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Coupled Fixed Point Theory over Quantale-Valued Quasi-Metric Spaces (QVQMS) with Applications in Generalized Metric Structures

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Abstract

In this study, we establish several coupled fixed point results in quantale-valued quasi-metric spaces, which constitutes a generalization of metric and probabilistic metric spaces. The obtained results will be illustrated with concrete examples. Furthermore, we introduce the concept of θ_s-completeness and, as an application of the main theorems, we derive some results in both quantale-valued partial metric spaces and probabilisic metric spaces.

Keywords: value-quantale; coupled fixed point; quantale-valued quasi-metric space; quantale-valued partial metric space; probabilistic metric space

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1. Introduction

Metric spaces serve as one of the cornerstone of topology and analysis, providing a framework for measuring distances between elements in a set. Extensions of this concept have been developed to address more general situations, such as quasi-metrics [8], partial metrics [17] and probabilistic metric spaces (PMSs) [20]. Quantale-valued generalized metric spaces introduced by Flagg [6], also known as continuity spaces, is a significant generalization of metric spaces. This generalized metric space notion provides a unified setting that combines order theoretic and metric notions. Quantale-valued metric space is obtained by changing the value set $[0, \infty]$ of a classical metric space with a quantale. Within the framework of quantale-valued generalized metric spaces, one can recover metric spaces and probabilistic metric spaces as special examples. Moreover, these structures play a significant role in quantitative domain theory and denotational semantics, see [9]. For more detailed work on these spaces, see [11–15,21,22].

The Banach Contraction Principle [2] led to significant attention for fixed point theory (FPT) which is regarded as a powerful instrument in various diciplines. A variety of extensions of the Banach Fixed Point Theorem have been developed, with the coupled fixed point theorem being a prominent example. This notion was initially presented by Bhaskar and Lakshmikantham [3] and was later expanded by Sabetghadam et al. [19] into the structure of complete cone metric spaces. While FPT has seen substantial progress across numerous generalized metric frameworks, research focusing on fixed point theorems within quantale-valued generalized metric spaces remains rather limited. In this paper, the term quantale-valued quasi-metric space (QVQMS) will be used to denote a separated quantale-valued generalized metric space.

We arrange the paper in the following mannner: In the second section, we present some preliminary definitions and properties that will be frequently used throughout the paper. In the third section, inspired by the works of [5] and [19], we establish coupled fixed point results in quantale-valued quasi-metric spaces. Some of results obtained [19] will be extended to this broader setting. In the final section, we extend the notion of 0-completeness introduced in [18] for quantale-valued partial metric spaces [16] and introduce the concept of θ_s-completeness. Motivated by the study of [10], which reveals that fixed point results established in partial metric spaces can be obtained as consequences of those in metric spaces, we apply our main theorem from section 2 to the quantale-valued partial metric space context. Moreover, since probabilistic metric spaces can be regarded as quantale-valued quasi-metric spaces, we end the paper by deriving a coupled fixed point results for probabilistic metric spaces. The newly obtained fixed point theorems are expected to fill a gap in the literature and open perspectives for applications denotational semantics and theoretical computer science.

2. Preliminaries

Let $(Q, \precsim)$ be a complete lattice, where every subset has a supremum and infimum. We will denote the bottom and top elements of the complete latice Q by θ and e, respectively. The well above relation defined below is an extension of the strict inequality relation in [6], formulated in the context of complete lattices.

Definition 1 (Well above relation, [6])**.** *Let $(Q, \precsim)$ be a complete lattice. Let $q_1, q_2 \in Q$. Then, p is said to be well above q, symbolized as $p \gg q$, if the following holds:*

$$\textit{If } q \succsim \inf U, \textit{ for any subset } U \subseteq Q, \textit{ then one can find } u \in U \textit{ with } p \succsim u.$$

Moreover, if the following condition holds for all $p \in Q$, then Q is called completely distributive lattice:

$$p = \inf\{q \in Q : q \gg p\}.$$

Definition 2 (Value quantale, [6])**.** *A value quantale is defined as a triple $(Q, \precsim, \oplus)$ satisfying the following requirements for all $\varrho \in Q$ and $\{q_i\} \subseteq Q$:*

- *$(Q, \precsim)$ is a completely distributive lattice;*
- *On Q, $\oplus$ serves as a binary operation satisfying associativity and commutativity;*
- *$\varrho \oplus \theta = \varrho$;*
- *$\varrho \oplus \inf_{i \in I}\{q_i\} = \inf_{i \in I}\{\varrho \oplus q_i\}$;*
- *$e \gg \theta$ and if $\varrho_1 \gg \theta$ and $\varrho_2 \gg \theta$, then $\varrho_1 \wedge \varrho_2 \gg \theta$.*

Let $\mu, \nu \in Q$. The mapping $: Q \times Q \to Q$ is given by:

$$\mu\nu = \inf\{\lambda \in Q : \nu \oplus \lambda \succsim \mu\},$$

for further details on its properties, see [6].

Definition 3 (Quantale-Valued Quasi Metric, see [6])**.** *Consider a nonempty set S and value quantale $(Q, \precsim, \oplus)$. A mapping $\delta : S \times S \to Q$ is referred as a quantale-valued quasi-metric whenever for all $\varrho, \upsilon, \varsigma \in S$, the conditions below are satisfied:*

- $\delta(\varrho, \varrho) = \theta,$ *(reflexive property)*
- $\delta(\varrho, \upsilon) = \delta(\upsilon, \varrho) = \theta \Rightarrow \varrho = \upsilon,$ *(separated property)*
- $\delta(\varrho, \upsilon) \precsim \delta(\varrho, \varsigma) \oplus \delta(\varsigma, \upsilon),$ *(transitive property)*

The pair (S, δ) is referred as quantale-valued quasi-metric space (also called a separated continuity space in [6]).

