A Unified Proof of the Extended, Generalized, and Grand Riemann Hypothesis Based on the General Properties of L-Functions

Weicun Zhang *

Posted Date: 1 April 2026

doi: 10.20944/preprints202506.0481.v14

Keywords: extended Riemann hypothesis; generalized Riemann hypothesis; grand Riemann hypothesis; Landau–Siegel zeros conjecture

 doi:10.20944/preprints202506.0481.v14

Article

A Unified Proof of the Extended, Generalized, and Grand Riemann Hypothesis Based on the General Properties of L-Functions

Weicun Zhang

University of Science and Technology Beijing, Beijing 100083, China; weicunzhang@ustb.edu.cn

Abstract

The Extended, Generalized, and Grand Riemann Hypotheses are proved under a unified framework, which is based on the general properties of L-functions, i.e., the divisibility of entire functions contained in the symmetric functional equation, where the uniqueness of zero multiplicities (although their specific values remain unknown) of a given non-zero entire function plays a critical role. Consequently, the existence of Landau-Siegel zeros is excluded, thereby confirming the Landau-Siegel zeros conjecture.

Keywords: extended riemann hypothesis; generalized riemann hypothesis; grand riemann hypothesis; landau–siegel zeros conjecture

---

1. Notation and Sketch of the Proof

Notation.

$\mathbb{R}$: the field of real numbers.
$\mathbb{C}$: the field of complex numbers (the complex plane).
$\mathbb{Q}$, the field of rational numbers.
$\mathbb{R}[x]$: the ring of real polynomials.

It should be noted that in this paper, 'j' is used to denote the imaginary unit ($j^2 = -1$), while 'i' and 'n' serve as natural number indices.

It is also worth noting that the following fact plays an important role in this paper: for any given non-zero entire function, the multiplicities of its zeros are finite and uniquely determined, hence invariant, even though their specific values may be unknown.

Sketch of the Proof.

Preliminary Lemmas: Three Lemmas are provided.
Foundational Theorems: Four Theorems are progressively proved from Theorem 1 to Theorem 4.
Main Results: Based on Theorem 4 as a unified framework, Theorem 5 provides a proof of the generalized Riemann Hypothesis (for Dirichlet L-functions); Theorem 6 provides a proof of the extended Riemann Hypothesis (for Dedekind zeta functions); Theorem 7 deals with the non-trivial zero distribution of cuspidal automorphic L-functions, which corresponds to grand Riemann Hypothesis (version 1); Theorem 8 deals with the non-trivial zero distribution of all kinds of L-functions, which corresponds to grand Riemann Hypothesis (version 2).

2. Preliminary Lemmas

Lemma 1: Let $m(x), g_1(x), ..., g_n(x) \in \mathbb{R}[x], n \geq 2$. If $m(x)$ is irreducible (prime) and divides the product $g_1(x) \cdots g_n(x)$, then $m(x)$ divides one of the polynomials $g_1(x), ..., g_n(x)$.

 doi:10.20944/preprints202506.0481.v14

Remark: The contents of Lemma 1 can be found in many textbooks of linear algebra, modern algebra, or abstract algebra.

Lemma 2: Let $f(x)$ be an entire function on $\mathbb{C}$. Suppose $f(x) = \prod_{i=1}^{\infty} g_i(x)$ converges absolutely on $\mathbb{C}$, where each $g_i(x) \in \mathbb{R}[x]$ is irreducible in $\mathbb{R}[x]$ of degree $d \in \{1, 2\}$. If $m(x) \in \mathbb{R}[x]$ is irreducible in $\mathbb{R}[x]$ of degree d and divides $f(x)$, then $m(x)$ divides $g_i(x)$ for some $i \geq 1$.

Proof: Let α be a root of $m(x)$, i.e., $m(\alpha) = 0$. Since $m(x) \mid f(x)$, we have $f(\alpha) = 0$. By absolute convergence of $\prod_{i=1}^{\infty} g_i(x)$, there exists at least one index $i \in \mathbb{N}$ such that $g_i(\alpha) = 0$, otherwise $g_i(\alpha) \neq 0$ for all i, then $\prod_{i=1}^{\infty} g_i(x)$ converges to a non-zero limit, contradicting $f(\alpha) = 0$.

As $g_i(x)$ and $m(x)$ are irreducible over $\mathbb{R}$ with $\deg(g_i(x)) = \deg(m(x)) = d$, they share the root α. Thus:

- If $d = 1$, then $g_i(x) = a(x - \alpha)$ and $m(x) = b(x - \alpha)$ for $a, b \neq 0$, so $m(x) \mid g_i(x)$.
- If $d = 2$, then both $g_i(x)$ and $m(x)$ have roots $\{\alpha, \bar{\alpha}\}$, so $g_i(x) = c \cdot m(x)$ for $c \neq 0$, hence $m(x) \mid g_i(x)$.

In both cases, $m(x)$ divides $g_i(x)$.

That completes the proof of Lemma 2.

Remark: Lemma 2 is actually Lemma 5 in Ref. [1], which is a preprint paper by this author.

Lemma 3: The infinite product $\prod_{i=1}^{\infty} \left(\frac{\beta_i^2}{\alpha_i^2+\beta_i^2}\right)^{m_i}$ converges to a non-zero constant, given the conditions: $0 \leq \alpha_i \leq k, k > 0$ is a real constant, $\beta_i \neq 0, \sum_{i=1}^{\infty} \frac{1}{\beta_i^2} < \infty$, and $1 \leq m_i < \infty$ is the multiplicity of zero $\alpha_i + j\beta_i$ of a given entire function.

Proof: First of all, we know that

$$\prod_{i=1}^{\infty} \left(\frac{\beta_i^2}{\alpha_i^2 + \beta_i^2}\right)^{m_i} = \prod_{i=1}^{\infty} \frac{\beta_i^2}{\alpha_i^2 + \beta_i^2}$$

where in the right side expression, i^{th} factor appears m_i times.

Let $a_i = \frac{\alpha_i^2}{\alpha_i^2+\beta_i^2}$, then $\frac{\beta_i^2}{\alpha_i^2+\beta_i^2} = 1 - \frac{\alpha_i^2}{\alpha_i^2+\beta_i^2} = 1 - a_i$.

Since $0 \leq \alpha_i \leq k$ and $\beta_i \neq 0$, we have: $0 \leq a_i < \frac{k^2}{\beta_i^2}$. Then $\sum_{i=1}^{\infty} \frac{1}{\beta_i^2} < \infty$ (given condition) implies $\sum_{i=1}^{\infty} |a_i| = \sum_{i=1}^{\infty} a_i < k^2 \sum_{i=1}^{\infty} \frac{1}{\beta_i^2} < \infty$.

Further, the absolute convergence of $\sum_{i=1}^{\infty} a_i$ guarantees that the product $\prod_{i=1}^{\infty}(1 - a_i) = \prod_{i=1}^{\infty} \frac{\beta_i^2}{\alpha_i^2+\beta_i^2} = \prod_{i=1}^{\infty} \left(\frac{\beta_i^2}{\alpha_i^2+\beta_i^2}\right)^{m_i}$ converges to a non-zero constant.

That completes the proof of Lemma 3.

3. Foundational Theorems

As pointed out in Ref.[2] (p.57), the non-trivial zeros of the zeta function can be enumerated in order of increasing absolute value of their imaginary parts; where zeros whose imaginary parts have the same absolute value are arranged arbitrarily. Consequently, the same enumeration rule applies to the zeros of entire functions in the following contents. We therefore drop the default assumption $|\beta_1| \leq |\beta_2| \leq \cdots$ as a condition hereafter for simplicity.

