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Description of the Electron in the Electromagnetic Field: The Dirac Type Equation and the Equation for the Wave Function in Spinor Coordinate Space

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Abstract: Physical processes are usually described using four-dimensional vector quantities - coordinate vector, momentum vector, current vector. But at the fundamental level they are characterized by spinors - coordinate spinors, momentum spinors, spinor wave functions. The propagation of fields and their interaction takes place at the spinor level, and since each spinor uniquely corresponds to a certain vector, the results of physical processes appear before us in vector form. For example, the relativistic Schrödinger equation and the Dirac equation are formulated by means of coordinate vectors, momentum vectors and quantum operators corresponding to them. In the Dirac equation a step forward is taken and the wave function is a spinor with complex components, but still coordinates and momentum are vectors. For a closed description of nature using only spinor quantities, it is necessary to have an equation similar to the Dirac equation in which momentum, coordinates and operators are spinors. It is such an equation that is presented in this paper. Using the example of the interaction between an electron and an electromagnetic field, we can see that the spinor equation contains more detailed information about the interaction than the vector equations. This is not new for quantum mechanics, since it describes interactions using complex wave functions, which cannot be observed directly, and only when measured goes to probabilities in the form of squares of the moduli of the wave functions. In the same way spinor quantities are not observable, but they completely determine observable vectors. In Section 2 of the paper, we analyze the quadratic form for an arbitrary four-component complex vector based on Pauli matrices. The form is invariant with respect to Lorentz transformations including any rotations and boosts. The invariance of the form allows us to construct on its basis an equation for a free particle combining the properties of the relativistic wave equation and the Dirac equation. For an electron in the presence of an electromagnetic potential it is shown that taking into account the commutation relations between the momentum and coordinate components allows us to obtain from this equation the known results describing the interactions of the electron spin with the electric and magnetic field. In the presence of a potential the momentum components cease to commute with each other. To neutralize this effect, the Schrödinger equation is supplemented by several equations with mixed derivatives on coordinates. In section 3 of the paper this quadratic form is expressed through momentum spinors, which makes it possible to obtain an equation for the spinor wave function in spinor coordinate space by replacing the momentum spinor components by partial derivative operators on the corresponding coordinate spinor component. Section 4 presents a modification of the theory of the path integral, which consists in considering the path integral in the spinor coordinate space. The Lagrangian densities for the scalar field and for the electron field, along with their corresponding propagators, are presented. An equation of motion for the electron is proposed that is relativistically invariant, in contrast to the Dirac equation, which lacks this invariance. This novel equation permitted the construction of an actually invariant procedure for the second quantization of the fermion field in spinor coordinate space. Furthermore, it is demonstrated that the field operators are a combination of plane waves in spinor or vector space, with the coefficients of which being pseudospinors or pseudovectors. Each of these pseudovectors or pseudospinors corresponds to one of the particles presented in the theory of electrodynamics. Furthermore, each plane wave possesses an additional coefficient in the form of a birth or annihilation operator. In vector space, these operators commute, whereas in spinor space they anticommutate. The paper presents the spinor and vector

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representations of the field operators in explicit form, comprising sets of 16 pseudospinors or 4 pseudovectors corresponding to particles represented in electrodynamics.

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1. Introduction

Nowadays, the interest to study applications of the Dirac equation to different situations and to find out the conditions of its generalization is not weakening. In particular, in [1] new versions of an extended Dirac equation and the associated Clifford algebra are presented. In [2] a study of the Schrödinger-Dirac covariant equation in the presence of gravity, where the non-commuting gamma matrices become space-time-dependent, is carried out. In [3] an idea is discussed that the visible properties of the electron, including rest mass and magnetic moment, are determined by a massless charge spinning at light speed within a Compton domain. In [4] some aspects of conformal rescaling in detail are explored and the role of the "quantum" potential is discussed as a natural consequence of non-inertial motion and is not exclusive to the quantum domain. Author establishes the fundamental importance of conformal symmetry, in which rescaling of the rest mass plays a vital role. Thus, the basis for a radically new theory of quantum phenomena based on the process of mass-energy flow is proposed. In [5] author has derived the covariant fourth-order/one-function equivalent of the Dirac equation for the general case of an arbitrary set of γ-matrices.

Supporting these search aspirations, in our work we propose a deeper understanding of the Dirac equation with an emphasis on the direct use of the principles of symmetry and invariance to Lorentz transformations. For the first time we present a formulation of the Dirac and Schrödinger equations in spinor coordinate space.

2. Generalized Dirac Type Equation

Let us introduce notations, which will be used further on. The speed of light and the rationalized Planck's constant will be considered as unity.

Pauli matrices

$$\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Matrices constructed from Pauli matrices

$$S_0 = \begin{pmatrix} \sigma_0 & 0 \\ 0 & \sigma_0 \end{pmatrix} \quad S_1 = \begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_1 \end{pmatrix} \quad S_2 = \begin{pmatrix} \sigma_2 & 0 \\ 0 & \sigma_2 \end{pmatrix} \quad S_3 = \begin{pmatrix} \sigma_3 & 0 \\ 0 & \sigma_3 \end{pmatrix}$$

A vector of matrices

$$\vec{\mathbf{S}}^T \equiv (S_1, S_2, S_3)$$

A set of arbitrary complex numbers and a vector of its three components

$$\mathbf{X}^T \equiv (X_0, X_1, X_2, X_3)$$

$$\vec{\mathbf{X}}^T \equiv (X_1, X_2, X_3)$$

Let us define a 2×2 matrix of Lorentz transformations given by the set of real rotation angles $(\alpha_1, \alpha_2, \alpha_3)$ and boosts $(\beta_1, \beta_2, \beta_3)$

$$n = exp\left(-\frac{1}{2}i\alpha_1\sigma_1\right) exp\left(\frac{1}{2}\beta_1\sigma_1\right) exp\left(-\frac{1}{2}i\alpha_2\sigma_2\right) exp\left(\frac{1}{2}\beta_2\sigma_2\right) exp\left(-\frac{1}{2}i\alpha_3\sigma_3\right) exp\left(\frac{1}{2}\beta_3\sigma_3\right)$$

and a similar 4×4 transformation matrix

$$N = exp\left(-\frac{1}{2}i\alpha_1 S_1\right) exp\left(\frac{1}{2}\beta_1 S_1\right) exp\left(-\frac{1}{2}i\alpha_2 S_2\right) exp\left(\frac{1}{2}\beta_2 S_2\right) exp\left(-\frac{1}{2}i\alpha_3 S_3\right) exp\left(\frac{1}{2}\beta_3 S_3\right)$$

We also define a 4×4 matrix of Lorentz transformations Λ, where μ and ν take values 0,1,2,3

$$\Lambda^{\mu}_{\nu} = \frac{1}{2}\mathrm{Tr}[\sigma_\mu n \sigma_\nu n^\dagger]$$

which can also be written explicitly using the 4×4 matrices of turn generators (R_1, R_2, R_3) and boosts (K_1, K_2, K_3)

$$\Lambda = exp(\alpha_1 R_1) exp(\beta_1 K_1) exp(\alpha_2 R_2) exp(\beta_2 K_2) exp(\alpha_3 R_3) exp(\beta_3 K_3)$$

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Let's define a 4×4 matrix

$$M^2 = (S_0X_0 - S_1X_1 - S_2X_2 - S_3X_3)(S_0X_0 + S_1X_1 + S_2X_2 + S_3X_3) =$$
$$\left(S_0X_0 - \vec{\mathbf{S}}^T\vec{\mathbf{X}}\right)\left(S_0X_0 + \vec{\mathbf{S}}^T\vec{\mathbf{X}}\right) =$$
$$S_0X_0S_0X_0 - S_1X_1S_1X_1 - S_2X_2S_2X_2 - S_3X_3S_3X_3 +$$
$$S_0X_0(S_1X_1 + S_2X_2 + S_3X_3) - S_1X_1(S_0X_0 + S_2X_2 + S_3X_3) -$$
$$S_2X_2(S_0X_0 + S_1X_1 + S_3X_3) - S_3X_3(S_0X_0 + S_1X_1 + S_2X_2)$$

In fact, we consider a quaternion with complex coefficients, which we multiply by its conjugate quaternion (due to the complexity of the coefficients, these are biquaternions, but we still use quaternionic conjugation, without complex conjugation).