Consider a quantale-valued quasi-metric space (S, δ). Let $\sigma, \upsilon \in S$. Then the mappings δ^{-1} (dual of δ) and δ^s (symmetrization of δ) are specified respectively by

$$\delta^{-1}(\sigma, \upsilon) = \delta(\upsilon, \sigma) \text{ and } \delta^s(\sigma, \upsilon) = \delta(\sigma, \upsilon) \vee \delta(\upsilon, \sigma),$$

see [5]. For the topological properties of this space, the reader may consult [4–7] for further details. Consider a net $(\varrho_\lambda)_{\lambda \in \Lambda}$ in S. $(\varrho_\lambda)_{\lambda \in \Lambda}$ is symmetrically convergent to ϱ whenever for any $\varepsilon \gg \theta$, there exists $\lambda_0 \in \Lambda$ such that for all $\lambda \succcurlyeq \lambda_0$, $\delta^s(\varrho_\lambda, \varrho) \precsim \epsilon$. A net $(\varrho_\lambda)_{\lambda \in \Lambda}$ will be termed Cauchy whenever for every $\varepsilon \gg \theta$ one can find $\lambda_0 \in \Lambda$ such that $\delta(\varrho_\alpha, \varrho_\beta) \precsim \epsilon$ holds for all indices $\alpha, \beta \succcurlyeq \lambda_0$, see [7]. Moreover, if every Cauchy net is symmetrically convergent, then S is called complete, see [7]. If we take a sequence instead of a net in the definition of completeness, then such a quantale-valued quasi-metric space is referred to as s-complete. Every complete quantale-valued quasi-metric space is necessarily s-complete.

Example 1 ([7])**.** *Let us consider the value quantale $(Q, \precsim, \oplus)$. The mapping $\delta : Q \times Q \to Q$ given by $\delta(\mu, \nu) = \nu\mu$ is a quantale-valued quasi-metric on Q and (Q, δ) is complete, see Theorem 4.9 in [7].*

Now, let us recall the definition of probabilistic metric space, see [20]. Let Φ be the collection of distribution functions satisfying monotonicity and left-continuity, that is $\psi : [0, \infty) \to [0, 1]$,

$$\forall x \in [0, \infty), \sup_{y<x} \psi(y) = \psi(x) \text{ (left-continuity).}$$

A mapping $\divideontimes : [0, 1] \times [0, 1] \to [0, 1]$ is regarded as a left continuous t-norm whenever the properties below hold:

1. the operation $\divideontimes$ possesses both associativity and commutativity;
2. $\lambda \divideontimes 1 = \lambda$, for all $\lambda \in [0, 1]$;
3. whenever $\lambda_1 \leq \lambda_2, \lambda_3 \leq \lambda_4$, it holds that $\lambda_1 \divideontimes \lambda_3 \leq \lambda_2 \divideontimes \lambda_4$;
4. $\lambda_1 \divideontimes \lambda_2 = \sup\{u \divideontimes v : 0 < u < \lambda_1, 0 < v < \lambda_2\}$ (left continuity),

see [20]. As stated in [20], a probabilistic metric is a mapping $\Psi : S \times S \to \Phi$ that meets the requirements given below for all $\alpha, \beta, \gamma \in S$ and $\lambda_1, \lambda_2 \in [0, 1]$:

1. $\Psi_{\alpha,\beta} = \Psi_{\beta,\alpha} = \epsilon_0 \Leftrightarrow \alpha = \beta$,
2. $\Psi_{\alpha,\beta} = \Psi_{\beta,\alpha}$,
3. $\Psi_{\alpha,\gamma}(\lambda_1 + \lambda_2) \geq \Psi_{\alpha,\beta}(\lambda_1) \divideontimes \Psi_{\beta,\gamma}(\lambda_2)$,

where $\Psi(\alpha, \beta) = \Psi_{\alpha,\beta}$ and ϵ_0 is the top element of Φ according to the pointwise order and defined by

$$\epsilon_0(\lambda) = \begin{cases} 0, & \lambda = 0 \\ 1, & \lambda \neq 0 \end{cases}$$

and $(S, \Psi, \divideontimes)$ is referred as a probabilistic metric space (PMS). A sequence $(\varrho_n)_{n\in\mathbb{N}}$ in S strongly converges to $\varrho \in S$ if $\zeta > 0$ is given, there exists $\tilde{n} \in \mathbb{N}$ beyond which

$$\Psi_{\varrho_n,\varrho}(\zeta) > 1 - \zeta \qquad \text{whenever } n \geq \tilde{n}$$

holds. It is termed a strong Cauchy when, for any $\zeta > 0$, some $\tilde{n} \in \mathbb{N}$ can be found with

$$\Psi_{\varrho_n,\varrho_m}(\zeta) > 1 - \zeta \qquad \text{whenever } m, n \geq \tilde{n}.$$

Strong completeness of $(S, \Psi, \divideontimes)$ means that any strong Cauchy sequence in S possesses a strong limit within S, as described in [20].

As shown in [6], PMSs can be obtained as a particular subclass within the framework of QVQMSs. In particular, consider the PMS $(S, \Psi, \divideontimes)$, where $\divideontimes$ be a left continuous t-norm. If we equip Φ the opposite pointwise order $\leq^{op}$ defined by:

$$\psi_1 \leq^{op} \psi_2 \Leftrightarrow \psi_1(x) \geq \psi_2(x), \text{ for all } x \in [0, \infty)$$

and take the binary operation $\circledast$ given by,

$$(\psi_1 \circledast \psi_2)(x) = \bigvee_{y+z=x} \psi_1(y) \divideontimes \psi_2(z) \tag{1}$$

as the quantale operation, then it follows that $(\Phi, \leq^{op}, \circledast)$ is indeed a value quantale and (S, Ψ) is a QVQMS. Moreover, strong complete PMSs corresponds to an s-complete QVQMSs. We end this section by recalling the definition of action introducen in [4]. We will consider the following definition of the action:

Definition 4. *Consider a value quantale* $(Q, \precsim, \oplus)$*. An action of* $[0, \infty]$ *on* Q *is a monotone mapping*

$$\otimes : [0, \infty] \times Q \to Q$$

such that, for all $\lambda, \gamma \in [0, \infty]$ *and* $\varrho_1, \varrho_2 \in Q$ *the following conditions hold:*

1. $1 \otimes \varrho_1 = \varrho_1$ *and* $\varrho_1 \succcurlyeq \lambda \otimes \varrho_1$ *for* $\lambda \neq 1$*;*
2. $(\lambda\gamma) \otimes \varrho_1 = \lambda \otimes (\gamma \otimes \varrho_1)$*;*
3. $(\lambda + \gamma) \otimes \varrho_1 = (\lambda \otimes \varrho_1) \oplus (\gamma \otimes \varrho_1)$*;*
4. $(\lambda \otimes \varrho_1) \oplus (\lambda \otimes \varrho_2) \precsim \lambda \otimes (\varrho_1 \oplus \varrho_2)$.

