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Article

New Polynomial Identities and Some Consequences

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Abstract: Using an elementary approach involving the Euler Beta function and the binomial theorem, we derive two polynomial identities; one of which is a generalization of a known polynomial identity. Two well-known combinatorial identities, namely Frisch's identity and Klamkin's identity, appear as immediate consequences of these polynomial identities. We subsequently establish several combinatorial identities, including a generalization of each of Frisch's identity and Klamkin's identity. Finally, we develop a scheme for deriving combinatorial identities associated with polynomial identities of a certain type.

Keywords: beta function; polynomial identity; frisch identity; klamkin identity; binomial coefficient; combinatorial identity

1. Introduction

Our purpose in writing this paper is to derive the following new polynomial identities in x :

$$\sum_{k=0}^n \binom{n}{k} \binom{k+r}{s}^{-1} x^k = s \sum_{k=0}^n \frac{(-1)^k}{k+s} \binom{n}{k} \binom{k+r}{k+s}^{-1} x^k (1+x)^{n-k} \quad (1)$$

and

$$\sum_{k=0}^n \binom{n}{k} \binom{r}{k+s}^{-1} x^k = (r+1) \sum_{k=0}^n \frac{(-1)^{n-k}}{r-k+1} \binom{n}{k} \binom{r-k}{r-s-n}^{-1} (1-x)^{n-k}. \quad (2)$$

In identities (1) and (2), n is a non-negative integer and x is a complex variable. Identity (1) holds for complex numbers r and s for which $\Re(r-s+1) > 0$ and s is not a non-positive integer, while (2) is valid for complex numbers r and s such that $\Re(r-n-s+1) > 0$ and s is not a negative integer.

Identity (1) is simpler than, and yet, generalizes Identity (4.13) of Gould's book [4, p.47], the latter corresponding to the case $r = s$ in (1).

At $x = -1$, identity (1) reduces to Frisch's identity [3], namely,

$$\sum_{k=0}^n (-1)^k \binom{n}{k} \binom{k+r}{s}^{-1} = \frac{s}{k+s} \binom{k+r}{k+s}^{-1}, \quad (3)$$

while at $x = 1$, identity (2) yields Klamkin's identity:

$$\sum_{k=0}^n \binom{n}{k} \binom{r}{k+s}^{-1} = \frac{r+1}{(r-n+1)} \binom{r-n}{s}^{-1}. \quad (4)$$

In a recent paper, Gould and Quaintance [6] employed the well-known formula of Gauss for the hypergeometric function to give new proofs of (3) and (4). More recently, Abel [1] used the Euler Beta function to give elementary short proofs of (3) and (4). Our approach also uses the Beta function and is quite similar to that of Abel. For more historical facts concerning Frisch's identity and Klamkin's identity, the reader is referred to Abel [1] and Gould and Quaintance [6]. In a recent paper, Adegoke and Frontczak [2] derived many harmonic and odd harmonic number identities from Frisch's identity.

Let m be a non-negative integer and let u be a complex number. Among other results, we will derive the following respective generalization of Frisch's identity and Klamkin's identity:

$$\sum_{k=0}^n (-1)^k \binom{n}{k} \binom{u+n-k}{u} \binom{k+r}{s}^{-1} = \sum_{k=0}^n \frac{s}{k+s} \binom{n}{k} \binom{u}{n-k} \binom{k+r}{k+s}^{-1}$$

and

$$\sum_{k=0}^n \binom{n}{k} \binom{u+n-k}{n-k} \binom{r}{k+s}^{-1} = (r+1) \sum_{k=0}^n \frac{1}{r-k+1} \binom{n}{k} \binom{u}{n-k} \binom{r-k}{s}^{-1};$$

and also establish the following extensions:

$$\begin{aligned} & \sum_{k=0}^n (-1)^k k^m \binom{n}{k} \binom{k+r}{s}^{-1} \\ &= \sum_{k=0}^m \frac{(n-k)^m s}{n-k+s} \binom{n}{k} \binom{n-k+r}{n-k+s}^{-1} \sum_{p=0}^k (-1)^p \binom{k}{p} \left(1 + \frac{p}{n-k}\right)^m, \end{aligned}$$

and

$$\begin{aligned} & \sum_{k=0}^n k^m \binom{n}{k} \binom{r}{k+s}^{-1} \\ &= (r+1) \sum_{k=0}^m \frac{(-1)^k}{r-n+k+1} \binom{n}{k} \binom{k+r-n}{k+s}^{-1} \sum_{p=0}^k (-1)^p \binom{k}{p} p^m. \end{aligned}$$

Binomial coefficients are defined, for non-negative integers i and j , by

$$\binom{i}{j} = \begin{cases} \frac{i!}{j!(i-j)!}, & i \geq j; \\ 0, & i < j; \end{cases}$$

the number of distinct sets of j objects that can be chosen from i distinct objects.

Generalized binomial coefficients are defined for complex numbers r and s by

$$\binom{r}{s} = \frac{\Gamma(r+1)}{\Gamma(s+1)\Gamma(r-s+1)},$$

where the Gamma function, $\Gamma(z)$, is defined for $\Re(z) > 0$ by

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} dt = \int_0^\infty (\log(1/t))^{z-1} dt,$$

and is extended to the rest of the complex plane, excluding the non-positive integers, by analytic continuation.

2. Proof of (1) and (2)

The integration formulas required for proving (1) and (2) are given in Lemma 1.

