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Not peer-reviewed version

Some Errors on Hesitant Fuzzy Set Theory

Shizhan Lu *

Posted Date: 13 November 2025

doi: 10.20944/preprints202511.0795.v1

Keywords: hesitant fuzzy sets; hesitant fuzzy topology spaces; hesitant fuzzy approximate spaces; hesitant fuzzy algebras

Article

Some Errors on Hesitant Fuzzy Set Theory

Shizhan Lu

School of Management, Jiangsu University, Zhengjiang 212013, China; lushizhan20140910@126.com

Abstract

Hesitant fuzzy set theory serves as a valuable framework that has been extensively applied across various domains, including decision-making, attribute reduction, linguistic perception, among others. Hesitant fuzzy elements are discrete arrays, and the intersection and union operations for hesitant fuzzy sets differ from those defined for fuzzy sets. Consequently, certain erroneous propositions have emerged in the literature on hesitant fuzzy sets. This review examines some incorrect propositions found in studies related to hesitant fuzzy topological spaces, hesitant fuzzy approximation spaces and hesitant fuzzy algebra, and provides corresponding counterexamples in each incorrect proposition. The advancement of a mathematical knowledge system must be free from errors, as inaccuracies can compromise the integrity of the theoretical framework. It is essential that researchers rigorously scrutinize the flawed propositions identified in this work when further investigating hesitant fuzzy sets and their mathematical structures, thereby promoting the robust development of hesitant fuzzy set theory.

Keywords: hesitant fuzzy sets; hesitant fuzzy topology spaces; hesitant fuzzy approximate spaces; hesitant fuzzy algebras

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1. Introduction

Fuzzy sets [1–4], interval-valued fuzzy sets [5,6], intuitionistic fuzzy sets [7], and hesitant fuzzy sets [8,9] are commonly used tools for dealing with uncertain information. Hesitant fuzzy sets with a membership degree represented by a discrete array are better suited for capturing the indecisiveness in real world compared to fuzzy sets, interval-valued fuzzy sets, and intuitionistic fuzzy sets. Hesitant fuzzy sets find extensive applications in various domains, including decision-making [10,11], attribute reduction [12], classification [13], linguistic perceptual [14], and forecasting [15], among others.

The hesitant fuzzy element is a discrete array, characterized by diversity and complexity. These attributes have led some scholars, in their research on hesitant fuzzy sets, to formulate imprecise or flawed propositions due to insufficient consideration. This article reviews some literatures on hesitant fuzzy sets, discusses some propositions of hesitant fuzzy topology spaces, hesitant fuzzy approximate spaces and hesitant fuzzy algebras, and provides counterexamples.

The construction of this paper is organized as follows. Section 2 introduces foundations of hesitant fuzzy sets. Section 3 discusses absorption law and distributive law of hesitant fuzzy sets. Section 4 discusses some propositions of hesitant fuzzy topology spaces. Section 5 discusses some propositions of hesitant fuzzy approximate spaces. Section 6 discusses some propositions of hesitant fuzzy algebras. Section 7 provides a conclusion.

2. Foundations of Hesitant Fuzzy Sets

This section reviews some foundations about hesitant fuzzy sets.

2.1. Reviews of Fuzzy Sets and Hesitant Fuzzy Sets

Definition 2.1. [1] A fuzzy set F on U is a mapping $F : U \to [0, 1]$.

Definition 2.2. [1] F_1 and F_2 are two fuzzy sets on U, F_1 is a subset of F_2 if $F_1(x) \leqslant F_2(x)$ for all $x \in U$, denoted as $F_1 \subset F_2$.

 doi:10.20944/preprints202511.0795.v1

Definition 2.3. [16] A hesitant fuzzy element is a non-empty, finite subset of $[0,1]$.

Definition 2.4. [8] A hesitant fuzzy set on U is defined as a function that when applied to U returns a subset of $[0,1]$.

In the following, $HF(U)$ denotes the set of all hesitant fuzzy sets defined over U.

Definition 2.5. [8] For each $x \in U$, and a hesitant fuzzy set H, the lower bound and upper bound of $H(x)$ are defined
lower bound $H^-(x) = inf\ H(x)$,
upper bound $H^+(x) = sup\ H(x)$.

Definition 2.6. [8] Given two hesitant fuzzy sets represented by their membership functions H_1 and H_2, their union and intersection are defined
union $(H_1 \cup H_2)(x) = \{h \in H_1(x) \cup H_2(x) : h \geqslant sup(H_1^-(x), H_2^-(x))\}$,
intersection $(H_1 \cap H_2)(x) = \{h \in H_1(x) \cup H_2(x) : h \leqslant inf(H_1^+(x), H_2^+(x))\}$.

Example 2.7. Let $U = \{x\}$, $H_1(x) = \{0.3, 0.3, 0.6\}$ and $H_2(x) = \{0.5, 0.8\}$.
$H_1^-(x) = 0.3$, $H_2^-(x) = 0.5$, $max\{H_1^-(x), H_2^-(x)\} = 0.5$. $(H_1 \cup H_2)(x) = \{0.5, 0.6, 0.8\}$.
$H_1^+(x) = 0.6$, $H_2^+(x) = 0.8$, $min\{H_1^+(x), H_2^+(x)\} = 0.6$. $(H_1 \cap H_2)(x) = \{0.3, 0.3, 0.5, 0.6\}$.

Definition 2.8. [8] For $x \in U$ and $H \in HF(U)$, the complement of H is denoted as H^c, where
$H^c(x) = \cup_{\gamma \in H(x)}\{1 - \gamma\}$.