Theorem 1: Given an entire function

$$F(s) = \prod_{i=1}^{\infty} \left(1 + \frac{(s - \alpha_i)^2}{\beta_i^2}\right)^{m_i}, \quad s \in \mathbb{C}, \tag{1}$$

which converges absolutely on $\mathbb{C}$.

Here

 doi:10.20944/preprints202506.0481.v14

- $\rho_i = \alpha_i + j\beta_i$ and $\bar{\rho}_i = \alpha_i - j\beta_i$ ($\beta_i \neq 0$) are the complex-conjugate zeros of $F(s)$;
- $m_i \geq 1$ is the multiplicity of ρ_i and $\bar{\rho}_i$;
- $0 \leq \alpha_i \leq k$, $\sum_{i=1}^{\infty} \frac{1}{|\rho_i|^2} < \infty$, $k > 0$ is a real constant.

Then

$$F(s) = F(k-s) \Longrightarrow \alpha_i = \frac{k}{2}\ (\forall i) \quad \text{and} \quad 0 < |\beta_1| < |\beta_2| < |\beta_3| < \cdots . \tag{2}$$

Remark: It should be noted that m_i is actually the multiplicity of quadruplets of zeros $(\rho_i, \bar{\rho}_i, k-\rho_i, k-\bar{\rho}_i)$ under the assumption $F(s) = F(k-s)$.

Proof. Considering

$$\left(1 + \frac{(s-\alpha_i)^2}{\beta_i^2}\right)^{m_i} = \underbrace{\left(1 + \frac{(s-\alpha_i)^2}{\beta_i^2}\right)\cdots}_{m_i \text{ times}}$$

we have from $F(s) = F(k-s)$ and Eq.(1)

$$\prod_{i=1}^{\infty}\left(1 + \frac{(s-\alpha_i)^2}{\beta_i^2}\right) = \prod_{i=1}^{\infty}\left(1 + \frac{(k-s-\alpha_i)^2}{\beta_i^2}\right) \tag{3}$$

where the i^{th} factor appears m_i times in any order.

Due to absolute convergence, both products $\prod_{i=1}^{\infty}\left(1 + \frac{(s-\alpha_i)^2}{\beta_i^2}\right)$ and $\prod_{i=1}^{\infty}\left(1 + \frac{(k-s-\alpha_i)^2}{\beta_i^2}\right)$ can be rearranged into any single quadratic polynomial factor times the remaining product, which stays absolutely convergent and hence defines an entire function. Then by the divisibility property of entire functions [3][4], Eq.(3) implies that each quadratic polynomial factor on either side divides the infinite product on the opposite side, i.e.,

$$\begin{cases} \left(1 + \frac{(s-\alpha_i)^2}{\beta_i^2}\right) \mid \prod_{i=1}^{\infty}\left(1 + \frac{(k-s-\alpha_i)^2}{\beta_i^2}\right) \\ \left(1 + \frac{(k-s-\alpha_i)^2}{\beta_i^2}\right) \mid \prod_{i=1}^{\infty}\left(1 + \frac{(s-\alpha_i)^2}{\beta_i^2}\right) \end{cases} \tag{4}$$

where "|" is the divisible sign, $i \in \mathbb{N} = \{1, 2, 3, \cdots\}$.

Both polynomials $1 + \frac{(s-\alpha_i)^2}{\beta_i^2}$ and $1 + \frac{(k-s-\alpha_i)^2}{\beta_i^2}$ have discriminant $\Delta = -4 \cdot \frac{1}{\beta_i^2} < 0$. Hence they are irreducible in $\mathbb{R}[x]$. By Lemma 2, Eq.(4) yields:

$$\begin{cases} \left(1 + \frac{(s-\alpha_i)^2}{\beta_i^2}\right) \mid \left(1 + \frac{(k-s-\alpha_l)^2}{\beta_l^2}\right) \\ \left(1 + \frac{(k-s-\alpha_i)^2}{\beta_i^2}\right) \mid \left(1 + \frac{(s-\alpha_l)^2}{\beta_l^2}\right) \\ i, l \in \mathbb{N} \end{cases} \tag{5}$$

For the special kind of polynomials in Eq.(5), "divisible" means "equal", which can be verified by comparing the like terms in the following equation to get $k' = 1$.

$$\left(1 + \frac{(s-\alpha_i)^2}{\beta_i^2}\right) = k'\left(1 + \frac{(k-s-\alpha_l)^2}{\beta_l^2}\right), k' \neq 0 \in \mathbb{R} \tag{6}$$

Further, due to the uniqueness of the multiplicity m_i, the only solution to Eq.(5) is $i = l$, otherwise, duplicated zeros (in quadruplets) with $\alpha_i + \alpha_l = k, \beta_i^2 = \beta_l^2, l \neq i$ would be generated to change m_i. Therefore we have from Eq.(5):

$$\left(1 + \frac{(s-\alpha_i)^2}{\beta_i^2}\right) = \left(1 + \frac{(k-s-\alpha_i)^2}{\beta_i^2}\right), i \in \mathbb{N} \tag{7}$$

 doi:10.20944/preprints202506.0481.v14

By comparing the like terms in Eq.(7), we obtain $\alpha_i = \frac{k}{2}(\forall i)$. Further, to ensure the uniqueness of m_i while $\alpha_i = \frac{k}{2}$, we need limit the β_i values to be distinct, i.e., $0 < |\beta_1| < |\beta_2| < |\beta_3| < \cdots$.
That completes the proof of Theorem 1. □

Theorem 2: Given an entire function

$$G(s) = \prod_{i=1}^{\infty} \left(1 - \frac{s}{\rho_i}\right), s \in \mathbb{C} \tag{8}$$

which converges absolutely on $\mathbb{C}$ in the form of

$$\begin{aligned} G(s) &= \prod_{i=1}^{\infty} \left(\frac{\beta_i^2}{\alpha_i^2 + \beta_i^2} + \frac{(s-\alpha_i)^2}{\alpha_i^2 + \beta_i^2}\right)^{m_i} \\ &= \prod_{i=1}^{\infty} \left(1 - \frac{s}{\rho_i}\right)^{m_i} \left(1 - \frac{s}{\bar{\rho}_i}\right)^{m_i} \end{aligned} \tag{9}$$

Here

- $\rho_i = \alpha_i + j\beta_i$ and $\bar{\rho}_i = \alpha_i - j\beta_i$ $(\beta_i \neq 0)$ are the complex-conjugate zeros of $G(s)$;
- $m_i \geq 1$ is the multiplicity of ρ_i and $\bar{\rho}_i$;
- $0 \leq \alpha_i \leq k, \sum_{i=1}^{\infty} \frac{1}{|\rho_i|^2} < \infty, k > 0$ is a real constant.