Lemma 1. If r, k and s are complex numbers and x is a complex variable, then

$$\int_0^1 y^{r+k-s}(1-y)^{s-1} dy = \frac{1}{s} \binom{k+r}{s}^{-1}, \quad \Re(r+k-s+1) > 0 \text{ and } 0 \neq s \notin \mathbb{Z}^-; \quad (5)$$

$$\int_0^1 y^{r-s}(1-y)^{k+s-1} dy = \frac{1}{k+s} \binom{k+r}{k+s}^{-1}, \quad \Re(r-s+1) > 0 \text{ and } \Re(k+s) > 0; \quad (6)$$

$$\int_0^1 y^{k+s}(1-y)^{r-k-s} dy = \frac{1}{r+1} \binom{r}{k+s}^{-1}, \quad \Re(k+s+1) > 0 \text{ and } \Re(r-k-s+1) > 0, \quad (7)$$

and

$$\begin{aligned} \int_0^1 y^{n-k+s}(1-y)^{r-n-s} \\ = \frac{1}{r-k+1} \binom{r-k}{r-s-n}, \quad \Re(n-k+s+1) > 0 \text{ and } \Re(r-n-s+1) > 0. \end{aligned} \quad (8)$$

Proof. The integrals in (5)–(8) are immediate consequences of the Beta function, $B(r, s)$, defined, as usual, for complex numbers r and s such that $\Re(r) > 0$ and $\Re(s) > 0$, by

$$B(r, s) = B(s, r) = \int_0^1 y^{r-1}(1-y)^{s-1}.$$

With the help of the Gamma function, the integral is evaluated as

$$B(r, s) = \frac{\Gamma(r)\Gamma(s)}{\Gamma(r+s)} = \frac{1}{s} \binom{r+s-1}{s}^{-1} = \frac{1}{r} \binom{r+s-1}{r}^{-1}.$$

Note that in obtaining (7) and (8), we also used

$$\binom{u+1}{v+1} = \frac{u+1}{v+1} \binom{u}{v},$$

an identity which we will often use without comment in this paper. $\square$

Theorem 1. If n is a non-negative integer, x is a complex variable and r and s are complex numbers such that $\Re(r-s+1) > 0$ and s is not a non-positive integer, then

$$\sum_{k=0}^n \binom{n}{k} \binom{k+r}{s}^{-1} x^k = s \sum_{k=0}^n \frac{(-1)^k}{k+s} \binom{n}{k} \binom{k+r}{k+s}^{-1} x^k (1+x)^{n-k}.$$

Proof. Application of the binomial theorem to both sides of

$$1 + xy = -x(1-y) + 1 + x$$

gives

$$\sum_{k=0}^n \binom{n}{k} x^k y^k = \sum_{k=0}^n \binom{n}{k} (-1)^k x^k (1-y)^k (1+x)^{n-k},$$

which upon multiplying through by $y^{r-s}(1-y)^{s-1}$ can also be written as

$$\begin{aligned} \sum_{k=0}^n \binom{n}{k} y^{r+k-s}(1-y)^{s-1} x^k \\ = \sum_{k=0}^n (-1)^k \binom{n}{k} x^k y^{r-s}(1-y)^{s+k-1} (1+x)^{n-k}. \end{aligned}$$

Thus, term-wise integration gives

$$\begin{aligned} \sum_{k=0}^n \binom{n}{k} x^k \int_0^1 y^{r+k-s} (1-y)^{s-1} dy \\ = \sum_{k=0}^n (-1)^k \binom{n}{k} x^k (1+x)^{n-k} \int_0^1 y^{r-s} (1-y)^{s+k-1} dy, \end{aligned}$$

from which (1) follows by (5) and (6). $\square$

Theorem 2. If n is a non-negative integer, x is a complex variable and r and s are complex numbers such that $\Re(r-n-s+1) > 0$ and s is not a negative integer, then

$$\sum_{k=0}^n \binom{n}{k} \binom{r}{k+s}^{-1} x^k = (r+1) \sum_{k=0}^n \frac{(-1)^{n-k}}{r-k+1} \binom{n}{k} \binom{r-k}{r-s-n}^{-1} (1-x)^{n-k}.$$

Proof. Raising both sides of

$$xy + 1 - y = y(x-1) + 1$$

to power n and expanding via the binomial theorem gives

$$\sum_{k=0}^n \binom{n}{k} y^k (1-y)^{n-k} x^k = \sum_{k=0}^n (-1)^{n-k} \binom{n}{k} y^{n-k} (1-x)^{n-k}$$

which, after multiplying through by $y^s (1-y)^{r-n-s}$, leads to

$$\sum_{k=0}^n \binom{n}{k} y^{k+s} (1-y)^{r-k-s} x^k = \sum_{k=0}^n (-1)^{n-k} \binom{n}{k} y^{n-k+s} (1-y)^{r-n-s} (1-x)^{n-k}.$$

Performing term-wise integration, we therefore have

$$\begin{aligned} \sum_{k=0}^n \binom{n}{k} x^k \int_0^1 y^{k+s} (1-y)^{r-k-s} dy \\ = \sum_{k=0}^n (-1)^{n-k} \binom{n}{k} (1-x)^{n-k} \int_0^1 y^{n-k+s} (1-y)^{r-n-s} dy; \end{aligned}$$

and hence (2) by (7) and (8). $\square$

3. Combinatorial Identities

In this section we discuss some combinatorial identities that are consequences of (1) and (2) and related identities. In particular, we will derive a generalization of each of Frisch's identity and Klamkin's identity.