2.2. Inclusion Definitions of Hesitant Fuzzy Sets

For two hesitant fuzzy sets H_1 and H_2, the references [17–21] introduced that H_1 is a hesitant fuzzy subset of H_2 if $H_1(x) \subset H_2(x)$ for all $x \in U$. However, there is no clear description for $H_1(x) \subset H_2(x)$ in references [17–21]. For example, $H_1(x) = \{0.1, 0.9\}$ and $H_2(x) = \{0.6, 0.8\}$, how to determine $H_1(x) \subset H_2(x)$ or $H_2(x) \subset H_1(x)$ based on the values of $\{0.1, 0.9\}$ and $\{0.6, 0.8\}$?

The hesitant fuzzy element contains multiple values, a single inclusion relationship is not sufficient to fully describe the various relationships between hesitant fuzzy sets. There should be multiple inclusion relationships among various hesitant fuzzy sets. Lu and Xu et al [9] presented several kinds of inclusion relationships among various hesitant fuzzy sets and provided an example to describe the various relationships (shown in Example 2.9).

Example 2.9. [9] Let $U = \{x_1, x_2, x_3, x_4, x_5, x_6\}$ be a set of decision-making schemes and H be an expert team that consists of three experts. $H(U) = \frac{\{0.9,0.2\}}{x_1} + \frac{\{0.6,0.6,0.5\}}{x_2} + \frac{\{0.7,0.5,0.5\}}{x_3} + \frac{\{0.8,0.6,0.5\}}{x_4} + \frac{\{0.9,0.3,0.1\}}{x_5} + \frac{\{0.9,0.8,0.7\}}{x_6}$ are the estimated values of schemes provided by experts, in which the estimated values for x_1 are 0.9 and 0.2 that are obtained through the evaluations made by experts. One of the three experts fails to evaluate the scheme x_1.

(1) Here, x_1 has an estimated value of 0.9, which is greater than or equal to all the estimated values for x_2, it is possible that the scheme x_1 is better than the scheme x_2, denoted as $H(x_2) \subset_p H(x_1)$.

(2) Here, $0.55 = mean[H(x_1)] < mean[H(x_2)] = 0.567$, where $mean[\cdot]$ is the mean value operator. In comparing the mean values of estimated values, we find that the scheme x_2 is better than the scheme x_1, denoted as $H(x_1) \subset_m H(x_2)$.

(3) The best estimated value of x_3 is 0.7, which is greater than or equal to the best estimated value of x_2. On the other hand, the worst estimated value of x_3 is 0.5, which is greater than or equal to the worst estimated value of x_2. To compare the respective best and worst cases of schemes x_2 and x_3, it is acceptable that the scheme x_3 is better than the scheme x_2, denoted as $H(x_2) \subset_a H(x_3)$.

(4) To compare the estimated values for schemes x_3 and x_4 one by one ($0.7 \leqslant 0.8$; $0.5 \leqslant 0.6$; and $0.5 \leqslant 0.5$), it is strongly credible that the scheme x_4 is better than the scheme x_3, denoted as $H(x_3) \subset_s H(x_4)$.

(5) We can obtain $H(x_1) \subset_s H(x_5)$ after truncating the tail estimated value of x_5, i.e., deleting the estimated value 0.1 of x_5. This case is denoted as $H(x_1) \subset_{st} H(x_5)$ and is recorded briefly as $H(x_1) \subset_t H(x_5)$.

(6) The worst estimated value of x_6 is greater than or equal to the best estimated value of x_3. Thus, it is necessary that the scheme x_6 is better than the scheme x_3, which is denoted as $H(x_3) \subset_n H(x_6)$.

Definition 2.10. [9] Let H_1 and H_2 be two hesitant fuzzy sets on U. Several kinds of inclusion relationships of two hesitant fuzzy sets are defined as follows,

(1) If $H_1^+(x) \leqslant H_2^+(x)$, then $H_1(x) \subset_p H_2(x)$. If $H_1(x) \subset_p H_2(x)$ for all $x \in U$, then $H_1 \subset_p H_2$. If $H_1 \subset_p H_2$ and $H_2 \subset_p H_1$, then $H_1 =_p H_2$.

(2) If $H_1^+(x) \leqslant H_2^+(x)$ and $H_1^-(x) \leqslant H_2^-(x)$, then $H_1(x) \subset_a H_2(x)$. If $H_1(x) \subset_a H_2(x)$ for all $x \in U$, then $H_1 \subset_a H_2$. If $H_1 \subset_a H_2$ and $H_2 \subset_a H_1$, then $H_1 =_a H_2$.

(3) If $mean[H_1(x)] \leqslant mean[H_2(x)]$, then $H_1(x) \subset_m H_2(x)$. If $H_1(x) \subset_m H_2(x)$ for all $x \in U$, then $H_1 \subset_m H_2$. If $H_1 \subset_m H_2$ and $H_2 \subset_m H_1$, then $H_1 =_m H_2$.

(4) Let $H_1(x) = V = \{v_1, v_2, \cdots, v_k\}$ and $H_2(x) = W = \{w_1, w_2, \cdots, w_l\}$ be two descending sequences. If $k \geqslant l$ and $w_i \geqslant v_i$ for $1 \leqslant i \leqslant l$, then $H_1(x) \subset_s H_2(x)$. If $H_1(x) \subset_s H_2(x)$ for all $x \in U$, then $H_1 \subset_s H_2$. If $H_1 \subset_s H_2$ and $H_2 \subset_s H_1$, then $H_1 =_s H_2$.