Example 2. *[4] Given* $\lambda \in (0, \infty]$ *and* $\psi \in \Phi$*, set*

$$(\lambda \otimes \psi)(x) = \psi(\frac{x}{\lambda}), \text{ for all } x \in [0, \infty). \tag{2}$$

Then $\otimes$ *defines an action.*

3. Results on Coupled Fixed Points within QVQMS

For convenience, we shall denote the structure $(Q, \precsim, \oplus)$ endowed with the action $\otimes$ by $(Q, \precsim, \oplus, \otimes)$ and we shall refer to $(Q, \precsim, \oplus, \otimes)$ as quantale-action structure in the paper.

As introduced in [3], the coupled fixed point has the following QVQMS-version:

Definition 5. *Consider a value quantale* $(Q, \precsim, \oplus)$*. Let* (S, δ) *be a QVQMS. The pair* (ϱ, ς) *is a coupled fixed point of* $P : S \times S \to S$ *whenever*

$$P(\varrho, \varsigma) = \varrho, \qquad P(\varsigma, \varrho) = \varsigma.$$

Theorem 1. *Consider a quantale-action structure* $(Q, \precsim, \oplus, \otimes)$*, and let* (S, δ) *be an s-complete QVQMS. Consider a mapping* $P : S \times S \to S$ *satisfies the following condition for all* $\varrho, \varsigma, \xi, \tau \in S$*:*

$$\delta(P(\varrho, \varsigma), P(\xi, \tau)) \precsim \lambda \otimes [\delta(\varrho, \xi) \vee \delta(\varsigma, \tau)], \tag{3}$$

where $\lambda \in [0, 1]$*. Suppose that there exists a couple* (ϱ_0, ς_0) *satisfying the following requirement:*

$$\bigwedge_{n \in \mathbb{N}} \bigvee_{m \geq n} \lambda^m \otimes [\delta^s(\varrho_0, P(\varrho_0, \varsigma_0)) \vee \delta^s(\varsigma_0, P(\varsigma_0, \varrho_0))] = \theta. \tag{4}$$

Then, P admits a unique coupled fixed point in S.

Proof. Let (ϱ_0, ς_0) be a point satisfying condition (4) and set

$$\varrho_1 = P(\varrho_0, \varsigma_0), \varrho_2 = P(\varrho_1, \varsigma_1), \ldots, \varrho_{n+1} = P(\varrho_n, \varsigma_n)$$

and

$$\varsigma_1 = P(\varsigma_0, \varrho_0), \varsigma_2 = P(\varsigma_1, \varrho_1), \ldots, \varsigma_{n+1} = P(\varsigma_n, \varrho_n).$$

Then, by condition (3), it follows that

$$\begin{aligned}\delta(\varrho_n, \varrho_{n+1}) &= \delta(P(\varrho_{n-1}, \varsigma_{n-1}), P(\varrho_n, \varsigma_n)) \\ &\precsim \lambda \otimes [\delta(\varrho_{n-1}, \varrho_n) \vee \delta(\varsigma_{n-1}, \varsigma_n)].\end{aligned}$$

Analogously, we have

$$\begin{aligned}\delta(\varsigma_n, \varsigma_{n+1}) &= \delta(P(\varsigma_{n-1}, \varrho_{n-1}), P(\varsigma_n, \varrho_n)) \\ &\precsim \lambda \otimes [\delta(\varsigma_{n-1}, \varsigma_n) \vee \delta(\varrho_{n-1}, \varrho_n)].\end{aligned}$$

Define $\zeta_n := \delta(\varrho_n, \varrho_{n+1}) \vee \delta(\varsigma_n, \varsigma_{n+1})$. Using the above inequalities and the monotonicity of the action $\otimes$, we have

$$\begin{aligned}\zeta_n &= \delta(\varrho_n, \varrho_{n+1}) \vee \delta(\varsigma_n, \varsigma_{n+1}) \\ &\precsim \lambda \otimes \big[\delta(\varrho_{n-1}, \varrho_n) \vee \delta(\varsigma_{n-1}, \varsigma_n)\big] \\ &= \lambda \otimes \zeta_{n-1} \\ &\vdots \\ &\precsim \lambda^n \otimes \zeta_0.\end{aligned}$$

When $\zeta_0 = \theta$, one can readily see that (ϱ_0, ς_0) is a coupled fixed point. Therefore, let us assume that $\zeta_0 \gg \theta$. By the definition of the well-above relation and inequality (4), for any $\epsilon \gg \theta$ one can find $n_0 \in \mathbb{N}$ such that, whenever $n_0 \leq m < n$, we obtain:

$$\delta(\varrho_m, \varrho_n) \precsim \delta(\varrho_m, \varrho_{m+1}) \oplus \ldots \oplus \delta(\varrho_{n-1}, \varrho_n)$$

and

$$\delta(\varsigma_m, \varsigma_n) \precsim \delta(\varsigma_m, \varsigma_{m+1}) \oplus \ldots \oplus \delta(\varsigma_{n-1}, \varsigma_n).$$

