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Maria del Mar Bibiloni-Femenias and Oscar Valero *

Posted Date: 17 May 2024

doi: 10.20944/preprints202405.1133.v1

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# Modular Quasi-Pseudo Metrics and the Aggregation Problem

**Maria del Mar Bibiloni-Femenias [1,2,†] and Oscar Valero [1,2,*,†]**

[1] Mathematics and Computer Science Department, Universitat de les Illes Balears, Carr. Valldemossa km. 7.5, Palma, 07122, Balearic Islands, Spain.
[2] Health Research Institute of the Balearic Islands (IdISBa), Hospital Universitari Son Espases, Palma, 10587, Balearic Islands, Spain.

* Correspondence: o.valero@uib.es

† These authors contributed equally to this work.

**Abstract:** The applicability of the distance aggregation problem has attracted the interest of many authors. Motivated by this fact, in this paper we face the modular quasi-(pseudo-)metric aggregation problem. We characterize those functions that allow merging a collection of modular quasi-(pseudo-)metrics into a single one. Specifically, a description of such functions in terms of triangle triplets is given and, in addition, the relationship between modular quasi-(pseudo-)metric aggregation functions and modular (pseudo-)metric aggregation functions is discussed. Such characterizations are illustrated with appropriate examples. A few methods to construct modular quasi-(pseudo-)metrics are yielded. Several properties of modular quasi-(pseudo-)metric aggregation functions are explored and used to develop quick tests for discarding candidate functions to aggregate modular quasi-(pseudo-)metrics. Moreover, a characterization of those modular quasi-(pseudo-)metric aggregation functions that preserve modular quasi-(pseudo-)metrics is also provided. Furthermore, the relationship between modular quasi-(pseudo-)metric aggregation functions and quasi-(pseudo-)metric aggregation functions is studied in such a way that significative differences are displayed.

**Keywords:** modular quasi-pseudo metric; quasi-pseudo metric; aggregation; monotony; subadditivity; triangle triplet

**MSC:** 54E25; 54E35

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## 0. Introduction

The aggregation of different pieces of information that comes from several sources is a common practice in applied sciences. The importance of the aggregation process is given by the fact that such pieces of information are transformed into a unique numerical value that is used to make a decision, which will allow solving the problem under consideration. In many problems the aforementioned pieces of information correspond to distances between different elements in our space. Thus the aggregation can be understood as a way to induce a global distance from a collection of particular distances. A typical situation occurs in path planning in robotics. Indeed, an autonomous robot must go from a start point until a target point in such a way that all obstabcles encountered on the path are adequately avoided. The robot must simultaneously calculate distances to different obstacles and then merge them in order to make the best decision.

The applicability and the relevance of aggregating has motivated that many authors have studied the distance aggregation problem. In this problem, the objective is to find the conditions that a function must satisfy in order to merge a collection of distances defined on the same set into a single one. Concretely the problem could be formulated as follows: given $n \in \mathbb{N}$ ($\mathbb{N}$ stands for the set of positive integers numbers), a function $F : [0, +\infty]^n \to [0, +\infty]$ is said to be a distance aggregation function provided if, for each collection of distances $\{d_i\}_{i=1}^n$ defined on the (non-empty) set $X$, the function $F \circ \widetilde{d}$ is a distance on $X$, where $\widetilde{d} : X \times X \to [0, +\infty]^n$ is defined, for all $x, y \in X$, by $\widetilde{d}(x,y) = (d_1(x,y), \ldots, d_n(x,y))$. Notice that we allow that the distance can take the $+\infty$ value. Some authors call this type of distances as extended distances (see [1,2]).

The distance aggregation problem has been explored when the distance is exactly a pseudo-metric in [3,4]. Following [1], let us recall that a pseudo-metric on a (non-empty) set $X$ is a function $d : X \times X \to [0, \infty[$ such that for all $x, y, z \in X$ :

**(pm1)** $d(x,x) = 0$,
**(pm2)** $d(x,y) = d(y,x)$,
**(pm3)** $d(x,z) \leq d(x,y) + d(y,z)$.

A pseudo-metric $d$ on $X$ is a metric when it satisfies the condition $(m1)$ given, for all for all $x, y \in X$, as follows:

**(m1)** $d(x,y) = 0 \Leftrightarrow x = y$.

A characterization of pseudo-metric aggregation functions was provided [3,4]. With the aim of introducing such a characterization, let us recall that, following [5], $(a,b,c) \in [0, +\infty]^n$ forms an $n$-triangular triplet whenever $a \preceq b + c$, $b \preceq a + c$ and $c \preceq a + b$. Notice that we denote by $\leq$ the usual partial order on the extended set of real numbers and that $a \preceq b \Leftrightarrow a_i \leq b_i$ for all $i = 1, \ldots, n$. According to the nomenclature in [6], if $(a,b,c) \in (0, +\infty]^n$ is a $n$-triangular triplet, then we will say that it is a positive $n$-triangular triplet. Moreover, a function $F : [0, +\infty]^n \to [0, +\infty]$ transforms $n$-triangular triplets into a 1-triangular triplet provided that $(F(a), F(b), F(c))$ is a 1-triangular triplet when $(a,b,c)$ is a $n$-triangular triplet. From now on, $0_n$ will denote the element of $[0, +\infty]^n$ given by $0_n = (0, \ldots, 0)$.

From now on, if the distances under consideration are not extended then we will consider functions $F : [0, +\infty[^n \to [0, +\infty[$ which can be understood as functions $F : [0, +\infty]^n \to [0, +\infty]$ such that $F([0, +\infty[) \subseteq [0, +\infty[$.

The following result was obtained in [3,4].

**Theorem 1.** *Let $n \in \mathbb{N}$ and let $F : [0, +\infty[^n \to [0, +\infty[$ be a function. Then the following assertions are equivalent:*

***(1)*** *$F$ is a pseudo-metric aggregation function.*
***(2)*** *$F$ satisfies the following conditions:*

***(2.1)*** $F(0_n) = 0$,
***(2.2)*** *$F$ transforms $n$-triangular triplets into a 1-triangular triplet.*

The problem of metric aggregation has been explored in [6]. Those functions that merge a collection of metrics into a single one were characterized in the spirit of Theorem 1 as follows:

**Theorem 2.** *Let $n \in \mathbb{N}$ and let $F : [0, +\infty[^n \to [0, +\infty[$ be a function. The following assertions are equivalent:*

***(1)*** *$F$ is a metric aggregation function.*
***(2)*** *$F$ satisfies the following conditions:*

***(2.1)*** $F(0_n) = 0$,
***(2.2)*** *$F$ transforms positive $n$-triangular triplets into a positive 1-triangular triplet,*
***(2.3)*** *If $a \in [0, +\infty[^n$ and $F(a) = 0$, then $\min\{a_1, \ldots, a_n\} = 0$.*

The relationhip between pseudo-metric aggregation functions and metric aggregation functions has not been explored. However, it not hard to see that there are pseudo-metric aggregation functions which are not metric aggregation functions as, for instance, the function constantly equals zero. Moreover, a slight modification of Example 10 in [6] shows that there are metric aggregation functions that are not pseudo-metric aggregation functions. Indeed, consider the function $F : [0,\infty[^2 \to [0,\infty[$ given by $F(a) = 0$ if $\min\{a_1, a_2\} = 0$ and otherwise $F(a) = 2$. Then $F$ satisfies all conditions in Theorem 2 an, thus it is a metric aggregation function. However, $F$ does not transform 2-triangular triplets into a 1-triangular triplet because $(a, b, c \in [0,+\infty[^2)$ forms a 2-triangular triplet when we consider $a = (1,0)$, $b = (0,1)$ and $c = (1,1)$ but $(F(a), F(b), F(c))$ is not a 1-triangular triple.

The symmetry inherent to pseudo-metrics limits their applicability to describing real problems and for this reason the notion of quasi-pseudo metric was introduced in the literature (see, for instance, [7,8]). According to [7,8], a quasi-pseudo-metric on a (non-empty) set $X$ is a function $q : X \times X \to [0,\infty[$ such that for all $x, y, z \in X$ :

**(qp1)** $q(x,x) = 0$,
**(qp2)** $q(x,z) \leq q(x,y) + q(y,z)$.

A quasi-pseudo-metric $d$ on $X$ is called a quasi-metric when it satisfies the condition (qm1) given, for all for all $x, y \in X$, as follows:

**(qm1)** $q(x,y) = q(y,x) = 0 \Leftrightarrow x = y$.

It must be stressed that every (pseudo-)metric is a quasi-(pseudo-)metric which satisfies in addition the symmetry.

In [9] an extension of Theorem 2 was obtained in the framework of quasi-metrics. Thus, several characterizations of quasi-metric aggregation functions were provided. In order to state them we recall that a function $F : [0,+\infty]^n \to [0,+\infty]$ is said to be monotone if $F(a) \leq F(b)$ for each $a, b \in [0,+\infty]^n$ with $a \preceq b$. Moreover, a function $F : [0,+\infty]^n \to [0,+\infty]$ is said to be subadditive if $F(a+b) \leq F(a) + F(b)$ for each $a, b \in [0,+\infty]^n$, where we have used the symbol $+$ for the usual addition on $[0,+\infty]^n$ and on $[0,+\infty]$ simultaneously.