(5) Let $H_1(x) = V = \{v_1, v_2, \cdots, v_k\}$ and $H_2(x) = W = \{w_1, w_2, \cdots, w_l\}$ be two descending sequences. If $k < l$ and $w_i \geqslant v_i$ for $1 \leqslant i \leqslant k$, then $H_1(x) \subset_t H_2(x)$. If $H_1(x) \subset_t H_2(x)$ for all $x \in U$, then $H_1 \subset_t H_2$. It is obvious that $H_1 \subset_t H_2$ and $H_2 \subset_t H_1$ cannot hold simultaneously.

(6) If $H_1^+(x) \leqslant H_2^-(x)$, then $H_1(x) \subset_n H_2(x)$. If $H_1(x) \subset_n H_2(x)$ for all $x \in U$, then $H_1 \subset_n H_2$. If $H_1 \subset_n H_2$ and $H_2 \subset_n H_1$, then $H_1 =_n H_2$.

3. Absorption Law and Distributive Law of Hesitant Fuzzy Sets

The hesitant fuzzy sets do not satisfy the absorption law and distributive law. We have provided two counterexamples, shown in Example 3.1 and 3.2.

3.1. Absorption Law

The following example demonstrates that the hesitant fuzzy sets do not satisfy the absorption law.

Example 3.1. Let $U = \{x\}$, $A = \frac{0.3,0.6}{x}$ and $B = \frac{0.55,0.7}{x}$.

$A \cup B = \frac{0.55,0.6,0.7}{x}$, $(A \cup B) \cap A = \frac{0.3,0.55,0.6,0.6}{x}$. $0.55 \in ((A \cup B) \cap A)(x)$, however, $0.55 \notin A(x)$. Hence, $(A \cup B) \cap A \neq A$.

$A \cap B = \frac{0.3,0.55,0.6}{x}$, $(A \cap B) \cup A = \frac{0.3,0.3,0.55,0.6,0.6}{x}$. $0.55 \in ((A \cap B) \cup A)(x)$, however, $0.55 \notin A(x)$. Hence, $(A \cap B) \cup A \neq A$.

3.2. Distributive Law

The Theorem 2.8 in [22] and the Theorem 1 in [23] show that hesitant fuzzy sets satisfy the distributive laws about intersection and union operations. We raise doubts about this proposition. A counterexample is shown as below.

Example 3.2. Let $U = \{x\}$, $A = \frac{\{0.1,0.35\}}{x}$, $B = \frac{\{0.55,0.7\}}{x}$, $C = \frac{\{0.3,0.6\}}{x}$.

$((A \cap B) \cup C)(x) = \{0.3, 0.35, 0.6\}$, $((A \cup C) \cap (B \cup C))(x) = \{0.3, 0.35, 0.55, 0.6, 0.6\}$.

$0.55 \notin ((A \cap B) \cup C)(x)$, however, $0.55 \in ((A \cup C) \cap (B \cup C))(x)$, then $(A \cap B) \cup C \neq (A \cup C) \cap (B \cup C)$.

$((A \cup B) \cap C)(x) = \{0.3, 0.55, 0.6\}$, $((A \cap C) \cup (B \cap C))(x) = \{0.3, 0.3, 0.35, 0.55, 0.6\}$.

$0.35 \notin ((A \cup B) \cap C)(x)$, however, $0.35 \in ((A \cap C) \cup (B \cap C))(x)$, then $(A \cup B) \cap C \neq (A \cap C) \cup (B \cap C)$.

4. Hesitant Fuzzy Topology Spaces

Topology [24] is an important foundation of mathematics and computer science, it is widely applied in various fields, such as precision feeding system [25], object detection and localization [26], structural analysis [27].

This section presents our viewpoints on the possible errors that may exist in the literatures on hesitant fuzzy topological spaces. If the researchers need to cite the propositions mentioned in this section in their subsequent work, they should verify them again.

4.1. Empty Hesitant Fuzzy Set and Full Hesitant Fuzzy Set

In current literatures of hesitant fuzzy sets, there are two different definitions for empty hesitant fuzzy set H^0 and full hesitant fuzzy set H^1 in hesitant fuzzy topology spaces, (i) $H^0(x) = \{0\}$ and $H^1(x) = \{1\}$ for all $x \in U$ [8,21], (ii) $H^0(x) = \emptyset$ and $H^1(x) = [0,1]$ for all $x \in U$ [18–20,28]. The references [18–20,28] cited the reference [8] or [21] to obtain the definitions of empty hesitant fuzzy set and full hesitant fuzzy set. However, the references [18–20,28] generated a new different type of definitions to compare with the cited references [8,21].

First, let's review the construction of classical sets [29] and fuzzy sets [1,30–33]. Let $U = \{x_1, x_2, x_3\}$ be the universe of discourse, two classical sets $A = \{x_1\}$ and $B = \{x_2, x_3\}$ can be described as the form of fuzzy sets, $A = \{\frac{1}{x_1}, \frac{0}{x_2}, \frac{0}{x_3}\}$ and $B = \{\frac{0}{x_1}, \frac{1}{x_2}, \frac{1}{x_3}\}$. $A = \{\frac{1}{x_1}, \frac{0}{x_2}, \frac{0}{x_3}\}$ indicates that the probability of x_1 belonging to set A is 100%, while the probabilities of x_2 and x_3 belonging to set A are both 0%. The range of probability values, which was originally $\{0,1\}$, is extended to $[0,1]$, resulting in a fuzzy set, such as $F = \{\frac{0.3}{x_1}, \frac{0.8}{x_2}, \frac{0.5}{x_3}\}$. To compared with fuzzy sets, hesitant fuzzy sets have an additional element of hesitation, such as $H = \{\frac{0.3,0.5}{x_1}, \frac{0.8,0.9}{x_2}, \frac{0.5,0.6,0.8}{x_3}\}$. When the values in $H(x)$ are the same values, the hesitant fuzzy sets degenerate into fuzzy sets, such as $H = \{\frac{0,0}{x_1}, \frac{1,1,1}{x_2}, \frac{0.8,0.8}{x_3}\} = \{\frac{0}{x_1}, \frac{1}{x_2}, \frac{0.8}{x_3}\}$.