Then

$$G(s) = G(k-s) \Longrightarrow \alpha_i = \frac{k}{2}\ (\forall i) \quad \text{and} \quad 0 < |\beta_1| < |\beta_2| < |\beta_3| < \cdots . \tag{10}$$

Proof. According to Theorem 1 and Lemma 3, we have

$$\begin{aligned} & F(s) = F(k-s) \\ & \Leftrightarrow \\ & \prod_{i=1}^{\infty} \left(\frac{\beta_i^2}{\alpha_i^2 + \beta_i^2}\right)^{m_i} F(s) = \prod_{i=1}^{\infty} \left(\frac{\beta_i^2}{\alpha_i^2 + \beta_i^2}\right)^{m_i} F(k-s) \\ & \Leftrightarrow \\ & \prod_{i=1}^{\infty} \left(\frac{\beta_i^2}{\alpha_i^2 + \beta_i^2} + \frac{(s-\alpha_i)^2}{\alpha_i^2 + \beta_i^2}\right)^{m_i} = \prod_{i=1}^{\infty} \left(\frac{\beta_i^2}{\alpha_i^2 + \beta_i^2} + \frac{(k-s-\alpha_i)^2}{\alpha_i^2 + \beta_i^2}\right)^{m_i} \\ & \Leftrightarrow \\ & G(s) = G(k-s) \end{aligned} \tag{11}$$

Then we know that

$$G(s) = G(k-s) \Longrightarrow \alpha_i = \frac{k}{2}\ (\forall i) \quad \text{and} \quad 0 < |\beta_1| < |\beta_2| < |\beta_3| < \cdots . \tag{12}$$

That completes the proof of Theorem 2. □

Theorem 3: Given an entire function represented by its Hadamard product:

$$\Lambda(\lambda, s) = e^{A(\lambda) + B(\lambda)s} \prod_{i=1}^{\infty} \left(1 - \frac{s}{\rho_i}\right) e^{\frac{s}{\rho_i}}, s \in \mathbb{C} \tag{13}$$

were λ denotes a mathematical object, $\overline{\lambda}$ is the dual of λ

- $\rho_i = \alpha_i + j\beta_i$ and $\bar{\rho}_i = \alpha_i - j\beta_i$ $(\beta_i \neq 0)$ are the complex-conjugate zeros of $\Lambda(\lambda, s)$;
- $0 \leq \alpha_i \leq k, \sum_{i=1}^{\infty} \frac{1}{|\rho_i|^2} < \infty, k > 0$ is a real constant.

 doi:10.20944/preprints202506.0481.v14

Then

$$\Lambda(\lambda,s)=\varepsilon(\lambda)\Lambda(\overline{\lambda},k-s)\Longrightarrow\alpha_i=\frac{k}{2}\ (\forall i)\quad\text{and}\quad 0<|\beta_1|<|\beta_2|<|\beta_3|<\cdots. \tag{14}$$

where $\varepsilon(\lambda)$ is a complex number of absolute value 1.

Remark: For more details about $\varepsilon(\lambda)$, see Ref.[5] (p.94).

Proof. First, we have

$$\begin{aligned}\Lambda(\lambda,s)&=e^{A(\lambda)+B(\lambda)s}\prod_{i=1}^{\infty}\Big(1-\frac{s}{\rho_i}\Big)e^{\frac{s}{\rho_i}}\\&=e^{A(\lambda)+B(\lambda)s}\prod_{i=1}^{\infty}\Big(1-\frac{s}{\rho_i}\Big)\Big(1-\frac{s}{\bar{\rho}_i}\Big)e^{\frac{2\alpha_i s}{\alpha_i^2+\beta_i^2}}\\&=e^{A(\lambda)+B(\lambda)s}e^{s\cdot\sum_{i=1}^{\infty}\frac{2\alpha_i}{\alpha_i^2+\beta_i^2}}\prod_{i=1}^{\infty}\Big(1-\frac{s}{\rho_i}\Big)\Big(1-\frac{s}{\bar{\rho}_i}\Big)\end{aligned} \tag{15}$$

Noticing that $\sum_{i=1}^{\infty}\frac{2\alpha_i}{\alpha_i^2+\beta_i^2}\le 2k\cdot\sum_{i=1}^{\infty}\frac{1}{|\rho_i|^2}<\infty$, then we have $\sum_{i=1}^{\infty}\frac{2\alpha_i}{\alpha_i^2+\beta_i^2}=c, c\in\mathbb{R}, c\neq 0$. Further, taking into account the multiplicity $m_i\ge 1$ of each zero, we rewrite the product as

$$\Lambda(\lambda,s)=e^{A(\lambda)+[B(\lambda)+c]s}\prod_{i=1}^{\infty}\Big(1-\frac{s}{\rho_i}\Big)^{m_i}\Big(1-\frac{s}{\bar{\rho}_i}\Big)^{m_i} \tag{16}$$

Accordingly

$$\Lambda(\overline{\lambda},k-s)=e^{A(\overline{\lambda})+[B(\overline{\lambda})+c](k-s)}\prod_{i=1}^{\infty}\Big(1-\frac{k-s}{\rho_i}\Big)^{m_i}\Big(1-\frac{k-s}{\bar{\rho}_i}\Big)^{m_i} \tag{17}$$

Because $e^{A(\lambda)+[B(\lambda)+c]s}$ and $\varepsilon(\lambda)e^{A(\overline{\lambda})+[B(\overline{\lambda})+c](k-s)}$ are units in the ring of entire functions, they have no zeros and therefore do not affect the complex zeros related divisibility in the functional equation $\Lambda(\lambda,s)=\varepsilon(\lambda)\Lambda(\overline{\lambda},k-s)$. We thus conclude that Theorem 3 holds by Theorem 2, whose detailed conditions match those of Theorem 3.

In addition to Eq.(14), we still have (by canceling the complex zeros related polynomial factors on both sides of functional equation: $\Lambda(\lambda,s)=\varepsilon(\lambda)\Lambda(\overline{\lambda},k-s)$, since we already have $\alpha_i=\frac{k}{2}$)

$$\Lambda(\lambda,s)=\varepsilon(\lambda)\Lambda(\overline{\lambda},k-s)\Longrightarrow e^{A(\lambda)+[B(\lambda)+c]s}=e^{A(\overline{\lambda})+[B(\overline{\lambda})+c](k-s)} \tag{18}$$

That completes the proof of Theorem 3.

□

In the following Theorem 4, we make further efforts to lay a foundation for the study of completed L-functions that possess both real and complex zeros, denoted by $\rho\in\mathcal{Z}_r$ ($\Im(\rho)=0$) and $\rho\in\mathcal{Z}_c$ ($\Im(\rho)\neq 0$), respectively. When these two zero sets have no common elements, we express their disjointness by: $\mathcal{Z}_r\cap\mathcal{Z}_c=\varnothing$, which is automatically satisfied by our definitions of $\mathcal{Z}_r$ and $\mathcal{Z}_c$ because $\mathcal{Z}_r$ and $\mathcal{Z}_c$ are mutually exclusive sets, i.e., if $\rho\in\mathcal{Z}_r$, then $\Im(\rho)=0$; if $\rho\in\mathcal{Z}_c$, then $\Im(\rho)\neq 0$.

The reason we need to consider this case is that, so far, we cannot rule out the existence of exceptional zeros (or Landau-Siegel zeros), although their numbers are very limited even if they do exist.

Denote the set of real zeros as

$$\mathcal{Z}_r=\{a_n\in\mathbb{R}\mid 0<a_n<k,\ n=1,2,\ldots,N\} \tag{19}$$

where N is a finite natural number. This finiteness follows from the Identity Theorem, which implies that any non-zero entire function cannot have infinitely many zeros in a bounded region.

 doi:10.20944/preprints202506.0481.v14

Remark: In the definition of $\mathcal{Z}_r$, $a_n = 0$ (violating $\rho \neq 0$) and $a_n = k$ (violating $\rho \neq k$) are excluded according to Lemma 5.5 and Theorem 9 to be presented later.