Proposition 1. If n is a non-negative integer and r and s are complex numbers such that $\Re(r-s+1) > 0$ and s is not a non-positive integer, then

$$\sum_{k=0}^n \frac{(-1)^k}{k+s} \binom{n}{k} \binom{k+r}{k+s}^{-1} = \frac{1}{s} \binom{n+r}{s}^{-1}. \quad (9)$$

In particular,

$$\sum_{k=0}^n \frac{(-1)^k}{k+r} \binom{n}{k} = \frac{1}{r} \binom{n+r}{r}^{-1}. \quad (10)$$

Proof. By writing $1/x$ for x , identity (1) can also be written as

$$\sum_{k=0}^n \binom{n}{k} \binom{k+r}{s}^{-1} x^{n-k} = s \sum_{k=0}^n \frac{(-1)^k}{k+s} \binom{n}{k} \binom{k+r}{k+s}^{-1} (1+x)^{n-k}, \quad (11)$$

from which (9) is obtained by evaluating at $x = 0$. $\square$

Remark 1. Identity (9) is the binomial transform of (3).

Proposition 2. If n is a non-negative integer and r and s are complex numbers such that $\Re(r - n - s + 1) > 0$ and s is not a negative integer, then

$$\sum_{k=0}^n \frac{(-1)^k}{r-k+1} \binom{n}{k} \binom{r-k}{r-s-n}^{-1} = \frac{(-1)^n}{r+1} \binom{r}{s}^{-1}. \quad (12)$$

Proof. Set $x = 0$ in (2). $\square$

Obviously, Frisch-type and Klamkin-type combinatorial identities are associated with polynomial identities having the following form:

$$\sum_{k=l_1}^{l_2} f(k) x^{p(k)} = \sum_{k=n_1}^{n_2} g(k) (1-x)^{q(k)}, \quad (13)$$

where $f(k)$ and $g(k)$ are sequences, $p(k)$ and $q(k)$ are sequences of non-negative integers and l_1, l_2, n_1 and n_2 are non-negative integers.

3.1. Frisch-type combinatorial identities

Theorem 3. Let r and s be complex numbers such that $\Re(r + \min(p(l_1), p(l_2)) - s + 1) > 0$ and $\Re(\min(q(n_1), q(n_2)) + s) > 0$; where $\min(a, b)$ picks the smaller of a and b . If a polynomial identity has the form (13), then the following combinatorial identities hold:

$$\sum_{k=l_1}^{l_2} f(k) \binom{p(k)+r}{s}^{-1} = s \sum_{k=n_1}^{n_2} \frac{g(k)}{q(k)+s} \binom{q(k)+r}{q(k)+s}^{-1}, \quad (14)$$

$$\sum_{k=n_1}^{n_2} g(k) \binom{q(k)+r}{s}^{-1} = s \sum_{k=l_1}^{l_2} \frac{f(k)}{p(k)+s} \binom{p(k)+r}{p(k)+s}^{-1}. \quad (15)$$

Proof. Multiply through (13) by $x^{r-s}(1-x)^{s-1}$ and perform term-wise integration with respect to x , making use of (5) and (6), thereby obtaining (14). Identity (13) can also be written as

$$\sum_{k=n_1}^{n_2} g(k) x^{q(k)} = \sum_{k=l_1}^{l_2} f(k) (1-x)^{p(k)},$$

so that identities derived from (13) remain valid under the following transpositions:

$$p(k) \leftrightarrow q(k), \quad f(k) \leftrightarrow g(k), \quad n_1 \leftrightarrow l_1, \quad n_2 \leftrightarrow l_2;$$

and hence identity (15). $\square$

We now illustrate Theorem 3 by deriving the Frisch-type identity associated with an identity of Simons.

Lemma 2 (Simons [8]). *If n is a non-negative integer and x is a complex variable, then*

$$\sum_{k=0}^n (-1)^k \binom{n}{k} \binom{n+k}{k} x^k = \sum_{k=0}^n (-1)^{n-k} \binom{n}{k} \binom{n+k}{k} (1-x)^k. \quad (16)$$

Proposition 3. *If n is a non-negative integer and r and s are complex numbers such that $\Re(r-s+1) > 0$ and s is not a non-positive integer, then*

$$\sum_{k=0}^n (-1)^k \binom{n}{k} \binom{n+k}{k} \binom{k+r}{s}^{-1} = \sum_{k=0}^n \frac{(-1)^{n-k} s}{k+s} \binom{n}{k} \binom{n+k}{k} \binom{k+r}{k+s}^{-1}. \quad (17)$$

Proof. Comparing (13) and (16), we find

$$f(k) = (-1)^k \binom{n}{k} \binom{n+k}{k}, \quad g(k) = (-1)^{n-k} \binom{n}{k} \binom{n+k}{k}, \quad (18)$$

$p(k) = k = q(k)$; and $l_1 = 0 = n_1$ and $l_2 = n = n_2$.

Using these in (14) gives (17). $\square$

We can obtain a Frisch-type identity with two binomial coefficients in the denominator, directly from (1).