$H^0(x) = \emptyset$ indicates that the probability domain for x belonging to H^0 is $\emptyset$, i.e., it is meaningless to discuss whether x belongs to H^0. $H^1(x) = [0,1]$ In a classical topological space, all elements of the domain belong to the full set. $H^1(x) = \{1\}$ for all $x \in U$ means that the probability that x belongs to H^1 is 100% for all $x \in U$. However, $H^1(x) = [0,1]$ means that the probability that x belongs to H^1 is a random number in $[0,1]$. The definitions in [8,21] are more reasonable than the other.

4.2. The Intersection and Union Operations of the Elements in Hesitant Fuzzy Topology

For a hesitant fuzzy topology τ, (i) if $H, G \in \tau$, then $H \cap G \in \tau$; (ii) if $H_\alpha \in \tau$ for $\alpha \in \Gamma$, then $\cup_{\alpha\in\Gamma} H_\alpha \in \tau$. However, it is very difficult to obtain a hesitant fuzzy topology to satisfy the conditions (i) and (ii).

For instance, the Example 1 in reference [18], $H_1(b) = \{0.3, 0.6, 0.9\}$, $H_2(b) = \{0.3, 0.6, 0.8\}$, $(H_1 \cap H_2)(b) = \{0.3, 0.3, 0.6, 0.6, 0.8\}$. $(H_1 \cap H_2)(b) \neq H^0(b)$, $(H_1 \cap H_2)(b) \neq H_1(b)$, $(H_1 \cap H_2)(b) \neq H_2(b)$, $(H_1 \cap H_2)(b) \neq H_3(b) = \{0.3, 0.6\}$, $(H_1 \cap H_2)(b) \neq H_4(b) = \{0.3, 0.6, 0.8, 0.9\}$, $(H_1 \cap H_2)(b) \neq H^1(b)$, i.e., $H_1 \cap H_2 \notin \tau$, where $\tau = \{H^0, H_1, H_2, H_3, H_4, H^1\}$ [18].

The conditions (i) and (ii) above can be weakened to obtain hesitant fuzzy m-topology and a-topology as follows.

Definition 4.1. Let U be a nonempty set, and let $\tau \subset HF(U)$, τ is called a hesitant fuzzy m-topology on U if it satisfies the following axioms:

(1) $H^0, H^1 \in \tau$.

(2) For $H, G \in \tau$, there is $H' \in \tau$ such that $H \cap G =_m H'$.

(3) For $\alpha \in \Gamma$ and $H_\alpha \in \tau$, there is $H'' \in \tau$ such that $\cup_{\alpha\in\Gamma} H_\alpha =_m H''$.

Definition 4.2. Let U be a nonempty set, and let $\tau \subset HF(U)$, τ is called a hesitant fuzzy a-topology on U if it satisfies the following axioms:

(1) $H^0, H^1 \in \tau$.

(2) For $H, G \in \tau$, there is $H' \in \tau$ such that $H \cap G =_a H'$.

(3) For $\alpha \in \Gamma$ and $H_\alpha \in \tau$, there is $H'' \in \tau$ such that $\cup_{\alpha\in\Gamma} H_\alpha =_a H''$.

Example 4.3. Let $U = \{x, y\}$, $H^0 = \{\frac{0}{x}, \frac{0}{y}\}$, $H_1 = \{\frac{0.1,0.2}{x}, \frac{0.6,0.7}{y}\}$, $H_2 = \{\frac{0.1,0.15,0.2}{x}, \frac{0.6,0.65,0.7}{y}\}$, $H_3 = \{\frac{0.3,0.4}{x}, \frac{0.8,0.9}{y}\}$, $H_4 = \{\frac{0.1,0.2,0.3,0.4}{x}, \frac{0.6,0.7,0.8,0.9}{y}\}$, and $H^1 = \{\frac{1}{x}, \frac{1}{y}\}$.

(1) Let $\tau = \{H^0, H_1, H_2, H^1\}$. $H_1 \cap H_2 = \{\frac{0.1,0.1,0.15,0.2,0.2}{x}, \frac{0.6,0.6,0.65,0.7,0.7}{y}\} = H_1 \cup H_2$, $H_1 \cap H_2 =_m H_1$ and $H_1 \cup H_2 =_m H_1$. Obvious, $H_1 \cap H_2 \notin \tau$. Then, τ is a hesitant fuzzy m-topology not a hesitant fuzzy topology.

(2) Let $\tau = \{H^0, H_1, H_3, H_4, H^1\}$. $H_1 \cap H_4 =_a H_1$, $H_1 \cup H_4 =_a H_4$, $H_3 \cap H_4 =_a H_4$, $H_3 \cup H_4 =_a H_3$. Obvious, $H_3 \cup H_4 \notin \tau$. Then, τ is a hesitant fuzzy a-topology not a hesitant fuzzy topology.