Let us subject the set of complex numbers to the Lorentz transformation

$$\mathbf{X}' = \Lambda\mathbf{X}$$

Let us write a relation whose validity for an arbitrary set of complex numbers can be checked directly

$$(S_0X_0{}' - S_1X_1{}' - S_2X_2{}' - S_3X_3{}')(S_0X_0{}' + S_1X_1{}' + S_2X_2{}' + S_3X_3{}')$$
$$= (S_0X_0 - S_1X_1 - S_2X_2 - S_3X_3)(S_0X_0 + S_1X_1 + S_2X_2 + S_3X_3) = M^2$$

The matrix M^2 in the simplest case is diagonal with equal complex elements on the diagonal equal to the square of the length of the vector $\mathbf{X}$ in the metric of Minkowski space, which we denote m^2. Both M^2 and m^2 do not change under any rotations and boosts, in physical applications the invariance of m^2 is usually used, in particular, for the four-component momentum vector this quantity is called the square of mass.

Since the matrices S_μ anticommutate with each other, for a vector $\mathbf{X}$ whose components commute with each other, we have just the simplest case with a diagonal matrix with m^2 on the diagonal. But if the components of vector $\mathbf{X}$ do not commute, the matrix M^2 already has a more complex structure and carries additional physical information compared to m^2. For example, the vector $\mathbf{X}$ may include the electron momentum vector and the electromagnetic potential vector. The four-component potential vector is a function of the four-dimensional coordinates of Minkowski space. The components of the four-component momentum do not commute with the components of the coordinate vector, respectively, and the coordinate function does not commute with the momentum components, and their commutator is expressed through the partial derivative of this function by the corresponding coordinate. If the components of the vector $\mathbf{X}$ do not commute, the matrix M^2 will no longer be invariant with respect to Lorentz transformations.

Suppose that the complex numbers we consider commute with all matrices, and note that the squares of all matrices are equal to the unit 4×4 matrix I

$$M^2 = (X_0X_0 - X_1X_1 - X_2X_2 - X_3X_3)I + (S_1X_0X_1 + S_2X_0X_2 + S_3X_0X_3)$$
$$- (S_1X_1X_0 + S_1S_2X_1X_2 + S_1S_3X_1X_3) - (S_2X_2X_0 + S_2S_1X_2X_1 + S_2S_3X_2X_3)$$
$$- (S_3X_3X_0 + S_3S_1X_3X_1 + S_3S_2X_3X_2)$$
$$= (X_0X_0 - X_1X_1 - X_2X_2 - X_3X_3)I + S_1(X_0X_1 - X_1X_0) + S_2(X_0X_2 - X_2X_0) + S_3(X_0X_3$$
$$- X_3X_0) - (S_1S_2X_1X_2 + S_1S_3X_1X_3) - (S_2S_1X_2X_1 + S_2S_3X_2X_3) - (S_3S_1X_3X_1 + S_3S_2X_3X_2)$$
$$= (X_0X_0 - X_1X_1 - X_2X_2 - X_3X_3)I + S_1(X_0X_1 - X_1X_0) + S_2(X_0X_2 - X_2X_0) + S_3(X_0X_3$$
$$- X_3X_0) - (S_1S_2X_1X_2 + S_2S_1X_2X_1) - (S_2S_3X_2X_3 + S_3S_2X_3X_2) - (S_3S_1X_3X_1 + S_1S_3X_1X_3)$$
$$= (X_0X_0 - X_1X_1 - X_2X_2 - X_3X_3)I + S_1(X_0X_1 - X_1X_0) + S_2(X_0X_2 - X_2X_0) + S_3(X_0X_3$$
$$- X_3X_0) - (S_1S_2X_1X_2 + S_2S_1X_1X_2 + S_2S_1(X_2X_1 - X_1X_2))$$
$$- (S_2S_3X_2X_3 + S_3S_2X_2X_3 + S_3S_2(X_3X_2 - X_2X_3))$$
$$- (S_3S_1X_3X_1 + S_1S_3X_3X_1 + S_1S_3(X_1X_3 - X_3X_1))$$

Taking into account anticommutative properties of matrices and expressions for their pairwise products we obtain

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$$M^2 = (X_0X_0 - X_1X_1 - X_2X_2 - X_3X_3)I + S_1(X_0X_1 - X_1X_0) + S_2(X_0X_2 - X_2X_0) + S_3(X_0X_3 - X_3X_0)$$
$$- S_2S_1(X_2X_1 - X_1X_2) - S_3S_2(X_3X_2 - X_2X_3) - S_1S_3(X_1X_3 - X_3X_1)$$
$$= (X_0X_0 - X_1X_1 - X_2X_2 - X_3X_3)I + S_1(X_0X_1 - X_1X_0) + S_2(X_0X_2 - X_2X_0) + S_3(X_0X_3$$
$$- X_3X_0) + iS_3(X_2X_1 - X_1X_2) + iS_1(X_3X_2 - X_2X_3) + iS_2(X_1X_3 - X_3X_1)$$
$$= (X_0X_0 - X_1X_1 - X_2X_2 - X_3X_3)I + S_1(X_0X_1 - X_1X_0) + iS_1(X_3X_2 - X_2X_3) + S_2(X_0X_2$$
$$- X_2X_0) + iS_2(X_1X_3 - X_3X_1) + S_3(X_0X_3 - X_3X_0) + iS_3(X_2X_1 - X_1X_2)$$

Consider the case when $\mathbf{X}$ is the sum of the momentum vector and the electromagnetic potential vector, which is a function of coordinates

$$\mathbf{X} = \mathbf{P} + \mathbf{A}$$
$$\mathbf{P}^T \equiv (P_0, P_1, P_2, P_3)$$
$$\mathbf{A}^T \equiv (A_0, A_1, A_2, A_3)$$
$$\vec{\mathbf{P}}^T \equiv (P_1, P_2, P_3)$$
$$\vec{\mathbf{A}}^T \equiv (A_1, A_2, A_3)$$

$$M^2 = I[(P_0 + A_0)(P_0 + A_0) - (P_1 + A_1)(P_1 + A_1) - (P_2 + A_2)(P_2 + A_2) - (P_3 + A_3)(P_3 + A_3)] +$$
$$S_1[(P_0 + A_0)(P_1 + A_1) - (P_1 + A_1)(P_0 + A_0)] + iS_1[(P_3 + A_3)(P_2 + A_2) - (P_2 + A_2)(P_3 + A_3)] +$$
$$S_2[(P_0 + A_0)(P_2 + A_2) - (P_2 + A_2)(P_0 + A_0)] + iS_2[(P_1 + A_1)(P_3 + A_3) - (P_3 + A_3)(P_1 + A_1)] +$$
$$S_3[(P_0 + A_0)(P_3 + A_3) - (P_3 + A_3)(P_0 + A_0)] + iS_3[(P_2 + A_2)(P_1 + A_1) - (P_1 + A_1)(P_2 + A_2)]$$

For now, we'll stick with the Heisenberg approach, that is, we will consider the components of the momentum vector P_0, P_1, P_2, P_3 as operators for which there are commutation relations with coordinates or coordinate functions such as A_0, A_1, A_2, A_3. In this approach, the operators do not have to act on any wave function.