Applying the supremum to each side of the above two inequalities, we deduce:

$$\begin{aligned}\delta(\varrho_m, \varrho_n) \vee \delta(\varsigma_m, \varsigma_n) &\precsim (\delta(\varrho_m, \varrho_{m+1}) \vee \delta(\varsigma_m, \varsigma_{m+1})) \oplus \ldots \oplus (\delta(\varrho_{n-1}, \varrho_n) \vee \delta(\varsigma_{n-1}, \varsigma_n)) \\ &\precsim (\lambda^m \otimes \zeta_0) \oplus \ldots \oplus (\lambda^{n-1} \otimes \zeta_0) \\ &\precsim (\lambda^m + \ldots + \lambda^{n-1}) \otimes \zeta_0 \\ &\precsim \frac{\lambda^m}{1-\lambda} \otimes \zeta_0 \precsim \frac{\lambda^m}{1-\lambda} \otimes [\delta^s(\varrho_0, \varrho_1) \vee \delta^s(\varsigma_0, \varsigma_1)] \precsim \epsilon.\end{aligned}$$

Thus, we have $\delta^s(\varrho_m, \varrho_n) \precsim \epsilon$ and $\delta^s(\varsigma_m, \varsigma_n) \precsim \epsilon$. Hence, the sequences (ϱ_n) and (ς_n) are Cauchy. Since (S, δ) is s-complete, there exists points $\varrho*, \varsigma* \in S$ such that $\varrho_n \to \varrho*$ and $\varsigma_n \to \varsigma*$ with respect to ρ^s. According to Theorem 2.9 in [6], for each $\epsilon \gg \theta$ one can find $\mu \gg \theta$ with $2\mu = \mu \oplus \mu \ll \epsilon$ and $n_0 \in \mathbb{N}$ such that, for all $n \geq n_0$, $\delta^s(\varrho_n, \varrho*) \precsim \mu$ and $\delta^s(\varsigma_n, \varsigma*) \precsim \mu$. We now show that $(\varrho*, \varsigma*)$ is a coupled fixed point of P:

$$\begin{aligned}\delta^s(P(\varrho*, \varsigma*), \varrho*) &\precsim \delta^s(P(\varrho*, \varsigma*), \varrho_{n+1}) \oplus \delta^s(\varrho_{n+1}, \varrho*) \\ &\precsim \delta^s(P(\varrho*, \varsigma*), P(\varrho_n, \varsigma_n)) \oplus \delta^s(\varrho_{n+1}, \varrho*) \\ &\precsim [\lambda \otimes (\delta^s(\varrho*, \varrho_n) \vee \delta^s(\varsigma*, \varsigma_n))] \oplus \delta^s(\varrho_{n+1}, \varrho*) \\ &\precsim (\lambda \otimes \mu) \oplus \mu \\ &\precsim \mu \oplus \mu \precsim \epsilon.\end{aligned}$$

Thus, we obtain $P(\varrho*,\varsigma*) = \varrho*$. Moreover, in a similar way, $P(\varsigma*,\varrho*) = \varsigma*$. What is left is to prove that the uniqueness. Suppose, to the contrary, that (ϱ',ς') is a distinct coupled fixed point of P. Then, we have

$$\begin{aligned}\delta(\varrho',\varrho*) &= \delta(P(\varrho',\varsigma'),P(\varrho*,\varsigma*)) \\ &\precsim \lambda\otimes[\delta(\varrho',\varrho*)\vee\delta(\varsigma',\varsigma*)]\end{aligned}$$

and

$$\begin{aligned}\delta(\varsigma',\varsigma*) &= \delta(P(\varsigma',\varrho'),P(\varsigma*,\varrho*)) \\ &\precsim \lambda\otimes[\delta(\varsigma',\varsigma*)\vee\delta(\varrho',\varrho*)].\end{aligned}$$

From the above inequalities, we obtain

$$\delta(\varrho',\varrho*)\vee\delta(\varsigma',\varsigma*) \precsim \lambda\otimes[\delta(\varsigma',\varsigma*)\vee\delta(\varrho',\varrho*)].$$

Since, $\lambda < 1$, this yields a contradiction. Consequently, P has exactly one coupled fixed point. □

Corollary 1. *Consider a quantale-action structure* $(Q,\precsim,\oplus,\otimes)$, *and let* (S,ρ) *be an s-complete QVQMS. Consider a mapping* $P : S\times S\to S$ *satisfies the following condition for all* $\varrho,\varsigma,\xi,\tau\in S$:

$$\delta(P(\varrho,\varsigma),P(\xi,\tau)) \precsim (\alpha\otimes\delta(\varrho,\xi))\oplus(\beta\otimes\delta(\varsigma,\tau)), \tag{5}$$

where $\alpha,\beta\geq 0$ *and* $\alpha+\beta<1$. *Suppose that there exists a couple* (ϱ_0,ς_0) *satisfying the following requirement:*

$$\bigwedge_{n\in\mathbb{N}}\bigvee_{m\geq n}\lambda^m\otimes[\delta^s(\varrho_0,P(\varrho_0,\varsigma_0))\vee\delta^s(\varsigma_0,P(\varsigma_0,\varrho_0))] = \theta. \tag{6}$$

Then, P admits a unique coupled fixed point in S.

