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Article

Best Proximity Points for Geraghty-Type Non-Self Mappings

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Abstract

We study Geraghty-type non-self mappings within the framework of best proximity point theory. By introducing auxiliary functions with subsequential convergence, we establish general conditions ensuring the existence and uniqueness of best proximity points. Our results extend and unify earlier work on proximal and Kannan-type contractions under a Geraghty setting, and we provide counterexamples showing that the auxiliary assumptions are essential. As an illustration, we construct an explicit non-self alignment mapping on subsets of $\mathbb{R}^2$ for which all hypotheses can be verified and the unique best proximity point, as well as the convergence of the associated proximal iteration, can be computed in closed form.

Keywords: best proximity point; Geraghty-type contraction; proximal mapping; Kannan–Geraghty mapping; auxiliary function; subsequential convergence; alignment mapping

1. Introduction and Preliminaries

Fixed point theory plays a fundamental role in nonlinear analysis, with applications ranging from differential equations to optimization problems. Classical results such as the Banach contraction principle provide powerful tools for self-mappings, but in many practical situations one encounters non-self mappings, where the appropriate substitute is the notion of best proximity points.

The development of the field has proceeded through several key milestones. In 1968, Kannan introduced a celebrated generalization of Banach’s contraction principle by proving that for a complete metric space (X, d) , any mapping $T : X \rightarrow X$ satisfying

$$d(Tx, Ty) \leq \alpha [d(x, Tx) + d(y, Ty)], \quad \forall x, y \in X,$$

with $\alpha \in [0, \frac{1}{2})$, admits a unique fixed point [1]. A remarkable feature of Kannan’s theorem is that no continuity assumption on T is required. Later, Ariza-Ruiz and Jiménez-Melado (2010) extended this result by introducing *weakly Kannan maps*, in which the contractive coefficient depends on the points x, y . More precisely, there exists $\alpha : X \times X \rightarrow [0, 1)$ such that

$$d(Tx, Ty) \leq \alpha(x, y) [d(x, Tx) + d(y, Ty)], \quad \forall x, y \in X,$$

with the uniform bound $\sup\{\alpha(x, y) : a \leq d(x, y) \leq b\} < 1$ for all $0 < a \leq b$ [1]. This flexibility allows one to capture the behavior of a wider class of nonlinear operators.

The move from fixed point results to best proximity point results was initiated by Sadiq Basha (2011), who introduced the concept of *proximal contractions* and established best proximity point theorems for non-self mappings [7]. The key idea is to characterize conditions under which, even in the absence of fixed points, one can still guarantee the existence of a point x^* such that the distance between x^* and its image Tx^* is minimal. Best proximity point results for Geraghty-type proximal contraction mappings were later developed in [6].

Subsequent work by Moradi provided a different perspective by introducing the use of an auxiliary function. Specifically, for mappings $T, S : X \rightarrow X$ in a metric space (X, d) , it was shown in [5] that if S is one-to-one, continuous, and sequentially convergent, and if

$$d(STx, STy) \leq k d(Sx, Sy), \quad \forall x, y \in X, \quad (1)$$

for some constant $k \in (0, 1)$, then T possesses a best proximity point.

Building on these foundations, the present work introduces two novel classes of mappings: *proximal Geraghty* and *proximal Kannan–Geraghty* mappings of the first kind. We establish existence and uniqueness theorems for these mappings and illustrate, by examples, that the auxiliary conditions we impose are both natural and necessary. Before turning to our main results, we recall some fundamental concepts from best proximity theory.

Definition 1 ([8]). Let A, B be nonempty subsets of a metric space (X, d) . The best proximity sets A_0 and B_0 are defined as

$$\begin{aligned} A_0 &:= \{x \in A : d(x, y) = d(A, B) \text{ for some } y \in B\}, \\ B_0 &:= \{y \in B : d(x, y) = d(A, B) \text{ for some } x \in A\}, \end{aligned}$$

where

$$d(A, B) := \inf\{d(x, y) : x \in A, y \in B\}$$

denotes the minimal distance between A and B .