Taking into account the commutation relations of the components of the momentum vector and the coordinate vector, the commutator of the momentum component and the coordinate function is expressed through the derivative of this function by the corresponding coordinate, e.g.

$$[(P_2 + A_2)(P_1 + A_1) - (P_1 + A_1)(P_2 + A_2)] = P_2A_1 - A_1P_2 - (P_1A_2 - A_2P_1) = -i\frac{\partial A_1}{\partial x_2} - \left(-i\frac{\partial A_2}{\partial x_1}\right)$$

As a result, we obtain

$$M^2 = I[(P_0 + A_0)(P_0 + A_0) - (P_1 + A_1)(P_1 + A_1) - (P_2 + A_2)(P_2 + A_2) - (P_3 + A_3)(P_3 + A_3)]$$
$$+ S_1\left[-i\frac{\partial A_1}{\partial x_0} + i\frac{\partial A_0}{\partial x_1}\right] + iS_1\left[-i\frac{\partial A_2}{\partial x_3} + i\frac{\partial A_3}{\partial x_2}\right] + S_2\left[-i\frac{\partial A_2}{\partial x_0} + i\frac{\partial A_0}{\partial x_2}\right]$$
$$+ iS_2\left[-i\frac{\partial A_3}{\partial x_1} + i\frac{\partial A_1}{\partial x_3}\right] + S_3\left[-i\frac{\partial A_3}{\partial x_0} + i\frac{\partial A_0}{\partial x_3}\right] + iS_3\left[-i\frac{\partial A_1}{\partial x_2} + i\frac{\partial A_2}{\partial x_1}\right]$$
$$= I[(P_0 + A_0)(P_0 + A_0) - (P_1 + A_1)(P_1 + A_1) - (P_2 + A_2)(P_2 + A_2)$$
$$- (P_3 + A_3)(P_3 + A_3)] - iS_1\left[\frac{\partial A_1}{\partial x_0} - \frac{\partial A_0}{\partial x_1}\right] + S_1\left[\frac{\partial A_2}{\partial x_3} - \frac{\partial A_3}{\partial x_2}\right] - iS_2\left[\frac{\partial A_2}{\partial x_0} - \frac{\partial A_0}{\partial x_2}\right]$$
$$+ S_2\left[\frac{\partial A_3}{\partial x_1} - \frac{\partial A_1}{\partial x_3}\right] - iS_3\left[\frac{\partial A_3}{\partial x_0} - \frac{\partial A_0}{\partial x_3}\right] + S_3\left[\frac{\partial A_1}{\partial x_2} - \frac{\partial A_2}{\partial x_1}\right]$$
$$= I[(P_0 + A_0)(P_0 + A_0) - (P_1 + A_1)(P_1 + A_1) - (P_2 + A_2)(P_2 + A_2)$$
$$- (P_3 + A_3)(P_3 + A_3)] - iS_1F_{01} + S_1F_{32} - iS_2F_{02} + S_2F_{13} - iS_3F_{03} + S_3F_{21}$$
$$= I[(P_0 + A_0)(P_0 + A_0) - (P_1 + A_1)(P_1 + A_1) - (P_2 + A_2)(P_2 + A_2)$$
$$- (P_3 + A_3)(P_3 + A_3)] - iS_1E_x + S_1B_x - iS_2E_y + S_2B_y - iS_3E_z + S_3B_z$$

where

$$F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu$$
$$\partial_\mu \equiv \frac{\partial}{\partial x^\mu}$$
$$F_{\mu\nu} = \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & -B_z & B_y \\ -E_y & B_z & 0 & -B_x \\ -E_z & -B_y & B_x & 0 \end{pmatrix}$$

As a result, we have the expression

$$M^2 = I[(P_0 + A_0)(P_0 + A_0) - (P_1 + A_1)(P_1 + A_1) - (P_2 + A_2)(P_2 + A_2) - (P_3 + A_3)(P_3 + A_3)] +$$
$$\vec{\mathbf{S}}^T\vec{\mathbf{B}} - i\vec{\mathbf{S}}^T\vec{\mathbf{E}}$$

$$\vec{\mathbf{B}}^T \equiv (B_x, B_y, B_z) \equiv (B_1, B_2, B_3)$$

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$$\vec{\mathbf{E}}^T \equiv (E_x, E_y, E_z) \equiv (E_1, E_2, E_3)$$

Similarly, it can be shown that

$$(S_0P_0 - S_1P_1 - S_2P_2 - S_3P_3)(S_0A_0 + S_1A_1 + S_2A_2 + S_3A_3) + (S_0A_0 - S_1A_1 - S_2A_2 - S_3A_3)(S_0P_0 + S_1P_1 + S_2P_2 + S_3P_3) = 2I(P_0A_0 - P_1A_1 - P_2A_2 - P_3A_3) + \vec{\mathbf{S}}^T\vec{\mathbf{B}} - i\vec{\mathbf{S}}^T\vec{\mathbf{E}}$$

The matrix

$$M^2 - \{\vec{\mathbf{S}}^T\vec{\mathbf{B}} - i\vec{\mathbf{S}}^T\vec{\mathbf{E}}\} = I\{(P_0 + A_0)(P_0 + A_0) - (P_1 + A_1)(P_1 + A_1) - (P_2 + A_2)(P_2 + A_2) - (P_3 + A_3)(P_3 + A_3)\} \equiv Id^2$$

does not change under Lorentz transformations involving any rotations and boosts.

$$Id^2 = \big(S_0(P_0 + A_0) - S_1(P_1 + A_1) - S_2(P_2 + A_2) - S_3(P_3 + A_3)\big)\big(S_0(P_0 + A_0) + S_1(P_1 + A_1) + S_2(P_2 + A_2) + S_3(P_3 + A_3)\big) - \{\vec{\mathbf{S}}^T\vec{\mathbf{B}} - i\vec{\mathbf{S}}^T\vec{\mathbf{E}}\}$$
$$= \left(S_0(P_0 + A_0) - \vec{\mathbf{S}}^T(\vec{\mathbf{P}} + \vec{\mathbf{A}})\right)\left(S_0(P_0 + A_0) + \vec{\mathbf{S}}^T(\vec{\mathbf{P}} + \vec{\mathbf{A}})\right) - \{\vec{\mathbf{S}}^T\vec{\mathbf{B}} - i\vec{\mathbf{S}}^T\vec{\mathbf{E}}\}$$

Taking into account the electron charge we have

$$\mathbf{X} = \mathbf{P} - e\mathbf{A}$$

$$Id^2 = \left(S_0(P_0 - eA_0) - \vec{\mathbf{S}}^T(\vec{\mathbf{P}} + \vec{\mathbf{A}})\right)\left(S_0(P_0 - eA_0) + \vec{\mathbf{S}}^T(\vec{\mathbf{P}} + \vec{\mathbf{A}})\right) + e\{\vec{\mathbf{S}}^T\vec{\mathbf{B}} - i\vec{\mathbf{S}}^T\vec{\mathbf{E}}\}$$

Let us summarize our consideration. There is a correlation

$$Id^2 = M^2 + e\{\vec{\mathbf{S}}^T\vec{\mathbf{B}} - i\vec{\mathbf{S}}^T\vec{\mathbf{E}}\}$$

where

$$M^2 \equiv \left(S_0(P_0 - eA_0) - \vec{\mathbf{S}}^T(\vec{\mathbf{P}} - e\vec{\mathbf{A}})\right)\left(S_0(P_0 - eA_0) + \vec{\mathbf{S}}^T(\vec{\mathbf{P}} - e\vec{\mathbf{A}})\right)$$

$$Id^2 \equiv I\{(P_0 - eA_0)^2 - (P_1 - eA_1)^2 - (P_2 - eA_2)^2 - (P_3 - eA_3)^2\}$$
$$= I[(P_0 - eA_0)(P_0 - eA_0) - (P_1 - eA_1)(P_1 - eA_1) - (P_2 - eA_2)(P_2 - eA_2) - (P_3 - eA_3)(P_3 - eA_3)] = I\left[(P_0 - eA_0)(P_0 - eA_0) - (\vec{\mathbf{P}} - e\vec{\mathbf{A}})^T(\vec{\mathbf{P}} - e\vec{\mathbf{A}})\right]$$
$$= I\left\{(P_0 - eA_0)^2 - (\vec{\mathbf{P}} - e\vec{\mathbf{A}})^2\right\}$$

