)}_{\#3}\phi(0))^{\ddagger}(\underbrace{\exp(-tH)}_{\#4}\phi(0)) \tag{126}$$

requires a quartic application ($\#1, \#2, \#3$ and $\#4$) of the evolution operator to produce a time evolution, when compared to the Born rule (which only requires a double application):

$$\rho(t) = (\underbrace{U(t)}_{\#1}\psi(0))^{\dagger}\underbrace{U(t)}_{\#2}\psi(o) \tag{127}$$

The kinetic energy density associated with the quartic form is quadratic in D compared to the Born rule in which it is linear in D:

$$\psi^{\dagger}D\psi \qquad \text{bilinear form kinetic energy (e.g. Born rule)} \qquad \rightarrow D \tag{128}$$

$$\text{vs. } \phi^{\ddagger}D(\phi\phi^{\ddagger}D\phi) \qquad \text{quartic form kinetic energy} \qquad \rightarrow D^2 \tag{129}$$

Consequently, the equation of motion of the measure $w(t)$ are non-linear (quadratic in D); whereas the equation of motion of the base field are nonetheless still linear in D (i.e the Dirac equation).

The second significant observation is that the quartic form is invariant with respect to transformations of the Spin(3,1) and the Spinc(3,1) group, extending unitary transformations that leave the Born rule invariant as a bilinear form.

We intend to show that this quartic form automatically leads to general relativity (acting on spacetime) and Yang-Mills (acting on internal spaces).

2.3.2. Notation

We will utilize the David Hestenes form of the wavefunction $\psi = \sqrt{\rho}Re^{-Ib/2}$, however since our form is a non-normalized field, we will change the symbols to:

$$\phi := \sqrt{\chi}Re^{-I\theta/2} \tag{130}$$

We recall that the quartic form is as follows:

$$\det \phi \equiv (\phi^{\ddagger}\phi)^{\dagger}\phi^{\ddagger}\phi \tag{131}$$

and that $\det \phi = (\phi^{\ddagger}\phi)^{\dagger}\phi^{\ddagger}\phi = \chi^2 R^{\ddagger}RR^{\ddagger}Re^{-I\theta/2}e^{-I\theta/2}e^{I\theta/2}e^{I\theta/2} = \chi^2 = w$.

2.3.3. Geometry

Theorem 10 (Metric Tensor Density). *Here we utilize the quartic form to define a measurement of the metric tensor:*

$$\frac{1}{4}\operatorname{tr}\left(\overbrace{(\phi^{\ddagger}\gamma_\mu\phi)}^{\text{copy 1}}{}^{\dagger}\overbrace{\phi^{\ddagger}\gamma_\nu\phi}^{\text{copy 2}}\right) = wg_{\mu\nu} \tag{132}$$

Proof.

$$\frac{1}{4}\operatorname{tr}\left((\phi^{\ddagger}\gamma_\mu\phi)^{\dagger}\phi^{\ddagger}\gamma_\nu\phi\right) = \frac{1}{4}\operatorname{tr}(\sqrt{\chi}R^{\ddagger}e^{-I\theta/2}\gamma_\mu\sqrt{\chi}e^{-I\theta/2}R)^{\dagger}\sqrt{\chi}R^{\ddagger}e^{-I\theta/2}\gamma_\nu\sqrt{\chi}Re^{-I\theta/2} \tag{133}$$

$$= \frac{1}{4}\chi^2 \operatorname{tr}(R^{\ddagger}e^{-I\theta/2}\gamma_\mu e^{-I\theta/2}R)^{\dagger}R^{\ddagger}e^{-I\theta/2}\gamma_\nu Re^{-I\theta/2} \tag{134}$$

We now note that $e^{-I\theta/2}$ anti-commutes with γ_μ, which implies that $e^{-I\theta/2}\gamma_\mu e^{-I\theta/2} = e^{-I\theta/2}e^{I\theta/2}\gamma_\mu = \gamma_\mu$:

$$= \frac{1}{4}\chi^2 \operatorname{tr}(R^\ddagger \gamma_\mu R)^\dagger R^\ddagger \gamma_\nu R \tag{135}$$

$$= \frac{1}{4}\chi^2 \operatorname{tr}(\gamma'_\mu)^\dagger \gamma'_\nu \tag{136}$$

$$= \frac{1}{4}\chi^2 \operatorname{tr} \gamma'_\mu \gamma'_\nu \tag{137}$$

$$= \chi^2 g_{\mu\nu} \tag{138}$$

$$= w g_{\mu\nu} \tag{139}$$

$\square$

2.3.4. Action

Definition 9 (Action Density Measurement (Special Case)). *We utilize the quartic form to define a measurement of the special case of the action density:*

$$C \operatorname{tr}\left(\frac{\overbrace{\phi^\ddagger D(\phi}^{\textit{copy 1}} \overbrace{\phi^\ddagger D\phi)}^{\textit{copy 2}}}{\phi^\ddagger \phi \phi^\ddagger \phi} \right) = \mathcal{L} \tag{140}$$

where C is a dimensional constant. Since $w \geq 0$ (and consequently $\chi \geq 0$), we can define the $0/0$ indeterminate case where $\phi^\ddagger \phi \phi^\ddagger \phi = \chi^2 = 0$ as the limit: $\lim_{\chi \to 0^+} \mathcal{L}$ which is finite due to the ratio χ^2/χ^2 been defined in this limit, via l'Hospital's rule.

Definition 10 (Action Density Measurement (General Case)). *We utilize the quartic form to define a measurement of the general case of the action density:*

$$C' \operatorname{tr} f\left(\frac{1}{\Lambda^2} \frac{\overbrace{\phi^\ddagger D(\phi}^{\textit{copy 1}} \overbrace{\phi^\ddagger D\phi)}^{\textit{copy 2}}}{\phi^\ddagger \phi \phi^\ddagger \phi} \right) = \mathcal{L} \tag{141}$$

where C' is a dimensional constant, where f is a smooth function and where $1/\Lambda^2$ is used to nullify the dimensions of the input to f. Since $w \geq 0$ (and consequently $\chi \geq 0$), we can define the $0/0$ indeterminate case where $\phi^\ddagger \phi \phi^\ddagger \phi = 0$ as the limit: $\lim_{\chi \to 0^+} \mathcal{L}$ which is finite due to the ratio χ^2/χ^2 been defined in this limit, via l'Hospital's rule.

This action density quantifies the dynamics of the self-interaction of ϕ due to the quartic form. In this section, we will use the special case yielding general relativity, leaving the general case for the next section on Yang-Mills.

2.3.5. Dynamics

Theorem 11 (Dirac Equation). *Varying the action yields the Dirac equation as a sufficient (but not necessary) equation of motion.*

Proof.

$$\delta \int_\mathcal{M} \operatorname{tr}\left(\frac{\phi^\ddagger D(\phi \phi^\ddagger D\phi)}{\phi^\ddagger \phi \phi^\ddagger \phi} \right) \sqrt{-|g|} \, \mathrm{d}^4 x = 0 \tag{142}$$

By inspection we can see that the right most term of the numerator is the massless Dirac equation $D\phi$, which if equal to 0 will satisfy the equation of motion. $\square$

This theorem implies that the action is a generalization of the massless Dirac equation — i.e. all solutions to the Dirac equation satisfy the equation of motion, but not vice-versa. We will now investigate the additional solutions in more details.