(3) Let $\tau = \{H^0, H_1, H_3, H^1\}$. τ is a hesitant fuzzy topology, hesitant fuzzy m-topology and hesitant fuzzy a-topology.

4.3. Inferences Based on Inclusion Relationships

The proof of the mathematical theory is logically rigorous and clear. The logical relationship of the definitions used in the proof process is also clear. Unclear or inexact logical relationships can lead to incorrect proofs and the construction of erroneous theorems that are then cited by later generations.

For example, the Definition 2 in [18] is not clear enough, "$h_1 \subset h_2$ if $h_1(x) \subset h_2(x)$ for each $x \in X$". It do not express the condition of $h_1(x) \subset h_2(x)$. $h_1(x)$ and $h_2(x)$ do not necessarily have to be two real numbers; they could be two discrete arrays, then the condition of $h_1(x) \subset h_2(x)$ needs a clear expression. The logical relationship of Definition 2 in [18] in not clear enough. However, the proof of Theorem 1 in [18] used the inclusion relationships "$B \subset B_1 \tilde{\cap} B_2$".

The similar situations where the definition is unclear also occurs in Definition 2.2 of reference [19], Definition 2 of reference [20], Definition 2.14 of reference [21], etc.

5. Hesitant Fuzzy Approximate Spaces

This section presents our viewpoints on the possible errors that may exist in the literatures on hesitant fuzzy approximate spaces. If the researchers need to cite the propositions mentioned in this section in their subsequent work, they should verify them again.

5.1. Discussions on $\underline{R}(A \cap B)$ and $\underline{R}(A) \cap \underline{R}(B)$

Are $\underline{R}(A \cap B)$ and $\underline{R}(A) \cap \underline{R}(B)$ equal? The references [23,34,35] give this proposition $\underline{R}(A \cap B) = \underline{R}(A) \cap \underline{R}(B)$. This review challenges this proposition and provides a counterexample as follows. Subsequent researchers should verify this proposition again before citing it.

Example 5.1. Let $U = \{x_1, x_2\}$, $A = \frac{\{0.3,0.6\}}{x_1} + \frac{\{0.3,0.5\}}{x_2}$ and $B = \frac{\{0.4,0.7\}}{x_1} + \frac{\{0.55,0.6\}}{x_2}$. $R(U,U)$ is shown as follows,

$$R(U,U) = \begin{bmatrix} \{1\} & \{0.3,0.4,0.7\} \\ \{0.3,0.4,0.7\} & \{1\} \end{bmatrix}.$$

$A \cap B = \frac{\{0.3,0.4,0.6\}}{x_1} + \frac{\{0.3,0.5\}}{x_2}$.

$\underline{R}(A \cap B)(x_1) = \bigcap_{y \in U} [R^c(x_1,y) \cup (A \cap B)(y)] = [\{0\} \cup \{0.3,0.4,0.6\}] \cap [\{0.3,0.6,0.7\} \cup \{0.3,0.5\}] = \{0.3,0.4,0.6\} \cap \{0.3,0.3,0.5,0.6,0.7\} = \{0.3,0.3,0.3,0.4,0.5,0.6,0.6\}$.

$\underline{R}(A)(x_1) = \bigcap_{y \in U} [R^c(x_1,y) \cup A(y)] = [\{0\} \cup \{0.3,0.6\}] \cap [\{0.3,0.6,0.7\} \cup \{0.3,0.5\}] = \{0.3,0.6\} \cap \{0.3,0.3,0.5,0.6,0.7\} = \{0.3,0.3,0.3,0.5,0.6,0.6\}$.

$\underline{R}(B)(x_1) = \bigcap_{y \in U} [R^c(x_1,y) \cup B(y)] = [\{0\} \cup \{0.4,0.7\}] \cap [\{0.3,0.6,0.7\} \cup \{0.55,0.6\}] = \{0.4,0.7\} \cap \{0.55,0.6,0.6,0.7\} = \{0.4,0.55,0.6,0.6,0.7,0.7\}$.

$(\underline{R}(A) \cap \underline{R}(B))(x_1) = \{0.3,0.3,0.3,0.5,0.6,0.6\} \cap \{0.4,0.55,0.6,0.6,0.7,0.7\} = \{0.3,0.3,0.3,0.4,0.5,0.55, 0.6,0.6,0.6,0.6\}$.

$0.55 \in (\underline{R}(A) \cap \underline{R}(B))(x_1)$ and $0.55 \notin \underline{R}(A \cap B)(x_1)$, then $(\underline{R}(A) \cap \underline{R}(B))(x_1) \neq \underline{R}(A \cap B)(x_1)$. Hence, $\underline{R}(A) \cap \underline{R}(B) \neq \underline{R}(A \cap B)$.

5.2. Discussions on $\overline{R}(A \cup B)$ and $\overline{R}(A) \cup \overline{R}(B)$

Are $\overline{R}(A \cup B)$ and $\overline{R}(A) \cup \overline{R}(B)$ equal? The references [23,34,35] give this proposition $\overline{R}(A \cup B) = \overline{R}(A) \cup \overline{R}(B)$. This review challenges this proposition and provides a counterexample as follows. Subsequent researchers should verify this proposition again before citing it.

Example 5.2. Let $U = \{x_1, x_2\}$, $A = \frac{\{0.3,0.6\}}{x_1} + \frac{\{0.3,0.45\}}{x_2}$ and $B = \frac{\{0.4,0.7\}}{x_1} + \frac{\{0.5,0.6\}}{x_2}$. $R(U,U)$ is shown as follows,

$$R(U,U) = \begin{bmatrix} \{1\} & \{0.3,0.4,0.7\} \\ \{0.3,0.4,0.7\} & \{1\} \end{bmatrix}.$$

$A \cup B = \frac{\{0.4,0.6,0.7\}}{x_1} + \frac{\{0.5,0.6\}}{x_2}$.