**Theorem 3.** *Let $n \in \mathbb{N}$ and let $F : [0,\infty[^n \to [0,\infty[$ be a function. Then the following assertions are equivalent:*

**(1)** *F is a quasi-metric aggregation function.*
**(2)** *F satisfies the following conditions:*
  - **(2.1)** $F(0_n) = 0$,
  - **(2.2)** *If* $F(a) = 0$, *then* $\min\{a_1, \ldots, a_n\} = 0$,
  - **(2.3)** $F(a) \leq F(b) + F(c)$ *for each* $a, b, c \in [0,\infty)^n$ *with* $a \preceq b + c$.

**(3)** *F satisfies the following conditions:*
  - **(2.1)** $F(0_n) = 0$,
  - **(2.2)** *If* $F(a) = 0$, *then* $\min\{a_1, \ldots, a_n\} = 0$,
  - **(2.3)** *F is monotone and subadditive.*

Although Theorem 1 was not extended to the context of quasi-pseuod-metrics in [9], it is not hard to check that the following characterization holds.

**Theorem 4.** *Let $n \in \mathbb{N}$ and let $F : [0,\infty[^n \to [0,\infty[$ be a function. Then the following assertions are equivalent:*

**(1)** *F is a quasi-pseudo-metric aggregation function.*
**(2)** *F satisfies the following conditions:*

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*(2.1)* $F(0_n) = 0$,
*(2.2)* $F(a) \leq F(b) + F(c)$ *for each* $a, b, c \in [0, \infty)^n$ *with* $a \preceq b + c$.

*(3)* *F satisfies the following conditions:*

*(2.1)* $F(0_n) = 0$,
*(2.2)* *F is monotone and subadditive.*

It is clear that every quasi-(pseudo-)metric aggregation function is a (pseudo)-metric aggregation function. Nevertheless, the converse is not true such as Example 7 in [6] shows. Concretely, such an example provides a (pseudo-)metric aggregation function which is not monotone and, thus, it is not a quasi-(pseudo-)metric aggregation function.

In certain physical applications, the classical notion of (pseudo-)metric is not appropriate and it is necessary to incorporate a positive parameter in the metric axioms. This gives rise to the concept of modular metric. Following [2] (see also [10]), a modular (pseudo-)metric on a non-empty set $X$, is a function $w : ]0, +\infty[ \times X \times X \to [0, +\infty]$ which satisfies, for all $x, y, z \in X$, the conditions below:

**(MPM1)** $w(\lambda, x, x) = 0$ for all $\lambda > 0$,
**(MPM2)** $w(\lambda, x, y) = w(\lambda, y, x)$ for all $\lambda > 0$,
**(MPM3)** $w(\lambda + \mu, x, z) \leq w(\lambda, x, y) + w(\lambda, y, z)$ for all $\lambda, \mu > 0$.

When the axiom (MPM1) is replaced by the following one,

**(MM1)** $w(\lambda, x, y) = 0$ for all $\lambda > 0 \Leftrightarrow x = y$,

then $w$ is called a modular metric on $X$.

A typical example of modular (pseudo-)metric $w_d$ can be constructed from a given (pseudo-)metric $d$ on a non-empty set $X$ in the following way: $w_d(\lambda, x, y) = \frac{d(x,y)}{\lambda}$ for all $x, y \in X$ and for all $\lambda \in ]0, +\infty[$. Observe that the value $w_d(\lambda, x, y)$ provided by the modular pseudo-metric $w_d$ can be interpreted as the mean velocity between the point $x$ and $y$ in time $\lambda$. Thus, considering the collection $\{w_{d,\lambda} : \lambda \in ]0, +\infty[\}$, where $w_{d,\lambda}(x, y) = w_d(\lambda, x, y)$, one has the velocity field which is nonlinear in general.

It must be pointed out that the value $w(\lambda, x, y)$ can be understood in general as the distance between $x$ and $y$ with respect to the parameter $\lambda \in ]0, +\infty[$.

Many topological and metric studies on modular (pseudo-)metric spaces have been developed in the last years (see, for instance, [2,11,12] and references therein). Many applications of modular (pseudo-)metrics to operator theory have been given in the last years. A few works in this direction can be found, for instance, in [10,13–17].

Motivated by the growing interest in the distance aggregation problem, a characterization of those functions that merge a collection of modular (pseudo-)metrics has been provided in [18]. With the aim of recalling such a result we introduce the notion of modular (pseudo-)metric aggregation function.

On account of [18], given $n \in \mathbb{N}$, a function $F : [0, +\infty]^n \to [0, +\infty]$ is a modular (pseudo-)metric aggregation function provided that, for each collection of modular (pseudo-)metrics $\{w_i\}_{i=1}^n$ defined on the same set $X$, the function $F \circ \widetilde{w}$ is a modular (pseudo-)metric on $X$, where $\widetilde{w} : ]0, +\infty[ \times X \times X \to [0, +\infty]^n$ is given, for all $x, y \in X$ and for all $\lambda > 0$, by $\widetilde{w}(\lambda, x, y) = (w_1(\lambda, x, y), \ldots, w_n(\lambda, x, y))$.

Notice that aggregation problem in the context of modular (pseudo-)metrics is a particular case of the general distance aggregation problem stated at the beginning of this section.

In the light of the preceding notions we can recall the aforementioned characterization.

**Theorem 5.** *Let* $n \in \mathbb{N}$ *and let* $F : [0, +\infty]^n \to [0, +\infty]$ *be a function. Then the following assertions are equivalent:*

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**(1)** *$F$ is a modular pseudo-metric aggregation function.*
**(2)** *$F(0_n) = 0$, $F$ is monotone and subadditive.*
**(3)** *$F(0_n) = 0$ and $F(c) \leq F(a) + F(b)$ for all $a, b, c \in [0, +\infty]^n$ with $c \preceq a + b$.*
**(4)** *$F(0_n) = 0$, $F$ is monotone and transforms $n$-triangular triplets into $1$-triangular triplet.*

According to [18], in the case of modular metrics the preceding characterization can be stated as follows:

**Theorem 6.** *Let $n \in \mathbb{N}$ and let $F : [0, +\infty]^n \to [0, +\infty]$ be a function. The following assertions are equivalent:*

**(1)** *$F$ is a modular metric aggregation function.*
**(2)** *$F(0_n) = 0$, $F$ is monotone and subadditive. Moreover, if $a \in [0, +\infty]^n$ and $F(a) = 0$, then $a_i = 0$ for some $i = 1, \ldots, n$.*
**(3)** *$F(0_n) = 0$ and, in addition, $F(c) \leq F(a) + F(b)$ for all $a, b, c \in [0, +\infty]^n$ with $c \preceq a + b$. Moreover, if $a \in [0, +\infty]^n$ and $F(a) = 0$, then $a_i = 0$ for some $i = 1, \ldots, n$.*
**(4)** *$F(0_n) = 0$, $F$ is monotone and transforms $n$-triangular triplets into a $1$-triangular triplet. Moreover, if $a \in [0, +\infty]^n$ and $F(a) = 0$, then $a_i = 0$ for some $i = 1, \ldots, n$.*

Of course, every modular metric aggregation function is always a modular pseudo-metric aggregation function (this fact was stated as Corollary 1 in [18]). The following instance, which can be found in Example 1 in [18], shows that the reciprocal implication does not hold true. Consider the function $F : [0, +\infty]^n \to [0, +\infty]$ defined by $F(a) = 0$ for all $a \in [0, +\infty]^n$. Then $F$ satisfies all assumptions in the statement of Theorem 5 but $F(a) = 0$ when $a \in ]0, +\infty]^n$ which implies that it does not satisfy all conditions in the statement of Theorem 6.

According to [11], the fact that the value $w_d(\lambda, x, y)$, yielded by the modular (pseudo-)metric $w_d$, introduced above, can be interpreted as the mean velocity between the points $x$ and $y$ in time $\lambda$ and, in addition, the fact that such a velocity is typically asymmetric, suggest that there is not a sound reasoning to impose symmetry in the definition of modular distances (axiom (MPM2)). Hence, the notion of modular quasi-(pseudo-)metric is introduced and explored in [11].

Following [11], a modular quasi-pseudo-metric on a non-empty set $X$ is a function $w : ]0, +\infty[ \times X \times X \to [0, +\infty]$ which satisfies, for all $x, y, z \in X$, the conditions below:

**(MQPM1)** $w(\lambda, x, x) = 0$ for all $\lambda > 0$,
**(MQPM2)** $w(\lambda + \mu, x, z) \leq w(\lambda, x, y) + w(\lambda, y, z)$ for all $\lambda, \mu > 0$.

When the axiom (MQPM1) is replaced by the following one,

**(MQPM1)** $w(\lambda, x, y) = w(\lambda, y, x) = 0$ for all $\lambda > 0 \Leftrightarrow x = y$,

then $w$ is called a modular quasi-metric on $X$.

Note that the concept of modular (pseudo-)metric space can be retrieved from the notion of quasi-(pseudo-)metric when the symmetry is imposed.