Proposition 4. *If n is a non-negative integer and r, s, t and u are complex numbers such that $\Re(r-s+1) > 0$, $\Re(t-u+1) > 0$ and s and u are not negative integers, then*

$$\begin{aligned} & \sum_{k=0}^n (-1)^k \binom{n}{k} \binom{k+r}{s}^{-1} \binom{k+t}{u}^{-1} \\ &= su \sum_{k=0}^n \frac{1}{(k+s)(n-k+u)} \binom{n}{k} \binom{k+r}{k+s}^{-1} \binom{n+t}{n-k+u}^{-1}. \end{aligned} \quad (19)$$

Proof. Write $-x$ for x in (1), multiply through by $x^{t-u}(1-x)^{u-1}$ and integrate with respect to x from 0 to 1. $\square$

In particular,

$$\begin{aligned} & \sum_{k=0}^n (-1)^k \binom{n}{k} \binom{k+r}{s}^{-2} \\ &= s^2 \sum_{k=0}^n \frac{1}{(k+s)(n-k+s)} \binom{n}{k} \binom{k+r}{k+s}^{-1} \binom{n+r}{n-k+s}^{-1} \end{aligned} \quad (20)$$

and

$$\sum_{k=0}^n (-1)^k \binom{n}{k} \binom{k+r}{r}^{-2} = r^2 \sum_{k=0}^n \frac{1}{(k+r)(n-k+r)} \binom{n}{k} \binom{n+r}{n-k+r}^{-1}. \quad (21)$$

The reader is invited to discover the combinatorial identity having two binomial coefficients in the denominator associated with (11) by making appropriate substitutions in Theorem 3.

3.1.1. A generalization of Frisch's identity

In Theorem 4, we derive a generalization of Frisch's identity. We require the following known polynomial identity.

Lemma 3 ([4, Identity (3.18), p.24]). If n is a non-negative integer, u is a complex number and x and y are complex variables, then

$$\sum_{k=0}^n \binom{n}{k} \binom{u}{k} x^{n-k} y^k = \sum_{k=0}^n \binom{n}{k} \binom{u+k}{k} (x-y)^{n-k} y^k. \quad (22)$$

Theorem 4. If n is a non-negative integer, u is a complex number and r and s are complex numbers such that $\Re(r-s+1) > 0$ and s is not a negative integer, then

$$\sum_{k=0}^n \binom{n}{k} \binom{u}{n-k} \binom{k+r}{s}^{-1} = \sum_{k=0}^n \frac{(-1)^k s}{k+s} \binom{n}{k} \binom{u+n-k}{u} \binom{k+r}{k+s}^{-1}, \quad (23)$$

$$\sum_{k=0}^n (-1)^k \binom{n}{k} \binom{u+n-k}{u} \binom{k+r}{s}^{-1} = \sum_{k=0}^n \frac{s}{k+s} \binom{n}{k} \binom{u}{n-k} \binom{k+r}{k+s}^{-1}. \quad (24)$$

Proof. Setting $y = 1$ in (22) gives

$$\sum_{k=0}^n \binom{n}{k} \binom{u}{k} x^{n-k} = \sum_{k=0}^n (-1)^{n-k} \binom{n}{k} \binom{u+k}{k} (1-x)^{n-k},$$

so that comparing with (13), we choose

$$f(k) = \binom{n}{k} \binom{u}{k}, \quad g(k) = (-1)^{n-k} \binom{n}{k} \binom{u+k}{k}, \quad (25)$$

$p(k) = n-k = q(k)$, $l_1 = 0 = n_1$ and $l_2 = n = n_2$.

Substituting these in (14) and (15) gives

$$\sum_{k=0}^n \binom{n}{k} \binom{u}{k} \binom{n-k+r}{s}^{-1} = \sum_{k=0}^n \frac{(-1)^{n-k} s}{n-k+s} \binom{n}{k} \binom{u+k}{k} \binom{n-k+r}{r-s}^{-1}$$

and

$$\sum_{k=0}^n (-1)^{n-k} \binom{n}{k} \binom{u+k}{k} \binom{n-k+r}{s}^{-1} = \sum_{k=0}^n \frac{s}{n-k+s} \binom{n}{k} \binom{u}{k} \binom{n-k+r}{r-s}^{-1},$$

which can also be written in the equivalent forms (23) and (24) $\square$

Remark 2. Frisch's identity (3) is obtained by setting $u = -1$ in (23) or $u = 0$ in (24).

3.2. Klamkin-type combinatorial identities

Theorem 5. Let r and s be complex numbers such that s is not a negative integer, $\Re(r - \max(q(n_1), q(n_2)) - s + 1) > 0$ and $\Re(r - \max(p(l_1), p(l_2)) - s + 1) > 0$; where $\max(a, b)$ picks the greater of a and b . If a polynomial identity has the form (13), then the following combinatorial identities hold:

$$\sum_{k=n_1}^{n_2} (-1)^{q(k)} g(k) \binom{r}{q(k)+s}^{-1} = (r+1) \sum_{k=l_1}^{l_2} \frac{f(k)}{r-p(k)+1} \binom{r-p(k)}{s}^{-1}, \quad (26)$$

$$\sum_{k=l_1}^{l_2} (-1)^{p(k)} f(k) \binom{r}{p(k)+s}^{-1} = (r+1) \sum_{k=n_1}^{n_2} \frac{g(k)}{r-q(k)+1} \binom{r-q(k)}{s}^{-1}. \quad (27)$$