Theorem 4: Given an entire function represented by its Hadamard product:

$$\begin{aligned}\Lambda(\lambda, s) &= e^{A(\lambda)+B(\lambda)s} \prod_{\rho} \left(1 - \frac{s}{\rho}\right) e^{\frac{s}{\rho}}, s \in \mathbb{C} \\ &= e^{A(\lambda)+B(\lambda)s} \prod_{\rho \in \mathcal{Z}_r} \left(1 - \frac{s}{\rho}\right) e^{\frac{s}{\rho}} \prod_{\rho \in \mathcal{Z}_c} \left(1 - \frac{s}{\rho}\right) e^{\frac{s}{\rho}}\end{aligned} \tag{20}$$

where λ denotes a mathematical object, $\overline{\lambda}$ is the dual of λ

- $\rho_i = \alpha_i + j\beta_i$ and $\bar{\rho}_i = \alpha_i - j\beta_i$ ($\beta_i \neq 0$) are the complex-conjugate zeros of $\Lambda(\lambda, s)$, $a_n, n = 1, 2, \ldots, N$ are the real zeros of $\Lambda(\lambda, s)$;
- $0 \leq \alpha_i \leq k$, $\sum_{i=1}^{\infty} \frac{1}{|\rho_i|^2} < \infty$, $0 < a_n < k$, $k > 0$ is a real constant.
- $\mathcal{Z}_r \cap \mathcal{Z}_c = \varnothing$.

Then

$$\Lambda(\lambda, s) = \varepsilon(\lambda)\Lambda(\overline{\lambda}, k - s) \Longrightarrow \begin{cases} \alpha_i = \frac{k}{2} \ (\forall i); \\ 0 < |\beta_1| < |\beta_2| < |\beta_3| < \cdots; \\ a_n = \frac{k}{2}, n = 1 \end{cases} \tag{21}$$

where $\varepsilon(\lambda)$ is a complex number of absolute value 1.

i.e., all the zeros (both real and complex) of $\Lambda(\lambda, s)$ in the critical strip $0 \leq \Re(s) \leq k$ lie on the critical line $\Re(s) = \frac{k}{2}$.

Proof. By Theorem 3, to determine the distribution of the complex zeros of $\Lambda(\lambda, s)$, it suffices to show that the newly introduced factors $\prod_{\rho \in \mathcal{Z}_r} \left(1 - \frac{s}{\rho}\right) e^{\frac{s}{\rho}}$ and $\prod_{\rho \in \mathcal{Z}_r} \left(1 - \frac{k-s}{\rho}\right) e^{\frac{k-s}{\rho}}$ do not affect the complex zeros related divisibility in the functional equation $\Lambda(\lambda, s) = \varepsilon(\lambda)\Lambda(\overline{\lambda}, k - s)$. This follows from the given condition $\mathcal{Z}_r \cap \mathcal{Z}_c = \varnothing$, which implies that $\prod_{\rho \in \mathcal{Z}_r} \left(1 - \frac{s}{\rho}\right) e^{\frac{s}{\rho}}$ and $\prod_{\rho \in \mathcal{Z}_c} \left(1 - \frac{k-s}{\rho}\right) e^{\frac{k-s}{\rho}}$ are co-prime, $\prod_{\rho \in \mathcal{Z}_r} \left(1 - \frac{k-s}{\rho}\right) e^{\frac{k-s}{\rho}}$ and $\prod_{\rho \in \mathcal{Z}_c} \left(1 - \frac{s}{\rho}\right) e^{\frac{s}{\rho}}$ are co-prime, according to Ref.[3] (p.174, p.208) or Ref.[4] (see its THEOREM 4).

Thus, we conclude by Theorem 3 that

$$\Lambda(\lambda, s) = \varepsilon(\lambda)\Lambda(\overline{\lambda}, k - s) \Rightarrow \begin{cases} \alpha_i = \frac{k}{2}, i = 1, 2, 3, \cdots \\ 0 < |\beta_1| < |\beta_2| < |\beta_3| < \cdots \end{cases} \tag{22}$$

Next, we consider the real zeros of $\Lambda(\lambda, s)$.

By canceling the complex zeros related polynomial factors on both sides of $\Lambda(\lambda, s) = \varepsilon(\lambda)\Lambda(\overline{\lambda}, k - s)$, we have

$$e^{A(\lambda)+[B(\lambda)+c]s} \prod_{n=1}^{N} \left(1 - \frac{s}{a_n}\right) e^{\frac{s}{a_n}} = \varepsilon(\lambda) e^{A(\overline{\lambda})+[B(\overline{\lambda})+c](k-s)} \prod_{n=1}^{N} \left(1 - \frac{k-s}{a_n}\right) e^{\frac{k-s}{a_n}} \tag{23}$$

where constant c is the same as in the proof of Theorem 3.

Further Eq.(23) is equivalent to

$$(-1)^N e^{A(\lambda)+[B(\lambda)+c']s} \prod_{n=1}^{N} \left(s - a_n\right) = \varepsilon(\lambda) e^{A(\overline{\lambda})+[B(\overline{\lambda})+c'](k-s)} \prod_{n=1}^{N} \left(s - (k - a_n)\right) \tag{24}$$

where $c' = c + \sum_{n=1}^{N} \frac{1}{a_n}$.

 doi:10.20944/preprints202506.0481.v14

Suppose the multiplicity of zero $s = a_n$ is m_n ($m_n \geq 1$) that is finite and uniquely determined although unknown. Then Eq.(24) becomes

$$(-1)^N e^{A(\lambda)+[B(\lambda)+c']s} \prod_{t=1}^{T} \Big(s - a_t\Big)^{m_t} = \varepsilon(\lambda) e^{A(\overline{\lambda})+[B(\overline{\lambda})+c'](k-s)} \prod_{t=1}^{T} \Big(s - (k - a_t)\Big)^{m_t} \tag{25}$$

where $\sum_{t=1}^{T} m_t = N$.

Considering $(s - a_t)$ and $(s - (k - a_t))$ are irreducible in $\mathbb{R}[x]$, then by Lemma 1, Eq.(25) means

$$\begin{cases} (s - a_t) \mid (s - (k - a_l) \\ (s - (k - a_t)) \mid (s - a_l) \\ t, l = 1, 2, 3, \cdots, T \end{cases} \tag{26}$$

The only solution to Eq.(26) is $t = l, a_t = \frac{k}{2}, t = 1, \cdots, T$, otherwise the uniqueness of m_t would be violated with $a_t + a_l = k, t \neq l$. To avoid changing the multiplicity of m_t while $a_t = \frac{k}{2}$, we need to limit $T = 1$. Thus we get

$$\Lambda(\lambda, s) = \varepsilon(\lambda)\Lambda(\overline{\lambda}, k - s) \Rightarrow a_t = \frac{k}{2}, t = 1, m_1 = N \tag{27}$$

Putting Eq.(22) and Eq.(27) together, we proved Eq.(21).

In addition to Eq.(21), we still have (by canceling the real zeros related polynomial factors on both sides of Eq.(25))

$$\Lambda(\lambda, s) = \varepsilon(\lambda)\Lambda(\overline{\lambda}, k - s) \Longrightarrow (-1)^N e^{A(\lambda)+[B(\lambda)+c']s} = \varepsilon(\lambda) e^{A(\overline{\lambda})+[B(\overline{\lambda})+c'](k-s)}$$

That completes the proof of Theorem 4. □

We have the following supplementary remarks regarding Theorem 4.

1): If $\Lambda(\lambda, s)$ has poles at $s = 0, k$ with multiplicity $m \geq 1$, then $s^m(k - s)^m\Lambda(\lambda, s)$ is an entire function. This adjustment does not affect the results of Theorem 4, since we have $s^m(k - s)^m\Lambda(\lambda, s) = \varepsilon(\lambda)(k - s)^m s^m \Lambda(\bar{\lambda}, k - s) \Leftrightarrow \Lambda(\lambda, s) = \varepsilon(\lambda)\Lambda(\bar{\lambda}, k - s)$.