A paradigmatic example of modular quasi-(pseudo-)metric which is not a modular (pseudo-)metric is is given by the function $w_u : ]0, +\infty[ \times [0, +\infty[ \to [0, +\infty[$ defined by $w_u(\lambda, x, y) = \frac{\max\{y-x, 0\}}{\lambda}$ for all $\lambda \in ]0, +\infty[$ and for all $x, y \in [0, +\infty[$.

In the light of all exposed facts, as a natural line of work, in this paper we focus our efforts on the modular quasi-(pseudo-)metric aggregation problem. Thus we provide a new characterization, in the spirit of Theorems 5 and 6, of those functions that allow merging a collection of modular quasi-(pseudo-)metrics into a single one. In particular, a description of such functions in terms of triangle triplets is given and, in addition, the relationship between modular quasi-(pseudo-)metric aggregation

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functions and modular (pseudo-)metric aggregation functions is discussed. The aforementioned characterizations are illustrated with appropriate examples. A few methods to construct modular quasi-(pseudo-)metrics are yielded. Several properties, common in aggregation theory, are explored and used to develop quick tests for discarding candidate functions to aggregate modular quasi-(pseudo-)metrics. Moreover, a characterization of those modular quasi-(pseudo-)metric aggregation functions that preserve modular quasi-(pseudo-)metrics is also provided. Furthermore, as a natural question, the connection between modular quasi-(pseudo-)metric aggregation functions and quasi-(pseudo-)metric aggregation functions is studied in such a way that significative differences are displayed. Finally, conlusions of the developed work and an exposition of future research lines are detailed.

## 1. The Modular Quasi-(Pseudo-)Metric Aggregation Problem

In this section we face the problem of merging a collection of modular quasi-(pseudo-)metrics as a natural extension of the problems, exposed in the preceding section, of aggregating a collection of modular (pseudo-)metrics.

To this end, let us introduce the notion of modular quasi-(pseudo-)metric aggregation function. Thus, given $n \in \mathbb{N}$, a function $F : [0, +\infty]^n \to [0, +\infty]$ is said to be a modular quasi-(pseudo-)metric aggregation function provided that, for each collection of modular quasi-(pseudo-)metrics $\{w_i\}_{i=1}^n$ defined on the same set $X$, the function $F \circ \widetilde{w}$ is a modular quasi-(pseudo-)metric on $X$, where $\widetilde{w} : ]0, +\infty[ \times X \times X \to [0, +\infty]^n$ is given, for all $x, y \in X$, and for all $\lambda > 0$, by $\widetilde{w}(\lambda, x, y) = (w_1(\lambda, x, y), \ldots, w_n(\lambda, x, y))$.

The next result was proved in [18, Theorem 1]. However, since it will be of great importance to our objective, we will include its proof for the sake of completeness.

**Theorem 7.** *Let $n \in \mathbb{N}$. If $F : [0, +\infty]^n \to [0, +\infty]$ is a monotone function such that $F(0_n) = 0$, then the following assertions are equivalent:*

- **(1)** *$F$ is subadditive.*
- **(2)** *$F(c) \leq F(a) + F(b)$ for all $a, b, c \in [0, +\infty]^n$ with $c \preceq a + b$.*
- **(3)** *$F$ transforms $n$-triangular triplets into a 1-triangular triplet.*

**Proof.** (1) ⇒ (2). Let $a, b \in [0, +\infty]^n$ with $c \preceq a + b$. Then

$$F(c) \leq F(a + b) \leq F(a) + F(b).$$

Notice that first inequality is derived from the monotony of $F$ and the second one is due to the subadditivity of $F$.

(2) ⇒ (3). It is a straightforward verification.

(3) ⇒ (1). Consider $a, b \in [0, +\infty]^n$ . Then $(a, b, c)$ forms an $n$-dimensional triangular triplet, where $c = a + b$. Since $F$transforms $n$-triangular triplets into a 1-triangular triplet we deduce that $(F(a), F(b), F(c))$ is a 1-triangular triplet. So $F(c) = F(a + b) \leq F(a) + F(b)$ and, thus, $F$ is subad-ditve. □

Next we focus our attention on getting a characterization of modular quasi-(pseudo-)metric aggregation functions. With this aim, we note that every modular quasi-(pseudo-)metric aggregation function is a modular (pseudo-)metric aggregation function such as the following result shows.

**Proposition 1.** *Let $n \in \mathbb{N}$. If $F : [0, +\infty]^n \to [0, +\infty]$ is a modular quasi-(pseudo-)metric aggregation function, then $F$ is a modular (pseudo-)metric aggregation function.*

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**Proof.** Consider a collection of modular (pseudo-)metrics $\{w_i\}_{i=1}^n$ on a non-empty set $X$. Then $\{w_i\}_{i=1}^n$ is a collection of modular quasi-(pseudo-)metrics on $X$ and, thus, $F \circ \widetilde{w}$ is a modular quasi-(pseudo-)metric on $X$. Moreover, for all $\lambda \in ]0, +\infty[$ and for all $x, y \in X$, we have that

$$F \circ \widetilde{w}(\lambda, x, y) = F(w_1(\lambda, x, y), \dots, w_n(\lambda, x, y))$$
$$= F((w_1(\lambda, y, x), \dots, w_n(\lambda, y, x)) = F \circ \widetilde{w}(\lambda, y, x),$$

since $w_i(\lambda, x, y) = w_i(\lambda, y, x)$ for all $i \in \{1, \dots, n\}$. So $F \circ \widetilde{w}$ is a modular (pseudo-)metric on $X$. Whence we conclude that $F$ is a modular (pseudo-)metric aggregation function. □

In the ligt of the preceding result, Theorems 5, 6 and 7 we immediately obtain the following statements.

**Proposition 2.** *Let $n \in \mathbb{N}$ and let $F : [0, +\infty]^n \to [0, +\infty]$ be a modular quasi-(pseudo-)metric aggregation function. Then the following assertions hold:*

**(1)** *$F(0_n) = 0$, $F$ is monotone and subadditive.*
**(2)** *$F(0_n) = 0$ and $F(c) \leq F(a) + F(b)$ for all $a, b, c \in [0, +\infty]^n$ with $c \preceq a + b$.*
**(3)** *$F(0_n) = 0$, $F$ is monotone and transforms $n$-triangular triplets into 1-triangular triplet.*

The characterization of modular quasi-pseudo-metric aggregation functions can be sated as follows:

**Theorem 8.** *Let $n \in \mathbb{N}$ and let $F : [0, +\infty]^n \to [0, +\infty]$ be a function. Then the following assertions are equivalent:*

**(1)** *$F$ is a modular quasi-pseudo-metric aggregation function.*
**(2)** *$F$ is a modular pseudo-metric aggregation function.*
**(3)** *$F(0_n) = 0$, $F$ is monotone and subadditive.*
**(4)** *$F(0_n) = 0$ and $F(c) \leq F(a) + F(b)$ for all $a, b, c \in [0, +\infty]^n$ with $c \preceq a + b$.*
**(5)** *$F(0_n) = 0$, $F$ is monotone and transforms $n$-triangular triplets into 1-triangular triplets.*

**Proof.** (1) $\Rightarrow$ (2). follows from Proposition 1. The equivalences (2) $\Leftrightarrow$ (3) $\Leftrightarrow$ (4) $\Leftrightarrow$ (5) are provided by Theorem 5. Now we prove that (4) $\Rightarrow$ (1). To this end, consider a collection $\{w_i\}_{i=1}^n$ of modular quasi-pseudo-metrics on a non-empty set $X$. Then $w_i(\lambda + \mu, x, z) \leq w_i(\lambda, x, y) + w_i(\mu, y, z)$ for all $\lambda, \mu \in ]0, +\infty[$, for all $x, y \in X$ and for all $i \in \{1, \dots, n\}$. Whence we obtain that $a, b, c \in [0, +\infty]^n$ satisfies that $c \preceq a + b$ with $c_i = w_i(\lambda + \mu, x, z)$, $a_i = w_i(\lambda, x, y)$ and $b_i = w_i(\mu, y, z)$ for all $i \in \{1, \dots, n\}$. It follows that

$$F \circ \widetilde{w}(\lambda + \mu, x, z) = F(w_1(\lambda + \mu, x, z), \dots, w_n(\lambda + \mu, x, z)) = F(c) \leq F(a) + F(b)$$
$$= F(w_1(\lambda, x, y), \dots, w_n(\lambda, x, y)) + F(w_1(\mu, y, z), \dots, w_n(\mu, y, z))$$
$$= F \circ \widetilde{w}(\lambda, x, y) + F \circ \widetilde{w}(\mu, y, z).$$

Hence, condition (MQPM2) is satisfied. We still need to verify that condition (MQPM1) is also satisfied. Since $w_i(\lambda, x, x) = 0$ for all $\lambda \in ]0, +\infty[$ and for all $x \in X$ we have that

$$F \circ \widetilde{w}(\lambda, x, x) = F(w_1(\lambda, x, x), \dots, w_n(\lambda, x, x)) = F(0_n) = 0.$$

Therefore $F \circ \widetilde{w}$ is a modular quasi-(pseudo-)metric on $X$ and, hence, $F$ is a modular quasi-(pseudo-)metric aggregation function. □

The fact that every modular metric aggregation function is always a modular pseudo-metric aggregation function provides the following consequence.