Proof. Write $1/x$ for x in (13) to obtain

$$\sum_{k=n_1}^{n_2} (-1)^{q(k)} g(k) x^{q(k)} (1-x)^{r-q(k)} = \sum_{k=l_1}^{l_2} f(k) (1-x)^{r-p(k)},$$

from which, by multiplying through with $x^s(1-x)^{r-s}$, we get

$$\sum_{k=n_1}^{n_2} (-1)^{q(k)} g(k) x^{q(k)+s} (1-x)^{r-q(k)-s} = \sum_{k=l_1}^{l_2} f(k) x^s (1-x)^{r-p(k)-s},$$

and hence (26) after term-wise integration. $\square$

Proposition 5. If n is a non-negative integer and r, s, t and u are complex numbers such that $\Re(r-n-s) > 0$, $\Re(t-n-u+1) > 0$ and s and t are not negative integers, then

$$\begin{aligned} \sum_{k=0}^n \frac{1}{t-k+1} \binom{n}{k} \binom{t-k}{t-u-n}^{-1} \binom{r}{n-k+s}^{-1} \\ = \frac{r+1}{t+1} \sum_{k=0}^n \frac{1}{r-k+1} \binom{n}{k} \binom{t}{k+u}^{-1} \binom{r-k}{s}^{-1} \end{aligned} \quad (28)$$

and

$$\begin{aligned} \sum_{k=0}^n (-1)^k \binom{n}{k} \binom{t}{k+u}^{-1} \binom{r}{k+s}^{-1} \\ = (r+1)(t+1) \sum_{k=0}^n \frac{(-1)^{n-k} \binom{n}{k}}{(t-k+1)(r-n+k+1)} \binom{t-k}{t-u-n}^{-1} \binom{r-n+k}{s}^{-1}. \end{aligned} \quad (29)$$

Proof. Write (2) as

$$\sum_{k=0}^n \binom{n}{k} \binom{t}{k+u}^{-1} x^k = (t+1) \sum_{k=0}^n \frac{(-1)^{n-k}}{t-k+1} \binom{n}{k} \binom{t-k}{t-u-n}^{-1} (1-x)^{n-k}.$$

Consider (13) and identify

$$f(k) = \binom{n}{k} \binom{t}{k+u}^{-1}, \quad g(k) = \frac{(-1)^{n-k}(t+1)}{t-k+1} \binom{n}{k} \binom{t-k}{t-u-n}^{-1},$$

$p(k) = k$, $q(k) = n-k$ and $l_1 = n_1 = 0$ and $l_2 = n_2 = n$.

Use these in (26) to obtain (28).

$\square$

Proposition 6. If n is a non-negative integer and r and s are complex numbers such that $\Re(r-s-n+1) > 0$ and s is not a negative integer, then

$$\begin{aligned} \sum_{k=0}^n (-1)^k \binom{n}{k} \binom{r}{k+s}^{-2} \\ = \frac{(r+1)^2}{(r-s-n+1)(s+1)} \sum_{k=0}^n (-1)^{n-k} \binom{n}{k} \binom{r-k+1}{r-s-n+1}^{-1} \binom{r-n+k+1}{s+1}^{-1}. \end{aligned} \quad (30)$$

Proof. Set $t = r$ and $u = s$ in (29). $\square$

Our next result is a Klamkin-type identity derived from the identity of Simons (16).

Proposition 7. Let n be a non-negative integer. If r and s are complex numbers such that $\Re(r - n - s + 1) > 0$ and s is not a negative integer, then

$$\begin{aligned} \sum_{k=0}^n \binom{n}{k} \binom{n+k}{k} \binom{r}{k+s}^{-1} \\ = (r+1)(-1)^n \sum_{k=0}^n \frac{(-1)^k}{r-k+1} \binom{n}{k} \binom{n+k}{k} \binom{r-k}{s}^{-1}. \end{aligned} \quad (31)$$

Proof. Use $f(k)$, $g(k)$, etc. given in (18) in Theorem 5. $\square$

The reader is invited to discover the combinatorial identity having two binomial coefficients in the denominator associated with (11) by making appropriate substitutions in Theorem 5.

3.2.1. A generalization of Klamkin's identity

We close this section by giving a generalization of Klamkin's identity.

Theorem 6. If n is a non-negative integer, u is a complex number and r and s are complex numbers such that $\Re(r - n - s + 1) > 0$ and s is not a negative integer, then

$$\sum_{k=0}^n \binom{n}{k} \binom{u+n-k}{n-k} \binom{r}{k+s}^{-1} = (r+1) \sum_{k=0}^n \frac{1}{r-k+1} \binom{n}{k} \binom{u}{n-k} \binom{r-k}{s}^{-1}, \quad (32)$$

$$\sum_{k=0}^n (-1)^k \binom{n}{k} \binom{u}{n-k} \binom{r}{k+s}^{-1} = (r+1) \sum_{k=0}^n \frac{(-1)^k}{r-k+1} \binom{n}{k} \binom{u+n-k}{n-k} \binom{r-k}{s}^{-1}. \quad (33)$$

Proof. Use $f(k)$, $g(k)$, etc. given in (25) in Theorem 5.

$\square$

Remark 3. Klamkin's identity (4) is obtained by setting $u = 0$ in (32), while $u = 0$ in (33) recovers (12).

4. Extensions of Frisch's identity and Klamkin's identity

In this section we derive a closed form for the following combinatorial sums:

$$\sum_{k=0}^n (-1)^k k^m \binom{n}{k} \binom{k+r}{s}^{-1}, \quad \sum_{k=0}^n (-1)^k k^m \binom{n}{k} \binom{r}{k+s}^{-1},$$

where m is a non-negative integer; thereby providing an extension for each of Frisch's identity (3) and Klamkin's identity (4), the latter corresponding to $m = 0$. For this purpose we require the results stated in Lemma 4.