When A and B are closed subsets of a normed linear space with $d(A, B) > 0$, it is known that both A_0 and B_0 lie on the boundaries of their respective sets [7].

Definition 2 ([8]). Given a mapping $T : A \rightarrow B$, a point $x^* \in A$ is called a best proximity point of T if

$$d(x^*, Tx^*) = d(A, B).$$

Definition 3 ([7]). A mapping $S : A \rightarrow B$ is called a proximal contraction of the first kind if there exists $\alpha \in [0, 1)$ such that for all $x_1, x_2, u_1, u_2 \in A$,

$$[d(u_1, Sx_1) = d(A, B), d(u_2, Sx_2) = d(A, B)] \implies d(u_1, u_2) \leq \alpha d(x_1, x_2).$$

For self-maps, this reduces exactly to the Banach contraction principle; for non-self maps, proximal contractions need not be contractions in the usual sense.

Corollary 1 ([4]). Let A, B be nonempty closed subsets of a complete metric space such that A_0 and B_0 are nonempty. Suppose $T : A \rightarrow B$ and $g : A \rightarrow A$ satisfy:

1. g is one-to-one and continuous, and $g^{-1} : g(A) \rightarrow A$ is uniformly continuous;
2. T is a proximal contraction of the first kind with $T(A_0) \subseteq B_0$.

Then there exists a unique $x^* \in A$ such that $d(gx^*, Tx^*) = d(A, B)$. Moreover, for any $x_0 \in A_0$, the sequence defined by $d(gx_{n+1}, Tx_n) = d(A, B)$ converges to x^* .

When $g = \text{Id}_A$, we simply say that $T : A \rightarrow B$ is a proximal contraction if condition (b) holds.

Definition 4 ([4]). Let (A, B) be a nonempty pair of subsets of a metric space (X, d) . A mapping $T : A \rightarrow B$ is called a proximal Kannan non-self mapping if there exists $\alpha \in [0, \frac{1}{2})$ such that for all $u, v, x, y \in A$ with

$$d(u, Tx) = d(A, B), \quad d(v, Ty) = d(A, B),$$

we have

$$d(u, v) \leq \alpha [d^*(x, Tx) + d^*(y, Ty)], \quad (2)$$

where

$$d^*(x, Tx) := d(x, Tx) - d(A, B) \geq 0.$$

The class of proximal Kannan non-self mappings properly contains the class of Kannan non-self mappings.

Definition 5 (Weak Proximal Kannan Non-Self Mapping [4]). Let (X, d) be a metric space, and let $A, B \subseteq X$. A mapping $T : A \rightarrow B$ is called a weak proximal Kannan non-self mapping if there exists $\alpha \in (0, \frac{1}{2})$ such that for all $u, v, x, y \in A$ with

$$d(u, Tx) = d(A, B), \quad d(v, Ty) = d(A, B),$$

the implication

$$\frac{1}{r} d^*(x, Tx) \leq d(x, y) \implies d(u, v) \leq \alpha [d^*(x, Tx) + d^*(y, Ty)]$$

holds, where $r = \frac{\alpha}{1-\alpha}$.

Theorem 1 ([4]). Let (A, B) be a nonempty pair of subsets of a complete metric space (X, d) such that A_0 is nonempty and closed. If $T : A \rightarrow B$ is a weak proximal Kannan non-self mapping with $T(A_0) \subseteq B_0$, then there exists a unique $x^* \in A$ such that $d(x^*, Tx^*) = d(A, B)$. Moreover, if $\{x_n\} \subseteq A$ satisfies $d(x_{n+1}, Tx_n) = d(A, B)$, then $x_n \rightarrow x^*$.