2): As pointed out in Ref.[5] (p.102), if $\rho_i = \alpha_i + j\beta_i$ is a zero of $\Lambda(\lambda, s)$, then $\bar{\rho}_i = \alpha_i - j\beta_i$ is a zero of $\Lambda(\bar{\lambda}, s)$. Therefore, to use Theorem 4 while $\lambda \neq \bar{\lambda}$, we need to construct a new symmetric functional equation $\Lambda(\bar{\lambda}, s)\Lambda(\lambda, s) = \varepsilon(\lambda)\varepsilon(\overline{\lambda})\Lambda(\lambda, k - s)\Lambda(\bar{\lambda}, k - s)$ to ensure that the conjugate zeros appear together in the related Hadamard products.

3): In the symmetric functional equation $\Lambda(\lambda, s) = \varepsilon(\lambda)\Lambda(\bar{\lambda}, k - s)$, $\varepsilon(\lambda)$ can be relaxed to $\varepsilon(s, \lambda)$ if only it has no zeros.

4. Main Results

We will make use of Theorem 4 to prove the extended Riemann Hypothesis (for Dedekind zeta functions), the generalized Riemann Hypothesis (for Dirichlet L-functions), and the grand Riemann Hypothesis (for cuspidal automorphic L-functions, and for any kind of L-functions, repectively).

In the following contents, the critical line means: $\Re(s) = \frac{1}{2}$, or more generally, $\Re(s) = \frac{k}{2}, k > 0$ is a real constant.

To begin with, we provide a general property of L-functions, which was labeled Lemma 5.5 in Ref.[5] (p.101).

Lemma 5.5 [5]: Let $L(f, s)$ be an L-function. All zeros ρ of $\Lambda(f, s)$ are in the critical strip $0 \leq \sigma \leq 1$. For any $\epsilon > 0$, we have

$$\sum_{\rho \neq 0,1} |\rho|^{-1-\epsilon} < +\infty.$$

 doi:10.20944/preprints202506.0481.v14

In Lemma 5.5, $\Lambda(f,s)$ is the completed L-function corresponding to $L(f,s)$, σ is the real part of ρ, and f is identical to λ in this paper as a symbol representing a mathematical object (e.g., Dirichlet character, modular form, automorphic representation).

To adapt to the wider situation of critical strip, i.e., $0 \leq \Re(s) \leq k$, Lemma 5.5 has been extended to Theorem 9 (positioned after Theorem 8).

Another general property of L-functions is as follows.

The zeros of the completed L-function $\Lambda(\lambda,s)$ are precisely the non-trivial zeros of $L(\lambda,s)$. See Ref.[5] (p.96) for more details (with f replaced by λ).

Thus, we can discuss the non-trivial zeros of L-functions based on the zeros of the corresponding completed L-functions.

4.1. Dirichlet L-function

Definition: The Dirichlet L-function associated with a Dirichlet character χ modulo q is defined for $\Re(s) > 1$ by the series:

$$L(\chi,s) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s} \tag{28}$$

For the principal (trivial) character χ_0 (where $\chi_0(n) = 1$ if $\gcd(n,q) = 1$ and $\chi_0(n) = 0$ otherwise), the L-function is related to the Riemann zeta function by:

$$L(\chi_0,s) = \zeta(s) \prod_{p|q} \left(1 - \frac{1}{p^s}\right) \tag{29}$$

Remark: The Riemann zeta-function $\zeta(s)$ is a special case of $L(\chi,s)$ with $\chi(n) \equiv 1$ ($q \equiv 1$).

Completed L-function: Let χ be a primitive Dirichlet character modulo $q > 1$. The completed Dirichlet L-function is defined as:

$$\Lambda(\chi,s) = \left(\frac{q}{\pi}\right)^{\frac{s+a}{2}} \Gamma\left(\frac{s+a}{2}\right) L(\chi,s) \tag{30}$$

where $a = 0$ if $\chi(-1) = 1$ (even character) and $a = 1$ if $\chi(-1) = -1$ (odd character).

Functional Equation: The completed Dirichlet L-function satisfies the functional equation:

$$\Lambda(\chi,s) = W(\chi)\Lambda(\overline{\chi}, 1-s) \tag{31}$$

where $W(\chi)$ is the Gauss sum:

$$W(\chi) = \frac{\tau(\chi)}{j^a\sqrt{q}} \tag{32}$$

where $j = \sqrt{-1}, \tau(\chi) = \sum_{n=1}^{q} \chi(n) e^{2\pi jn/q}$ is the Gauss sum associated with χ.

Hadamard Product: For primitive non-principal character χ, the completed L-function $\Lambda(\chi,s)$ is an entire function of order 1 and has the Hadamard product:

$$\Lambda(\chi,s) = e^{A(\chi)+B(\chi)s} \prod_{\rho} \left(1 - \frac{s}{\rho}\right) e^{s/\rho} \tag{33}$$

where the product is over all zeros ρ of $\Lambda(\chi,s)$, and $A(\chi)$ and $B(\chi)$ are constants depending on χ.

For principal (trivial) character χ_0, $\Lambda(\chi_0,s)$ carries a simple pole at $s = 1$ (and also at $s = 0$ precisely when $q = 1$). Then the Hadamard product is applied to $s(1-s)\Lambda(\chi_0,s)$. But this (trivial) situation makes no difference to our results under the symmetric functional equation, i.e.,

$$\Lambda(\chi_0,s) = W(\chi_0)\Lambda(\overline{\chi}_0, 1-s) \Longrightarrow s(1-s)\Lambda(\chi_0,s) = W(\chi_0)(1-s)s\Lambda(\overline{\chi}_0, 1-s)$$

 doi:10.20944/preprints202506.0481.v14

Thus this (trivial) situation is omitted in the proof of Theorem 5.

Next we prove the generalized Riemann Hypothesis.

Theorem 5: The non-trivial zeros of Dirichlet L-functions in the critical strip $0 < \Re(s) < 1$ lie on the critical line.

Remark: It suffices to prove that all the zeros of $L(\chi, s)$ in the critical strip $0 \leq \Re(s) \leq 1$ lie on the critical line, since $0 < \Re(s) < 1$ is a subset of $0 \leq \Re(s) \leq 1$.

Remark: The non-trivial zero problem of $L(\chi, s)$ for arbitrary non-principal characters reduces to that for primitive non-principal characters, so we only consider primitive non-principal characters in the following proof.

Proof. We conduct the proof in two cases.

CASE 1: $\chi = \bar{\chi}$ (self-dual)

It suffices to verify that the properties of $\Lambda(\chi, s)$ match the conditions of Theorem 4 with $\lambda = \chi$, $\chi = \bar{\chi}$, $\varepsilon(\lambda) = W(\chi)$, $k = 1$:

1) Symmetric functional equation;

2) Hadamard product expression;

3) The complex zeros appear in pairs $(\rho_i, \bar{\rho}_i) \in \mathcal{Z}_c$ in the Hadamard product expression;

4) $0 \leq \alpha_i \leq 1$, $\sum_{i=1}^{\infty} \frac{1}{|\rho_i|^2} < \infty$, $0 < a_n < 1$;

5) $\mathcal{Z}_r \cap \mathcal{Z}_c = \varnothing$.