Lemma 4. If m , u and v are non-negative integers, then

$$\left. \frac{d^m}{dx^m} ((1-e^x)^u e^{vx}) \right|_{x=0} = \sum_{p=0}^u (-1)^p \binom{u}{p} (v+p)^m. \quad (34)$$

Proof. Since

$$(1-e^x)^u e^{vx} = \sum_{p=0}^u (-1)^p \binom{u}{p} e^{(v+p)x},$$

we have

$$\frac{d^m}{dx^m} ((1-e^x)^u e^{vx}) = \sum_{p=0}^u (-1)^p \binom{u}{p} (v+p)^m e^{(v+p)x};$$

and hence (34).

$\square$

4.1. An extension of Frisch's identity

Proposition 8. If m and n are non-negative integers and r and s are complex numbers such that $\Re(r - s + 1) > 0$ and s is not a non-positive integer, then

$$\begin{aligned} \sum_{k=0}^n (-1)^k k^m \binom{n}{k} \binom{k+r}{s}^{-1} \\ = \sum_{k=0}^m \frac{s}{n-k+s} \binom{n}{k} \binom{n-k+r}{n-k+s}^{-1} \sum_{p=0}^k (-1)^p \binom{k}{p} (n-k+p)^m. \end{aligned} \quad (35)$$

In particular,

$$\sum_{k=0}^n (-1)^k k \binom{n}{k} \binom{k+r}{s}^{-1} = \frac{ns}{n+r} \frac{s-r-1}{n+s-1} \binom{n+r-1}{n+s-1}^{-1} \quad (36)$$

and

$$\sum_{k=0}^n (-1)^k k^2 \binom{n}{k} \binom{k+r}{s}^{-1} = \frac{ns}{n+r} \frac{r-s+1}{r+n-1} \frac{n(r-s+1)-r}{n+s-2} \binom{n+r-2}{n+s-2}^{-1}, \quad (37)$$

with the special values

$$\sum_{k=0}^n (-1)^k k \binom{n}{k} \binom{k+r}{r}^{-1} = -\frac{nr}{(n+r-1)(n+r)} \quad (38)$$

and

$$\sum_{k=0}^n (-1)^k k^2 \binom{n}{k} \binom{k+r}{r}^{-1} = \frac{nr(n-r)}{(n+r)(n+r-1)(n+r-2)}. \quad (39)$$

Proof. Write $- \exp x$ for x in (1) and differentiate m times with respect to x to obtain

$$\begin{aligned} \sum_{k=0}^n (-1)^k k^m \binom{n}{k} \binom{k+r}{s}^{-1} e^{kx} \\ = \sum_{k=0}^n \frac{s}{k+s} \binom{n}{k} \binom{k+r}{k+s}^{-1} \frac{d^m}{dx^m} ((1-e^x)^{n-k} e^{kx}) \\ = \sum_{k=0}^n \frac{s}{n-k+s} \binom{n}{k} \binom{n-k+r}{n-k+s}^{-1} \frac{d^m}{dx^m} ((1-e^x)^k e^{(n-k)x}). \end{aligned}$$

Evaluation at $x = 0$ gives

$$\begin{aligned} \sum_{k=0}^n (-1)^k k^m \binom{n}{k} \binom{k+r}{s}^{-1} \\ = \sum_{k=0}^n \frac{s}{n-k+s} \binom{n}{k} \binom{n-k+r}{n-k+s}^{-1} \frac{d^m}{dx^m} (1-e^x)^k e^{(n-k)x} \Big|_{x=0} \\ = \sum_{k=0}^m \frac{s}{n-k+s} \binom{n}{k} \binom{n-k+r}{n-k+s}^{-1} \frac{d^m}{dx^m} (1-e^x)^k e^{(n-k)x} \Big|_{x=0}, \end{aligned}$$

and hence (35), in view of Lemma 4.

□

4.2. An extension of Klamkin's identity

Proposition 9. If m and n are non-negative integers and r and s are complex numbers such that $\Re(r - n - s + 1) > 0$ and s is not a negative integer, then

$$\begin{aligned} \sum_{k=0}^n k^m \binom{n}{k} \binom{r}{k+s}^{-1} \\ = (r+1) \sum_{k=0}^m \frac{(-1)^k}{r-n+k+1} \binom{n}{k} \binom{k+r-n}{k+s}^{-1} \sum_{p=0}^k (-1)^p \binom{k}{p} p^m. \end{aligned} \quad (40)$$

In particular,

$$\sum_{k=0}^n k \binom{n}{k} \binom{r}{k+s}^{-1} = \frac{n(r+1)}{(r-n+2)} \binom{r-n+1}{s+1}^{-1}, \quad (41)$$

and

$$\sum_{k=0}^n k^2 \binom{n}{k} \binom{r}{k+s}^{-1} = \frac{(r+1)n(n(s+1)+r-s+1)}{(n-r-2)(n-r-3)} \binom{r-n+1}{s+1}^{-1}. \quad (42)$$

Proof. Write $\exp x$ for x in (2) and proceed as in the proof of Proposition 8. $\square$