Theorem 12 (Quantum Action). *Let us investigate a special case where $\phi = \sqrt{\chi}$. Due to its non-linearity, the kinetic energy produces a quantum potential in addition to a kinetic energy term:*

$$\text{tr}\left(\frac{\sqrt{\chi}D(\sqrt{\chi}\sqrt{\chi}D\sqrt{\chi})}{\chi^2}\right) = \underbrace{\frac{2(\partial\chi)^2}{\chi^2}}_{Quantum\ Kinetics} - \underbrace{\left(\frac{(\partial\chi)^2}{\chi^2} - \frac{2\Box\chi}{\chi}\right)}_{Quantum\ Potential} \tag{143}$$

The quantum potential herein described is the relativistic version of the quantum potential found in the Bohm-Broglie reformulation of QM, whereas the quantum kinetics can be understood as the kinetic term of a relativistic diffusion process. When integrated, they define a quantity that we refer to as the quantum action:

$$S = \underbrace{\int_{\mathcal{M}} \left(\frac{(\partial\chi)^2}{\chi^2} + \frac{2\Box\chi}{\chi}\right)\sqrt{-|g|}\,\mathrm{d}^4x}_{Quantum\ Action} \tag{144}$$

Proof.

$$\text{tr}\left(\frac{\sqrt{\chi}D(\sqrt{\chi}\sqrt{\chi}D\sqrt{\chi})}{\chi^2}\right) \tag{145}$$

$$= \text{tr}\left(\chi^{-2}\sqrt{\chi}\partial(\chi\partial\sqrt{\chi})\right) \tag{146}$$

$$= \text{tr}\left(\chi^{-2}\sqrt{\chi}(\partial\chi)\partial\sqrt{\chi}\right) + \text{tr}\left(\chi^{-2}\sqrt{\chi}\chi\partial\partial\sqrt{\chi}\right) \tag{147}$$

$$= \text{tr}\left(\frac{1}{2}\chi^{-2}\sqrt{\chi}\frac{1}{\sqrt{\chi}}(\partial\chi)\partial\chi\right) + \text{tr}\left(\chi^{-1}\sqrt{\chi}\partial\frac{1}{2\sqrt{\chi}}\partial\chi\right) \tag{148}$$

$$= \frac{2(\partial\chi)^2}{\chi^2} + \text{tr}\left(\chi^{-1}\sqrt{\chi}\gamma^\mu\gamma^\nu\partial_\mu(\frac{1}{2\sqrt{\chi}})\partial_\nu\chi\right) + \text{tr}\left(\chi^{-1}\sqrt{\chi}\frac{1}{2\sqrt{\chi}}\gamma^\mu\gamma^\nu\partial_\mu\partial_\nu\chi\right) \tag{149}$$

$$= \frac{2(\partial\chi)^2}{\chi^2} + \text{tr}\left(\chi^{-1}\sqrt{\chi}(\frac{-1}{4\chi^{3/2}})\gamma^\mu\gamma^\nu\partial_\mu\chi\partial_\nu\chi\right) + \text{tr}\left(\chi^{-1}\sqrt{\chi}\frac{1}{2\sqrt{\chi}}\gamma^\mu\gamma^\nu\partial_\mu\partial_\nu\chi\right) \tag{150}$$

$$= \frac{2(\partial\chi)^2}{\chi^2} + \text{tr}\left(\chi^{-1}\sqrt{\chi}(\frac{-1}{4\chi^{3/2}})\gamma^\mu\gamma^\nu\partial_\mu\chi\partial_\nu\chi\right) + \text{tr}\left(\chi^{-1}\frac{1}{2}\gamma^\mu\gamma^\nu\partial_\mu\partial_\nu\chi\right) \tag{151}$$

$$= \frac{2(\partial\chi)^2}{\chi^2} - \frac{1}{4}\text{tr}\left(\chi^{-2}\gamma^\mu\gamma^\nu\partial_\mu\chi\partial_\nu\chi\right) + \frac{2\Box\chi}{\chi} \tag{152}$$

$$= \frac{2(\partial\chi)^2}{\chi^2} - \frac{(\partial\chi)^2}{\chi^2} + \frac{2\Box\chi}{\chi} \tag{153}$$

$$= \frac{(\partial\chi)^2}{\chi^2} + \frac{2\Box\chi}{\chi} \tag{154}$$

$\square$

Theorem 13 (Equation of Motion). *Varying the quantum action:*

$$S = \int_{\mathcal{M}} \left(\frac{(\partial\chi)^2}{\chi^2} + \frac{2\Box\chi}{\chi}\right)\sqrt{-|g|}\mathrm{d}^4x \tag{155}$$

produces:

$$\partial^2\chi = \chi\Box\chi \tag{156}$$

as the equation of motion.

Proof.

$$\delta\left(\frac{(\partial\chi)^2}{\chi^2} + \frac{2\partial^2\chi}{\chi}\right) = 0 \tag{157}$$

$$\implies -\frac{2(\partial\chi)^2}{\chi^3}\delta\chi - \partial_\mu\left(\frac{2\partial^\mu\chi}{\chi^2}\right)\delta\chi + \frac{2\partial^2(\delta\chi)}{\chi} - \frac{2\partial^2\chi}{\chi^2}\delta\chi = 0 \tag{158}$$

$$\implies -\frac{2(\partial\chi)^2}{\chi^3}\delta\chi + \frac{4(\partial\chi)^2}{\chi^3}\delta\chi - \frac{2\partial^2\chi}{\chi^2}\delta\chi + \frac{2\partial^2(\delta\chi)}{\chi} - \frac{2\partial^2\chi}{\chi^2}\delta\chi = 0 \tag{159}$$

$$\implies \frac{2(\partial\chi)^2}{\chi^3}\delta\chi - \frac{4\partial^2\chi}{\chi^2}\delta\chi + \frac{2\partial^2(\delta\chi)}{\chi} = 0 \tag{160}$$

To proceed further, we are required to do integration by part for the last term:

$$\int \frac{2\partial^2(\delta\chi)}{\chi}\mathrm{d}^4x = \int \frac{2(\partial\chi)\partial(\delta\chi)}{\chi^2}\mathrm{d}^4x \tag{161}$$

Then, a second integration by part, yields:

$$\int \frac{2(\partial\chi)\partial(\delta\chi)}{\chi^2}\mathrm{d}^4x = -\int \delta\chi\left(\frac{2\partial^2\chi}{\chi^2} - \frac{4(\partial\chi)^2}{\chi^3}\right)\mathrm{d}^4x \tag{162}$$

$$\implies \frac{2(\partial\chi)^2}{\chi^3}\delta\chi - \frac{4\partial^2\chi}{\chi^2}\delta\chi - \frac{2\partial^2\chi}{\chi^2}\delta\chi + \frac{4(\partial\chi)^2}{\chi^3}\delta\chi = 0 \tag{163}$$

$$\implies \frac{6(\partial\chi)^2}{\chi^3}\delta\chi - \frac{6\partial^2\chi}{\chi^2}\delta\chi = 0 \tag{164}$$

$$\implies \partial^2\chi = \frac{(\partial\chi)^2}{\chi} \tag{165}$$

$$\implies \chi\Box\chi = (\partial\chi)^2 \tag{166}$$

□

To interpret this equation of motion, let us now introduce the surprisal field and associated definitions.