Eq.(31) shows the symmetric functional equation; Hadamard product Eq.(33) is equivalent to Eq.(20) in Theorem 4 by separating all zeros into two sets $\mathcal{Z}_r$ and $\mathcal{Z}_c$; $\chi = \bar{\chi}$ guarantees that the complex conjugate zeros of $\Lambda(\chi, s)$ appear in pairs; The condition $0 \leq \alpha_i \leq 1$, $\sum_{i=1}^{\infty} \frac{1}{|\rho_i|^2} < \infty$, $0 < a_n < 1$ can be assured by Lemma 5.5, considering that $\sum_{i=1}^{\infty} \frac{1}{|\rho_i|^2}$ is a subseries of $\sum_{\rho \neq 0,1} \frac{1}{|\rho|^2}$; The condition $\mathcal{Z}_r \cap \mathcal{Z}_c = \varnothing$ holds because $\mathcal{Z}_r$ and $\mathcal{Z}_c$ are mutually exclusive sets, i.e., if $\rho \in \mathcal{Z}_r$, then $\Im(\rho) = 0$; if $\rho \in \mathcal{Z}_c$, then $\Im(\rho) \neq 0$.

Therefore, by Theorem 4 with $\lambda = \chi$, $\chi = \bar{\chi}$, $\varepsilon(\lambda) = W(\chi)$, $k = 1$, we know that all zeros (real, if any, and complex) of $\Lambda(\chi, s)$ in the critical strip $0 \leq \Re(s) \leq 1$ lie on the critical line.

CASE 2: $\chi \neq \bar{\chi}$

In this case, the complex conjugate zeros do not appear together in Eq.(33), because if ρ is a zero of $\Lambda(\chi, s)$, then $\bar{\rho}$ is a zero of $\Lambda(\bar{\chi}, s)$.

Thus, we need to extend Eq.(31) to another form, i.e.,

$$\Lambda(\bar{\chi}, s) = W(\bar{\chi})\Lambda(\chi, 1 - s) \tag{34}$$

Combining (34) with (31), we get a new symmetric functional equation

$$\Lambda(\bar{\chi}, s)\Lambda(\chi, s) = W(\bar{\chi})W(\chi)\Lambda(\chi, 1 - s)\Lambda(\bar{\chi}, 1 - s) \tag{35}$$

The product $\Lambda(\bar{\chi}, s)\Lambda(\chi, s)$ is an entire function of finite order ≤ 1, as this holds for the product of any two entire functions of order 1. Moreover, the zero set of $\Lambda(\bar{\chi}, s)\Lambda(\chi, s)$—being the union of the zeros of the individual factors—possesses the conjugate zeros paring property.

Further, based on Eq.(33), we have the following Hadamard product

$$\Lambda(\chi, s)\Lambda(\bar{\chi}, s) = e^{A(\chi)+A(\bar{\chi})+[B(\chi)+B(\bar{\chi})+c]s} \prod_{\rho \in \mathcal{Z}_r} \left(1 - \frac{s}{\rho}\right) e^{\frac{s}{\rho}} \prod_{i=1}^{\infty} \left(1 - \frac{s}{\rho_i}\right)\left(1 - \frac{s}{\bar{\rho}_i}\right) \tag{36}$$

where $c = \sum_{i=1}^{\infty} \frac{2\alpha_i}{\alpha_i^2 + \beta_i^2}$.

The other conditions required by Theorem 4 hold for the same reasons as in CASE 1.

Therefore, according to Theorem 4, all zeros (real, if any, and complex) of $\Lambda(\chi, s)\Lambda(\bar{\chi}, s)$, and consequently of $\Lambda(\chi, s)$, in the critical strip $0 \leq \Re(s) \leq 1$, lie on the critical line for $\chi \neq \bar{\chi}$.

 doi:10.20944/preprints202506.0481.v14

Combining CASE 1 and CASE 2, we conclude that Theorem 5 holds as a specific case of Theorem 4 with $\lambda = \chi$, $\varepsilon(\lambda) = W(\chi)$, $k = 1$, and $0 < \Re(s) < 1$ is a subset of $0 \leq \Re(s) \leq 1$. □

Remark: According to Theorem 5: All non-trivial zeros (both real and complex) of Dirichlet L-functions, in the critical strip $0 < \Re(s) < 1$, lie on the critical line $\Re(s) = \frac{1}{2}$. This implies the non-existence of Landau-Siegel zeros. Hence, the Landau-Siegel zeros conjecture is justified.

4.2. *Dedekind Zeta Function*

Definition: For a number field K with ring of algebraic integers $\mathcal{O}_K$, the Dedekind zeta function is defined for $\Re(s) > 1$ by:

$$\zeta_K(s) = \sum_{\mathfrak{a}} \frac{1}{N(\mathfrak{a})^s} \tag{37}$$

where the sum is over all non-zero ideals $\mathfrak{a}$ of $\mathcal{O}_K$, and $N(\mathfrak{a})$ is the norm of the ideal.

Remark: The Riemann zeta-function $\zeta(s)$ is a special case of $\zeta_K(s)$ with $K = \mathbb{Q}$, where $\mathbb{Q}$ denotes the field of rational numbers.

Completed Zeta Function: The completed Dedekind zeta function is defined as:

$$\Lambda_K(s) = |D_K|^{s/2} \left(\pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\right)^{r_1} \left(2(2\pi)^{-s}\Gamma(s)\right)^{r_2} \zeta_K(s) \tag{38}$$

where D_K is the discriminant of K, r_1 is the number of real embeddings of K, r_2 is the number of pairs of complex embeddings of K.

Functional Equation: The completed Dedekind zeta function satisfies:

$$\Lambda_K(s) = \varepsilon(K)\Lambda_K(1-s) \tag{39}$$

where $\varepsilon(K) = 1$ for all number fields K, showing the symmetry of the functional equation.

Hadamard Product: The completed Dedekind zeta function has a simple pole at $s = 1$ with residue $\frac{2^{r_1}(2\pi)^{r_2} h_K R_K}{w_K \sqrt{|D_K|}}$, where h_K is the class number, R_K is the regulator, and w_K is the number of roots of unity in K. The function $s(s-1)\Lambda_K(s)$ is an entire function of order 1 and has the Hadamard product:

$$s(s-1)\Lambda_K(s) = e^{A_K + B_K s} \prod_{\rho \neq 0,1} \left(1 - \frac{s}{\rho}\right) e^{s/\rho} \tag{40}$$

where the product runs over all zeros ρ of $\Lambda_K(s)$ except $\rho = 0$ and $\rho = 1$, and A_K and B_K are constants depending on K.

For more details of Dedekind zeta functions, please be referred to Ref.[5] (Chapter 5.10) and Ref.[6] (Section 10.5.1).

Next, we prove the extended Riemann Hypothesis.

Theorem 6: The non-trivial zeros of Dedekind zeta functions in the critical strip $0 < \Re(s) < 1$ lie on the critical line.

Remark: It suffices to prove that all the zeros of $\Lambda_K(s)$ in the critical strip $0 \leq \Re(s) \leq 1$ have real part $\frac{1}{2}$, i.e., all the zeros of $\Lambda_K(s)$ lie on the critical line.

Proof. It suffices to show that the properties of $\Lambda_K(s)$ match the conditions of Theorem 4 with $\lambda = K$, $\varepsilon(\lambda) = 1, k = 1$.

 doi:10.20944/preprints202506.0481.v14

Actually, symmetric functional equation Eq.(39) and Hadamard product Eq.(40) guarantee that

$$e^{A_K+B_K s}\prod_{\rho\neq 0,1}\left(1-\frac{s}{\rho}\right)e^{s/\rho}=e^{A_K+B_K(1-s)}\prod_{\rho\neq 0,1}\left(1-\frac{1-s}{\rho}\right)e^{1-s/\rho} \tag{41}$$

where $\prod_{\rho\neq 0,1}\left(1-\frac{s}{\rho}\right)e^{s/\rho}=\prod_{\rho\in\mathcal{Z}_r}\left(1-\frac{s}{\rho}\right)e^{s/\rho}\prod_{\rho\in\mathcal{Z}_c}\left(1-\frac{s}{\rho}\right)e^{s/\rho}$.