Proof. We only need to verify that the contraction assumption (5) yields the one in (3). Let $\alpha,\beta\geq 0$ be constants satisfying $\alpha+\beta<1$. Take arbitrary elements $\varrho,\varsigma,\xi,\tau\in S$ satisfying (5). Then we have

$$\begin{aligned}\delta(P(\varrho,\varsigma),P(\xi,\tau)) &\precsim (\alpha\otimes\delta(\varrho,\xi))\oplus(\beta\otimes\delta(\varsigma,\tau)) \\ &\precsim [\alpha\otimes(\delta(\varrho,\xi)\vee\delta(\varsigma,\tau))]\oplus[\beta\otimes(\delta(\varrho,\xi)\vee\delta(\varsigma,\tau))] \\ &\precsim (\alpha+\beta)\otimes[\delta(\varrho,\xi)\vee\delta(\varsigma,\tau)].\end{aligned}$$

Hence, the proof follows from Theorem 1. □

Corollary 2. *Consider a quantale-action structure* $(Q,\precsim,\oplus,\otimes)$, *and let* (S,δ) *be an s-complete QVQMS. Consider a mapping* $P : S\times S\to S$ *satisfies the following condition for all* $\varrho,\varsigma,\xi,\tau\in S$:

$$\delta(P(\varrho,\varsigma),P(\xi,\tau)) \precsim \frac{\lambda}{2}\otimes[\delta(\varrho,\xi)\oplus\delta(\varsigma,\tau)], \tag{7}$$

where $\lambda\in[0,1)$. *Suppose that there exists a couple* (ϱ_0,ς_0) *satisfying the following requirement:*

$$\bigwedge_{n\in\mathbb{N}}\bigvee_{m\geq n}\lambda^m\otimes[\delta^s(\varrho_0,P(\varrho_0,\varsigma_0))\vee\delta^s(\varsigma_0,P(\varsigma_0,\varrho_0))] = \theta. \tag{8}$$

Then, P admits a unique coupled fixed point in S.

Proof. Take arbitrary $\varrho, \varsigma, \xi, \tau \in S$ satisfying (7). Let $\lambda \in [0,1)$. Then, we obtain the following:

$$\begin{aligned}\delta(P(\varrho,\varsigma),P(\xi,\tau)) &\precsim \frac{\lambda}{2} \otimes [\delta(\varrho,\xi) \oplus \delta(\varsigma,\tau)] \\ &\precsim [\frac{\lambda}{2} \otimes \delta(\varrho,\xi)] \oplus [\frac{\lambda}{2} \otimes \delta(\varsigma,\tau))] \\ &\precsim [\frac{\lambda}{2} \otimes (\delta(\varrho,\xi) \vee \delta(\varsigma,\tau))] \oplus [\frac{\lambda}{2} \otimes (\delta(\varsigma,\tau) \vee \delta(\varrho,\xi))] \\ &= \lambda \otimes (\delta(\varrho,\xi) \vee \delta(\varsigma,\tau)).\end{aligned}$$

This completes the proof by the Theorem 1. ☐

Example 3. *Let $S = \{e, \theta, \alpha, \beta\}$. Consider a partial order $\precsim$ defined by $\precsim = \{(e,e),(\theta,\theta),(\alpha,\alpha),(\beta,\beta),(\alpha,\beta)\}$. Then, $(S,\precsim)$ is a complete lattice. Moreover, let us define the quantale operation $\oplus$ as follows:*

$\oplus$	θ	e	α	β
θ	θ	e	α	β
e	e	e	e	e
α	α	e	α	β
β	β	e	β	β

Take the action $\otimes$ on S defined by $\lambda \otimes \varrho = \varrho$, for all $\lambda \in [0,\infty), \varrho \in S$. Then, $(S, \precsim, \oplus, \otimes)$ is a quantale-action structure. Furthermore, equip S with the QVQMS δ given by $\delta(\varrho,\varsigma) = \varsigma\varrho$. From Theorem 4.9 in [7], one can get that (S,δ) s-complete. According to this metric, the distances between the points can be easily computed, and they are presented in the table below:

$\delta(\cdot,\cdot)$	e	θ	α	β
e	θ	θ	θ	θ
θ	e	θ	α	β
α	e	θ	θ	θ
β	e	θ	β	θ

Now, consider a mapping $P : S \times S \to S$ defined by

$$P(\varrho,\varsigma) = \begin{cases} \alpha \oplus (\varrho \wedge \varsigma), & \varrho \precsim \varsigma \\ \varrho \vee \varsigma, & \textit{otherwise} \end{cases}$$

Obviously, there exists $(e,\theta) \in S \times S$ which satisfies the inequality (4). Therefore, the hypotheses of Theorem 1 hold, and (α,β) is the unique coupled fixed point.

Remark 1. *Let $Q = [0,\infty]$ be equipped with the usual order on the extended non-negative real numbers and with the ordinary adition as the quantale operation. In this case, quantale-valued quasi-metric spaces reduce to classical quasi-metric spaces. If a quasi-metric space is complete with respect to its symmetrization, then it is called a bicomplete quasi-metric space, see [1]. Hence, every bicomplete quasi-metric space can be viewed as a s-complete quantale-valued quasi-metric space. Moreover, consider the action $\otimes$ of $[0,1)$ on $[0,\infty]$ given by*

$$\lambda \otimes r = \begin{cases} \lambda r, & r < \infty \\ \infty, & r = \infty, \end{cases}$$

for $\lambda \in [0,1)$ and $r \in [0,\infty]$, see [5]. Then, the following corollaries follow directly from the main results.

Corollary 3. *Consider a bicomplete quasi-metric space* (S,δ). *Assume that the mapping* $P : S \times S \to S$ *satisfies the following condition for all* $\varrho,\varsigma,\xi,\tau \in S$:

$$\delta(P(\varrho,\varsigma),P(\xi,\tau)) \leq \lambda \max\{\delta(\varrho,\xi),\delta(\varsigma,\tau)\},$$

where $\lambda \in [0,1]$. *Then,* P *admits a unique coupled fixed point in* S.

Corollary 4. *Consider a bicomplete quasi-metric space* (S,δ). *Assume that the mapping* $P : S \times S \to S$ *satisfies the following condition for all* $\varrho,\varsigma,\xi,\tau \in S$:

$$\delta(P(\varrho,\varsigma),P(\xi,\tau)) \leq \alpha\delta(\varrho,\xi) + \beta\delta(\varsigma,\tau),$$

where $\alpha,\beta \geq 0$ *and* $\alpha+\beta < 1$. *Then,* P *admits a unique coupled fixed point in* S.