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**Corollary 1.** *Let $n \in \mathbb{N}$ and let $F : [0, +\infty]^n \to [0, +\infty]$ be a function. If $F$ is a modular metric aggregation function, then $F$ is a modular quasi-pseudo-metric aggregation function.*

**Proof.** It follows immediately from Theorems 6 and 8. □

It seems natural to wonder whether the converse of the preceding corollary is also true. Nevertheless, the answer to the posed question in negative. Indeed, observe that the classes of modular quasi-pseudo-metric aggregation functions and modular pseudo-metric aggregation functions are the same. Then the function $F : [0, +\infty]^n \to [0, +\infty]$ constantly equals 0, as commented in Section 0, satisfies all assumptions in the statement of Theorem 8 and, thus, it is a modular quasi-pseudo-metric aggregation function. However, $F(a) = 0$ when $a \in ]0, +\infty]^n$ which implies that it does not satisfy all conditions in the statement of Theorem 6 and, hence, it is not a modular metric aggregation function.

Next we provide several examples of modular quasi-pseudo-metric aggregation functions.

**Example 1.** *Let $n \in \mathbb{N}$. The following functions $F : [0, +\infty]^n \longrightarrow [0, +\infty]$ are, for all $w_1, \ldots, w_n \in [0, \infty[$, modular quasi-pseudo-metric aggregation functions:*

*(1)* $F(a) = \begin{cases} 0 & \text{if } a = 0_n, \\ +\infty & \text{otherwise.} \end{cases}$
*(2)* $F(a) = \sum_{i=1}^n w_i a_i$.
*(3)* $F(a) = \max\{w_1 a_1, \ldots, w_n a_n\}$.
*(4)* $F(a) = (\sum_{i=1}^n (w_i a_i)^p)^{\frac{1}{p}}$ *for all* $p \in [1, \infty[$.
*(5)* $F(a) = \min\{c, \sum_{i=1}^n w_i a_i\}$ *with* $c \in (0, \infty)$.

The following examples show functions which are not modular quasi-pseudo-metric aggregation functions.

**Example 2.** *Let $k \in (0, \infty)$. Define the functions $F_k : [0, +\infty]^n \to [0, +\infty]$ as follows:*

$$F_{(k)}(a) = \begin{cases} 0 & \text{if } a = 0_n, \\ k & \text{if } \min\{a_1, \ldots a_n\} > 0, \\ +\infty & \text{otherwise.} \end{cases}$$

*It is evident that $F$ is not monotone, since $k = F(a) < F(b) = +\infty$ with $(1, 0, \ldots, 0) = b \preceq a = (1, \ldots, 1)$. Theorem 8 warranties that $F$ is not a modular quasi-pseudo-metric aggregation function.*

**Example 3.** *Let the function $F : [0, +\infty]^n \to [0, +\infty]$ defided as follows:*

$$F(a) = \begin{cases} \frac{1}{2} & \text{if } a = (0, 0), \\ 1 & \text{otherwise.} \end{cases}$$

It is clear that $F(0, 0) \neq 0$, so Theorem 8 ensures that $F$ is not a modular quasi-pseudo-metric aggregation function.

Once the modular quasi-pseudo-metric aggregation problem has been studied, in the following we face a refinement of the aforementioned problem. Concretely, we try to describe those functions that are able to merge a collection of modular quasi-metrics into a single one. Accordingly, we are interested in getting an appropriate version of Theorem 8 extending Theorem 6.

The result below will play a crucial role in order to achieve our target. It must be stressed that it is an slight adaptation of a result given in [9, Lemma 2.5]. However, we have decided to include the proof, which remains the same, in order to help the reader.

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**Lemma 1.** *Let $n \in \mathbb{N}$ and let $F : [0,+\infty]^n \to [0,+\infty]$ be a subadditive function. Then the following assertions are equivalent:*

**(1)** *There exists $i_0 \in \{1,\ldots,n\}$ satisfying the following: for each $a \in [0,+\infty]^n$ with $F(a) = 0$ we have that $a_{i_0} = 0$.*
**(2)** *If $a \in [0,+\infty]^n$ such that $F(a) = 0$, then $\min\{a_1,\ldots,a_n\} = 0$.*

**Proof.** (1) $\Rightarrow$ (2). It is obvious.

(2) $\Rightarrow$ (1). Suppose for the purpose of contradiction that for each $i \in \{1,\ldots,n\}$ there exists $a^i \in [0,\infty)^n$ with $F(a^i) = 0$ and $a^i_i > 0$. From the fact that $F$ is subadditive we deduce that $F(a^1 + \cdots + a^n) \leq F(a^1) + \cdots + F(a^n) = 0$. Whence we obtain the existence of $c \in [0,\infty)^n$ such that $c = a^1 \cdots + a^n$ and, in addition, $F(c) = 0$. Nevertheless, $c_i > 0$ for all $i \in \{1,\ldots,n\}$, which is a contradiction because $\min\{c_1,\cdots,c_2\} = 0$. □

Taking into account that Proposition 1 gives that every modular metric aggregation function is in fact a modular quasi-metric aggregation function we have the following characterization.

**Theorem 9.** *Let $n \in \mathbb{N}$ and let $F : [0, +\infty]^n \to [0, +\infty]$ be a function. The following assertions are equivalent:*

**(1)** *$F$ is a modular quasi-metric aggregation function.*
**(2)** *$F$ is a modular metric aggregation function.*
**(3)** *$F(0_n) = 0$, $F$ is monotone and subadditive. Moreover, if $a \in [0, +\infty]^n$ and $F(a) = 0$, then $a_i = 0$ for some $i = 1,\ldots,n$.*
**(4)** *$F(0_n) = 0$ and, in addition, $F(c) \leq F(a) + F(b)$ for all $a,b,c \in [0, +\infty]^n$ with $c \preceq a + b$. Moreover, if $a \in [0, +\infty]^n$ and $F(a) = 0$, then $a_i = 0$ for some $i = 1,\ldots,n$.*
**(5)** *$F(0_n) = 0$, $F$ is monotone and transforms $n$-triangular triplets into a $1$-triangular triplet. Moreover, if $a \in [0, +\infty]^n$ and $F(a) = 0$, then $a_i = 0$ for some $i = 1,\ldots,n$.*

**Proof.** (1) $\Rightarrow$ (2). follows from Proposition 1. The equivalences (2) $\Rightarrow$ (3) $\Leftrightarrow$ (4) $\Leftrightarrow$ (5) are provided by Theorem 6. Now we prove that (4) $\Rightarrow$ (1). To this end, consider a collection $\{w_i\}_{i=1}^n$ of modular quasi-metrics on a non-empty set $X$. The same arguments to those applied to the prove of Theorem 8 gives that $F \circ \widetilde{w}$ satisfies condition (MPQM2). It remains to prove that condition (MQM1) is hold.

Since $w_i(\lambda, x, x) = 0$ for all $\lambda \in ]0,+\infty[$ and for all $x \in X$ we have that

$$F \circ \widetilde{w}(\lambda, x, x) = F(w_1(\lambda, x, x),\ldots, w_n(\lambda, x, x)) = F(0_n) = 0.$$

Now assume that we have that $F \circ \widetilde{w}(\lambda, x, y) = F \circ \widetilde{w}(\lambda, y, x) = 0$ for any $x, y \in X$ and for all $\lambda \in ]0,+\infty[$. Then $F(w_1(\lambda, x, y),\ldots, w_n(\lambda, x, y)) = 0$ and $F(w_1(\lambda, y, x),\ldots, w_n(\lambda, y, x)) = 0$. By Lemma 1, there exists $i_0 \in \{1,\ldots,n\}$ such that $w_{i_0}(\lambda, x, y) = w_{i_0}(\lambda, y, x) = 0$ for all $\lambda \in ]0,+\infty[$. The fact that $w_i$ is a modular quasi-metric on $X$ yields that $x = y$. So $F$ is a modular quasi-metric aggregation function. □

The following example gives instances of modular quasi-metric aggregation functions.

**Example 4.** *Let $n \in \mathbb{N}$. The following functions $F : [0,+\infty]^n \longrightarrow [0,+\infty]$ are, for all $w_1,\ldots,w_n \in ]0,\infty[$, modular quasi-metric aggregation functions:*

**(1)** *$F(a) = \sum_{i=1}^n w_i a_i$. Observe that this instance contains the class of weighted arithmetic means, and thus the arithmetic mean (see [19]).*
**(2)** *$F(a) = \max\{w_1 a_1,\ldots, w_n a_n\}$.*
**(3)** *$F(a) = (\sum_{i=1}^n (w_i a_i)^p)^{\frac{1}{p}}$ for all $p \in [1,\infty[$. This instance contains those root-mean-powers such that $p \geq 1$ (see [19]).*

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**(4)** *$F(a) = \sum_{i=1}^{n} w_i a_{(i)}$ with $w_i \geq w_j$ for $i < j$, where $a_{(i)}$ is the ith largest of the $a_1, \ldots, a_n$. Of course OWA operators with decreasing weights belong to this class of functions (see, for instance, [19,20]).*

**(5)** *$F(a) = \min\{c, \sum_{i=1}^{n} w_i a_i\}$ with $c \in (0, \infty)$.*

**(6)** $F(a) = \begin{cases} 0 & \text{if } \min\{a_1, \ldots, a_n\} = 0, \\ c & \text{otherwise,} \end{cases}$ *with $c \in (0, \infty)$.*

Example 2 again shows a function which is not a modular quasi-metric aggregation function, since it is not monotone. In the same way, the fuction exposed in Example 3 is not a modular quasi-metric aggregation function. Notice that in the aforementioned example the image of $0_n$ is not zero.