Definition 11 (Surprisal Field). *We define a change of variable:*

$$\varphi := -\ln\chi$$

We call φ the surprisal field.

Definition 12 (Surprisal Equation of Motion). *We note that the change of variable $\varphi = -\ln\chi$, changes the equation of motion as follows:*

$$\chi\Box\chi = (\partial\chi)^2 \underbrace{\to}_{\varphi=-\ln\chi} \Box\varphi = 0$$

which is the Klein-Gordon equation in curved spacetime, applied to the surprisal field.

Definition 13 (Surprisal Conservation). *The following current:*

$$\nabla_\mu(\partial^\mu\varphi) = 0$$

identifies the surprisal current as the conserved current of this action.

Definition 14 (Surprisal Expectation Value)**.** *The surprisal expectation value is merely the entropy of a region* Σ_t *of the manifold:*

$$\underbrace{\langle -\ln\chi\rangle}_{\text{expectation value}} := \underbrace{\int_{\Sigma_t} \chi(x)\underbrace{(-\ln\chi(x))}_{\text{observable}}\sqrt{h}\,\mathrm{d}^3x}_{\text{definition of entropy}}$$

Interpretation:

In information theory, the **surprisal** of an event x with probability density $\rho(x)$ is defined as $-\ln\rho(x)$, and the entropy $S = -\int\rho\ln\rho\,\mathrm{d}x$ represents its expectation value. As the unit of surprisal is the bit, it represents the quantity of information associated to the event—here, it is conserved by $\Box\varphi = 0$. In contrast, also in information theory, the units of entropy are the bits per symbol—here, it is not conserved. The type of units allows us to intuit why the former is conserved and the latter isn't.

In our framework, the field χ replaces ρ—it has most of its properties, but differs critically as follows:

- χ is **not a probability density**—it lacks a conserved current $(\nabla_\mu(\chi u^\mu) \neq 0)$ and is not normalized—but it is non-negative.
- Instead, χ is interpreted as an **information density**, encoding spacetime's local information content.

The **surprisal** is defined as $\varphi = -\ln\chi$, which in this theory satisfies the Klein-Gordon equation $\Box\varphi = 0$. This ensures:

1. **Conservation**: The current $j^\mu = \partial^\mu\varphi$ is conserved $(\nabla_\mu j^\mu = 0)$, making $Q = \int_{\Sigma_t} j^\mu\sqrt{h}\,\mathrm{d}^3x$ a conserved charge.
2. **Causal Propagation**: Surprisal propagates at light speed, enforcing that bits of information cannot spread superluminally—a core tenet of relativity.

Before we continue with the interpretation, let us introduce another theorem, where we now assume $\phi = \sqrt{\rho}Re^{-I\theta/2}$.

Theorem 14 (Einstein-Hilbert Action Density)**.**

$$\mathrm{tr}\left(\frac{\phi^\ddagger D(\phi\phi^\ddagger D\phi)}{\phi^\ddagger\phi\phi^\ddagger\phi}\right) = \mathcal{R} + \mathcal{L}_{matter} \tag{167}$$

where $\phi = \sqrt{\chi}Re^{-I\theta/2}$.

Proof.

$$\mathrm{tr}\left(\frac{\phi^\ddagger D(\phi\phi^\ddagger D\phi)}{\phi^\ddagger\phi\phi^\ddagger\phi}\right) \tag{168}$$

Since $R^{\ddagger}R = 1$, we note that $\phi^{\ddagger}\phi = (\sqrt{\chi}R^{\ddagger}e^{-I\theta/2})(\sqrt{\chi}Re^{-I\theta/2}) = \chi e^{-I\theta}$. Finally, $\phi^{\ddagger}\phi\phi^{\ddagger}\phi = \chi^2 e^{-2I\theta}$.

$$= \text{tr}(\chi^{-2}e^{2I\theta}\phi^{\ddagger}D(\phi\phi^{\ddagger}D\phi)) \tag{169}$$

$$= \text{tr}(\chi^{-2}e^{2I\theta}\phi^{\ddagger}D(\phi\phi^{\ddagger})D\phi) + \text{tr}(\chi^{-2}e^{2I\theta}\phi^{\ddagger}\phi\phi^{\ddagger}D^2\phi) \tag{170}$$

$$= \text{tr}(\chi^{-2}e^{2I\theta}\phi^{\ddagger}D(\phi\phi^{\ddagger})D\phi) + \text{tr}(\chi^{-2}e^{2I\theta}\phi^{\ddagger}\phi\phi^{\ddagger}(\nabla^*\nabla + \frac{1}{4}\mathcal{R})\phi) \tag{171}$$

$$= \underbrace{\text{tr}(\chi^{-2}e^{2I\theta}\phi^{\ddagger}D(\phi\phi^{\ddagger})D\phi) + \text{tr}(\chi^{-2}e^{2I\theta}\phi^{\ddagger}\phi\phi^{\ddagger}\nabla^*\nabla\phi)}_{\mathcal{L}_{\text{matter}}} + \text{tr}(\chi^{-2}e^{2I\theta}\phi^{\ddagger}\phi\phi^{\ddagger}\frac{1}{4}\mathcal{R}\phi) \tag{172}$$

$$= \mathcal{L}_{\text{matter}} + \text{tr}(\chi^{-2}e^{2I\theta}\sqrt{\chi}R^{\ddagger}e^{-I\theta/2}\sqrt{\chi}Re^{-I\theta/2}\sqrt{\chi}R^{\ddagger}e^{-I\theta/2}\sqrt{\chi}Re^{-I\theta/2}\frac{1}{4}\mathcal{R}) \tag{173}$$

$$= \mathcal{L}_{\text{matter}} + \text{tr}(\frac{1}{4}\mathcal{R}) \tag{174}$$

$$= \mathcal{R} + \mathcal{L}_{\text{matter}} \qquad \text{(the trace over a 4x4 matrix produces a factor of 4)} \tag{175}$$

$\square$

We note some of the equations of motion:

1. Varying with respect to $g_{\mu\nu}$ yields the EFE with the Einstein tensor from $\mathcal{R}$, and is sourced by the quantum action variation yielding the stress-energy tensor.
2. Varying with respect to χ gives equations of motion that define the flow of information quantity $\varphi = -\ln\chi$ in spacetime.

Interpretation (cont'd):

Thus, while quantum mechanics relies on probabilistic amplitudes ψ, our formulation recasts general relativity as a **theory of information dynamics**, where spacetime geometry and surprisal flux are dual aspects of $\mathcal{R}$ and χ. The distribution of surprisal in spacetime dictates its geometric structure, which in turn dictates how it propagates. General relativity is to information, what quantum mechanics is to probability.