Let's analyze the obtained equality

$$M^2 = Id^2 - e\{\vec{\mathbf{S}}^T\vec{\mathbf{B}} - i\vec{\mathbf{S}}^T\vec{\mathbf{E}}\}$$

Note that the quantity d^2 is invariant to the Lorentz transformations irrespective of whether the momentum and field components commute or not. To solve this equation, we have to make additional simplifications. For example, to arrive at an equation similar to the Dirac equation, we must equate M^2 with the matrix Im^2, where m^2 is the square of the mass of a free electron. Then

$$Im^2 = Id^2 - e\{\vec{\mathbf{S}}^T\vec{\mathbf{B}} - i\vec{\mathbf{S}}^T\vec{\mathbf{E}}\}$$

$$Id^2 - Im^2 - e\{\vec{\mathbf{S}}^T\vec{\mathbf{B}} - i\vec{\mathbf{S}}^T\vec{\mathbf{E}}\} = 0$$

$$I\left\{(P_0 - eA_0)^2 - (\vec{\mathbf{P}} - e\vec{\mathbf{A}})^2\right\} - Im^2 - e\{\vec{\mathbf{S}}^T\vec{\mathbf{B}} - i\vec{\mathbf{S}}^T\vec{\mathbf{E}}\} = 0$$

With this substitution the generalized equation almost coincides with the equation [[6], formula (43.25)], the difference is that there is a plus sign before $e\vec{\mathbf{S}}^T\vec{\mathbf{B}}$, and instead of $i\vec{\mathbf{S}}^T\vec{\mathbf{E}}$ there is $i\vec{\boldsymbol{\alpha}}^T\vec{\mathbf{E}}$, in which the matrices $\boldsymbol{\alpha}$ have the following form

$$\vec{\boldsymbol{\alpha}}^T \equiv (\alpha_1, \alpha_2, \alpha_3)$$

$$\alpha_1 = \begin{pmatrix} 0 & \sigma_1 \\ \sigma_1 & 0 \end{pmatrix} \quad \alpha_2 = \begin{pmatrix} 0 & \sigma_2 \\ \sigma_2 & 0 \end{pmatrix} \quad \alpha_3 = \begin{pmatrix} 0 & \sigma_3 \\ \sigma_3 & 0 \end{pmatrix}$$

A similar equation is given by Dirac in [[7], Paragraph 76, Equation 24]; he does not use the matrices $\vec{\boldsymbol{\alpha}}$, only the matrices $\vec{\mathbf{S}}$, but the signs of the contributions of the magnetic and electric fields are the same.

Along with the original form

$$M^2 = \left(S_0(P_0 - eA_0) - \vec{\mathbf{S}}^T(\vec{\mathbf{P}} - e\vec{\mathbf{A}})\right)\left(S_0(P_0 - eA_0) + \vec{\mathbf{S}}^T(\vec{\mathbf{P}} - e\vec{\mathbf{A}})\right) = d^2 - e\{\vec{\mathbf{S}}^T\vec{\mathbf{B}} - i\vec{\mathbf{S}}^T\vec{\mathbf{E}}\}$$

it is possible to consider the form with a different order of the factors. It can be shown that this leads to a change in the sign of the electric field contribution

$$M^2 = \left(S_0(P_0 - eA_0) + \vec{\mathbf{S}}^T(\vec{\mathbf{P}} - e\vec{\mathbf{A}})\right)\left(S_0(P_0 - eA_0) - \vec{\mathbf{S}}^T(\vec{\mathbf{P}} - e\vec{\mathbf{A}})\right) = d^2 - e\{\vec{\mathbf{S}}^T\vec{\mathbf{B}} + i\vec{\mathbf{S}}^T\vec{\mathbf{E}}\}$$

Since Id^2, unlike M^2, is invariant to Lorentz transformations, it would be logical to replace it by Im^2. At least both these matrices are diagonal, and in the case of a weak field their diagonal elements

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are close. Nevertheless, the approach based on the Dirac equation leads to solutions consistent with experiment.

The matrix M^2 in the general case has complex elements and is not diagonal, and in the Dirac equations instead of it is substituted the product of the unit matrix by the square of mass m^2, the physical meaning of such a substitution is not obvious. Apparently it is implied that it is the square of the mass of a free electron. But the square of the length of the sum of the lengths of the electron momentum vectors and the electromagnetic potential vector is not equal to the sum of the squares of the lengths of these vectors, that is, it is not equal to the square of the mass of the electron, even if the square of the length of the potential vector were zero. But, for example, in the case of an electrostatic central field, even the square of the length of one potential vector is not equal to zero. Therefore, it is difficult to find a logical justification for using the mass of a free electron in the Dirac equation in the presence of an electromagnetic field. Due to the noted differences, the solutions of the generalized equation can differ from the solutions arising from the Dirac equation.

In the case when there is a constant magnetic field directed along the z-axis, we can write down

$$A_0 = 0 \qquad A_1 = -\frac{1}{2}B_3x_2 \qquad A_2 = \frac{1}{2}B_3x_1 \qquad A_3 = 0$$

$$(S_0P_0)^2 - M^2 - \left(\vec{\mathbf{P}} - e\vec{\mathbf{A}}\right)^T\left(\vec{\mathbf{P}} - e\vec{\mathbf{A}}\right)I - eS_3B_3 = 0$$

$$(S_0P_0)^2 - M^2 - (P_1 - eA_1)(P_1 - eA_1)I - (P_2 - eA_2)(P_2 - eA_2)I - eS_3B_3 = 0$$

$$(S_0P_0)^2 - M^2 - {P_0}^2I - {P_3}^2I - {P_1}^2 - (eA_1)^2 - {P_2}^2 - (eA_2)^2 + \frac{e}{2}B_3(x_1P_2 - x_2P_1 + x_1P_2 - x_2P_1) - eS_3B_3 = 0$$

$${P_0}^2I - M^2 - {P_0}^2I - {P_3}^2I - {P_1}^2I - (eA_1)^2I - {P_2}^2I - (eA_2)^2I + eB_3(x_1P_2 - x_2P_1)I - eS_3B_3 = 0$$

$$I\left(-{P_1}^2 - {P_2}^2 - {P_3}^2 - (eA_1)^2 - (eA_2)^2\right) - M^2 - eB_3\begin{pmatrix} L_3+1 & 0 & 0 & 0 \\ 0 & L_3-1 & 0 & 0 \\ 0 & 0 & L_3+1 & 0 \\ 0 & 0 & 0 & L_3-1 \end{pmatrix} = 0$$

Here $(x_1P_2 - x_2P_1) \equiv L_3$. Only when the field is directed along the z-axis, the matrix M^2 is diagonal and real because the third Pauli matrix is diagonal and real. And if the field is weak, M^2 can be approximated by the m^2I matrix. This is probably why it is customary to illustrate the interaction of electron spin with the magnetic field by choosing its direction along the z-axis. In any other direction M^2 is not only non-diagonal, but also complex, so that it is difficult to justify the use of m^2I.

When the influence of the electromagnetic field was taken into account, no specific characteristics of the electron were used. When deriving a similar result using the Dirac equation, it is assumed that since the electron equation is used, the result is specific to the electron. In our case Pauli matrices and commutation relations are used, apparently these two assumptions or only one of them characterize the properties of the electron, distinguishing it from other particles with non-zero masses.