Furthermore, the complex conjugate zeros of $\Lambda_k(s)$ appear in pairs since it is self-dual. The condition $0\leq\alpha_i\leq 1$, $\sum_{i=1}^{\infty}\frac{1}{|\rho_i|^2}<\infty$, $0<a_n<1$ and condition $\mathcal{Z}_r\cap\mathcal{Z}_c=\varnothing$ hold for the same reasons as in CASE 1 in the proof of Theorem 5.

Therefore, by Theorem 4, we know that all zeros (real, if any, and complex) of $\Lambda_K(s)$ in the critical strip $0\leq\Re(s)\leq 1$ lie on the critical line. Thus, Theorem 6 holds as a specific case of Theorem 4 with $\lambda=K$, $\varepsilon(\lambda)=1$, $k=1$, and $0<\Re(s)<1$ is a subset of $0\leq\Re(s)\leq 1$. □

4.3. Automorphic L-Function

Definition: Let $\pi=\otimes_v'\pi_v$ be a cuspidal automorphic representation of $\mathrm{GL}_m(\mathbb{A}_{\mathbb{Q}})$, where $\mathbb{A}_{\mathbb{Q}}$ denotes the adele ring of $\mathbb{Q}$, v ranges over all places of $\mathbb{Q}$. The associated standard L-function is defined for $\Re(s)>1$ by

$$L(s,\pi)=\prod_{p<\infty}L_p(s,\pi)=\sum_{n=1}^{\infty}\frac{\lambda_\pi(n)}{n^s}, \tag{42}$$

where $L_p(s,\pi)$ is the local L-factor at the prime p. If π_p is unramified with Satake parameters $\{\alpha_{1,p},\ldots,\alpha_{m,p}\}$, then

$$L_p(s,\pi)=\prod_{i=1}^{m}\left(1-\alpha_{i,p}p^{-s}\right)^{-1}. \tag{43}$$

Remark: The Riemann zeta-function $\zeta(s)$ is a special case of $L(\pi,s)$ with $m=1$, $\pi=1$.

Completed L-function: The completed automorphic L-function is defined by

$$\Lambda(s,\pi)=Q_\pi^{s/2}L_\infty(s,\pi)L(s,\pi), \tag{44}$$

where $Q_\pi\geq 1$ (integer) is the arithmetic conductor of π, and the archimedean factor $L_\infty(s,\pi)$ is a product of gamma factors determined by the Langlands parameters of π_∞.

Functional Equation: The completed L-function satisfies the functional equation

$$\Lambda(s,\pi)=\varepsilon(s,\pi)\Lambda(1-s,\tilde{\pi}), \tag{45}$$

where $\tilde{\pi}$ is the contragredient representation of π, $\varepsilon(s,\pi)=Q_\pi^{s-1/2}\varepsilon(\pi)$, $\varepsilon(\pi)$ is the root number with $|\varepsilon(\pi)|=1$.

Hadamard Product: For cuspidal π, $\Lambda(s,\pi)$ is an entire function of order 1, and hence admits the Hadamard product expansion

$$\Lambda(s,\pi)=e^{A(\pi)+B(\pi)s}\prod_{\rho}\left(1-\frac{s}{\rho}\right)e^{s/\rho}, \tag{46}$$

where the product runs over all zeros ρ of $\Lambda(s,\pi)$, and $A(\pi)$ and $B(\pi)$ are constants depending on π.

For more details of automorphic L-functions, please be referred to Refs.[5][7][8], particularly Ref.[9] for the details of coefficients $\lambda_\pi(n)$.

The grand Riemann Hypothesis (GRH) has many versions of statements. To be specific, we adopt the following two versions.

 doi:10.20944/preprints202506.0481.v14

GRH (version 1)[8]: The zeros of any $\Lambda(\pi, s)$ have real part equal to $\frac{1}{2}$.

GRH (version 2): The non-trivial zeros of any L-function in the critical strip $0 < \Re(s) < 1$ lie on the critical line.

GRH (version 1) corresponds to Theorem 7, GRH (version 2) corresponds to Theorem 8.

Theorem 7: The non-trivial zeros of cuspidal automorphic L-Functions in the critical strip $0 < \Re(s) < 1$ lie on the critical line.

Remark: It suffices to prove that all the zeros of $\Lambda(\pi, s)$ in the critical strip $0 \leq \Re(s) \leq 1$ have real part $\frac{1}{2}$, i.e., all the zeros of $\Lambda(\pi, s)$ lie on the critical line.

Proof. We conduct the proof in two cases.

CASE 1: $\pi = \tilde{\pi}$ (self-dual)

It suffices to show that the properties of $\Lambda(\pi, s)$ match the conditions of Theorem 4 with $\lambda = \pi = \tilde{\pi} = \bar{\lambda}, k = 1$. Eq.(45) shows the symmetric functional equation; Hadamard product Eq.(46) is equivalent to Eq.(20) in Theorem 4 by separating all zeros into two sets $\mathcal{Z}_r$ and $\mathcal{Z}_c$; $\pi = \tilde{\pi}$ guarantees that the complex conjugate zeros of $\Lambda(\pi, s)$ appear in pairs; The condition $0 \leq \alpha_i \leq 1, \sum_{i=1}^{\infty} \frac{1}{|\rho_i|^2} < \infty$, $0 < a_n < 1$ can be assured by Lemma 5.5. The condition $\mathcal{Z}_r \cap \mathcal{Z}_c = \varnothing$ holds as explained in the proof of Theorem 5.

Therefore, by Theorem 4, we know that all zeros (real, if any, and complex) of $\Lambda(\pi, s)$ in the critical strip $0 \leq \Re(s) \leq 1$ lie on the critical line.

CASE 2: $\pi \neq \tilde{\pi}$

To deal with this case $\pi \neq \tilde{\pi}$, we need first to extend Eq.(45) to another form, i.e.,

$$\Lambda(\tilde{\pi}, s) = Q_{\tilde{\pi}}^{s-1/2} \varepsilon(\tilde{\pi}) \Lambda(\pi, 1-s) \tag{47}$$

Combining Eq.(47) with Eq.(45), we get a new symmetric functional equation

$$\Lambda(\tilde{\pi}, s)\Lambda(\pi, s) = Q_{\pi}^{s-1/2} Q_{\tilde{\pi}}^{s-1/2} \varepsilon(\pi)\varepsilon(\tilde{\pi})\Lambda(\pi, 1-s)\Lambda(\tilde{\pi}, 1-s) \tag{48}$$

Both sides of Eq.(48) are the products of entire functions of order 1, thus they are still entire functions of order ≤ 1.

Then all complex zeros of $\Lambda(\tilde{\pi}, s)\Lambda(\pi, s)$ come in conjugate pairs.

Based on Eq.(46), we have the following Hadamard product

$$\Lambda(\pi, s)\Lambda(\tilde{\pi}, s) = e^{A(\pi)+A(\tilde{\pi})+[B(\pi)+B(\tilde{\pi})+c]s} \prod_{\rho \in \mathcal{Z}_r} \left(1 - \frac{s}{\rho}\right) e^{\frac{s}{\rho}} \prod_{i=1}^{\infty} \left(1 - \frac{s}{\rho_i}\right)\left(1 - \frac{s}{\bar{\rho}_i}\right) \tag{49}$$

where $c = \sum_{i=1}^{\infty} \frac{2\alpha_i}{\alpha_i^2 + \beta_i^2}$.