4. Applications within QVPMS and PMS

QVPMs were defined in [16] as follows:

Definition 6 (Kopperman et al., see [16]). *Consider a nonempty set* S *and value quantale* $(Q,\precsim,\oplus)$. *A mapping* $\delta_p : S \times S \to Q$ *is referred as a quantale-valued partial metric whenever for all* $\varrho,\upsilon,\varsigma \in S$, *the conditions below are satisfied:*

$QP_1)$ $\delta_p(\varrho,\varsigma) \succsim \delta_p(\varrho,\varrho)$,
$QP_2)$ $\delta_p(\varrho,\varsigma) = \delta_p(\varsigma,\varrho)$,
$QP_3)$ $\varrho = \varsigma$ *iff* $\delta_p(\varrho,\varsigma) = \delta_p(\varrho,\varrho) = \delta_p(\varsigma,\varsigma)$,
$QP_4)$ $\delta_p(\varrho,\upsilon) \precsim \delta_p(\varrho,\varsigma) \oplus [\delta_p(\varsigma,\upsilon)\delta_p(\varsigma,\varsigma)]$.

The pair (S,δ_p) *is referred as quantale-valued partial metric space (QVPMS).*

The notion of 0-complete partial metric, originally given by Romaguera in [18], will be carried over to the framework QVPMSs as follows:

Definition 7. *Consider a nonempty set* S *and value-quantale* $(Q,\precsim,\oplus)$. *Let* (S,δ_p) *be a QVQMS. A sequence* $(\varrho_n) \in S$ *is called* θ*-Cauchy if*

$$\lim_{n,m\to\infty} \delta_p(\varrho_n,\varrho_m) = \theta,$$

that is, for any $\epsilon \gg \theta$ *there exists* $n_0 \in \mathbb{N}$ *such that for all* $m,n \geq n_0$, $\delta_p(\varrho_m,\varrho_n) \precsim \epsilon$. *If every* θ*-Cauchy sequence* (ϱ_n) *converges to a point* $\varrho* \in S$ *such that* $\delta_p(\varrho*,\varrho*) = \theta$, *i.e., for any* $\epsilon \gg \theta$, *there exists* $n_0 \in \mathbb{N}$ *such that for all* $n \geq n_0$, $\delta_p(\varrho_n,\varrho*) \precsim \epsilon$, *then* (S,δ_p) *is called* θ_s*-complete QVPMS.*

Example 4. *Consider a value-quantale* $(Q,\precsim,\oplus)$. *Let us define a mapping* $\delta_p : Q \times Q \to Q$ *by* $\delta_p(\varrho,\varsigma) = \varrho \vee \varsigma$, *for all* $\varrho,\varsigma \in S$. *Then,* (Q,δ_p) *is* θ_s*-complete QVPMS. Indeed, we first establish that* (Q,δ_p) *is a QVPMS. Let* $\varrho,\varsigma,\nu \in Q$:

$QP_1)$ $\delta_p(\varrho,\varrho) = \varrho \precsim \delta_p(\varrho,\varsigma) = \varrho \vee \varsigma$,
$QP_2)$ $\delta_p(\varrho,\varsigma) = \varrho \vee \varsigma = \varsigma \vee \varrho = \delta_p(\varsigma,\varrho)$,
$QP_3)$ $\delta_p(\varrho,\varsigma) = \delta_p(\varrho,\varrho) = \delta_p(\varsigma,\varsigma) \Leftrightarrow \varrho = \varrho \vee \varsigma = \varsigma \Leftrightarrow \varrho = \varsigma$,
$QP_4)$ *It is clear that the following inequality can be derived from the definition of supremum and condition* (q_4) *of Definition 2.1 in* [6]:

$$\varrho \precsim (\varrho \vee \varsigma) \oplus [(\varsigma \vee \nu)(\varsigma \vee \varsigma)]. \tag{9}$$

Moreover, from the (2) *property of Theorem 2.2 in* [6]*, one can obtain:*

$$\begin{aligned}\nu \precsim \nu \vee \varsigma &\precsim (\varrho \vee \varsigma) \oplus [(\varsigma \vee \nu)(\varsigma \vee \varrho)] \\ &\precsim (\varrho \vee \varsigma) \oplus [(\varsigma \vee \nu)(\varsigma \vee \varsigma)].\end{aligned} \tag{10}$$

If the supremum of both sides of inequalities (7) *and* (8) *is taken, the following is obtained:*

$$\varrho \vee \nu \precsim (\varrho \vee \varsigma) \oplus [(\varsigma \vee \nu)(\varsigma \vee \varsigma)].$$

Therefore, we conclude that δ_p *is a quantale-valued partial metric. Next, we prove that* (Q, δ_p) *is* θ_s*-complete. Let* (ϱ_n) *be a* θ*-Cauchy sequence. Then, for any* $\epsilon \gg \theta$*, there exists* $n_0 \in \mathbb{N}$ *such that for all* $n_0 \leq m, n$*,*

$$\delta_p(\varrho_m, \varrho_n) = \varrho_m \vee \varrho_n \precsim \epsilon.$$

Since $\epsilon \gg \theta$ *is arbitrary, it follows that* $\varrho_n, \varrho_m = \theta$ *for all* $m, n \geq n_0$*. Consequently,* (ϱ_n) *converges to* θ*, and since* $\delta_p(\theta, \theta) = \theta$*, we deduce that* (Q, δ_p) *is a* θ_s*-complete QVPMS.*

In the subsequent results, we draw inspiration from the work presented in [10]. Our first aim is to demonstrate that Proposition 2.1 in [10] can be generalized to the framework of QVPMSs.