The proposed equation echoes the Dirac equation, at least from it one can obtain the same formulas for the interaction of spin and electromagnetic field as with the Dirac equation, and in the absence of a field the proposed equation is invariant to the Lorentz transformations. In contrast, to prove the invariance of the Dirac equation even in the absence of a field, the infinitesimal Lorentz transformations are used, but the invariance at finite angles of rotations and boosts is not demonstrated. The proof of invariance of the Dirac equation is based on the claim that a combination of rotations at finite angles can be represented as a combination of infinitesimal rotations. But this is true only for rotations or boosts around one axis, and if there are at least two axes, this statement is not true because of non-commutability of Pauli matrices, which are generators of rotations, so that the exponent of the sum is not equal to the product of exponents if the sum includes generators of rotations or boosts around different axes. By a direct check we can verify that the invariance of the Dirac equation takes place at any combination of rotations, but only under the condition of zero boosts, i.e., only in a rest frame of reference, any boost violates the invariance.

A test case for any theory is the model of the central electrostatic field used in the description of the hydrogen atom, in which the components of the vector potential are zero

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$$\left(S_0(P_0-eA_0)-\vec{\mathbf{S}}^T\vec{\mathbf{P}}\right)\left(S_0(P_0-eA_0)+\vec{\mathbf{S}}^T\vec{\mathbf{P}}\right)=I\left[(P_0-eA_0)^2-{P_1}^2-{P_2}^2-{P_3}^2\right]+ie\vec{\mathbf{S}}^T\vec{\mathbf{E}}$$

If again we equate the left part with Im^2, we obtain

$$I\left[(P_0-eA_0)^2-{P_1}^2-{P_2}^2-{P_3}^2\right]-Im^2+ie\vec{\mathbf{S}}^T\vec{\mathbf{E}}=0$$

$$I\left[(P_0-eA_0)^2-{P_1}^2-{P_2}^2-{P_3}^2-m^2\right]-ie\left(S_1\frac{\partial A_0}{\partial x_1}+S_2\frac{\partial A_0}{\partial x_2}+S_3\frac{\partial A_0}{\partial x_3}\right)=0$$

Introducing the notations ($A_0\equiv\varphi(r)=Q/r,\ P_0\equiv E,\ r=1/\sqrt{{x_1}^2+{x_2}^2+{x_3}^2}$), we obtain

$$I\left[\left(E-\frac{eQ}{r}\right)^2-{P_1}^2-{P_2}^2-{P_3}^2-m^2\right]-ie\left(S_1\frac{\partial\varphi(r)}{\partial x_1}+S_2\frac{\partial\varphi(r)}{\partial x_2}+S_3\frac{\partial\varphi(r)}{\partial x_3}\right)=0$$

$$I\left[\left(E-\frac{eQ}{r}\right)^2-{P_1}^2-{P_2}^2-{P_3}^2-m^2\right]+i\frac{eQ}{r^3}(S_1x_1+S_2x_2+S_3x_3)=0$$

If we substitute operators acting on the wave function instead of momentum components into the equation, we obtain a generalized analog of the relativistic Schrödinger equation, in which the wave function has four components and changes as a spinor under Lorentz transformations. Using the substitutions

$$P_0\to i\frac{\partial}{\partial t}\qquad P_1\to -i\frac{\partial}{\partial x_1}\qquad P_2\to -i\frac{\partial}{\partial x_2}\qquad P_3\to -i\frac{\partial}{\partial x_3}$$

the equation for the four-component wave function $\boldsymbol{\psi}$ before all transformations has the form

$$\left(S_0\left(\frac{\partial}{\partial t}-eA_0\right)+\vec{\mathbf{S}}^T(\boldsymbol{\nabla}-e\vec{\mathbf{A}})\right)\left(S_0\left(\frac{\partial}{\partial t}-eA_0\right)-\vec{\mathbf{S}}^T(\boldsymbol{\nabla}-e\vec{\mathbf{A}})\right)\boldsymbol{\psi}+M^2\boldsymbol{\psi}=0$$

and after transformations

$$\left\{\left(S_0(P_0-eA_0)\right)^2-\left(\vec{\mathbf{P}}-e\vec{\mathbf{A}}\right)^2I-e\vec{\mathbf{S}}^T\vec{\mathbf{B}}+ie\vec{\mathbf{S}}^T\vec{\mathbf{E}}\right\}\boldsymbol{\psi}=M^2\boldsymbol{\psi}$$

Once again, note that the matrix M^2 is not diagonal and real.

All the above deductions are also valid when replacing 4×4 matrices S_μ by 2×2 matrices σ_μ, since their commutative and anticommutative properties are the same. The corresponding generalized equation is of the form

$$\left(\sigma_0(P_0-eA_0)\right)^2-M^2-\left(\vec{\mathbf{P}}-e\vec{\mathbf{A}}\right)^2I-e\vec{\boldsymbol{\sigma}}^T\vec{\mathbf{B}}+ie\vec{\boldsymbol{\sigma}}^T\vec{\mathbf{E}}=0$$

where

$$\vec{\boldsymbol{\sigma}}^T\equiv(\sigma_1,\sigma_2,\sigma_3)$$

and the equation for the now two-component wave function looks like

$$\left(\sigma_0\left(\frac{\partial}{\partial t}-eA_0\right)+\vec{\boldsymbol{\sigma}}^T(\boldsymbol{\nabla}-e\vec{\mathbf{A}})\right)\left(\sigma_0\left(\frac{\partial}{\partial t}-eA_0\right)-\vec{\boldsymbol{\sigma}}^T(\boldsymbol{\nabla}-e\vec{\mathbf{A}})\right)\boldsymbol{\psi}+M^2\boldsymbol{\psi}=0$$

In deriving his equation, Dirac [[7], Paragraph 74] noted that as long as we are dealing with matrices with two rows and columns, we cannot obtain a representation of more than three anticommuting quantities; to represent four anticommuting quantities, he turned to matrices with four rows and columns. In our case, however, three anticommuting matrices are sufficient, so the wave function can also be two-component. Dirac also explains that the presence of four components results in twice as many solutions, half of which have negative energy. In the case of a two-component wave function, however, no negative energy solutions are obtained. Particles with negative energy in this case also exist, but they are described by the same equation in which the signs of all four matrices S or σ are reversed.

One would seem to expect similar results from other representations of the momentum operator, e.g., [6, formula (24.15)]

$$\omega_0=\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}\quad \omega_1=\frac{1}{\sqrt{2}}\begin{pmatrix}0&1&0\\1&0&1\\0&1&0\end{pmatrix}\quad \omega_2=\frac{1}{\sqrt{2}}\begin{pmatrix}0&-i&0\\i&0&-i\\0&i&0\end{pmatrix}\quad \omega_3=\begin{pmatrix}1&0&0\\0&0&0\\0&0&-1\end{pmatrix}$$

under the assumption that this representation can describe a particle with spin one. But this expectation is not justified, since the last three matrices do not anticommutate, and therefore the quadratic form constructed on their basis is not invariant under Lorentz transformations.

If one consistently adheres to the Heisenberg approach and does not involve the notion of wave function, it is not very clear how to search for solutions of the presented equations. The Schrödinger

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approach with finding the eigenvalues of the M^2matrix and their corresponding eigenfunctions can help here.

$$\left\{\left(S_0(P_0 - eA_0)\right)^2 - (\vec{\mathbf{P}} - e\vec{\mathbf{A}})^2 I - e\vec{\mathbf{S}}^T\vec{\mathbf{B}} + ie\vec{\mathbf{S}}^T\vec{\mathbf{E}}\right\}\boldsymbol{\psi} = M^2\boldsymbol{\psi}$$

In the left-hand side are the operators acting on the wave function, and in the right-hand side is a constant matrix on which the wave function is simply multiplied. This equality must be satisfied for all values of the four-dimensional coordinates (t, x_1, x_2, x_3) at once. Then M^2 is not fixed but can take a set of possible values, finding all these values is the goal of solving the equation.