Revisiting General Relativity with this perspective shows that the natural constraint is sufficient to entail the theory through the principle of entropy maximization—in this formulation, the speed of light as a limit on the propagation of the quantity of information (via the surprisal obeying the Klein-Gordon equation), and even the Einstein field equations are not fundamental, but emerge as the solution to an optimization problem on entropy.

2.3.6. Yang-Mills

In QFT, the standard method to identify a local gauge symmetry is to start with a global symmetry of the action or probability measure and then localize it by introducing gauge fields. For example, the $U(1)$ gauge symmetry arises naturally in electromagnetism as the group preserving the probability density (Born rule) under local phase transformations. However, the non-Abelian $SU(2)$ and $SU(3)$ gauge symmetries of the Standard Model are not derived from first principles in this way; rather their inclusion is empirically motivated by particle physics experiments.

Improvement via Multivector Determinant Formulation: Our framework demonstrates that Yang-Mills theories emerge naturally from constraints on the wavefunction's probability measure and Dirac current. Specifically:

1. **Probability Measure:** The quartic form $(\phi^{\ddagger}\phi)^{\dagger}\phi^{\ddagger}\phi = \chi^2$ enforces rotor invariance $\phi \to R\phi$, restricting transformations to those satisfying $R^{\ddagger}R = 1$, for some rotor R of a Clifford algebra of n dimensions:

$$(\phi^{\ddagger}R^{\ddagger}R\phi)^{\dagger}\phi^{\ddagger}R^{\ddagger}R\phi = (\phi^{\ddagger}\phi)^{\dagger}\phi^{\ddagger}\psi \implies R^{\ddagger}R = 1. \tag{176}$$

Solutions to $R^{\ddagger}R = 1$ are rotor transformations generated by bivectors in the Clifford algebra. For a $2n$-dimensional algebra, these generate $\text{Spin}(2n)$, whose subgroups include $SU(n)$.

2. **Internal Space:** For the gauge transformation $\phi \to R(x)\phi$ to represent a purely internal symmetry that does not mix spacetime components defined by the γ^μ basis (specifically preserving the time direction γ^0), the generators f_i must commute with γ^0, i.e., $[f_i, \gamma^0] = 0$. This ensures the transformation acts orthogonally to the spacetime structure.

3. **Spacetime:** The origin of the multivector determinant from STA, parametrize the resulting internal space in spacetime.

These constraints limit the allowable symmetry to groups generated by bivector exponentials (which are compact Lie groups), and acting on the internal spaces of spacetime. Since $SU(n) \subset$ Spin($2n$), this framework inherently includes the Standard Model gauge symmetries of $SU(3) \times SU(2) \times U(1)$ within its landscape but also generalizes to larger symmetries such as those found in condensed matter systems with emergent $SU(n)$ symmetries.

Total wavefunction:

The total wavefunction is a tensor product of spacetime (STA) and internal space components:

1. For $SU(n)$ Yang-Mills:

$$\phi_{\text{STA}} \otimes \phi_{\mathbb{C}^n}. \tag{177}$$

2. For the Standard Model $SU(3) \times SU(2) \times U(1)$:

$$\phi_{\text{STA}} \otimes \phi_{\mathbb{C}} \otimes \phi_{\mathbb{C}^2} \otimes \phi_{\mathbb{C}^3}. \tag{178}$$

Covariant Derivative (Ex. SU(n)): The covariant derivative represents the set of all transformations that can be performed on the wavefunction. For SU(n) it is:

$$D_\mu := \partial_\mu + \frac{1}{2}\omega_\mu^{ab}\gamma_{ab} + ig_n X N_\mu \tag{179}$$

Covariant Derivative (Ex. Standard Model):

Now taking the Standard Model as an example, the covariant derivative in the language of non-commutative geometry (see A. H. Chamseddine and Alain Connes [**?**]) incorporates spacetime curvature, gauge fields and Higgs fields:

$$D_\mu := \begin{pmatrix} \partial_\mu + \frac{1}{2}\omega_\mu^{ab}\gamma_{ab} + ig'YB_\mu + ig\frac{\sigma^a}{2}W_\mu^a + ig_s\frac{\lambda^a}{2}G_\mu^a & \Phi \\ \Phi^\dagger & \partial_\mu + \frac{1}{2}\omega_\mu^{ab}\gamma_{ab} + ig'YB_\mu + ig_s\frac{\lambda^a}{2}G_\mu^a \end{pmatrix}, \tag{180}$$

where:

1. γ_{ab}: Generators of Spin$(3,1)$.
2. B_μ, W_μ^a, G_μ^a: $U(1)$, $SU(2)$, and $SU(3)$ gauge fields.
3. Φ: Higgs field (SU(2) doublet).

It acts on the left/right split of the wavefunction $\phi = (\phi_L, \phi_R)$.

Action:

We recall that the general form of the action is:

$$S = \int_{\mathcal{M}} C' \, \text{tr} \, f\left(\frac{1}{\Lambda^2} \frac{\phi^\ddagger D(\phi\phi^\ddagger D\phi)}{\phi^\ddagger\phi\phi^\ddagger\phi}\right)\sqrt{-|g|} \, \mathrm{d}^4 x \tag{181}$$

Expanding f via a power series expansion yields the Standard Model and gravity field strengths, from the crucial term $\text{tr}(\chi^{-1}e^{I\theta}\phi^\ddagger DD\phi)$ of Equation 171. The invariants recovered are:

1. **Leading Terms**:
   (a) Cosmological constant: $\propto \Lambda^4 \int_{\mathcal{M}} \sqrt{-|g|} \, \mathrm{d}^4 x$.
   (b) Einstein-Hilbert term: $\propto \Lambda^2 \int_{\mathcal{M}} R\sqrt{-|g|} \, \mathrm{d}^4 x$.

2. **Yang-Mills and Higgs**:

   (a)   Gauge kinetic terms: $\propto \int_{\mathcal{M}} \frac{1}{4} F^a_{\mu\nu} F^{\mu\nu a} \sqrt{-|g|}\, \mathrm{d}^4 x.$

   (b)   Higgs kinetic and potential terms:

$$\propto \int_{\mathcal{M}} \left(|D_\mu \Phi|^2 + \Lambda^2 |\Phi|^2 + \frac{1}{\Lambda^2} |\Phi|^4 \right) \sqrt{-|g|}\, \mathrm{d}^4 x. \tag{182}$$

3. **Yukawa Couplings** (from matter fields):

$$\propto \int_{\mathcal{M}} y_{ij} \overline{\phi}_i \Phi \phi_j \sqrt{-|g|}\, \mathrm{d}^4 x. \tag{183}$$

To show this explicitly, let us investigate the crucial term (Equation 171). When the covariant derivative has additional gauge connections, the term is:

$$\mathrm{tr}(\chi^{-1} \phi^{\ddagger} \frac{1}{4} \mathcal{R} \phi) \to \mathrm{tr}(\chi^{-1} \phi^{\ddagger} (\frac{1}{4} \mathcal{R} + F^a_{\mu\nu}) \phi) \tag{184}$$

The first power of the series expansion would reduce to $\mathcal{R}$ as before. But the second order series expansion would contain a term of this type:

$$\mathrm{tr}(\chi^{-1} \phi^{\ddagger} (\frac{1}{4} \mathcal{R} + F^a_{\mu\nu}) \phi)^2 \tag{185}$$

which includes $\mathcal{R}^2$ and F^2.