Definition 6 ([9]). Let (X, d) be a metric space. A mapping $T : X \rightarrow X$ is called a Geraghty contraction if there exists $\beta \in \Gamma$ such that

$$d(Tx, Ty) \leq \beta(d(x, y)) d(x, y), \quad \forall x, y \in X,$$

where Γ denotes the class of functions $\beta : [0, \infty) \rightarrow [0, 1)$ with the property that

$$\beta(t_n) \rightarrow 1 \implies t_n \rightarrow 0.$$

Theorem 2 ([9]). Let (X, d) be a complete metric space and $T : X \rightarrow X$ a Geraghty contraction. Then T has a unique fixed point.

According to [12], let (X, d) be a metric space. A mapping $f : X \rightarrow X$ is said to be a Kannan–Geraghty self-mapping if there exists a function $\beta \in \Gamma$ such that, for all $x, y \in X$,

$$d(f(x), f(y)) \leq \beta(d(x, y)) \cdot \frac{1}{2} [d(x, f(x)) + d(y, f(y))].$$

This definition combines the features of Kannan mappings with the flexibility of Geraghty-type control functions. Motivated by this, we now develop the non-self, proximal analogues.

2. Main Results

In this section, we introduce S-proximal contraction non-self mappings and S-proximal Kannan non-self mappings, and establish sufficient conditions for the existence and uniqueness of best proximity points in complete metric spaces. Before stating the main theorems, we recall the notion of an auxiliary function.

Definition 7. Let (X, d) be a metric space, and let $A, B \subseteq X$. A mapping $S : A \cup B \rightarrow A \cup B$ is called an auxiliary function if $S(A) \subseteq A$ and $S(B) \subseteq B$.

Theorem 3 (Extended Proximal Geraghty of the First Kind). *Let (A, B) be a pair of nonempty subsets of a complete metric space (X, d) such that A_0 and B_0 are nonempty and closed. Let S be an auxiliary function that is continuous on A and B , one-to-one, subsequentially convergent, and satisfies $S(A_0) \subseteq A_0$ and $S(B_0) \subseteq B_0$. Suppose $T : A \rightarrow B$ is a mapping such that $T(A_0) \subseteq B_0$, and assume that for all $u, v, x, y \in A$,*

$$d(Su, STx) = d(Sv, STy) = d(A, B) \implies d(Su, Sv) \leq \beta(d(Sx, Sy)) d(Sx, Sy),$$

where $\beta \in \Gamma$ (that is, T is an S -proximal Geraghty contraction).

Then there exists a unique $x^* \in A$ such that

$$d(Sx^*, STx^*) = d(A, B).$$

Moreover, if $\{x_n\} \subseteq A$ is a sequence satisfying

$$d(Sx_{n+1}, STx_n) = d(A, B), \quad \forall n \in \mathbb{N},$$

then $x_n \rightarrow x^*$.

Proof. We first introduce the images of the proximity sets under S :

$$S(A_0) := \{Sx \mid d(Sx, Sy) = d(A, B) \text{ for some } y \in B\},$$

$$S(B_0) := \{Sy \mid d(Sx, Sy) = d(A, B) \text{ for some } x \in A\}.$$

Let $x_0 \in A_0$. Since $T(A_0) \subseteq B_0$, we have $Tx_0 \in B_0$ and hence $STx_0 \in S(B_0)$. Thus, there exists $x_1 \in A_0$ with

$$d(Sx_1, STx_0) = d(A, B).$$

Iterating this construction, we obtain a sequence $\{x_n\} \subseteq A_0$ such that

$$d(Sx_{n+1}, STx_n) = d(A, B), \quad \forall n \in \mathbb{N}. \quad (3)$$

Set

$$d_n := d(Sx_n, Sx_{n-1}), \quad n \geq 1.$$

For each $n \geq 1$, apply the S -proximal Geraghty condition with

$$u = x_{n+1}, \quad x = x_n, \quad v = x_n, \quad y = x_{n-1}.$$

Then, by (3),

$$d(Su, STx) = d(Sx_{n+1}, STx_n) = d(A, B), \quad d(Sv, STy) = d(Sx_n, STx_{n-1}) = d(A, B),$$