Inspired by Example 2 we give a method to construct modular quasi-(pseudo-)metric aggregation functions in the following result.

**Proposition 3.** *Let $n \in \mathbb{N}$ and $F : [0, +\infty[^n \to [0, +\infty[$ be a monotone and subadditive function. Consider the function $G : [0, +\infty]^n \to [0, +\infty]$ defined by*

$$G(a) = \begin{cases} F(a) & \text{if } a \in [0, +\infty[^n, \\ \infty & \text{otherwise.} \end{cases}$$

*Then the following assertions hold:*

**(1)** *$G : [0, +\infty]^n \to [0, +\infty]$ is a modular quasi-pseudo-metric aggregation function provided that $F(0_n) = 0$.*

**(2)** *$G : [0, +\infty]^n \to [0, +\infty]$ is a modular quasi-metric aggregation function provided that $F(0_n) = 0$ and that $F$ satisfies the following property: if $a \in [0, +\infty]^n$ and $F(a) = 0$, then $a_i = 0$ for some $i = 1, \ldots, n$.*

**Proof.** We first show that $G$ is monotone. Let $a, b \in [0, +\infty]^n$ such as $a \preceq b$. Indeed, let us distinguish two possible cases.

Case 1. There exists $i \in \{1, \ldots, n\}$ such that $a_i = +\infty$. Then $b_i = +\infty$ and, thus, $G(a) = G(b) = +\infty$. So $G(a) \leq G(b)$.

Case 2. $a_i \neq \infty$ for all $i \in \{1, \ldots, n\}$. Then $G(a) = F(a) \leq G(b)$.

Next we prove that $G$ is subadditibve. With this aim, consider $a, b \in [0, +\infty]^n$. Again, we distinguish two possible cases:

Case 1. There exists $i \in \{1, \ldots, n\}$ such that either $a_i = +\infty$ or $b_i = +\infty$. Then $G(a + b) = +\infty$, and either $G(a) = +\infty$ or $G(b) = +\infty$. So $G(a + b) \leq G(a) + G(b)$.

Case 2. $a_i \neq \infty$ and $b_i \neq \infty$ for all $i \in \{1, \ldots, n\}$. Then $a + b \in [0, +\infty[^n$ and $G(a + b) = F(a + b) \leq F(a) + F(b) = G(a) + G(b)$.

Therefore $G$ is subadditve.

Assume that $F(0_n) = 0$. Then $G(0_n) = F(0_n) = 0$ and, thus, Theorrem 8 gives that $G$ is a modular quasi-pseudo-metric aggregation function. Finally, assume that $F$ satisfies the property: if $a \in [0, +\infty]^n$ and $F(a) = 0$, then $a_i = 0$ for some $i = 1, \ldots, n$.. Suppose that $G(a) = 0$. It follows that it must necessarily that $F(a) = 0$. Then $a_i = 0$ for some $i = 1, \ldots, n$.. Consequently, by Theorem 9, we obtain that $F$ is a modular quasi-metric aggregation function. □

The fact that a function $F : [0, +\infty[^n \to [0, +\infty[$ satisfying all assumptions in the statement of Proposition 3 is either a quasi-pseudo-metric aggregation function or a quasi-metric aggregation function, suggests us to explore the relationship between the aforementioned functions and the modular quasi-(pseudo-)metric aggregation functions.

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First of all we show that there are quasi-(pseudo-)metric aggregation functions which are not modular quasi-(pseudo-)metric aggregation functions. The following example warranties such a statement.

**Example 5.** *Let* $F : [0, +\infty[^2 \to [0, +\infty[$ *be the function defined by*

$$F(a) = \begin{cases} 0 & \text{if } a = (0,0), \\ 1 & \text{otherwise.} \end{cases}$$

*Clearly F satisfies all assumptions in Theorem 3 and thus, in Theorem 4. Whence we deduce that F is a quasi-(pseudo-)metric aggregation function. Now consider the collection of modular quasi-(pseudo-)metrics* $\{w_i\}_1^n$ *on* $\mathbb{R}$ *where* $w_i = w$ *for all* $i \in \{1, \ldots, n\}$ *with* $w(\lambda, x, x) = 0$ *for all* $\lambda \in ]0, +\infty[$ *and* $w(\lambda, x, y) = +\infty$ *for all* $\lambda \in ]0, +\infty[$ *and for all* $x, y \in \mathbb{R}$ *such that* $x \neq y$*. Then* $F \circ \widetilde{w}$ *is not a modular (quasi-)pseudo-metric aggregation function because* $F \circ \widetilde{w}(1, 2, 3)$ *is not defined (observe that the value* $F(+\infty, \ldots, +\infty)$ *is not defined).*

Notice that Example 5 also shows that there are (pseudo-)metric aggregation functions which are not modular (pseudo-)metric aggregation functions. This fact was not studied in [18].

The example below gives an instance of modular quasi-(pseudo-)metric aggregation functions which is not a quasi-(pseudo-)metric aggregation function.

**Example 6.** *Let* $n \in \mathbb{N}$*. Consider the function* $F : [0, +\infty]^n \to [0, +\infty]$ *defined by* $F(0_n) = 0$ *and* $F(a) = +\infty$ *for all* $a \neq 0_n$*. It is a simple matter to check that F is a modular quasi-(pseudo-)metric aggregation function but is not a quasi-(pseudo-)metric aggregation function.*

Notice that Example 6 also shows that there are modular (pseudo-)metric aggregation functions which are not a (pseudo-)metric aggregation function. This fact was not explored in [18].

The instances of modular quasi-metric aggregation function given in Example 4 inspires the following method to construct such functions.

**Proposition 4.** *Let* $g : [0, +\infty]^n \longrightarrow [0, +\infty]$ *be a subadditive, monotone function such that* $g(a) = 0$ *if and only if* $a = 0_n$*. Let* $W : [0, +\infty]^n \to [0, +\infty]^n$ *be a function such that* $W(0_n) = 0_n$ *and satisfying the following conditions:*

***(1)*** *If* $W(a) = 0_n$*, then* $\min\{a_1, \ldots, a_n\} = 0$.
***(2)*** $W(a) \preceq W(b)$ *whenever* $a \preceq b$*.*

*If the function* $g \circ W : [0, +\infty]^n \longrightarrow [0, +\infty]$ *is subadditive, then the function* $F : [0, +\infty]^n \to [0, +\infty]$ *defined by* $F(a) = g(W(a))$ *for all* $a \in [0, +\infty]^n$ *is a modular quasi-metric aggregation function.*

**Proof.** The subadditivity of $g \circ W$ gives that subadditivity of $F$. Moreover, the monotony of $F$ is directly derived from the monotony of $g$ and condition (2). Furthermore, $F(0_n) = g(W(0_n)) = g(0_n) = 0$. Now, assume that there is $a \in [0, +\infty]^n$ such that $F(a) = 0_n$. Then $g(W(a)) = 0_n$. Hence, $W(a) = 0_n$. It follows, from condition (1), that $\min\{a_1, \ldots, a_n\} = 0$. By Theorem 9 we have that $F$ is a modular quasi-metric aggregation function. □

The next example shows that the "separation" condition on $g$ can not be deleted from the statement of Proposition 4.

**Example 7.** *Consider the function* $W : [0, +\infty]^n \to [0, +\infty]^n$ *given by* $W(a) = (a_1, \ldots, a_n)$*. Then W satisfies all assumptions in the statement of Proposition 4. Fix* $i_0 \in \{1, \ldots, n\}$*. Define the function* $g : [0, +\infty]^n \longrightarrow$

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*$[0,+\infty]$ by $g(a) = a_{i_0}$ for all $a \in [0,+\infty]^n$. The $g$ is subadditive, monotone and satisfies that $g(0_n) = 0$. However, $g(0_{i_0}1) = 0$ but $0_{i_0}1 \neq 0_n$, where $0_{i_0}1$ stands for the element of $[0,+\infty]^n$ with the $i_0$th coordinate as $0$ and the $j$th coordinate with $j \neq i_0$ as $1$. Clearly, the function $F : [0,+\infty]^n \longrightarrow [0,+\infty]$ given by $F(a) = g(W(a))$ for all $a \in [0,+\infty]^n$ fulfills that $F(0_{i_0}1) = g(0_{i_0}1) = 0$ and, as a consequence, it is not a modular quasi-metric aggregation function.*

The next result clarifies when a modular (quasi-)pseudo-metric aggregation function is also a (quasi-)pseudo-metric aggregation function. In order to state it we will make use of the notion of finite modular quasi-pseudo-metric aggregation function. We will say that a modular (quasi-)(pseudo-)metric aggregation function $F : [0,+\infty]^n \to [0,+\infty]$ is a finite modular (quasi-)(pseudo-)metric aggregation function provided that, for each collection of modular (quasi-)(pseudo-)metrics $\{w_i\}_{i=1}^n$ defined on the same set $X$ such that $w_i(\lambda,x,y) < +\infty$ for all $\lambda \in ]0,+\infty[$, for all $x,y \in X$ and for all $i \in \{1,\ldots.n\}$, the function $F \circ \widetilde{w}$ is a modular (quasi-)(pseudo-)metric on $X$ with $F \circ \widetilde{w}(\lambda,x,y) < +\infty$ for all $\lambda \in ]0,+\infty[$ and for all $x,y \in X$.