$\overline{R}(A \cup B)(x_1) = \bigcup_{y \in U} [R(x_1,y) \cap (A \cup B)(y)] = [\{1\} \cap \{0.4,0.6,0.7\}] \cup [\{0.3,0.4,0.7\} \cap \{0.5,0.6\}] = \{0.4,0.6,0.7\} \cup \{0.3,0.4,0.5,0.6\} = \{0.4,0.4,0.5,0.6,0.6,0.7\}$.

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$\overline{R}(A)(x_1) = \bigcup_{y\in U}[R(x_1,y)\cap A(y)] = [\{1\}\cap\{0.3,0.6\}]\cup[\{0.3,0.4,0.7\}\cap\{0.3,0.45\}] = \{0.3,0.6\}\cup\{0.3,0.3,0.4,0.45\} = \{0.3,0.3,0.3,0.4,0.45,0.6\}$.

$\overline{R}(B)(x_1) = \bigcup_{y\in U}[R(x_1,y)\cap B(y)] = [\{1\}\cap\{0.4,0.7\}]\cup[\{0.3,0.4,0.7\}\cap\{0.5,0.6\}] = \{0.4,0.7\}\cup\{0.3,0.4,0.5,0.6\} = \{0.4,0.4,0.5,0.6,0.7\}$.

$(\overline{R}(A)\cup\overline{R}(B))(x_1) = \{0.3,0.3,0.3,0.4,0.45,0.6\}\cup\{0.4,0.4,0.5,0.6,0.7\} = \{0.4,0.4,0.4,0.45,0.5,0.6,0.6,0.7\}$.

$0.45 \in (\overline{R}(A)\cup\overline{R}(B))(x_1)$ and $0.45 \notin \overline{R}(A\cup B)(x_1)$, then $(\overline{R}(A)\cup\overline{R}(B))(x_1) \neq \overline{R}(A\cup B)(x_1)$. Hence, $\overline{R}(A)\cup\overline{R}(B) \neq \overline{R}(A\cup B)$.

6. Hesitant Fuzzy Algebras

This section presents our viewpoints on the possible errors that may exist in the literatures on hesitant fuzzy algebras. If the researchers need to cite the propositions mentioned in this section in their subsequent work, they should verify them again.

6.1. Inferences Based on Inclusion Relationships

The proof of the mathematical theory is logically rigorous and clear. The logical relationship of the definitions used in the proof process is also clear. The references [36–47] on hesitant fuzzy algebras have used the inclusion relationships of hesitant fuzzy sets, but there is no clear definition of the inclusion relationships of hesitant fuzzy sets.

6.2. Inferences Based on Union Operation of Hesitant Fuzzy Sets

The formula (3.3) in [36] describes that $G(a)\subset G(x)$ and $G(b)\subset G(y)$ then $G(a)\cup G(b)\not\subset G(x)\cup G(y)$. This proposition holds for $\subset_a$, but does not hold for $\subset_m$. For example, let $G(a)=\{0.1,0.9\}$, $G(x)=\{0.5,0.6\}$, $G(b)=\{0.5,0.6\}$ and $G(y)=\{0.5,0.7\}$. $G(a)\cup G(b)=\{0.5,0.6,0.9\}=0.666$, $G(y)=\{0.5,0.5,0.6,0.7\}=0.575$. $G(a)\subset_m G(x)$ and $G(b)\subset_m G(y)$, however, $G(a)\cup G(b)\not\subset_m G(x)\cup G(y)$. The reference [36] does not provide the definition of the inclusion relationships of hesitant fuzzy sets.

The proof of the Proposition 3.9 in [36] uses an inconclusive conclusion that $G(a)\subset G(x)$ and $G(b)\subset G(y)$ then $G(a)\cup G(b)\subset G(x)\cup G(y)$. Even though $G(x)\subset G(x)$ and $G(0*y)\subset G(0)\cup G(y)$ are true, but $G(x)\cup G(0*y)\subset G(x)\cup G(0)\cup G(y)$ may not be true. For example, let $G(x)=\{0.1,0.9\}$, $G(0*y)=\{0.4,0.5\}$ and $G(0)\cup G(y)=\{0.1,0.8\}$, then $G(x)\cup G(0*y)=\{0.4,0.5,0.9\}$ and $G(x)\cup G(0)\cup G(y)=\{0.1,0.1,0.8,0.9\}$. $G(x)\subset_m G(x)$ and $G(0*y)\subset_m G(0)\cup G(y)$, however $G(x)\cup G(0*y)\not\subset_m G(x)\cup G(0)\cup G(y)$. Hence, the propositions about $\cup$-hesitant fuzzy subalgebra G of a BCI-algebra X may need to be re-verified.

6.3. Inferences Based on Intersection Operation of Hesitant Fuzzy Sets

The proof of Proposition 3.3 in the reference [40] describes that $C_1\cap C_2\subset C_1$ and $C_1\cap C_2\subset C_2$. $C_1\cap C_2\subset_a C_1$ and $C_1\cap C_2\subset_a C_2$ hold simultaneously. However, $C_1\cap C_2\subset_m C_1$ and $C_1\cap C_2\subset_m C_2$ do not necessarily both hold at the same time. For example, let $U=\{x\}$, $C_1(x)=\{0.1,0.7\}$ and $C_2(x)=\{0.6,0.8\}$, then $(C_1\cap C_2)(x)=\{0.1,0.6,0.7\}$ and $(C_1\cap C_2)(x)\not\subset_m C_1(x)$. Then, the conclusion of the Proposition 3.3 in [40] that $C_1\cap C_2$ is a chained hesitant fuzzy $\mathcal{R}$-module needs to be re-verified.