Thus, we have arrived at an equation containing a matrix M^2 which is non-diagonal, complex and in general depends on the coordinates (t, x_1, x_2, x_3). After the standard procedure of separating the time and space variables, we can go to a stationary equation in which there will be no time dependence, but the dependence the matrix M^2 on the coordinates will remain. It is possible to ignore the dependence of M^2 on the coordinates and its non-diagonality and simply replace this matrix by a unit matrix with a coefficient in the form of the square of the free electron mass. Then the equation will give solutions coinciding with those of the Dirac equation. But this solution can be considered only approximate and the question remains how far we depart from strict adherence to the principle of invariance with respect to Lorentz transformations and how far we deviate from the hypothetical true solution, which is fully consistent with this principle. To find this solution, we need to approach this equation without simplifying assumptions and look for a set of solutions, each of which represents an eigenvalue matrix M^2 of arbitrary form and its corresponding four-component eigenfunction.

Let us return to the question of Lorentz invariance of the expression

$$(S_0X_0 - S_1X_1 - S_2X_2 - S_3X_3)(S_0X_0 + S_1X_1 + S_2X_2 + S_3X_3) = M^2$$

As it was noted, this expression does not change at rotations and boosts in Minkowski space only if the components of (X_0, X_1, X_2, X_3) commute with each other. If they do not commute, the matrix M^2 changes under Lorentz transformations. Two parts can be distinguished in this matrix

$$M^2 = (X_0X_0 - X_1X_1 - X_2X_2 - X_3X_3)I$$
$$+S_1(X_0X_1 - X_1X_0) + iS_1(X_3X_2 - X_2X_3) + S_2(X_0X_2 - X_2X_0)$$
$$+iS_2(X_1X_3 - X_3X_1) + S_3(X_0X_3 - X_3X_0) + iS_3(X_2X_1 - X_1X_2)$$

The first row represents the unit matrix multiplied by a value that still does not change under Lorentz transformations. All changes occur in the last two rows. In the particular case of electrodynamics, we have

$$M^2 = I[(P_0 + A_0)(P_0 + A_0) - (P_1 + A_1)(P_1 + A_1) - (P_2 + A_2)(P_2 + A_2) - (P_3 + A_3)(P_3 + A_3)]$$
$$+S_1[(P_0 + A_0)(P_1 + A_1) - (P_1 + A_1)(P_0 + A_0)] + iS_1[(P_3 + A_3)(P_2 + A_2) - (P_2 + A_2)(P_3 + A_3)]$$
$$+S_2[(P_0 + A_0)(P_2 + A_2) - (P_2 + A_2)(P_0 + A_0)] + iS_2[(P_1 + A_1)(P_3 + A_3) - (P_3 + A_3)(P_1 + A_1)]$$
$$+S_3[(P_0 + A_0)(P_3 + A_3) - (P_3 + A_3)(P_0 + A_0)] + iS_3[(P_2 + A_2)(P_1 + A_1) - (P_1 + A_1)(P_2 + A_2)]$$

Here the first line is invariant, but the last three are not. The only way to ensure complete invariance of M^2 is to require these three lines to be zero. Let us again consider the commutation relations, but now we will not assume that the momentum components commute with each other, only the potential components still commute with each other. Now we can write the relations of the form

$$[(P_2 + A_2)(P_1 + A_1) - (P_1 + A_1)(P_2 + A_2)]$$
$$= P_2(P_1 + A_1) - (P_1 + A_1)P_2 - \left(P_1(P_2 + A_2) - A_2(P_1 + A_1)\right)$$
$$= -i\frac{\partial(P_1 + A_1)}{\partial x_2} - \left(-i\frac{\partial(P_2 + A_2)}{\partial x_1}\right)$$

Such values as

$$\frac{\partial P_1}{\partial x_0} - \frac{\partial P_0}{\partial x_1}$$

always enter M^2 as a sum with the component of the field, in this case the electric one

$$\left(\frac{\partial P_1}{\partial x_0} - \frac{\partial P_0}{\partial x_1}\right) + \left(\frac{\partial A_1}{\partial x_0} - \frac{\partial A_0}{\partial x_1}\right) = \left(\frac{\partial P_1}{\partial x_0} - \frac{\partial P_0}{\partial x_1}\right) + E_x$$

If we formally define a new value

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$$V_1 \equiv \frac{P_1}{m}$$

and suppose that m does not change at rotations and boosts, and also to take into account the presence of charge at the electron, it is possible to require for this and all other similar sums the fulfilment of the condition

$$m\left(\frac{\partial V_1}{\partial x_0} - \frac{\partial V_0}{\partial x_1}\right) + eE_x = 0$$

The value V_1 can be regarded as a component of velocity, and velocity not in the usual sense, as a derivative of the spatial coordinate by time, but simply as a component of momentum divided by the inertial mass m. Then the above equality can be interpreted in the spirit of Newton's law, namely, that the acceleration multiplied by the mass is equal to the force acting on the side of the electric field. If all such equalities are fulfilled, only the first line will remain in the quantity M^2, and it will be invariant under Lorentz transformations. It is possible to go further, and to assume equality of the masses appearing here, namely

$$M^2 = Im^2$$

As a result, we obtain a system of equations

$$(P_0 + eA_0)(P_0 + eA_0) - (P_1 + eA_1)(P_1 + eA_1) - (P_2 + eA_2)(P_2 + eA_2) - (P_3 + eA_3)(P_3 + eA_3) = m^2$$
$$(\partial_\mu P_\nu - \partial_\nu P_\mu) + e(\partial_\mu A_\nu - \partial_\nu A_\mu) = 0$$

It is the fulfilment of these equations that causes the mass M^2 and m^2 to have the meaning we are accustomed to, that is, not only invariant under Lorentz transformations, but also unchanged by changes in momentum. We can introduce tensor notations

$$G_{\mu\nu} + eF_{\mu\nu} = 0$$

where

$$G_{\mu\nu} \equiv \partial_\mu P_\nu - \partial_\nu P_\mu$$

The resulting system of equations describes not only uniform but also accelerated motion. The presence of an external field leads to a change in momentum, and vice versa, any change in momentum perturbs the potential and generates an electromagnetic field.

For quantum mechanics we can replace the momentum components in all equations by the derivative operators

$$P_0 \rightarrow i\frac{\partial}{\partial x_0} \qquad P_1 \rightarrow -i\frac{\partial}{\partial x_1} \qquad P_2 \rightarrow -i\frac{\partial}{\partial x_2} \qquad P_3 \rightarrow -i\frac{\partial}{\partial x_3}$$

This also applies to equations from the second group, where mixed derivatives arise

$$\partial_\mu P_0 \rightarrow i\frac{\partial}{\partial x_\mu}\frac{\partial}{\partial x_0} \qquad \partial_\mu P_1 \rightarrow -\, i\frac{\partial}{\partial x_\mu}\frac{\partial}{\partial x_1}$$
$$\partial_\mu P_2 \rightarrow -\, i\frac{\partial}{\partial x_\mu}\frac{\partial}{\partial x_2} \qquad \partial_\mu P_3 \rightarrow -\, i\frac{\partial}{\partial x_\mu}\frac{\partial}{\partial x_3}$$

As a result, we obtain for the wave function a system of equations with second order derivatives, the innovation compared to the commonly used equations is the presence in the equations of mixed derivatives on all components of the coordinate vector.