so the hypothesis is satisfied and we obtain

$$d(Sx_{n+1}, Sx_n) \leq \beta(d(Sx_n, Sx_{n-1})) d(Sx_n, Sx_{n-1}),$$

that is,

$$d_{n+1} \leq \beta(d_n) d_n, \quad n \geq 1. \quad (4)$$

By the standard argument in Geraghty's fixed point theorem (see [9]), (4) implies that $d_n \rightarrow 0$ and that $\{Sx_n\}$ is a Cauchy sequence in X . Since X is complete and A_0 is closed, there exists $v \in A_0$ such that

$$\lim_{n \rightarrow \infty} Sx_n = v. \quad (5)$$

Because S is subsequentially convergent, $\{x_n\}$ has a subsequence $\{x_{n(k)}\}$ converging to some $u \in A_0$. By continuity of S ,

$$\lim_{k \rightarrow \infty} Sx_{n(k)} = Su.$$

Comparing with (5), we conclude $Su = v$.

We claim u is the unique best proximity point of T . Since $u \in A_0$ and $T(A_0) \subseteq B_0$, there exists $y^* \in A_0$ with $d(Sy^*, STu) = d(A, B)$. Combining this with (3) for the subsequence $\{x_{n(k)}\}$, we have

$$d(Sy^*, STu) = d(A, B), \quad d(Sx_{n(k)+1}, STx_{n(k)}) = d(A, B).$$

Applying the contractive condition with

$$u = x_{n(k)+1}, \quad x = x_{n(k)}, \quad v = y^*, \quad y = u,$$

we obtain

$$d(Sx_{n(k)+1}, Sy^*) \leq \beta(d(Sx_{n(k)}, Su)) d(Sx_{n(k)}, Su).$$

Letting $k \rightarrow \infty$, we obtain $d(Su, Sy^*) = 0$, hence $Su = Sy^*$. Since S is one-to-one, $u = y^*$. Thus u is a best proximity point of T .

Finally, for uniqueness, suppose $x_1, x_2 \in A$ are two distinct best proximity points, so that $d(Sx_i, STx_i) = d(A, B)$ for $i = 1, 2$. Applying the contractive condition with $u = x_1, x = x_1, v = x_2, y = x_2$ gives

$$d(Sx_1, Sx_2) \leq \beta(d(Sx_1, Sx_2)) d(Sx_1, Sx_2),$$

which is impossible unless $d(Sx_1, Sx_2) = 0$. Hence $x_1 = x_2$. $\square$

We emphasize that the assumption of subsequential convergence of S in Theorem 3 cannot be omitted. This is demonstrated by an example (see Example 1 below).

In the next result, we extend the notion of proximal Kannan non-self mappings introduced by Gabeleh [4] to a Geraghty-type setting.

Theorem 4 (Extended Proximal Kannan–Geraghty of the First Kind). *Let (A, B) be a pair of nonempty subsets of a complete metric space (X, d) such that A_0 and B_0 are nonempty and closed. Let S be an auxiliary function that is continuous on A and B , one-to-one, and subsequentially convergent, with $S(A_0) \subseteq A_0$ and $S(B_0) \subseteq B_0$. Suppose $T : A \rightarrow B$ is a mapping with $T(A_0) \subseteq B_0$, and assume that for all $u, v, x, y \in A$,*

$$\begin{aligned} d(Su, STx) &= d(Sv, STy) = d(A, B) \\ \implies d(Su, Sv) &\leq \beta(d(Sx, Sy)) [d^*(Sx, STx) + d^*(Sy, STy)], \end{aligned} \tag{6}$$

where $\beta \in \Gamma$ and

$$d^*(Sx, STx) := d(Sx, STx) - d(A, B) \geq 0.$$

Then there exists a unique point $x^* \in A$ such that

$$d(Sx^*, STx^*) = d(A, B).$$

Moreover, if $\{x_n\} \subseteq A$ is a sequence satisfying

$$d(Sx_{n+1}, STx_n) = d(A, B), \quad \forall n \in \mathbb{N},$$

then $x_n \rightarrow x^*$.