**Theorem 10.** *Let $n \in \mathbb{N}$ and let $F : [0,+\infty]^n \to [0,+\infty]$ be a modular quasi-pseudo-metric aggregation function. The following assertions are equivalent.*

- **(1)** *$F$ is a finite modular quasi-pseudo-metric aggregation function.*
- **(2)** *$F$ is a finite modular pseudo-metric aggregation function.*
- **(3)** *$F \restriction_{[0,+\infty[^n}$ is a quasi-pseudo-metric aggregation function.*
- **(4)** *$F \restriction_{[0,+\infty[^n}$ is a pseudo-metric aggregation function.*
- **(5)** *$F(a) = +\infty$ for some $a \in [0,+\infty]^n \Leftrightarrow a_i = +\infty$ for some $i \in \{1,\ldots,n\}$.*

**Proof.** (1) $\Leftrightarrow$ (2). It is evident.

(1) $\Rightarrow$ (3). Consider a collection of quasi-pseudo-metrics $\{q_i\}_{i=1}^n$ defined on a non-empty set $X$. Define on $X$ the collection $\{w_i\}_{i=1}^n$ by $w_i(\lambda,x,y) = q_i(x,y)$ for all $\lambda \in ]0,+\infty[$ and for all $x,y \in X$. Then $\{w_i\}_{i=1}^n$ is a collection of modular quasi-pseudo-metrics on $X$. Since $F$ is a modular quasi-pseudo-metric aggregation function we have that $F \circ \widetilde{w}$ is a modular quasi-pseudo-metric on $X$. Moreover, we have, on the one hand, that $F \circ \widetilde{w}(\lambda,x,y) < +\infty$ for all $\lambda \in ]0,+\infty[$ and for all $x,y \in X$ and, on the other hand, that $F \circ \widetilde{w}(\lambda,x,y) = F(q_1(x,y)\ldots,q_n(x,y))$ for all $\lambda \in ]0,+\infty[$ and for all $x,y \in X$. Then $F \circ \widetilde{w}_1$ is a quasi-pseudo-metric on $X$ with $F \circ \widetilde{w}_1(x,y) = F \circ \widetilde{w}(1,x,y) = F(q_1(x,y)\ldots,q_n(x,y))$ for all $x,y \in X$. Whence we deduce that $F \circ \widetilde{q}$ is a quasi-pseudo-metric on $X$. Therefore $F \restriction_{[0,+\infty[^n}$ is a quasi-pseudo-metric aggregation function.

(3) $\Leftrightarrow$ (4). The eqiuvalence is guaranteed by the fact that $F \restriction_{[0,+\infty[^n}$ is monotone and by Theorems 4 and 1.

(4) $\Rightarrow$ (5). For the purpose of contradiction, assume that there exists $a \in [0,+\infty]^n$ such that $F(a) = +\infty$ and $a_i \in [0,+\infty[$ for all $i \in \{1,\ldots,n\}$. Define the collection $\{d_i\}_{i=1}^n$ of pseudo-metrics on a non-empty set $X$ by $d_i(x,y) = a_i d_D(x,y)$ for all $x,y \in X$, where $d_D$ is the discrete pseudo-metric on X. Since $F \restriction_{[0,+\infty[^n}$ is a pseudo-metric aggregation function. It follows that $F \circ \widetilde{d}$ is a pseudo-metric on $X$. Hence $F \circ \widetilde{d}(x,y) = F(d_1(x,y),\ldots,d_n(x,y)) < +\infty$ for all $x,y \in X$. However, let $u,v \in X$ with $u \neq v$, $+\infty = F(a) = F(d_1(u,v),\ldots,d_n(u,v)) < +\infty$ which is a contradiction.

(5) $\Rightarrow$ (1). It is immediate, since $F([0,+\infty[^n) \subseteq [0,+\infty[$. $\square$

Similar arguments apply to the quasi-metic case.

**Theorem 11.** *Let $n \in \mathbb{N}$ and let $F : [0,+\infty]^n \to [0,+\infty]$ be a modular quasi-metric aggregation function. The following assertions are equivalent.*

- **(1)** *$F$ is a finite modular quasi-metric aggregation function.*

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**(2)** *$F$ is a finite modular metric aggregation function.*
**(3)** *$F\restriction_{[0,+\infty[^n}$ is a quasi-metric aggregation function.*
**(4)** *$F\restriction_{[0,+\infty[^n}$ is a metric aggregation function.*
**(5)** *$F(a) = +\infty$ for some $a \in [0,+\infty]^n \Leftrightarrow a_i = +\infty$ for some $i \in \{1,\ldots,n\}$.*

In the light of the preceding result, it is clear that every finite modular quasi-(pseudo-)metric aggregation function merge a collection of modular quasi-(pseudo-)metrics which do not take the $+\infty$ value into a modular quasi-(pseudo-)metric that also does not take the $+\infty$ value. This is the reason for the name. The functions (2), (3), (4) and (5) given in Example 1 are instances of finite modular quasi-pseudo-metric aggregation functions. Nevertheless, the function (1) provided in the aforementioned example is a modular quasi-pseudo-metric aggregation function that is not finite.

It must be pointed out that Theorems 10 and 11 stated in the modular framework are surprising due to the fact that there are (pseudo-)metric aggregation functions which are not quasi-(pseudo-)metric aggregation functions as exposed in Section 0.

It seems interesting to stress that the function (5) in Example 1 and (5) and (6) in Example 4 are instances of modular (quasi-)(pseudo-)metric aggregation functions which always merge a collection of modular (quasi-)(pseudo-)metrics into a modular (quasi-)(pseudo-)metric that does not take the $+\infty$ value. This fact inspires the possibility of describing such kind of functions.

In the following we characterize such functions. Before stating the characterization, let us recall that, given $a \in [0,+\infty]$, $a_i0$ will denote the element of $[0,+\infty]^n$ with the $i$th coordinate as $a$ and the $j$th coordinate with $j \neq i$ as 0.

**Proposition 5.** *Let $n \in \mathbb{N}$ and let $F : [0, +\infty]^n \to [0, +\infty]$ be a modular (quasi-)(pseudo-)metric aggregation function. Then the following assertions are equivalent.*

**(1)** *If $\{w_i\}_{i=1}^n$ is a collection of modular (quasi-)(pseudo-)metrics defined on a non-empty set $X$, then $F \circ \widetilde{w}(\lambda, x, y) < +\infty$ for all $\lambda \in ]0, +\infty[$ and for all $x, y \in X$.*
**(2)** *$F(+\infty,\ldots,+\infty) < +\infty$.*
**(3)** *$F(+\infty_i0) < +\infty$ for all $i \in \{1,\ldots,n\}$.*

**Proof.** (1) $\Rightarrow$ (2). For the purpose of contradiction we assume that $F(+\infty,\ldots,+\infty) = +\infty$. Now consider a non-empty set $X$ (with at least two different elements) and the collection of modular (quasi-)(pseudo-)metrics $\{w_i\}_{i=1}^n$ on $X$ such that $w_i = w$ for all $i \in \{1,\ldots,n,\}$ where $w$ is defined, for all $\lambda \in ]0,+\infty[$, by $w(\lambda,x,y) = 0$ if $x = y$ and $w(\lambda,x,y) = +\infty$ if $x \neq y$. Then $F \circ \widetilde{w}$ is a modular (quasi-)(pseudo-)metric such that $F \circ \widetilde{w}(1,x,y) = F(+\infty,\ldots,+\infty)$ provided that $x \neq y$. Whence we have that $+\infty = F(+\infty,\ldots,+\infty) = F \circ \widetilde{w}(1,x,y) < +\infty$, which is a contradiction. So $F(+\infty,\ldots,+\infty) < +\infty$.

(2) $\Rightarrow$ (3). Since $F$ is a modular (quasi-)(pseudo-)metric aggregation function we have that it is monotone. Thus $F(+\infty_i0) \leq F(+\infty,\ldots,+\infty) < +\infty$ for all $i \in \{1,\ldots,n\}$.

(3) $\Rightarrow$ (2). Since $F$ is a modular (quasi-)(pseudo-)metric aggregation function we have that it is subadditive. Hence $F(+\infty,\ldots,+\infty) \leq \sum_{i=1}^n F(+\infty_i0) < n \cdot +\infty = +\infty$.