The other conditions required by Theorem 4 hold for the same reasons as in CASE 1.

Therefore, by Theorem 4, all zeros (real, if any, and complex) of $\Lambda(\pi, s)\Lambda(\tilde{\pi}, s)$, and consequently of $\Lambda(\pi, s)$, in the critical strip $0 \leq \Re(s) \leq 1$, lie on the critical line for $\pi \neq \tilde{\pi}$.

Combining CASE 1 and CASE 2, we conclude that Theorem 7 holds as a specific case of Theorem 4 with $\lambda = \pi, \bar{\lambda} = \tilde{\pi}, k = 1$, and $0 < \Re(s) < 1$ is a subset of $0 \leq \Re(s) \leq 1$. □

Actually, from the above proofs of Theorem 5, Theorem 6, and Theorem 7, we can note that each proof does not depend on the specific definition of the L-function $L(\lambda, s)$, but rather relies on the following general properties of $\Lambda(\lambda, s)$ and $L(\lambda, s)$:

P1: Symmetric functional equation between $\Lambda(\lambda, s)$ and $\Lambda(\bar{\lambda}, k-s)$: $\Lambda(\lambda, s) = \varepsilon(\lambda)\Lambda(\bar{\lambda}, k-s)$;

P2: Hadamard product expression of entire function $\Lambda(\lambda, s)$ or $s^m(k-s)^m\Lambda(\lambda, s), m \geq 1$;

P3: The complex zeros in the relevant Hadamard products appear in pairs $(\rho, \bar{\rho})$;

P4: All zeros in the relevant Hadamard products lie in the critical strip $0 \leq \Re(\rho) \leq k$ and satisfy $\sum_{\rho \neq 0, k} \frac{1}{|\rho|^2} < \infty$;

 doi:10.20944/preprints202506.0481.v14

P5: The disjointness of real and complex non-trivial zero sets;
P6: The zeros of $\Lambda(\lambda, s)$ are precisely the non-trivial zeros of $L(\lambda, s)$.

Therefore, we have the following result on the non-trivial zero distribution of all kinds of L-functions.

Theorem 8: The non-trivial zeros of any L-function in the critical strip $0 < \Re(s) < k$ lie on the critical line if only the properties **P1**, **P2**, **P3**, **P4**, **P5**, and **P6** are satisfied.

Proof. It is not difficult to see that **P1**, **P2**, **P3**, **P4**, and **P5** cover all the conditions in Theorem 4. Thus, we know by Theorem 4 that all the zeros, both real (if any) and complex, of $\Lambda(\lambda, s)$ in the critical strip $0 \leq \Re(s) \leq k$ lie on the critical line. Furthermore, according to **P6**, we conclude that all the non-trivial zeros, both real (if any) and complex, of $L(\lambda, s)$ in the critical strip $0 \leq \Re(s) \leq k$, thus in the critical strip $0 < \Re(s) < k$, lie on the critical line.
That completes the proof of Theorem 8. □

Remark: Conditions **P1-P3** and **P6** are general properties of L-functions, see Chapter 5 of Ref.[5]. Condition **P4** follows from Theorem 9, which is an extended result of Lemma 5.5. Condition **P5** is automatically satisfied by our definitions of $\mathcal{Z}_r$ (the set of zeros with $\Im(\rho) = 0$) and $\mathcal{Z}_c$ (the set of zeros with $\Im(\rho) \neq 0$). Therefore, Theorem 8 is expected to serve as a guideline for constructing new zeta- or L-type functions.

Theorem 9: Let $L(\lambda, s)$ be an L-function, $\Lambda(\lambda, s)$, the corresponding completed L-function satisfying a functional equation of the form $\Lambda(\lambda, s) = \epsilon(\lambda)\Lambda(\bar{\lambda}, k - s)$, with $k > 0, k \in \mathbb{R}$. Then all zeros ρ of $\Lambda(\lambda, s)$ lie in the critical strip $0 \leq \Re(\rho) \leq k$. Moreover, for any $\epsilon > 0$, we have

$$\sum_{\rho \neq 0,k} |\rho|^{-1-\epsilon} < \infty.$$

where λ denotes a mathematical object, $\overline{\lambda}$ is the dual of λ; $\varepsilon(\lambda)$ is a complex number of absolute value 1, called the "root number" of L-function $L(\lambda, s)$.

Proof. Defining $s = kt$ and $F(\lambda, t) = \Lambda(\lambda, kt)$, the functional equation transforms as follows:

$$F(\lambda, t) = \Lambda(\lambda, kt) = \epsilon(\lambda)\Lambda(\bar{\lambda}, k - kt) = \epsilon(\lambda)\Lambda(\bar{\lambda}, k(1 - t)) = \epsilon(\lambda)F(\bar{\lambda}, 1 - t).$$

We thus obtain a new entire function $F(\lambda, t)$ with the standard functional equation $F(\lambda, t) = \epsilon(\lambda)F(\bar{\lambda}, 1 - t)$, to which the classical Lemma 5.5 applies.

Let τ be a zero of $F(\lambda, t)$. By definition, $\rho = k\tau$ is then a zero of $\Lambda(\lambda, s)$, and this correspondence is bijective. The classical Lemma 5.5 guarantees that $0 \leq \Re(\tau) \leq 1$, $\sum_{\tau \neq 0,1} |\tau|^{-1-\epsilon} < \infty$ (for any $\epsilon > 0$), which immediately implies $0 \leq \Re(\rho) \leq k$, $\sum_{\rho \neq 0,k} |\rho|^{-1-\epsilon} < \infty$ (for any $\epsilon > 0$).

That completes the proof of Theorem 9. □

Acknowledgments: My sincere gratitude to my master's degree supervisor, Professor Zhifang Zhang (retired from the University of Michigan at Ann Arbor). Special gratitude is extended to the PREPRINT platform preprints.org for making this work permanently available and citable.

References

1. Zhang, W. (2025). A Proof of the Riemann Hypothesis via a New Expression of $\xi(s)$, Preprints, https://www.preprints.org/manuscript/202108.0146
2. Karatsuba A. A., Nathanson M. B. (1993), Basic Analytic Number Theory, Springer, Berlin, Heidelberg.

 doi:10.20944/preprints202506.0481.v14

3. Conway, J. B. (1978), Functions of One Complex Variable I, Second Edition, New York: Springer-Verlag.
4. Olaf Helmer (1940), Divisibility properties of integral functions, Duke Mathematical Journal, 6(2): 345-356.
5. Iwaniec, H., Kowalski, E. (2004). Analytic Number Theory, American Mathematical Society Colloquium Publications, Vol. 53.
6. Cohen, Henri (2007), Number theory, Volume II: Analytic and modern tools, Graduate Texts in Mathematics, 240, New York: Springer, doi:10.1007/978-0-387-49894-2, ISBN 978-0-387-49893-5.
7. Cogdell, J. W. (2014). L-functions and functoriality. In Jianya Liu (Ed.), Automorphic forms and L-functions. Beijing: Higher Education Press.
8. Iwaniec H., Sarnak P. (2000), Perspectives on the analytic theory of L-functions. GAFA 2000 (Tel Aviv, 1999), Geom. Funct. Anal. 2000, Special Volume, Part II, 705-741
9. Yujiao Jiang, Guangshi Lv (2023), A survey on coefficients of automorphic L-functions, Advances in Mathematics (China), 52(2): 201-223.