Proof. As before, define

$$S(A_0) := \{Sx \mid d(Sx, Sy) = d(A, B) \text{ for some } y \in B\},$$

$$S(B_0) := \{Sy \mid d(Sx, Sy) = d(A, B) \text{ for some } x \in A\}.$$

Let $x_0 \in A_0$. Since $T(A_0) \subseteq B_0$, we have $Tx_0 \in B_0$, hence $STx_0 \in S(B_0)$. Thus there exists $x_1 \in A_0$ such that

$$d(Sx_1, STx_0) = d(A, B).$$

Proceeding inductively, we obtain a sequence $\{x_n\} \subseteq A_0$ with

$$d(Sx_{n+1}, STx_n) = d(A, B), \quad \forall n \in \mathbb{N}. \quad (7)$$

Set again $d_n := d(Sx_n, Sx_{n-1})$ for $n \geq 1$. For each $n \geq 1$, apply (6) with

$$u = x_{n+1}, \quad x = x_n, \quad v = x_n, \quad y = x_{n-1}.$$

Using (7), we obtain

$$d(Sx_n, Sx_{n+1}) \leq \beta(d(Sx_n, Sx_{n-1})) [d^*(Sx_n, STx_n) + d^*(Sx_{n-1}, STx_{n-1})].$$

By the triangle inequality and (7),

$$\begin{aligned} d^*(Sx_n, STx_n) &= d(Sx_n, STx_n) - d(A, B) \\ &\leq d(Sx_n, Sx_{n+1}) + d(Sx_{n+1}, STx_n) - d(A, B) = d(Sx_n, Sx_{n+1}) = d_{n+1}. \end{aligned}$$

and similarly $d^*(Sx_{n-1}, STx_{n-1}) \leq d(Sx_{n-1}, Sx_n) = d_n$. Hence

$$d_{n+1} \leq \beta(d_n) (d_{n+1} + d_n).$$

Rearranging,

$$d_{n+1}(1 - \beta(d_n)) \leq \beta(d_n) d_n,$$

and therefore

$$d_{n+1} \leq \frac{\beta(d_n)}{1 - \beta(d_n)} d_n, \quad n \geq 1. \quad (8)$$

Since $0 \leq \beta(t) < 1$ for all t , the right-hand side is finite. As in the standard Kannan–Geraghty argument (cf. [9,12]), (8) implies that $d_n \rightarrow 0$ and that $\{Sx_n\}$ is a Cauchy sequence in X . Because X is complete and A_0 is closed, there exists $v \in A_0$ such that

$$\lim_{n \rightarrow \infty} Sx_n = v. \quad (9)$$

Since S is subsequentially convergent, $\{x_n\}$ has a subsequence $\{x_{n(k)}\}$ converging to some $u \in A_0$. By continuity of S ,

$$\lim_{k \rightarrow \infty} Sx_{n(k)} = Su.$$

Comparing with (9), we conclude $Su = v$.

We claim u is the unique best proximity point of T . Since $u \in A_0$ and $T(A_0) \subseteq B_0$, there exists $y^* \in A_0$ with $d(Sy^*, STu) = d(A, B)$. Combining this with (7), for the subsequence $\{x_{n(k)}\}$ we have

$$d(Sy^*, STu) = d(A, B), \quad d(Sx_{n(k)+1}, STx_{n(k)}) = d(A, B).$$

Applying the contractive condition (6) with

$$u = x_{n(k)+1}, \quad x = x_{n(k)}, \quad v = y^*, \quad y = u,$$

we obtain

$$d(Sx_{n(k)+1}, Sy^*) \leq \beta(d(Sx_{n(k)}, Su)) [d^*(Sx_{n(k)}, STx_{n(k)}) + d^*(Su, STu)].$$

Letting $k \rightarrow \infty$ and using the convergence of $\{Sx_n\}$, we obtain $d(Su, Sy^*) = 0$, hence $Su = Sy^*$. Since S is one-to-one, $u = y^*$. Thus u is a best proximity point of T .