Proposition 1. *Consider a value-quantale* $(Q, \precsim, \oplus)$*. Suppose that* (S, δ_p) *is a* θ_s*-complete QVPMS. Define a mapping* $\delta^* : S \times S \to Q$ *by*

$$\delta^*(\varrho, \varsigma) = \begin{cases} \theta, & \varrho = \varsigma \\ \delta_p(\varrho, \varsigma), & \varrho \neq \varsigma \end{cases}$$

Then, (S, δ^*) *is a s-complete and symmetric QVQMS.*

Proof. First, we show that (S, δ^*) is a symmetric QVQMS. We only verify the triangle inequality, since the other conditions are straightforward. Let $\varrho, \varsigma, \nu \in S$. We now examine the following cases:

- **Case I.** Let $\varrho \neq \varsigma = \nu$. Then we have

$$\begin{aligned}\delta^*(\varrho, \varsigma) = \delta_p(\varrho, \varsigma) &\precsim \delta_p(\varrho, \nu) \oplus [\delta_p(\nu, \varsigma)\delta_p(\nu, \nu)] \\ &\precsim \delta^*(\varrho, \nu) \oplus [\delta_p(\nu, \nu)\delta_p(\nu, \nu)] \\ &\precsim \delta^*(\varrho, \nu) \oplus \theta = \delta^*(\varrho, \nu) \oplus \delta^*(\nu, \varsigma).\end{aligned}$$

- **Case II.** Let $\varrho = \nu \neq \varsigma$. It follows that

$$\begin{aligned}\delta^*(\varrho, \varsigma) = \delta_p(\varrho, \varsigma) &\precsim \delta_p(\varrho, \nu) \oplus [\delta_p(\nu, \varsigma)\delta_p(\nu, \nu)] \\ &\precsim \theta \oplus \delta_p(\nu, \varsigma) = \delta^*(\varrho, \nu) \oplus \delta^*(\nu, \varsigma).\end{aligned}$$

- **Case III.** Let $\varrho \neq \varsigma \neq \nu$. In this case, we obtain

$$\begin{aligned}\delta^*(\varrho, \varsigma) = \delta_p(\varrho, \varsigma) &\precsim \delta_p(\varrho, \nu) \oplus [\delta_p(\nu, \varsigma)\delta_p(\nu, \nu)] \\ &\precsim \delta^*(\varrho, \nu) \oplus [\delta^*(\nu, \varsigma)\delta^*(\nu, \nu)] \\ &\precsim \delta^*(\varrho, \nu) \oplus \delta^*(\nu, \varsigma).\end{aligned}$$

- **Case IV.** Finally, we consider when $\varrho = \varsigma$. Then it s easiy to see that:

$$\delta^*(\varrho, \varsigma) = \theta \precsim \delta^*(\varrho, \nu) \oplus \delta^*(\nu, \varsigma).$$

We now prove that S is s-complete. Let (ϱ_n) be a Cauchy sequence in (S,δ^*). It is sufficient to consider the case where $\varrho_n \neq \varrho_m$ holds whenever $n \neq m$. Then, $\delta^*(\varrho_n,\varrho_m) = \delta_p(\varrho_n,\varrho_m)$ and hence (ϱ_n) is a θ-Cauchy in (S,δ_p). Because S is θ_s-complete, we can quarantee the existence of a point $\varrho^* \in S$ such that the sequence converges to ϱ^* with respect to δ_p. Therefore, for any $\epsilon \gg \theta$, there exists $n_0 \in \mathbb{N}$ such that for all $m,n \geq n_0$, we have $\delta_p(\varrho_n,\varrho^*) \precsim \epsilon$, and hence $\delta^*(\varrho_n,\varrho^*) \precsim \epsilon, \delta^*(\varrho^*,\varrho_n) \precsim \epsilon$. Consequently, we get that (S,δ^*) is s-complete QVQMS. ☐

Corollary 5. *Consider a quantale-action structure* $(Q,\precsim,\oplus,\otimes)$*, and let* (S,δ_p) *be a* θ_s*-complete QVPMS. Let* $P: S\times S \to S$ *be a mapping satisfying the following condition for all* $\varrho,\varsigma,\xi,\tau \in S$*:*

$$\delta_p(P(\varrho,\varsigma),P(\xi,\tau)) \precsim (\alpha\otimes\delta_p(\varrho,\xi)) \oplus (\beta\otimes\delta_p(\varsigma,\tau)), \tag{11}$$

where $\alpha,\beta \geq 0$ *with* $\alpha+\beta<1$*. Suppose that there exists a couple* (ϱ_0,ς_0) *satisfying the following requirement:*

$$\bigwedge_{n\in\mathbb{N}}\bigvee_{m\geq n}\lambda^m \otimes [\delta_p(\varrho_0,P(\varrho_0,\varsigma_0)) \vee \delta_p(\varsigma_0,P(\varsigma_0,\varrho_0))] = \theta. \tag{12}$$

Then, P admits a unique coupled fixed point in S.

Proof. Proposition 1 ensures that (S,δ^*) is s-complete and symmetric QVQMS; accordingly, it suffices to show that contraction condition (11) implies condition (3), which completes the proof. Let $\varrho,\varsigma,\xi,\tau \in S$ and suppose that (11) holds. Then, we have

$$\begin{aligned}\delta_p(P(\varrho,\varsigma),P(\xi,\tau)) &\precsim (\alpha\otimes\delta_p(\varrho,\xi)) \oplus (\beta\otimes\delta_p(\varsigma,\tau))\\ &\precsim \lambda\otimes[\delta_p(\varrho,\xi)\vee\delta_p(\varsigma,\tau)],\end{aligned}$$

where $\lambda = \alpha+\beta$. If $P(\varrho,\varsigma) = P(\xi,\tau)$, the proof is straightforward; hence, we may assume that $P(\varrho,\varsigma) \neq P(\xi,\tau)$. Therefore, we will examine the following conditions:

- **Case I.** Let $\varrho \neq \xi \neq$ and $\varsigma \neq \tau$. Then, we obtain

$$\begin{aligned}\delta^*(P(\varrho,\varsigma),P(\xi,\tau)) = \delta_p(P(\varrho,\varsigma),P(\xi,\tau)) &\precsim \lambda\otimes[\delta_p(\varrho,\xi)\vee\delta_p(\varsigma,\tau)]\\ &\precsim \lambda\otimes[\delta^*(\varrho,\xi)\vee\delta^*(\varsigma,\tau)].\end{aligned}$$