The equations proposed here initially take into account the non-commutability of momentum components, their derivation relies only on the unconditional fulfilment (even in coupled systems) of the requirement of invariance to Lorentz transformations for the product of conjugate quaternions with arbitrary coefficients

$$(S_0X_0 - S_1X_1 - S_2X_2 - S_3X_3)(S_0X_0 + S_1X_1 + S_2X_2 + S_3X_3) = M^2$$

Putting all equations together, we write a truly relativistic system of equations

$$\left(\sigma_0\left(\frac{\partial}{\partial x_0} - eA_0\right) + \vec{\boldsymbol{\sigma}}^T\left(\boldsymbol{\nabla} - e\vec{\mathbf{A}}\right)\right)\left(\sigma_0\left(\frac{\partial}{\partial x_0} - eA_0\right) - \vec{\boldsymbol{\sigma}}^T\left(\boldsymbol{\nabla} - e\vec{\mathbf{A}}\right)\right)\boldsymbol{\psi} + M^2\boldsymbol{\psi} = 0$$
$$\left(i\frac{\partial}{\partial x_j}\frac{\partial}{\partial x_0} + i\frac{\partial}{\partial x_0}\frac{\partial}{\partial x_j}\right)\boldsymbol{\psi} + e\left(\frac{\partial}{\partial x_j}A_0 - \frac{\partial}{\partial x_0}A_j\right)\boldsymbol{\psi} = 0$$
$$\left(-\, i\frac{\partial}{\partial x_0}\frac{\partial}{\partial x_j} - i\frac{\partial}{\partial x_j}\frac{\partial}{\partial x_0}\right)\boldsymbol{\psi} + e\left(\frac{\partial}{\partial x_0}A_j - \frac{\partial}{\partial x_j}A_0\right)\boldsymbol{\psi} = 0$$

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This system is a generalization of the relativistic Schrödinger equation. The essence of the generalization consists not only in taking into account the spin of the electron, which takes place already in the Dirac equation, but also takes into account the non-commutability of the momentum components. It can be assumed that the solutions of this generalized system will give exact values for stationary electron energy levels in the atom, for which no radiative corrections will be needed.

If not to substitute the coordinate derivative instead of the momentum component and to remain in the framework of classical physics, the system of equations

$$(P_0 + eA_0)^2 - (P_1 + eA_1)^2 - (P_2 + eA_2)^2 - (P_3 + eA_3)^2 = m^2$$
$$\left(\partial_\mu P_\nu - \partial_\nu P_\mu\right) + e\left(\partial_\mu A_\nu - \partial_\nu A_\mu\right) = 0$$

describes the motion of a macroscopic charged particle in the presence of an electromagnetic or other potential field, for example, gravitational field. Let us note the nontrivial fact that even in classical physics, when an electric field acts on a charge, one should use the combination of derivatives $\frac{\partial P_i}{\partial x_0} - \frac{\partial P_0}{\partial x_i}$ instead of the simple acceleration $\frac{\partial P_i}{\partial x_0}$.

By means of the antisymmetric Levy-Civita symbol we transform antisymmetric tensors into dual tensors

$$\breve{F}^{\mu\nu} = \frac{1}{2}\varepsilon^{\mu\nu\rho\sigma}F_{\rho\sigma} \qquad \breve{G}^{\mu\nu} = \frac{1}{2}\varepsilon^{\mu\nu\rho\sigma}G_{\rho\sigma}$$

and we use Maxwell's equations written in compact form

$$\partial_\mu F^{\mu\nu} = j^\nu \qquad \partial_\mu \breve{F}^{\mu\nu} = 0$$

Let us apply the derivative operator to our proposed equations

$$\partial_\mu G^{\mu\nu} + e\partial_\mu F^{\mu\nu} = 0 \qquad \partial_\mu \breve{G}^{\mu\nu} + e\partial_\mu \breve{F}^{\mu\nu} = 0$$

then taking into account Maxwell's equations we obtain

$$\partial_\mu \breve{G}^{\mu\nu} = 0$$
$$\partial_\mu G^{\mu\nu} + ej^\nu = 0$$
$$\partial_\mu(\partial^\mu P^\nu - \partial^\nu P^\mu) + ej^\nu = 0$$
$$m\partial_\mu(\partial^\mu V^\nu - \partial^\nu V^\mu) + ej^\nu = 0$$
$$\frac{m}{e}\partial_\mu(\partial^\mu V^\nu - \partial^\nu V^\mu) = -j^\nu$$

These formulas describe the behavior of field and charge that results from their mutual influence.

If in the presence of an arbitrary potential there is no particle in the moving point, then our equations are the homogeneous Maxwell equations for an arbitrarily moving point. In the particular case of uniform motion, they transform into the ordinary Maxwell equations. If a charge is placed in the point, we obtain inhomogeneous Maxwell equations for an arbitrarily moving source.

The equations we propose can even be considered as a derivation of Maxwell's equations. Taking our equations as a basis and equating all derivatives of momentum to zero, we obtain as a residue exactly Maxwell's equations for a stationary or uniformly moving point.

The conditions expressed by the second line of our equations may be too strong, since they require that each pair of brackets with derivatives is zero. But invariance can also be achieved with a weaker requirement that only their sum as a whole is zero. That is, each pair of brackets can deviate from zero; the main thing is that these deviations are compensated in the total sum. This can work both in classical and quantum mechanics. A hint on the validity of this approach is given by Maxwell's equations, in which conditions are imposed not on individual derivatives, but on their sums. In addition, it is intuitively clear that the components having similarity to velocity should be considered in the sum in order not to depend on the rotations of the coordinate system.

3. Equation for the Spinor Coordinates Space

Let us consider the set of arbitrary complex numbers, for simplicity we will call it a vector

$$\mathfrak{X}^{\mathrm{T}} \equiv (\mathfrak{X}_0, \mathfrak{X}_1, \mathfrak{X}_2, \mathfrak{X}_3)$$

and let us consider arbitrary four-component complex spinors

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$$\mathbf{p}^{\mathbf{T}} \equiv (p_0, p_1, p_2, p_3)$$
$$\mathfrak{x}\mathbf{1}^{\mathbf{T}} \equiv (\mathfrak{x}1_0, \mathfrak{x}1_1, \mathfrak{x}1_2, \mathfrak{x}1_3)$$
$$\mathfrak{x}\mathbf{2}^{\mathbf{T}} \equiv (\mathfrak{x}2_0, \mathfrak{x}2_1, \mathfrak{x}2_2, \mathfrak{x}2_3)$$

Among all possible vectors, let us select a set of such vectors for which there is a representation of components through arbitrary complex spinors

$$\mathfrak{X}_\mu = \frac{1}{2}\mathfrak{x}\mathbf{1}^\dagger S_\mu \mathfrak{x}\mathbf{2}$$

and there is another way to calculate them

$$\mathfrak{X}_\mu = \frac{1}{2}Tr[\mathfrak{x}\mathbf{1}\mathfrak{x}\mathbf{2}^\dagger S_\mu]$$

Further we will assume that both spinors are identical, then the vector constructed from them is

$$\mathbf{P}^{\mathbf{T}} \equiv (P_0, P_1, P_2, P_3)$$

has real components, and we will assume that this is the electron momentum vector constructed from the complex momentum spinor **p**

$$P_\mu = \frac{1}{2}\mathbf{p}^\dagger S_\mu \mathbf{p}$$
$$P_\mu = \frac{1}{2}Tr[\mathbf{p}\mathbf{p}^\dagger S_\mu]$$