Uniqueness follows similarly: if $x_1, x_2 \in A$ are both best proximity points, then, using (6) with $u = x_1, x = x_1, v = x_2, y = x_2$, we get

$$d(Sx_1, Sx_2) \leq \beta(d(Sx_1, Sx_2)) [d^*(Sx_1, STx_1) + d^*(Sx_2, STx_2)],$$

but both defect terms vanish (since $d(Sx_i, STx_i) = d(A, B)$), so the right-hand side is zero. Hence $Sx_1 = Sx_2$, and injectivity of S yields $x_1 = x_2$. $\square$

Example 1. Consider the metric space

$$X := \{0, 1\} \times [0, \infty)$$

with the Euclidean metric. Define

$$A := \{(0, x) : x \in [0, \infty)\}, \quad B := \{(1, y) : y \in [0, \infty)\}.$$

For $(0, x) \in A$ and $(1, y) \in B$,

$$d((0, x), (1, y))^2 = 1 + (x - y)^2,$$

so

$$d(A, B) = 1, \quad A_0 = A, \quad B_0 = B.$$

Let $T : A \rightarrow B$ be given by

$$T(0, x) = (1, 2x + 1),$$

and let $S : X \rightarrow X$ be defined by

$$S(x, y) = (x, e^{-y}).$$

Clearly S is one-to-one, $S(A) \subseteq A$ and $S(B) \subseteq B$. Moreover, $T(A_0) \subseteq B_0$.

We first note that T has no best proximity point. Indeed, a best proximity point $x^* = (0, t)$ would have to satisfy

$$d((0, t), T(0, t)) = d(A, B) = 1.$$

However,

$$d((0, t), (1, 2t + 1))^2 = 1 + (t - (2t + 1))^2 = 1 + (t + 1)^2 > 1$$

for all $t \geq 0$, so no such t exists.

On the other hand, consider the sequence $\{(0, n)\}_{n \geq 1} \subseteq A$. We have

$$S(0, n) = (0, e^{-n}) \longrightarrow (0, 0) \in A,$$

so $\{S(0, n)\}$ converges in X , but $\{(0, n)\}$ has no convergent subsequence in A itself (the second coordinate diverges to $+\infty$). Hence S fails to be subsequentially convergent in the sense required by Theorems 3 and 4.

In this example, the S -proximal Geraghty (or Kannan–Geraghty) condition is satisfied only vacuously, since there are no quadruples $u, v, x, y \in A$ with

$$d(Su, STx) = d(Sv, STy) = d(A, B),$$

as one checks directly from the form of S and T . Nevertheless, the lack of subsequential convergence of S is accompanied by the failure of T to admit a best proximity point, showing that the subsequential convergence assumption on S is essential.

3. Application

In image registration and alignment problems, simplified geometric configurations are often used to analyze the behavior of matching schemes; see, for example, [10,11]. Motivated by this viewpoint, we present an explicit non-self alignment mapping on subsets of $\mathbb{R}^2$ in which all assumptions of our main results are verified directly.

3.1. Geometric Setting

Fix $\delta > 0$ and consider the subsets

$$A = \{(0, t) : t \in [0, 1]\}, \quad B = \{(\delta, s) : s \in [0, 1]\}$$

of $\mathbb{R}^2$ endowed with the Euclidean metric d . Since $A \cup B$ is compact, $(A \cup B, d)$ is a complete metric space.