4.3. Related identities derived from polynomial identities of a certain type

Theorem 7. Let a polynomial identity have the following form:

$$\sum_{k=0}^n f(k)x^k = \sum_{k=0}^n g(k)(1-x)^k, \quad (43)$$

where n is a non-negative integer, x is a complex variable and $f(k)$ and $g(k)$ are sequences. If m is a non-negative integer, then

$$\sum_{k=0}^n k^m f(k) = \sum_{k=0}^m g(k) \sum_{p=0}^k (-1)^p \binom{k}{p} p^m, \quad (44)$$

$$\sum_{k=0}^n k^m g(k) = \sum_{k=0}^m f(k) \sum_{p=0}^k (-1)^p \binom{k}{p} p^m, \quad (45)$$

$$\sum_{k=0}^n k^m f(n-k) = \sum_{k=0}^m (-1)^k g(k) \sum_{p=0}^k (-1)^p \binom{k}{p} (n-k+p)^m, \quad (46)$$

and

$$\sum_{k=0}^n k^m g(n-k) = \sum_{k=0}^m (-1)^k f(k) \sum_{p=0}^k (-1)^p \binom{k}{p} (n-k+p)^m. \quad (47)$$

Proof. Write $\exp x$ for x in (43), differentiate m times with respect to x and evaluate at $x = 0$, using Lemma 4; this gives (44). To drive (46), write $1/x$ for x in (43), multiply through by x^n , replace x by $\exp x$, differentiate m times with respect to x and evaluate at $x = 0$, using Lemma 4. Identities (45) and (47) follow from the $f(k) \leftrightarrow g(k)$ symmetry of (43). $\square$

5. More Combinatorial Identities

This section contains further identities based on the integration formulas in Lemma 1, the evaluated derivatives in Lemma 4 and the identities stated in Theorem 7.

5.1. Identities Derived from the Geometric Progression

Proposition 10. If n is a non-negative integer and r and s are complex numbers excluding the set of non-positive integers, then

$$\sum_{k=0}^n \binom{k+r}{s}^{-1} = \frac{s}{s-1} \left(\binom{r-1}{s-1}^{-1} - \binom{n+r}{s-1}^{-1} \right). \quad (48)$$

Proof. Multiply through the sum of the geometric progression:

$$\sum_{k=0}^n x^k = \frac{1-x^{n+1}}{1-x}, \quad (49)$$

by $x^{r-s}(1-x)^{s-1}$ to obtain

$$\sum_{k=0}^n x^{k+r-s}(1-x)^{s-1} = x^{r-s}(1-x)^{s-2} - x^{n+r-s+1}(1-x)^{s-2}, \quad (50)$$

and hence (48) by termwise integration. $\square$

Remark 4. The special case $s = r$ of (48) was proved by Rockett [7].

Proposition 11. If n is a non-negative integer, s is a complex number excluding the set of negative integers and r is a complex number such that $\Re(r-n-s+1) > 0$, then

$$\sum_{k=0}^n (-1)^k \binom{r}{k+s}^{-1} = \frac{r+1}{s+1} \binom{r+2}{s+1}^{-1} + (-1)^n \frac{r+1}{n+s+2} \binom{r+2}{n+s+2}^{-1}. \quad (51)$$

Proof. Write $-x$ for x in (49), replace x with $x/(1-x)$ and multiply through by $x^s(1-x)^{r-n-s}$ to obtain

$$\sum_{k=0}^n (-1)^k x^{k+s}(1-x)^{r-k-s} = x^s(1-x)^{r-s+1} + (-1)^n x^{n+s+1}(1-x)^{r-n-s},$$

and hence (51). $\square$

5.2. An identity derived from Waring's formula

Waring's formula is [5, Equation (22)]

$$\sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k \frac{n}{n-k} \binom{n-k}{k} (xy)^k (x+y)^{n-2k} = x^n + y^n, \quad (52)$$

and holds for a positive integer n and complex variables x and y .

Proposition 12. If n is a non-negative integer and r and s are complex numbers excluding the set of negative integers and such that $\Re(r-s+1) > 0$, then

$$\sum_{k=0}^{\lfloor n/2 \rfloor} \frac{(-1)^k n}{(n-k)(k+s)} \binom{n-k}{k} \binom{2k+r}{k+s}^{-1} = \frac{1}{s} \binom{n+r}{s}^{-1} + \frac{1}{n+s} \binom{n+r}{n+s}^{-1}. \quad (53)$$

Proof. Set $y = 1-x$ in (52) and multiply through by $x^{r-s}(1-x)^{s-1}$ to obtain

$$\sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k \frac{n}{n-k} \binom{n-k}{k} x^{k+r-s}(1-x)^{k+s-1} = x^{n+r-s}(1-x)^{s-1} + x^{r-s}(1-x)^{n+s-1},$$

from which (53) follows. $\square$

5.3. More Identities from an Identity of Simons

Proposition 13. If m and n are non-negative integers, then

$$\sum_{k=0}^n (-1)^k k^m \binom{n}{k} \binom{k+n}{k} = \sum_{k=0}^m (-1)^{n-k} \binom{n}{k} \binom{k+n}{k} \sum_{p=0}^k (-1)^p \binom{k}{p} p^m, \quad (54)$$

$$\sum_{k=0}^n (-1)^k k^m \binom{n}{k} \binom{2n-k}{n-k} = \sum_{k=0}^m \binom{n}{k} \binom{k+n}{k} \sum_{p=0}^k (-1)^p \binom{k}{p} (n-k+p)^m. \quad (55)$$

Proof. Use $f(k)$ and $g(k)$ from (18) in identities (44) and (46) of Theorem 7. $\square$