6.4. Inferences Based on Distributive Law of Hesitant Fuzzy Sets

The Example 3.2 shows that the hesitant fuzzy sets do not satisfy the distributive laws for intersection and union operations.

The Theorem 2.8 in reference [22] needs to be re-verified. The proof of the Theorem 4.2 (in [22]) uses the result of the Theorem 2.8 (in [22]), the Corollary 4.3 (in [22]) is obtained by the result of Theorem 4.2 (in [22]). Hence, the result in Corollary 4.3 (in [22]), i.e., $(\mathcal{G}(U,E),\widetilde{\Cap},\widetilde{\cup})$ is a distributive quasilattice, needs to be re-verified.

7. Conclusion

In the mathematical system, no errors can occur; otherwise, it will lead to the collapse of the mathematical knowledge system. This review raises some questions regarding some propositions about hesitant fuzzy sets and their mathematical structures. For instance, propositions about the absorption rate and distributive law of hesitant fuzzy sets, the universal set and empty set of hesitant fuzzy topological spaces, propositions about upper and lower approximations of hesitant fuzzy approximation spaces, and propositions about the distributive law of hesitant fuzzy algebras.

This review only randomly selects some propositions from the literature that may be incorrect for discussion. Regarding the correctness of other literature, scholars should be cautious when citing them.

Acknowledgments: The authors thank the editors and the anonymous reviewers for their helpful comments and suggestions that have led to this improved version of the paper. This work was supported by the National Social Science Foundation Youth Project of China (No. 24CTQ029).

References

1. Zadeh, L.A. Fuzzy sets. *Information and control* **1965**, *8*, 338–353.
2. Wu, X.; Fang, Y.; et al. Application of Near-Infrared Spectroscopy and Fuzzy Improved Null Linear Discriminant Analysis for Rapid Discrimination of Milk Brands. *Foods* **2023**, *12*, 3929.
3. Hu, C.; Xu, H.; et al. Non-destructive Identification of the geographical origin of red jujube by near-infrared spectroscopy and fuzzy clustering methods. *International Journal of Food Properties* **2023**, *26*, 3275–3290.
4. Jin, M.; Zhao, Z.; et al. Improved piezoelectric grain cleaning loss sensor based on adaptive neuro-fuzzy inference system. *Precision Agriculture* **2022**, *23*, 1174–1188.
5. Gorzalczany, M.B. A method of inference in approximate reasoning based on interval-valued fuzzy sets. *Fuzzy Sets and Systems* **1987**, *21*, 1–17.
6. Lu, S.; Cheng, L. Pure rational and non-pure rational decision methods on interval-valued fuzzy soft β-covering approximation spaces. *Expert Systems with Applications* **2025**, *288*, 128262.
7. Atanassov, K.T. Intuitionistic fuzzy sets. *Fuzzy Sets And Systems* **1986**, *32*, 87–96.
8. Torra, V. Hesitant fuzzy sets. *International Journal of Intelligent Systems* **2010**, *25*, 529–539.
9. Lu, S.; Xu, Z.; et al. Foundational theories of hesitant fuzzy sets and hesitant fuzzy information systems and their applications for multi-strength intelligent classifiers. *Information Sciences* **2025**, *714*, 122212.
10. Li, L.; Xu, Y. An extended hesitant fuzzy set for modeling multi-source uncertainty and its applications in multiple-attribute decision-making. *Expert Systems With Applications* **2024**, *238*, 121834.
11. Xin, G.; Ying, L. Multi-attribute decision-making based on comprehensive hesitant fuzzy entropy. *Expert Systems With Applications* **2024**, *237*, 121459.
12. Zhang, J.; Hu, Q.; et al. Hesitant fuzzy three-way concept lattice and its attribute reduction. *Applied Intelligence* **2024**, *54*, 2445–2457.
13. Peng, Z.; Zhang, X. A New Similarity Measure of Hesitant Fuzzy Sets and Its Application. *International Journal of Fuzzy Systems* **2024**, pp. 1–15.
14. Abuasaker, W.; Nguyen, J.; et al. Perceptual maps to aggregate assessments from different rating profiles: A hesitant fuzzy linguistic approach. *Applied Soft Computing* **2023**, *147*, 110803.
15. Pant, M.; Mehra, N. Strong (α, k)-cut and computational-based segmentation based novel hesitant fuzzy time series forecasting model. *Applied Soft Computing* **2024**, *153*, 111251.
16. Xia, M.; Xu, Z. Hesitant fuzzy information aggregation in decision making. *International Journal of Approximate Reasoning* **2011**, *52*, 395–407.
17. Babitha, K.V.; John, S.J. Hesitant fuzzy soft sets. *Journal of New Results in Science* **2013**, *3*, 98–107.
18. Lee, J.G.; Hur, K. Hesitant fuzzy topological spaces. *Mathematics* **2020**, *8*, 188.
19. Swaminathan, A.; Sivaraja, S. Hesitant fuzzy paraopen and hesitant fuzzy mean open sets. *Journal of Applied and Pure Mathematics* **2022**, *4*, 141–150.
20. Ibrahim, H. On some weaker hesitant fuzzy open sets. *Communications Faculty of Sciences University of Ankara Series A1 Mathematics and Statistics* **2021**, *70*, 888–899.
21. Kim, J.; Jun, Y.B.; et al. The category of hesitant H-fuzzy sets. *Annals of Fuzzy Mathematics and Informatics* **2022**, *18*, 57–74.
22. Zhou, X.; Li, Q. Hesitant fuzzy soft set and its lattice structures. *Journal of Computational Analysis & Applications* **2016**, *20*, 72.
23. Yang, X.; Song, X.; et al. Constructive and axiomatic approaches to hesitant fuzzy rough set. *Soft Computing* **2014**, *18*, 1067–1077.
24. Manetti, M. *Topology*; Vol. 153, Springer Nature, 2023.