(2) $\Rightarrow$ (1). Let $\{w_i\}_{i=1}^n$ be a collection of modular (quasi-)(pseudo-)metrics defined on a non-empty set $X$. Then $F \circ \widetilde{w}$ is a modular (quasi-)(pseudo)-metric on $X$. Since $F$ is monotone and the fact that $w_i(\lambda,x,y) \leq +\infty$ for all $\lambda \in ]0,+\infty[$, for all $x,y \in X$ and for all $i \in \{1,\ldots,n\}$, we obtain that $F \circ \widetilde{w}(\lambda,x,y) = F(w_1(\lambda,x,y),\ldots,w_n(\lambda,x,y)) \leq F(+\infty,\ldots,+\infty) < +\infty$. □

We end this section exploring a question that arise in a natural way. Since every modular (quasi-)(pseudo-)metric aggregation function fuses a collection of modular (quasi-)(pseudo-)metrics into a single one, it seems natural to wonder the following question: Do this type of functions preserve modular (quasi-)(pseudo-)metrics? Notice that by preserving we mean that when all modular (quasi-)(pseudo-)metrics in the collection to be fused are the same then the aggregation function gives such a modular (quasi-)(pseudo-)metric as the aggregated one.

The concept below plays a central role in order to answer the posed question.

Given $n \in \mathbb{N}$, a function $F : [0, +\infty]^n \to [0, +\infty]$ has $p \in [0, +\infty]$ as an idempotent element if $F(p, \ldots, p) = p$ ([19]). In addition, $F$ is said to be idempotent if it has every element in $[0, +\infty]$ as an idempotent element, that is $F(p, \ldots, p) = p$ for all $p \in [0, +\infty]$.

In the light of the preceding notion, the result below answers the query.

**Theorem 12.** *Let $n \in \mathbb{N}$ and let $X$ be a non-empty set. If $F : [0, +\infty]^n \to [0, +\infty]$ is modular (quasi-)(pseudo-)metric aggregation function, then the following assertions are equivalent:*

- **(1)** *$F \circ \widetilde{w} = w$ for all modular (quasi-)(pseudo-)metric on $X$.*
- **(2)** *$F$ is idempotent.*

**Proof.** (1) ⇒ (2). Let $a \in [0, +\infty]$. Fix $\lambda_0 \in ]0, +\infty[$. Consider the modular (quasi-)(pseudo-)metric on a non-empty set $X$ given by

$$w(\lambda, x, y) = \begin{cases} 0 & \text{if } x = y \text{ and } \lambda > 0, \\ a & \text{if } x \neq y \text{ and } 0 < \lambda < \lambda_0, \\ 0 & \text{if } x \neq y \text{ and } \lambda \geq \lambda_0. \end{cases}$$

Then $F \circ \widetilde{w}$ i a modular (quasi-)(pseudo-)metric on $X$ and $F \circ \widetilde{w} = w$. So, taking a $0 < \lambda < \lambda_0$, we obtain that $F(a, \ldots, a) = F(w(\lambda, x, y), \ldots, w(\lambda, x, y)) = F \circ \widetilde{w}(\lambda, x, y) = w(\lambda, x, y) = a$. Whence we conclude that $F$ is dempotent.

(2) ⇒ (1). Consider modular (quasi-)(pseudo-)metric $w$ on a non-empty set $X$. Since $F$ is idempotent we have that $F \circ \widetilde{w}(\lambda, x, y) = F(w(\lambda, x, y), \ldots, w(\lambda, x, y)) = w(\lambda, x, y)$ for all $\lambda \in [0, +\infty]^n$ and for all $x, y \in X$. So $F \circ \widetilde{w} = w$ as claimed. □

## 2. The Aggregation Problem: Discarding Functions

In this section we explore a few properties, common in aggregation theory (see, for instance, [19]) and inspired by those explored in [6,21], that modular quasi-(pseudo-)metric aggregation functions enjoy. In some sense such properties allow us to develop a quick test for discarding candidate functions to aggregate modular quasi-(pseudo-)metrics.

On account of [19], a function $F : [0, +\infty]^n \to [0, +\infty]$ has $u \in [0, +\infty]$ as an absorbent (or annihilator) element in its $i$-th variable when

$$F(a_1, \ldots, a_{i-1}, u, a_{i+1}, \ldots, a_n) = u$$

for all $a_1, \ldots, a_{i-1}, a_i, \ldots, a_n \in [0, +\infty]$.

In the light of the preceding notion we have the following result.

**Proposition 6.** *Let $n \in \mathbb{N}$. If $F : [0, +\infty]^n \to [0, +\infty]$ is a modular quasi-pseudo-metric aggregation function, then the following assertions hold:*

- **(1)** *$F$ has not $u \in ]0, +\infty[$ as an absorbent element in at least two variables when $F$ has $+\infty$ as an idempotent element.*
- **(2)** *$F$ has not 0 as an absorbent element in at least two variables when $F$ has $p \in ]0, +\infty]$ as an idempotent element.*

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*(3) $F$ has not $u \in ]0, +\infty[$ as an absorbent element in at least two variables when $F$ has $p \in ]0, \infty[$ as an idempotent element with $p > 2u$.*

**Proof.** (1). Supose that $F$ has $u \in ]0, +\infty[$ as an absorbent element in the two first variables. We have that $F(+\infty, \ldots, +\infty, ) = +\infty$. Moreover, $2u = F(u, +\infty, \ldots, +\infty) + F(+\infty, u, \ldots, u)$. The subadditivity of $F$ gives that

$$+\infty = F(+\infty, \ldots, +\infty) \leq F(u, +\infty, \ldots, +\infty) + F(+\infty, u, \ldots, u) = 2u,$$

which is a contradiction.

(2). Assume without loss of generality that 0 is an absorbent element in the two first variables. We have that $(p, \ldots, p) = (0, p, p, \ldots, p) + (p, 0, \ldots, 0)$ Since 0 is a an absorbent element in the two first variables we have that $F(0, p, p, \ldots, p) = F((p, 0, \ldots, 0) = 0$. Hence, by subadditivity of $F$, $F(p, \ldots, p) \leq F(0, p, p, \ldots, p) + F((p, 0, \ldots, 0) = 0$. So $p = F(p, \ldots, p) \leq 0$, which is a contradiction.

(3). Suppose that $F$ has $u$ as absorbent element, for instance, in the two first variables. Then $F(p, \ldots, p) = p > 2u$. Moreover,

$$2u = F(u, p-u, p-u, \ldots, p-u) + F(p-u, u, u, \ldots, u) \geq F(p, \ldots, p) > 2u,$$

which is a contradiction. □

In the quasi-metric case we have the following specific result.

**Proposition 7.** *Let $n \in \mathbb{N}$. If $F : [0, +\infty]^n \to [0, +\infty]$ is a modular quasi-metric aggregation function, then $F$ has not 0 as an absorbent element in at least two variables.*

**Proof.** For the purpose of contradiction, we suppose that 0 is an absorbent element of $F$ in its $i$-th and $j$-th variables with $i < j$. The subadditivity of $F$ gives that

$$F(1, \ldots, 1^{i)}, \ldots, 1^{j)}, \ldots, 1) \leq F(1, \ldots, 0^{i)}, \ldots, 1^{j)}, \ldots, 1) + F(0, \ldots, 1^{i)}, \ldots, 0^{j)}, \ldots, 0) = 0.$$

Since $F$ is a modular quasi-metric aggregation function we have that $F(a) > 0$ for all $a \in ]0, +\infty]^n$. Indeed, assume that there exists $a \in ]0, +\infty]^n$ such that $F(a) = 0$. Then $a_i = 0$ for some $i \in \{1, \ldots, n\}$, which is a contradiction. Hence, $0 < F(1, \ldots, 1)$. Whence we deduce that $0 < F(1, \ldots, 1) \leq 0$, which is imposible. □

Following [19], a function $F : [0, +\infty]^n \to [0, +\infty]$ is said to be conjunctive provided that $F(a) \leq \min\{a_1, \ldots, a_n\}$ for all $a \in [0, +\infty]^n$.

The following result will be useful later on.

**Proposition 8.** *Let $n \in \mathbb{N}$. If $F : [0, +\infty]^n \to [0, +\infty]$ is a conjunctive function, then $F$ has 0 as an absorbent element in at least two variables.*

**Proof.** Consider $a \in [0, +\infty]^n$. Let $i, j \in \{1, \ldots, n\}$ with $i < j$. We have that

$$F(a_1, \ldots, 0^{i)}, \ldots, a_n) \leq \min\{a_1, \ldots, 0^{i)}, \ldots, a_n\} = 0$$

and, in addition, that $F(a_1, \ldots, 0^{j)}, \ldots, a_n) \leq \min\{a_1, \ldots, 0^{j)}, \ldots, a_n\} = 0$. It follows that $F$ has 0 as an absorbent element in at least two variables. □

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As a consequence of the preceding result and Proposition 7 we obtain that every modular quasi-metric aggregation function is not conjuctive.