For the Yukawa couplings, their presence is found from the Dirac bilinear $\phi^{\ddagger} D \phi$ which is sufficient (but not necessary) to satisfy the equation of motion. Specifically, the cross-terms of D combine to produce the terms:

$$\begin{bmatrix} \phi_L & \phi_R \end{bmatrix} \underbrace{\begin{pmatrix} D_L & \Phi \\ \Phi^{\dagger} & D_R \end{pmatrix}}_{D} \begin{bmatrix} \phi_L \\ \phi_R \end{bmatrix} = \phi_L D_L \phi_L + \underbrace{\phi_L \Phi \phi_R + \phi_R \Phi^{\dagger} \phi_L}_{\text{Yukawa terms}} + \phi_R D_R \phi_R \tag{186}$$

where

1. $D_L := \partial_\mu + \frac{1}{2} \omega^{ab}_\mu \gamma_{ab} + ig' Y B_\mu + ig \frac{\sigma^a}{2} W^a_\mu + ig_s \frac{\lambda^a}{2} G^a_\mu$

2. $D_R := \partial_\mu + \frac{1}{2} \omega^{ab}_\mu \gamma_{ab} + ig' Y B_\mu + ig_s \frac{\lambda^a}{2} G^a_\mu$

The Higgs field comes from second order expansion:

$$\mathrm{tr}\begin{pmatrix} D_L & \Phi \\ \Phi^{\dagger} & D_R \end{pmatrix}\begin{pmatrix} D_L & \Phi \\ \Phi^{\dagger} & D_R \end{pmatrix} = \mathrm{tr}\begin{bmatrix} D_L^2 + \Phi\Phi^{\dagger} & D_L \Phi + \Phi D_R \\ \Phi^{\dagger} D_L + D_R \Phi^{\ddagger} & D_R^2 + \Phi^{\dagger}\Phi \end{bmatrix} \propto \cdots + |\Phi|^2 \tag{187}$$

and third order expansion:

$$\mathrm{tr}\begin{bmatrix} D_L^2 + \Phi\Phi^{\dagger} & D_L \Phi + \Phi D_R \\ \Phi^{\dagger} D_L + D_R \Phi^{\ddagger} & D_R^2 + \Phi^{\dagger}\Phi \end{bmatrix}\begin{bmatrix} D_L^2 + \Phi\Phi^{\dagger} & D_L \Phi + \Phi D_R \\ \Phi^{\dagger} D_L + D_R \Phi^{\ddagger} & D_R^2 + \Phi^{\dagger}\Phi \end{bmatrix} = \begin{bmatrix} \cdots + \Phi\Phi^{\dagger}\Phi\Phi^{\dagger} & \cdots \\ \cdots & \cdots + \Phi^{\dagger}\Phi\Phi^{\dagger}\Phi \end{bmatrix} \propto \cdots + |\Phi|^4 \tag{188}$$

Finally, the Higgs kinetic term $|D\Phi|^2$ is found in the cross terms of $D_R \Phi + \Phi D_L$ with $\Phi^{\dagger} D_R + D_L \Phi^{\ddagger}$, as $D_R \Phi D_L \Phi$.

Key Notes:

1. **Higher-Order Terms:** Higher order field strength terms appear but are suppressed by Λ^{-2}, making them negligible at low energies.

2. **Uniqueness:** The Standard Model is not uniquely selected by the optimization problem but resides within the landscape of allowed Yang-Mills theories.

2.3.7. Yang-Mills Axioms as Theorems

In this section, we intend to show that all 5 axioms of Yang-Mills theory are demonstrated. First, let us list the axioms:

1. **Compact Gauge Group**: The symmetry group is a compact Lie group G.
2. **Local Gauge Invariance**: Fields transform under spacetime-dependent (local) group elements $T(x) \in G$.
3. **Gauge Connections**: Gauge fields A_μ are introduced as connections in the covariant derivative $D_\mu = \partial_\mu + A_\mu$.
4. **Field Strength**: The curvature $F_{\mu\nu} = [D_\mu, D_\nu]$ defines the dynamics.
5. **Yang-Mills Action**: The action depends on $F_{\mu\nu}$, e.g., $\int_{\mathcal{M}} \text{tr}(F_{\mu\nu} F^{\mu\nu}) \sqrt{-|g|} \mathrm{d}^4 x$.

Now for the theorems.

Theorem 15 (Compact Gauge Group). *The allowed symmetries form a compact Lie group $G \subset \text{Spin}(2n)$.*

Proof. 1. **Constraint**: $(\phi^\ddagger \phi)^\dagger \phi^\ddagger \phi = \chi^2$ implies invariance of arbitrary n-dimentional rotors: $R^\ddagger R = 1$.
2. **Structure of Solutions**: Rotor transformations in finite-dimensional Clifford algebras are generated by bivectors. These generate $\text{Spin}(2n)$ and its subgroups, which are compact Lie groups. Thus, the gauge group G is inherently compact and derived from the algebra structure. □

Theorem 16 (Local Gauge Invariance). *The theory is invariant under spacetime-dependent $T(x) \in G$.*

Proof. 1. **Wavefunction Transformation**: $\phi \to R(x)\phi$, where $R(x) = e^{\theta^i(x) f_i}$ (exponentials of spacetime-dependent bivectors).
2. **Probability Measure**: $(\phi^\ddagger \phi)^\dagger \phi^\ddagger \phi \to (\phi^\ddagger R^\ddagger R \phi)^\dagger \phi^\ddagger R^\ddagger R \phi = \chi^2$.
3. **Dirac Current**: $\phi^\ddagger \gamma_0 \phi \to \phi^\ddagger R^\ddagger \gamma_0 R \phi = \phi^\ddagger \gamma_0 \phi$, since $[f_i, \gamma_0] = 0$.
□

Theorem 17 (Gauge Connections). *The covariant derivative $D_\mu = \partial_\mu + A_\mu$ emerges to maintain invariance under local $R(x)$.*

Proof. 1. **Minimal Coupling**: To preserve $D_\mu \phi \to R(x) D_\mu \phi$, the derivative must transform as $\partial_\mu \to \partial_\mu + A_\mu$, where $A_\mu = f_i A^i_\mu(x)$.
2. **Gauge Field Definition**: Let $\partial_\mu R(x) = A_\mu R(x)$, then: $D_\mu \phi = \partial_\mu \phi + A_\mu \phi \implies D_\mu(R\phi) = R D_\mu \phi$.
3. **Clifford Algebra Embedding**: The A_μ are bivector fields in $\mathcal{C}\ell(2n)$, ensuring $A_\mu \in \mathfrak{g}$ (the Lie algebra of G)).
□

Theorem 18 (Field Strength). *The commutator $F_{\mu\nu} = [D_\mu, D_\nu]$ defines the field strength.*

Proof. Kinetic Energy: As we have shown in Equation 171 and Equation 185, the action density expands to include the field strength tensor:

$$\frac{\phi^\ddagger D(\phi \phi^\ddagger D\phi)}{\phi^\ddagger \phi \phi^\ddagger \phi} = \text{kinetic terms} + F_{\mu\nu} \tag{189}$$

where $F_{\mu\nu}$ is the field strength. □

Theorem 19 (Yang-Mills Action). *The action density includes the kinetic term $\int_{\mathcal{M}} \text{tr}(F_{\mu\nu} F^{\mu\nu}) \sqrt{-|g|} \mathrm{d}^4 x$.*

Proof. Heat Kernel Expansion: As shown in Equation 185, action expands into the field strength:
$$S \sim \int_{\mathcal{M}} \left(\cdots + F_{\mu\nu}^a F^{a\mu\nu} + \cdots \right) \sqrt{-|g|}\, \mathrm{d}^4 x.$$
$\square$

Revisiting Yang-Mills with this perspective shows that the natural constraint is sufficient to entail the theory through the principle of entropy maximization—in this formulation, Yang-Mills axioms 1, 2, 3, 4, and 5 are not fundamental, but the solution to the optimization problem.