Explicit examples from (54) include

$$\sum_{k=0}^n (-1)^k k \binom{n}{k} \binom{k+n}{k} = (-1)^n n(n+1), \quad (56)$$

$$\sum_{k=0}^n (-1)^k k^2 \binom{n}{k} \binom{k+n}{k} = (-1)^n \frac{n^2(n+1)^2}{2}, \quad (57)$$

and

$$\sum_{k=0}^n (-1)^k k^3 \binom{n}{k} \binom{k+n}{k} = (-1)^n \frac{n^2(n+1)^2(n^2+n+1)}{6}; \quad (58)$$

while examples from (55) include

$$\sum_{k=0}^n (-1)^k k \binom{n}{k} \binom{2n-k}{n-k} = -n^2, \quad (59)$$

$$\sum_{k=0}^n (-1)^k k^2 \binom{n}{k} \binom{2n-k}{n-k} = \frac{n^2(n^2-2n-1)}{2}, \quad (60)$$

and

$$\sum_{k=0}^n (-1)^k k^3 \binom{n}{k} \binom{2n-k}{n-k} = -\frac{n^2(4n^2-6n^3+6n+1+n^4)}{6}. \quad (61)$$

5.4. Combinatorial identities from an identity of MacMahon

MacMahon's identity is (Gould [4, Identity (6.7)], also Sun [9])

$$\sum_{k=0}^n (-1)^k \binom{n}{k}^3 x^k = \sum_{k=0}^n (-1)^k \binom{n+k}{2k} \binom{2k}{k} \binom{n-k}{k} x^k (1-x)^{n-2k}. \quad (62)$$

Proposition 14. If n is a non-negative integer and r and s are complex numbers such that $\Re(r-s+1) > 0$ and s is not a non-positive integer, then

$$\begin{aligned} & \sum_{k=0}^n (-1)^k \binom{n}{k}^3 \binom{k+r}{s}^{-1} \\ &= \sum_{k=0}^n \frac{(-1)^k s}{n-2k+s} \binom{n+k}{2k} \binom{2k}{k} \binom{n-k}{k} \binom{n+r-k}{n-2k+s}^{-1}. \end{aligned} \quad (63)$$

Proof. Multiply through (62) by $x^{r-s}(1-x)^{s-1}$ and integrate from 0 to 1. $\square$

Proposition 15. If n is a non-negative integer and m is a positive integer, then

$$\begin{aligned} \sum_{k=0}^{2n} (-1)^k k^m \binom{2n}{k}^3 \\ = \sum_{k=n-m+1}^n (-1)^k \binom{2n+k}{2k} \binom{2k}{k} \binom{2n-k}{k} \sum_{p=0}^{2(n-k)} (-1)^p \binom{2(n-k)}{p} (k+p)^m. \end{aligned} \quad (64)$$

In particular,

$$\sum_{k=0}^{2n} (-1)^k k \binom{2n}{k}^3 = (-1)^n n \binom{2n}{n} \binom{3n}{n} \quad (65)$$

and

$$\sum_{k=0}^{2n} (-1)^k k^2 \binom{2n}{k}^3 = (-1)^n \frac{2}{3} n^2 \binom{2n}{n} \binom{3n}{n}. \quad (66)$$

Proof. Write $2n$ for n in (62) to obtain

$$\sum_{k=0}^{2n} (-1)^k \binom{2n}{k}^3 x^k = \sum_{k=0}^{2n} (-1)^k \binom{2n+k}{2k} \binom{2k}{k} \binom{2n-k}{k} x^k (1-x)^{2(n-k)}, \quad (67)$$

from which (64) follows after replacing x with $\exp x$, differentiating m times with respect to x and evaluating at $x = 0$. $\square$

Remark 5. Setting $x = 1$ in (62) gives

$$\sum_{k=0}^n (-1)^k \binom{n}{k}^3 = \begin{cases} (-1)^{n/2} \binom{n}{n/2} \binom{3n/2}{n/2}, & \text{if } n \text{ is even;} \\ 0, & \text{if } n \text{ is odd;} \end{cases} \quad (68)$$

which subsumes

$$\sum_{k=0}^{2n} (-1)^k \binom{2n}{k}^3 = (-1)^n \binom{2n}{n} \binom{3n}{n}. \quad (69)$$

References

1. U. Abel, A short proof of the binomial identities of Frisch and Klamkin, J. Integer Sequences 23 (2020), Article 20.7.1.
2. K. Adegoke and R. Frontczak, Some notes on an identity of Frisch, Open J. Math. Sci. 8 (2024), 216–226.
3. R. Frisch, Sur les semi-invariants et moments employés dans l'étude des distributions statistiques, Skrifter utgitt av Det Norske Videnskaps-Akademi i Oslo, II, Historisk-Filosofisk Klasse, 3 (1926), 1–87.
4. H. W. Gould, Combinatorial Identities, Morgantown, West Virginia, 1972.
5. H. W. Gould, The Girard-Waring power sum formulas for symmetric functions and Fibonacci sequences, Fibonacci Quart. 37:2 (1999), 135–140.
6. H. W. Gould and J. Quaintance, On the binomial identities of Frisch and Klamkin, J. Integer Sequences 19 (2016), Article 16.7.7.
7. A. M. Rockett, Sums of the inverses of binomial coefficients, Fibonacci Quart. 19:5 (1981), 433–437.
8. S. Simons, A curious identity, Math. Gaz. 85 (2001), 296–298.
9. Z. Sun, On sums involving products of three binomial coefficients, Acta Arith. 156:2 (2012), 123–141.

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