Let us recall that, according to [19], a function $F : [0, +\infty]^n \to [0, +\infty]$ has $e \in [0, +\infty]$ as a neutral element if $F(a_i e) = a$ for all $a \in [0, +\infty]$ and for all $i =\in \{1, \ldots, n\}$, where $a_i e$ denotes the element of $[0, +\infty]^n$ such that the $i$-th coordinate is $a$ and the $j$-th coordinate with $j \neq i$ is $e$.

Next we discuss the neutral elements of modular quasi-pseudo-metric aggregation functions.

**Proposition 9.** *Let $n \in \mathbb{N}$. If $F : [0, +\infty]^n \to [0, +\infty]$ is a modular quasi-pseudo-metric aggregation function, then the following assertions hold:*

**(1)** *$F(+\infty, \ldots, +\infty) = +\infty$ provided that either 0 or $+\infty$ is a neutral element.*
**(2)** *$F$ has not $e \in ]0, +\infty[$ as neutral element.*

**Proof.** (1). The case in which $+\infty$ is a neutral element, it is evident that $F(+\infty, \ldots, +\infty) = +\infty$. Assume that 0 is neutral element. Then $F(+\infty, 0, \ldots, 0) = +\infty$. Since $F(+\infty, 0, \ldots, 0) \leq F(+\infty, \ldots, +\infty)$ we conclude that $F(+\infty, \ldots, +\infty) = +\infty$.

(2). Suppose for the purpose of contradiction that $e \in ]0, +\infty[$ is a neutral element. Set $a = (e, \ldots, e)$, $b = (0, e, \ldots, .e)$ and $c = (e, 0, e, \ldots, e)$. Clearly $a = b + c$. Then, by subadditivity of $F$, $F(a) \leq F(b) + F(c)$. Since $e$ is a neutral element we obtain that $F(a) = e$ and $F(b) = F(c) = 0$. Consequently $0 < e \leq 0$, which is a contradiction. □

Following [22], a monotone and subadditive function $F : [0, +\infty]^n \to [0, +\infty]$ is an Aumann function whenever it satisfies that $F(a_i 0) = a$ for all $a \in [0, +\infty]$ and for all $i \in \{1, \ldots, n\}$.

**Corollary 2.** *Let $n \in \mathbb{N}$. If $F : [0, +\infty]^n \to [0, +\infty]$ is a modular quasi-pseudo-metric aggregation function with 0 as neutral element, then $F$ is an Aumann function.*

Disjuntive functions play a distinguished role in aggregation theory. Let us recall that, according to [19], a function $F : [0, +\infty]^n \to [0, +\infty]$ is disjunctive when $\max\{a_1, \ldots, a_n\} \leq F(a)$ for all $a \in [0, +\infty]^n$.

In the light of the preceding notion, the next result guarantees, among other things, that every modular quasi-pseudo-metric aggregation function is disjunctive.

**Proposition 10.** *Let $n \in \mathbb{N}$. If $F : [0, +\infty]^n \to [0, +\infty]$ is a modular quasi-pseudo-metric aggregation function that has 0 as neutral element, then $\frac{1}{n} \sum_{i=1}^n a_i \leq \max\{a_1, \ldots, a_n\} \leq F(a) \leq \sum_{i=1}^n a_i$ for all $a \in [0, +\infty]^n$.*

**Proof.** Let $a \in [0, +\infty]^n$. Then $a = \sum_{i=1}^n a_i 0$. Since 0 is a neutral we have that $F(a_i 0) = a_i$ for all $i \in \{1, \ldots, n\}$. The subadditivity of $F$ implies that $F(a) \leq \sum_{i=1}^n F(a_i 0) = \sum_{i=1}^n a_i$. Moreover, the monotony of $F$ gives that $a_i = F(a_i 0) \leq F(a)$ for all $i \in \{1, \ldots, n\}$. Thus $\frac{1}{n} \sum_{i=1}^n a_i \leq \max\{a_1, \ldots, a_n\} \leq F(a)$. □

As a consequence of the preceding result we deduce, for every collection of modular quasi-pseudo-metrics on a non-empty set $X$ with 0 as neutral element, that the following inequality holds for all $\lambda_0 \in ]0, +\infty[$ and for all $x, y \in X$:

$$\frac{1}{n} \sum_{i=1}^{n} w_i(\lambda, x, y) \leq F \circ \widetilde{w}(\lambda, x, y) \leq \sum_{i=1}^{n} w_i(\lambda, x, y).$$

Observe that the proof of Proposition 10 shows that every modular quasi-pseudo-metric aggregation function with 0 as a neutral element satisfies that $a_i \leq F(a)$ for all $a \in [0, +\infty]^n$ and for all $i \in \{1, \ldots, n\}$. This property allows us to prove the following one.

**Proposition 11.** *Let $n \in \mathbb{N}$. Let $F : [0,+\infty]^n \to [0,+\infty]$ be a modular quasi-pseudo-metric aggregation function that has 0 as neutral element. If $G : [0,+\infty]^n \to [0,+\infty]$ is a conjunctive function, then $G(a) \leq F(a)$ for all $a \in [0,+\infty]^n$.*

**Proof.** Let $a \in [0,+\infty]^n$. Since $G$ is conjunctive $G(a) \leq \min\{a_1,\ldots,a_n\}$. The fact that $a_i \leq F(a)$ yields that $G(a) \leq \min\{a_1,\ldots,a_n\} \leq F(a)$. □

We end the paper discussing a relevant property in aggregation theory, the so-called Lipschitz condition. Following [19], a function $F : [0,+\infty]^n \to [0,+\infty]$ will be said to be $k$ ($k \in ]0,\infty[$ ) Lipschitz with respect to an extended norm $||\cdot||$ (which satisfies all axioms of classical norms and, in addition, the $+\infty$ value is allowed, see [23]) on $[0,+\infty]^n$ when $|F(a)-F(b)| \leq k||a-b||$ for all $a,b \in [0,+\infty]^n$.

The result below shows that modular quasi-pseudo-metric aggregation functions are 1 Lipschitz with respect to the extended norm on $[0,+\infty]^n$ defined as follows: $||a|| = \sum_{i=1}^n a_i$ for all $a \in [0,+\infty]^n$.

**Proposition 12.** *Let $n \in \mathbb{N}$. Let $n \in \mathbb{N}$. If $F : [0,+\infty]^n \to [0,+\infty]$ is a modular quasi-pseudo-metric aggregation function that has 0 as neutral element, then, for all $a,b \in [0,+\infty]^n$, the following inequality holds:*

$$|F(a)-F(b)| \leq \sum_{i=1}^n |b_i - a_i|.$$

**Proof.** Let $a,b \in [0,+\infty]^n$. Take $c \in [0,+\infty]^n$ given by $c_i = |b_i - a_i|$ for all $i \in \{1,\ldots,n\}$. Clearly, $a \preceq c+b$ and $b \preceq a+c$. Then, by Theorem 8, we have that $F(a) \leq F(b)+F(c)$ and $F(b) \leq F(a)+F(c)$. Whence we have that $|F(a)-F(b)| \leq F(c) = F(|b_1-a_1|,\ldots,|b_n-a_n|)$. Now, Proposition 10 gives that $F(|b_1-a_1|,\ldots,|b_n-a_n|) \leq \sum_{i=1}^n |b_i-a_i|$. Therefore, $|F(a)-F(b)| \leq \sum_{i=1}^n |b_i-a_i|$, as claimed. □

## 3. Conclusions and Future Work

In this paper we have exposed the modular quasi-(pseudo-)metric aggregation problem. A description of those functions that allow merging a collection of modular quasi-(pseudo-)metrics into a single one has been given in terms of triangle triplets. Moreover, the relationship between modular quasi-(pseudo-)metric aggregation functions and modular (pseudo-)metric aggregation functions has been discussed. The displayed characterizations are illustrated with appropriate examples. In addition, several methods to construct modular quasi-(pseudo-)metrics have been yielded. In order to develop quick tests for discarding candidate functions to aggregate modular quasi-(pseudo-)metrics, we have studied the existence of absorbent and neutral elements of modular quasi-(pseudo-)metric aggregation functions. As a consequence of such a study we have obtained that every modular quasi-pseudo-metric aggregation function that has 0 as neutral element is always an Aumann function, is majored by the sum and satisfies the 1-Lipschitz condition. Moreover, a characterization of those modular quasi-(pseudo-)metric aggregation functions that preserve modular quasi-(pseudo-)metrics has been also provided. In particular, we have shown that such functions are idempotent. Finally, the relationship between modular quasi-(pseudo-)metric aggregation functions and quasi-(pseudo-)metric aggregation functions has been discussed in such a way that significative differences have been evidenced.

**Author Contributions:** All the authors contributed (wrote, reviewed and supervised) to the final manuscript. All authors have read and agreed to the published version of the manuscript.

**Funding:** This research is part of project PID2022-139248NB-I00 funded by MCIN/AEI/10.13039/501100011033 and "ERDF A way of making Europe".

**Acknowledgments:** In this section you can acknowledge any support given which is not covered by the author contribution or funding sections. This may include administrative and technical support, or donations in kind (e.g., materials used for experiments).

 doi:10.20944/preprints202405.1133.v1



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