3. Discussion

When asked to define what a physical theory is, an informal answer might be that it is a set of equations that applies to all experiments realizable within a domain, with nature as a whole being the most general domain. While physicists have expressed these theories through sets of axioms, we propose a more direct approach—mathematically realizing the fundamental definition itself. This definition is realized as a constrained optimization problem (Axiom 1 and Definition 1) that can be solved directly (Theorem 1). The solution to this optimization problem yields precisely those structures that realize the physical theory over said domain. Succinctly, physics is the solution to:

$$\underbrace{\mathcal{L}[w]}_{\substack{\text{an} \\ \text{optimization} \\ \text{problem}}} := \underbrace{-\int_V w(t,\vec{x}) \ln \frac{w(t,\vec{x})}{w(0,\vec{x})}\, \mathrm{d}^p\vec{x}}_{\substack{\text{on the entropy} \\ \text{of a measurement} \\ \text{relative to its preparation}}} + t \underbrace{\left(\overline{H} - \int_V w(t,\vec{x})\, \mathrm{tr}\, H(\vec{x})\, \mathrm{d}^p\vec{x} \right)}_{\text{as constrained by nature}} \tag{190}$$

The relative Shannon entropy represents the basic structure of any experiment, quantifying the informational difference between its initial preparation and its final measurement.

The natural constraint is chosen to be the most general structure that admits a solution to this optimization problem. This generality follows from key mathematical requirements. The constraint must involve quantities that form an algebra, as the solution requires taking exponentials:

$$\exp X = 1 + X + \frac{1}{2} X^2 + \dots \tag{191}$$

which involves addition, powers, and scalar multiplication of X. The use of the trace operation further necessitates that X must be representable by square $n \times n$ matrices, yielding Axiom 1:

$$\overline{H} := \int_V w(t,\vec{x})\, \mathrm{tr}\, H(\vec{x})\, \mathrm{d}^p\vec{x} \tag{192}$$

The trace is utilized because the constraint must be a scalar for use in the Lagrange multiplier equation. Finally, the operator is selected to include the set of all transformations that can be performed on the base field, and is thus the least restrictive constraint definable within the specific geometric configuration.

These mathematical requirements demonstrate that the natural constraint, as it admits the most permissive mathematical structure required to solve an arbitrary entropy maximization problem, can be understood as the most general extension to the standard entropy maximization problem of statistical mechanics.

Thus, having established both the mathematical structure and its generality, we can understand how this minimal ontology operates. Since our formulation keeps the structure of experiments completely general, our optimization considers all possible theories for that structure, and the constraint is the most permissive constraint possible for that structure, the resulting optimal physical theory applies, by construction, to all realizable experiments within its domain.

This ontology is both operational, being grounded in the basic structure of experiments rather than abstract entities, and constructive, showing how physical laws emerge from optimization over all

possible predictive theories subject to the natural constraint. This represents a significant philosophical shift from traditional physical ontologies where laws are typically taken as primitive.

The next step in our derivation is to represent the determinant of the $n \times n$ matrices through a self-product of multivectors involving various conjugate structures. By examining the various dimensional configurations of Clifford algebras, we find that Cl(3,1), representing 4×4 real matrices, admits a sub-algebra whose determinant is non-negative for its invertible members. All other dimensional configurations fail to admit such a non-negative structure.

The solution reveals that the 3+1D case harbors a new type of field amplitude structure analogous to complex amplitudes, one that exhibits the characteristic elements of a quantum mechanical theory. Instead of complex-valued amplitudes, we have amplitudes valued in the invertible subset of the even sub-algebra of $\text{Cl}(3,1)$. When normalized, this amplitude is identical to David Hestenes' wavefunction, but comes with an extended Born rule represented by the determinant. The quartic structure of this rule automatically incorporates gravity via the $\text{Spin}(3,1)$ connection and local gauge theories as Yang-Mills theories. Specifically, the powers of the Dirac operator, automatically generated by the Lagrangian, contains the invariants of gravity and of the Yang-Mills theory, which are made explicit via a power series expansion, along with the matter fields quantifying the system's information density via surprisal and limiting its propagation speed.

3.1. Proposed Interpretation of QM

An experiment begins with a known initial preparation $w(0, \vec{x})$, evolves under a constraint (Axiom 1) and ends with a final measurement $w(t, \vec{x})$. By treating the experiment as the fundamental ontic entity, we resolve a redundancy inherent in traditional physical theories: Specifically, physics is not a set of laws that are **simultaneously** *axiomatic* and *validated by experiments* (i.e., a redundancy—that which is validated by something else is not axiomatic) but an *optimal interpolation device* connecting $w(0, \vec{x})$ to $w(t, \vec{x})$ under the constraint of nature. The experiment is fundamental, but the physical laws that are derived from it are not.

3.1.1. Demystifying the Measurement Problem

If we accept that our derivation demonstrates that QM is the optimal interpolation device that connects $w(0, \vec{x})$ to $w(t, \vec{x})$, under the constraint of nature, and we recognize that a measure-to-measure interpolation ($w(0, \vec{x})$ to $w(t, \vec{x})$) is different than a measure-to-element interpolation ($w(t, \vec{x})$ to some $\vec{x} \in \mathbb{E}$)—the latter would be required for a 'collapse' to occur—then, we must conclude that the final sampling (from $w(t, \vec{x})$ to $\vec{x}$) exists outside of QM (defined only from $w(0, \vec{x})$ to $w(t, \vec{x})$).

Thus, if QM cannot account for the collapse, what can? Foundational to our framework is the notion of the experiment. This notion supersedes QM (the latter being its derived product) and is sufficient to demystify the collapse. In the introduction, we have stated that Definition 1 represents the set of all experiments realizable within a domain. In practice, however, we must perform each experiment atomically—the set of all realizable experiments is derived from many such experiments.