25. Chen, C.; Liu, X.; et al. Development of the precision feeding system for sows via a rule-based expert system. *International Journal of Agricultural and Biological Engineering* **2023**, *16*, 187–198.
26. Hu, T.; Wang, W.; et al. Research on apple object detection and localization method based on improved YOLOX and RGB-D images. *Agronomy* **2023**, *13*, 1816.
27. Cheng, M.; Hou, X.; et al. Structural Insights into the Catalytic Cycle of Inulin Fructotransferase: From Substrate Anchoring to Product Releasing. *Journal of Agricultural and Food Chemistry* **2024**, *72*, 17030–17040.
28. Swaminathan, A.; Ozel, C.; et al. Hesitant Fuzzy Compactness and Hesitant Fuzzy Regularity in Hesitant Fuzzy Topological Spaces. *European Journal of Pure and Applied Mathematics* **2023**, *16*, 1381–1388.
29. Patras, F. *The Essence of Numbers, Lecture Notes in Mathematics book series*; Springer Press, 2020.
30. Lin, Y.C.; Webster, A.J.; et al. Fuzzy SETS: acknowledging multiple membership of elements within social-ecological-technological systems (SETS) theory. *Ecology and Society* **2025**, *30*, 300122.
31. Wu, X.; Wang, Y.; et al. Classification of Fritillaria using a portable near-infrared spectrometer and fuzzy generalized singular value decomposition. *Industrial Crops and Products* **2024**, *218*, 119032.
32. Zhang, J.; Wu, X.; et al. Near-infrared spectroscopy combined with fuzzy improved direct linear discriminant analysis for nondestructive discrimination of chrysanthemum tea varieties. *Foods* **2024**, *13*, 1439.
33. Cui, B.; Cui, X.; et al. Design and Testing of a Tractor Automatic Navigation System Based on Dynamic Path Search and a Fuzzy Stanley Model. *Agriculture* **2024**, *14*, 2136.
34. Zhang, H.; Shu, L.; et al. On novel hesitant fuzzy rough sets. *Soft Computing* **2019**, *23*, 11357–11371.
35. Zhang, H.; He, Y.; et al. An approximation reduction approach in multi-granulation hesitant fuzzy decision information system. *Journal of Intelligent & Fuzzy Systems* **2019**, *37*, 1555–1567.
36. Song, S.Z.; Bordbar, H.; et al. A new type of hesitant fuzzy subalgebras and ideals in BCK/BCI-algebras. *Journal of Intelligent & Fuzzy Systems* **2017**, *32*, 2009–2016.
37. Jun, Y.B. A new type of p-ideals in BCI-algebras based on hesitant fuzzy sets. *Azerbaijan Journal of Mathematics* **2018**, *8*, 104–115.
38. Iampan, A.; Yamunadevi, S.; et al. A novel extension of hesitant fuzzy sets on UP (BCC)-algebras: neutrosophic hesitant fuzzy sets. *International Journal of Neutrosophic Science* **2023**, *20*, 78–92.
39. Mosrijai, P.; Iampan, A. Anti-type of hesitant fuzzy sets on UP-algebras. *European Journal of Pure and Applied Mathematics* **2018**, *11*, 976–1002.
40. Munshid, K.S.; Muttair, H.H.; et al. Chained Hesitant Fuzzy $\mathcal{R}$-Modules. *International Journal of Mathematics & Computer Science* **2025**, *20*, 439–446.
41. Jun, Y.B.; Lee, K.J.; et al. Hesitant fuzzy bi-ideals in semigroups. *Communications of the Korean Mathematical Society* **2015**, *30*, 143–154.
42. Rezaei, A.; Borumand, S.A. Hesitant fuzzy filters in BE-algebras. *International Journal of Computational Intelligence Systems* **2016**, *9*, 110–119.
43. Ali, A.; Khan, M.; et al. Hesitant fuzzy ideals in Abel-Grassmann's groupoid. *Italian journal of pure and applied mathematics* **2015**, *35*, 537–556.
44. Jun, Y.B.; Roh, E.H.; et al. Hesitant fuzzy p-ideals and quasi-associative ideals in BCI-algebras. *Honam Mathematical Journal* **2022**, *44*, 148–164.
45. Jun, Y.B.; Song, S.Z.; et al. Hesitant fuzzy semigroups with a frontier. *Journal of Intelligent & Fuzzy Systems* **2016**, *30*, 1613–1618.
46. Iampan, A.; Yamunadevi, S.; et al. Hesitant fuzzy set theory applied to hilbert algebras. *Kragujevac Journal of Mathematics* **2026**, *50*, 1119–1133.
47. Muhiuddin, G.; Aldhafeeri, S. Join hesitant fuzzy filters of residuated lattices. *Italian Journal of Pure and Applied Mathematics* **2020**, *43*, 100–114.