- **Case II.** Let $\varrho = \xi$ and $\varsigma \neq \tau$. This implies that

$$\begin{aligned}\delta^*(P(\varrho,\varsigma),P(\xi,\tau)) = \delta_p(P(\varrho,\varsigma),P(\xi,\tau)) &\precsim \lambda\otimes[\delta_p(\varrho,\xi)\vee\delta_p(\varsigma,\tau)]\\ &\precsim \lambda\otimes[\theta\vee\delta^*(\varsigma,\tau)]\\ &= \lambda\otimes[\delta^s(\varrho,\xi)\vee\delta^*(\varsigma,\tau)].\end{aligned}$$

- **Case III.** Let $\varrho \neq \xi$ and $\varsigma = \tau$. Then, we get

$$\begin{aligned}\delta^*(P(\varrho,\varsigma),P(\xi,\tau)) = \delta_p(P(\varrho,\varsigma),P(\xi,\tau)) &\precsim \lambda\otimes[\delta_p(\varrho,\xi)\vee\delta_p(\varsigma,\tau)]\\ &\precsim \lambda\otimes[\delta^*(\varrho,\xi)\vee\theta]\\ &= \lambda\otimes[\delta^*(\varrho,\xi)\vee\delta^s(\varsigma,\tau)].\end{aligned}$$

Consequently, Theorem 1 ensures that P possesses exactly one coupled fixed point. ☐

As mentioned in the Preliminaries, QVQMSs can be seen as a generalization of PMSs. We conclude our paper with the following result which serves as an application of Theorem 1 to PMSs.

Corollary 6. *Consider the strong complete PMS $(S, \Psi, \circledast)$, where $\circledast$ is assumed to be a left continuous t-norm. Assume that the mapping $P : S \times S \to S$ satisfies, for all $\varrho, \varsigma, \xi, \tau \in S$, the inequality*

$$\Psi_{P(\varrho,\varsigma),P(\xi,\tau)}(t) \geq \max\{\Psi_{\varrho,\xi}(\frac{t}{\lambda}), \Psi_{\varsigma,\tau}(\frac{t}{\lambda})\}, \tag{13}$$

where $t \in [0, \infty), \lambda > 0$. Suppose further that there exists a pair $(\varrho_0, \varsigma_0) \in S \times S$ such that

$$\lim_{t \to \infty} \max\{\Psi_{\varrho_0,P(\varrho_0,\varsigma_0)}(t), \Psi_{\varsigma_0,P(\varsigma_0,\varrho_0)}(t)\} = 1. \tag{14}$$

Then P admits a unique coupled fixed point.

Proof. Consider the quantale-action structure $(\Phi, \leq^{op}, \circledast, \otimes)$, where $(\Phi, \leq^{op}, \circledast)$ is defined on page 5 and $\otimes$ is given in (1). Then (S, Ψ) forms an s-complete QVQMS. We now verify that condition (13) implies condition (3). Let us take $\varrho, \varsigma, \xi, \tau \in S$ and $t \in [0, \infty), \lambda > 0$ such that (13) holds. Under these assumptions, we get

$$\Psi_{P(\varrho,\varsigma),P(\xi,\tau)}(t) \geq \max\{(\lambda \otimes \Psi_{\varrho,\xi})(t), (\lambda \otimes \Psi_{\varsigma,\tau})(t)\}$$

From the definition of the opposite order relation, we obtain

$$\begin{aligned} \Psi_{P(\varrho,\varsigma),P(\xi,\tau)} &\leq^{op} \max\{\lambda \otimes \Psi_{\varrho,\xi}, \lambda \otimes \Psi_{\varsigma,\tau}\} \\ &= \lambda \otimes [\max\{\Psi_{\varrho,\xi}, \Psi_{\varsigma,\tau}\}]. \end{aligned}$$

Furthermore, condition (14) ensures condition (4). Hence, by the Theorem 1, the existence of a unique coupled fixed point is guaranteed. □

5. Conclusions

Many generalizations of metric spaces have been obtained by modifying the classical metric axioms. For instance, by dropping the symmetry requirement, quasi-metric spaces were introduced, [8]. Since many distance functions arising in real-life situations are inherently non-symmetric, quasi-metrics play a significant role in the mathematical modelling of various applied problems. Another important generalized metric structure, particularly relevant in theoretical computer science, is that of partial metric spaces, [17]. Most of these generalizations have been achieved through suitable alterations of the standard metric axioms. In 1977, Flagg [6] introduced quantale-valued metric spaces by replacing the classical value set $[0, \infty]$ with a value-quantale. This structure provides an abstract algebraic model capturing the essential properties of the usual addition and usual order on the non-negative real numbers. Quantale-based frameworks have proven to be highly useful and flexible, especially in theoretical computer science. Indeed, many problems in computer science can be represented by functions assigning a solution to each problem instance; such functions are referred to as a solution operators, see [21]. In [21], Siedlecki employed quantale-valued metrics to introduce a generalized notion of solution operators, which has become an important tool in the study of computational problems modelled by the partially ordered sets. Motivated by these developments, in the present paper we have extended coupled fixed point theorems from classical metric spaces to the broader setting of quantale-valued quasi-metric spaces, within the framework of abstract analysis and without resorting to categorical language. We have shown that, under appropriate conditions, coupled fixed point results can still be established in these highly abstract structures. The theoretical results obtained are supported by illustrative examples, and their applicability is demonstrated through applications to QVPMSs and PMSs. As a direction for future research, we plan to investigate fixed point theorems for solution operators in quantale-valued quasi-metric spaces. In particular, we aim to develop best proximity point results in this framework and to explore their potential applications in theoretical computer science.

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Abbreviations

The following abbreviations are used in this manuscript:

QVQMS	Quantale-valued quasi-metric space
QVPMS	Quantale-valued partial metric
FPT	Fixed point theory
PMS	Probabilistic metric space

Use of Artificial Intelligence: The author declares that no Artificial Intelligence (AI) tools were used in the preparation of this manuscript.

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