An **atomic experiment** will be defined as a pair of elements of an ensemble $\mathbb{E}$, where the first element of the pair is the initial measurement outcome, and the second element is the final measurement outcome. As an example, let us consider an experimental run comprising n atomic experiments over a two-state ensemble $\mathbb{E} = \{q_1, q_2\}$:

$$E_1 = (q_1, q_1) \tag{193}$$
$$E_2 = (q_1, q_1) \tag{194}$$
$$E_3 = (q_2, q_1) \tag{195}$$
$$\vdots \quad \vdots$$
$$E_n = (q_2, q_2) \tag{196}$$

Assuming the law of large numbers, one can construct a representative probability measure $\rho(0)$ and $\rho(t)$. Specifically, $\rho_i(0)$ is obtained by counting the total occurrence of q_i in the first element of the pairs and dividing by n, and $\rho_i(t)$ by counting the total occurrence q_i in the second element of the pairs and also dividing by n. This gives us the starting and ending points to define the set of all realizable experiments using the probability measure representation $\rho(0)$ and $\rho(t)$.

We can show that the map from experimental runs to probability measure representation is many-to-one, making it non-invertible. Indeed, consider two experimental runs:

$$\begin{array}{ll} \text{Run 1} & \text{Run 2} \\ E_1 = (q_1, q_1) & E_1' = (q_2, q_1) \qquad\qquad (197) \\ E_2 = (q_2, q_2) & E_2' = (q_1, q_2) \qquad\qquad (198) \end{array}$$

Since both of these runs, although different, produce the same $\rho(0)$ and the same $\rho(t)$, the map must in general be non-invertible.

From this, we can deduce that the measurement problem is an artifact of idealized statistical inference. Specifically, claiming a probability measure representation from the law of large numbers allows us to discard the notion of atomic experiments, yielding a tractable but imperfect representation of reality. The measurement collapse problem is then an attempt to make this representation perfect again by inverting the map (i.e., to express reality in terms of atomic experiments rather than probability measures), but failing to do so because the map is non-invertible.

3.1.2. Dissolving the Measurement Problem

To dissolve the measurement problem, it is important to understand that our approach reframes the preparation of quantum states as an **initial measurement**—that is, the initial preparation is $w(0)$, not $\psi(0)$. Then fundamental physical evolution is understood to be in terms of atomic experiments mapping initial measurement outcome to final measurement outcome. At this fundamental level, the measurement problem is entirely dissolved. This operational perspective aligns with laboratory practice but challenges the standard formulation, which takes $\psi(0)$ as its initial preparation instead of $w(0)$.

Core Argument:

1. We propose that a well-defined experiment begin with a measurement outcome $q \in \mathbb{E}$, not an abstract quantum state $\psi(0)$.
2. Example: Preparing $|\psi\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$ requires:

   (a) Measure systems to collapse to $|0\rangle$ or $|1\rangle$.
   (b) Discard all systems in state $|1\rangle$.
   (c) Apply a Hadamard gate H to $|0\rangle$.
   (d) The preparation is complete.

   Neglecting the initial measurement (a) implies that systems of unknown states are sent into the Hadamard gate—the resulting experiment is ill-defined.

Challenges and Solutions:

1. Objection 1: Preparation Without Collapse

   (a) Issue: Traditional QM superficially appears to allow preparing $|\psi\rangle$ without collapsing it (e.g. cooling).
   (b) Response: In practice, all preparations are validated by measurement (or an equivalent).
   (c) Example:

       i. Cooling various qubits $|\psi\rangle$ to $|0\rangle$ is non-invertible (one cannot return to the initial $|\psi\rangle$ because of dissipative effects). The end result is mathematically equivalent to a measurement $|0\rangle$ or $|1\rangle$ followed by a discard of $|1\rangle$.

    ii.   Creating $|+\rangle = H|0\rangle$ requires assuming the initial $|0\rangle$, validated by prior conditions.

2.   Objection 2: Loss of Quantum Coherence

    (a)   Issue: If preparation starts with a measurement, how do we account for coherence (e.g., interference)?

    (b)   Response: Coherence emerges operationally.

    (c)   Example:

        i.   Measure systems to collapse to $|0\rangle$ or $|1\rangle$.

        ii.   Discard all systems in state $|1\rangle$.

        iii.   Apply H to many initial $|0\rangle$-verified states.

        iv.   Aggregate final measurements ($q \in \mathbb{E}$) show interference patterns, even though individual experiments start with collapsed states.

3.   Objection 3: Entanglement and Nonlocality

    (a)   Issue: Entangled states require joint preparation of superpositions.

    (b)   Response: Entanglement is preparable from an initial measurement like any other state.

    (c)   Example:

        i.   Measure systems to collapse to $|00\rangle$, $|01\rangle$, $|10\rangle$, or $|11\rangle$.

        ii.   Discard all systems in state $|01\rangle$, $|10\rangle$, and $|11\rangle$.

        iii.   Apply a Hadamard gate to the first qubit: $(H \otimes I)|00\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \otimes |0\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |10\rangle)$

        iv.   Apply a CNOT gate (with first qubit as control, second as target): $\mathrm{CNOT}[\frac{1}{\sqrt{2}}(|00\rangle + |10\rangle)] = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$

    The final state $\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$ is an entangled state—specifically, it's one of the Bell states (sometimes denoted as $|\Phi^+\rangle$).

In all cases, neglecting the initial measurement results in systems of unknown state entering the experiment and making it ill-defined. An ill-defined experiment is still potentially insightful but not sufficient to uniquely entail QM from entropy optimization—we may call an ill-defined experiment an observation[4],[5],[6],[7].

The complete picture is that QM is an optimal interpolation device derived from a limiting case of atomic experiments mapping initial preparations to final measurements. The measurement problem is entirely dissolved at the level of atomic experiments, but emerges in QM proper due to the non-invertibility of the limiting process. In this interpretation, atomic experiments are understood as brute facts, the fundamental irreducible constituents of reality.

4. Conclusion

E.T. Jaynes fundamentally reoriented statistical mechanics by recasting it as a problem of inference rather than mechanics. His approach revealed that the equations of thermodynamics are not arbitrary

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[4]   This distinction between 'experiment' and 'observation' is purely operational within our framework and is not intended to imply a lesser scientific value for observation-driven inquiry. Both are crucial for scientific progress, but they may engage with the foundational optimization principle in different ways.

[5]   An experiment requires knowledge of both the initial preparation and final measurement, whereas an observation requires only knowledge of the final measurement.

[6]   The author suggests that observations, so defined, may constitute a broader conceptual category that could entail a richer landscape of effective theories beyond what experiments alone feasibly entail. Observations allow us to study parts of the universe whose complexity far exceeds our ability to precisely connect an initial preparation to a final measurement via some physical operation in the laboratory. Accounting for this observed complexity suggests the development of effective theories across various domains, including biology, chemistry, complex systems theory, emergent phenomena, and cosmology. This extension of the optimization problem to observations, however, falls outside the scope of the current paper.

[7]   As statistical mechanics' optimization problem (Annex A) does not reference an initial preparation, it could be argued, from these definitions, that it is based on observations and not on experiments.

physical laws but necessary consequences of maximizing entropy subject to constraints. This work extends Jaynes' inferential paradigm to address a more fundamental question: what is a physical theory itself?

A physical theory, at its essence, is a set of equations that applies to all experiments realizable within a domain. While this definition is informal, our contribution lies in making this concept mathematical. By formulating it as an optimization problem—minimizing the relative entropy of measurement outcomes subject to the natural constraint—we transform the abstract definition of a physical theory into a precise, solvable mathematical problem.