For $(0, t) \in A$ and $(\delta, s) \in B$,

$$d((0, t), (\delta, s))^2 = \delta^2 + (t - s)^2,$$

and this distance equals $d(A, B)$ if and only if $s = t$. Hence

$$d(A, B) = \delta, \quad A_0 = A, \quad B_0 = B,$$

and both A_0 and B_0 are nonempty and closed.

Define $S : A \cup B \rightarrow A \cup B$ by

$$S = \text{Id}_{A \cup B}.$$

Then S is continuous, injective, $S(A_0) \subseteq A_0$, $S(B_0) \subseteq B_0$, and trivially subsequentially convergent.

3.2. Definition of the Non-Self Mapping

Let $\kappa \in (0, 1)$ and define $T : A \rightarrow B$ by

$$T(0, t) = (\delta, \kappa t).$$

Since $\kappa t \in [0, 1]$ for all $t \in [0, 1]$, it follows that $T(A_0) \subseteq B_0$.

3.3. Verification of the S -Proximal Geraghty Condition

Let $u = (0, u_2)$, $v = (0, v_2)$, $x = (0, x_2)$, $y = (0, y_2) \in A$. Assume

$$d(Su, STx) = d(A, B) = \delta, \quad d(Sv, STy) = d(A, B) = \delta.$$

Using $S = \text{Id}$ and $T(0, t) = (\delta, \kappa t)$, these equalities become

$$d((0, u_2), (\delta, \kappa x_2)) = \delta, \quad d((0, v_2), (\delta, \kappa y_2)) = \delta.$$

Since

$$d((0, a), (\delta, b))^2 = \delta^2 + (a - b)^2,$$

each equality holds if and only if $u_2 = \kappa x_2$ and $v_2 = \kappa y_2$. Thus

$$Su = (0, \kappa x_2), \quad Sv = (0, \kappa y_2),$$

and hence

$$d(Su, Sv) = \kappa |x_2 - y_2| = \kappa d(Sx, Sy).$$

Define $\beta : [0, \infty) \rightarrow [0, 1)$ by $\beta(t) \equiv \kappa$. Since $\kappa < 1$, we have $\beta \in \Gamma$, and therefore

$$d(Su, Sv) \leq \beta(d(Sx, Sy)) d(Sx, Sy)$$

holds for all $u, v, x, y \in A$ satisfying the proximal hypotheses. Hence T is an S -proximal Geraghty contraction.

3.4. Best Proximity Point

By Theorem 3, there exists a unique $x^* \in A$ such that

$$d(Sx^*, STx^*) = d(A, B) = \delta.$$

Writing $x^* = (0, t)$, we compute

$$d((0, t), (\delta, \kappa t))^2 = \delta^2 + (t - \kappa t)^2 = \delta^2 + (1 - \kappa)^2 t^2.$$

This equals δ^2 if and only if $t = 0$. Therefore,

$$x^* = (0, 0), \quad Tx^* = (\delta, 0),$$

is the unique best proximity pair.

3.5. Convergence of the Proximal Sequence

Let $x_0 = (0, t_0) \in A_0$ be arbitrary and define $\{x_n\} \subset A_0$ by

$$d(Sx_{n+1}, STx_n) = d(A, B) = \delta, \quad n \geq 0.$$

With $x_n = (0, t_n)$, this becomes

$$d((0, t_{n+1}), (\delta, \kappa t_n)) = \delta,$$

which implies $t_{n+1} = \kappa t_n$. Thus

$$t_n = \kappa^n t_0 \longrightarrow 0, \quad x_n \longrightarrow x^* = (0, 0).$$

3.6. Kannan–Geraghty Variant

For the setting of Theorem 4, one may take

$$T(0, t) = (\delta, 0), \quad t \in [0, 1].$$

The proximal conditions again force $u = v = (0, 0)$, and the defining inequality of the proximal Kannan–Geraghty condition holds trivially. The unique best proximity point is again $(0, 0)$.

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