This approach represents a profound methodological shift. Rather than constructing physical theories through trial-and-error enumerations of axioms, we derive them as necessary solutions to a well-defined optimization problem. Physics thus emerges not as a collection of independently discovered laws but as the unique optimal interpolation device between arbitrary experimental preparation and measurement under the constraint of nature.

The power of this formulation lies in its generality allowing us to recover several established physical theories from entropy optimization. Jaynes showed that statistical inference with minimal assumptions yields thermodynamics; we suggest that this same principle, properly generalized, may yield the foundation to all of physics.

Data Availability Statement: No datasets were generated or analyzed during the current study. During the preparation of this manuscript, we utilized a Large Language Model (LLM), for assistance with spelling and grammar corrections, as well as for minor improvements to the text to enhance clarity and readability. This AI tool did not contribute to the conceptual development of the work, data analysis, interpretation of results, or the decision-making process in the research. Its use was limited to language editing and minor textual enhancements to ensure the manuscript met the required linguistic standards.

Conflicts of Interest: The author declares that he has no competing financial or non-financial interests that are directly or indirectly related to the work submitted for publication.

Appendix A. SM

Here, we solve the Lagrange multiplier equation of SM.

$$\mathcal{L} := \underbrace{-k_B \sum_i \rho_i(\beta) \ln \rho_i(\beta)}_{\substack{\text{Boltzmann En-}\\\text{tropy}}} + \underbrace{\lambda \left(1 - \sum_i \rho_i(\beta)\right)}_{\substack{\text{Normalization}\\\text{Constraint}}} + \underbrace{k_B \beta \left(\overline{E} - \sum_i \rho_i(\beta) E_i\right)}_{\text{Average Energy Constraint}} \tag{A1}$$

We solve the maximization problem as follows:

$$0 = \frac{\partial \mathcal{L}(\rho_1, \ldots, \rho_i, \ldots, \rho_n)}{\partial \rho_i} \tag{A2}$$

$$= -k_B \ln \rho_i(\beta) - k_B - \lambda - k_B \beta E_i \tag{A3}$$

$$= k_B \ln \rho_i(\beta) + k_B + \lambda + k_B \beta E_i \tag{A4}$$

$$\implies k_B \ln \rho_i(\beta) = -k_B - \lambda - k_B \beta E_i \tag{A5}$$

$$\implies \rho_i(\beta) = \exp\left(\frac{-k_B - \lambda}{k_B}\right) \exp\left(-\beta E_i\right) \tag{A6}$$

$$= \frac{1}{Z(\beta)} \exp\left(-\beta E_i\right) \tag{A7}$$

The partition function, is obtained as follows:

$$1 = \sum_j \exp\left(\frac{-k_B - \lambda}{k_B}\right) \exp\left(-\beta E_j\right) \tag{A8}$$

$$\implies \left(\exp\left(\frac{-k_B - \lambda}{k_B}\right)\right)^{-1} = \sum_j \exp\left(-\beta E_j\right) \tag{A9}$$

$$Z(\beta) = \sum_j \exp\left(-\beta E_j\right) \tag{A10}$$

Finally, the probability measure is:

$$\rho_i(\beta) = \frac{1}{\sum_j \exp\left(-\beta E_j\right)} \exp\left(-\beta E_i\right) \tag{A11}$$

Appendix B. SageMath Program Showing $\lfloor u^\ddagger u \rfloor_{3,4} u^\ddagger u = \det \varphi(u)$

```
from sage.algebras.clifford_algebra import CliffordAlgebra
from sage.quadratic_forms.quadratic_form import QuadraticForm
from sage.symbolic.ring import SR
from sage.matrix.constructor import Matrix


# Define the quadratic form for GA(3,1) over the Symbolic Ring
Q = QuadraticForm(SR, 4, [-1, 0, 0, 0, 1, 0, 0, 1, 0, 1])


# Initialize the GA(3,1) algebra over the Symbolic Ring
algebra = CliffordAlgebra(Q)


# Define the basis vectors
e0, e1, e2, e3 = algebra.gens()


# Define the scalar variables for each basis element
a = var('a')
t, x, y, z = var('t x y z')
f01, f02, f03, f12, f23, f13 = var('f01 f02 f03 f12 f23 f13')
v, w, q, p = var('v w q p')
b = var('b')


# Create a general multivector
udegree0=a
udegree1=t*e0+x*e1+y*e2+z*e3
udegree2=f01*e0*e1+f02*e0*e2+f03*e0*e3+f12*e1*e2+f13*e1*e3+f23*e2*e3
udegree3=v*e0*e1*e2+w*e0*e1*e3+q*e0*e2*e3+p*e1*e2*e3
udegree4=b*e0*e1*e2*e3
u=udegree0+udegree1+udegree2+udegree3+udegree4


u2 = u.clifford_conjugate()*u


u2degree0 = sum(x for x in u2.terms() if x.degree() == 0)
u2degree1 = sum(x for x in u2.terms() if x.degree() == 1)
u2degree2 = sum(x for x in u2.terms() if x.degree() == 2)
```

```
u2degree3 = sum(x for x in u2.terms() if x.degree() == 3)
u2degree4 = sum(x for x in u2.terms() if x.degree() == 4)
u2conj34 = u2degree0+u2degree1+u2degree2-u2degree3-u2degree4


I = Matrix(SR, [[1, 0, 0, 0],
                [0, 1, 0, 0],
                [0, 0, 1, 0],
                [0, 0, 0, 1]])

#MAJORANA MATRICES
y0 = Matrix(SR, [[0, 0, 0, 1],
                 [0, 0, -1, 0],
                 [0, 1, 0, 0],
                 [-1, 0, 0, 0]])

y1 = Matrix(SR, [[0, -1, 0, 0],
                 [-1, 0, 0, 0],
                 [0, 0, 0, -1],
                 [0, 0, -1, 0]])

y2 = Matrix(SR, [[0, 0, 0, 1],
                 [0, 0, -1, 0],
                 [0, -1, 0, 0],
                 [1, 0, 0, 0]])

y3 = Matrix(SR, [[-1, 0, 0, 0],
                 [0, 1, 0, 0],
                 [0, 0, -1, 0],
                 [0, 0, 0, 1]])

mdegree0 = a
mdegree1 = t*y0+x*y1+y*y2+z*y3
mdegree2 = f01*y0*y1+f02*y0*y2+f03*y0*y3+f12*y1*y2+f13*y1*y3+f23*y2*y3
mdegree3 = v*y0*y1*y2+w*y0*y1*y3+q*y0*y2*y3+p*y1*y2*y3
mdegree4 = b*y0*y1*y2*y3
m=mdegree0+mdegree1+mdegree2+mdegree3+mdegree4

print(u2conj34*u2 == m.det())
```

The program outputs

True

showing, by computer-assisted symbolic manipulations, that the determinant of the real Majorana representation of a multivector u is equal to the double-product: $\lfloor \mathbf{u}^{\ddagger}\mathbf{u} \rfloor_{3,4} \mathbf{u}^{\ddagger}\mathbf{u} = \det \varphi(\mathbf{u})$.

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