Consider the complex quantity

$$(\mathbf{p},\mathbf{x}) \equiv \mathbf{p}^T \Sigma_{MM} \mathbf{x} = (p_0, p_1, p_2, p_3)\begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}\begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{pmatrix} = (p_0, p_1, p_2, p_3)\begin{pmatrix} x_1 \\ -x_0 \\ x_3 \\ -x_2 \end{pmatrix}$$
$$= p_0x_1 - p_1x_0 + p_2x_3 - p_3x_2$$

where we introduce one more complex spinor, which in the future we will give the meaning of the complex coordinate spinor

$$\mathbf{x}^{\mathbf{T}} \equiv (x_0, x_1, x_2, x_3)$$

and

$$\Sigma_{MM} = \begin{pmatrix} \sigma_M & 0 \\ 0 & \sigma_M \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \quad \sigma_M = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

Coordinate vector of the four-dimensional Minkowski space

$$\mathbf{X}^{\mathbf{T}} \equiv (X_0, X_1, X_2, X_3)$$

is obtained from the coordinate spinor by the same formulas

$$X_\mu = \frac{1}{2}\mathbf{x}^\dagger S_\mu \mathbf{x}$$
$$X_\mu = \frac{1}{2}Tr[\mathbf{x}\mathbf{x}^\dagger S_\mu]$$

Thus, the vector in the Minkowski space is not a set of four arbitrary real numbers, but only such that are the specified bilinear combinations of components of completely arbitrary complex spinors

$$X_0 = \frac{1}{2}(\overline{x_0}x_0 + \overline{x_1}x_1 + \overline{x_2}x_2 + \overline{x_3}x_3)$$
$$X_1 = \frac{1}{2}(\overline{x_0}x_1 + \overline{x_1}x_0 + \overline{x_2}x_3 + \overline{x_3}x_2)$$
$$X_2 = \frac{1}{2}(-i\overline{x_0}x_1 + i\overline{x_1}x_0 - i\overline{x_2}x_3 + i\overline{x_3}x_2)$$
$$X_3 = \frac{1}{2}(\overline{x_0}x_0 - \overline{x_1}x_1 + \overline{x_2}x_2 - \overline{x_3}x_3)$$

Accordingly, the components of the vector in Minkowski space are interdependent, from this dependence automatically follow the relations of the special theory of relativity between space and time. For the same reason, the coordinates of Minkowski space cannot serve as independent variables in the equations. From the commutative properties of S_μ matrices, which are generators of rotations and boosts with respect to which the length of vectors is invariant, quantum mechanics automatically follows. Indeed, the commutation relations between the components of momenta are related to the noncommutativity of rotations in some way, and from them the commutation relations

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between the components of coordinates and momenta are directly deduced. And from these relations the differential equations are derived.

And since we do not doubt the truth of the theory of relativity and quantum mechanics, we cannot doubt the reality of spinor space, which by means of the simplest arithmetic operations generates our space and time.

The quantity $\mathbf{p}^{\boldsymbol{T}}\Sigma_{MM}\mathbf{x}$ is invariant under the Lorentz transformation simultaneously applied to the momentum and coordinate spinor, which automatically transforms both corresponding vectors as well

$$\mathbf{p}' = N\mathbf{p}$$
$$P'_\mu = \frac{1}{2}Tr[\mathbf{p}'\mathbf{p}'^\dagger S_\mu]$$
$$P'_\mu = \frac{1}{2}\mathbf{p}'^\dagger S_\mu \mathbf{p}'$$
$$\mathbf{P}' = \Lambda\mathbf{P}$$
$$\mathbf{x}' = N\mathbf{x}$$
$$X'_\mu = \frac{1}{2}Tr[\mathbf{x}'\mathbf{x}'^\dagger S_\mu]$$
$$X'_\mu = \frac{1}{2}\mathbf{x}'^\dagger S_\mu \mathbf{x}'$$
$$\mathbf{X}' = \Lambda\mathbf{X}$$

This quantity does not change for any combination of turns and boosts

$$\mathbf{p}'^{\boldsymbol{T}}\Sigma_{MM}\mathrm{x}' = \mathbf{p}^{\boldsymbol{T}}\Sigma_{MM}\mathbf{x}$$

Accordingly, the exponent

$$exp(\mathbf{p}^{\boldsymbol{T}}\Sigma_{MM}\mathbf{x}) = exp(p_0x_1 - p_1x_0 + p_2x_3 - p_3x_2)$$

characterizes the propagation process of a plane wave in spinor space with phase invariant to Lorentz transformations.

Let us apply the differential operator to the spinor analog of a plane wave

$$\left(\frac{\partial}{\partial x_1}\frac{\partial}{\partial x_2} - \frac{\partial}{\partial x_0}\frac{\partial}{\partial x_3}\right) exp(p_0x_1 - p_1x_0 + p_2x_3 - p_3x_2)$$
$$= (p_0(-p_3) - (-p_1)p_2)\, exp(p_0x_1 - p_1x_0 + p_2x_3 - p_3x_2) =$$
$$= (p_1p_2 - p_0p_3)\, exp(p_0x_1 - p_1x_0 + p_2x_3 - p_3x_2)$$

Applying this operator at another definition of the phase gives the same eigenvalue

$$\left(\frac{\partial}{\partial x_1}\frac{\partial}{\partial x_2} - \frac{\partial}{\partial x_0}\frac{\partial}{\partial x_3}\right) exp(p_0x_0 + p_1x_1 + p_2x_2 + p_3x_3) = (p_1p_2 - p_0p_3)\, exp(p_0x_0 + p_1x_1 + p_2x_2 + p_3x_3)$$

that is, two different eigenfunctions correspond to this eigenvalue, but in the second case the phase in the exponent is not invariant with respect to the Lorentz transformation, so we will use the first definition.

Since

$$(p_0, p_1)^{\mathrm{T}} \text{ and } (p_2, p_3)^{\mathrm{T}}$$

are complex spinors, which, under the transformation

$$\mathbf{p}' = N\mathbf{p} = \begin{pmatrix} n & 0 \\ 0 & n \end{pmatrix}\mathbf{p}$$

is affected by the same matrix n, then the complex quantity

$$m \equiv p_1p_2 - p_0p_3$$

is invariant under the action on the momentum spinor $\mathbf{p}$ of the transformation N. m is an eigenvalue of the differential operator, and the plane wave is the corresponding m eigenfunction, which is a solution of the equation

$$\left(\frac{\partial}{\partial x_1}\frac{\partial}{\partial x_2} - \frac{\partial}{\partial x_0}\frac{\partial}{\partial x_3}\right)\psi(x_0, x_1, x_2, x_3) = m\,\psi(x_0, x_1, x_2, x_3)$$

Here $\psi(x_0, x_1, x_2, x_3)$ denotes the complex function of complex spinor coordinates.

When substantiating the Schrödinger equation for a plane wave in four-dimensional vector space, an assumption is made (further confirmed in the experiment) about its applicability to an arbitrary wave function. Let us make a similar assumption about the applicability of the reduced spinor equation to an arbitrary function of spinor coordinates, that is, we will consider this equation as universal and valid for all physical processes.

Let us clarify that by the derivative on a complex variable from a complex function we here understand the derivative from an arbitrary stepped complex function using the formula that is valid at least for any integer degrees

$$\frac{\partial z^k}{\partial z} = kz^{k-1}$$

In particular, this is true for the exponential function, which is an infinite power series.

It is very important to emphasize that we consider the complex variable and the variable conjugate to it to be independent, so when finding the derivative of a complex variable from some function, we treat all the quantities which are conjugate to our variable and which are included in this function, as ordinary constants.

It is not by chance that we denote the eigenvalue by the symbol m, because if we form the momentum vector from the momentum spinor $\mathbf{p}$ included in the expression for the plane wave

$$P_\mu = \frac{1}{2}\mathbf{p}^\dagger S_\mu \mathbf{p}$$

then for the square of its length the following equality will be satisfied

$$P_0{}^2 - P_1{}^2 - P_2{}^2 - P_3{}^2 = \bar{m}m = m^2$$

That is the square of the modulus m has the sense of the square of the mass of a free particle, which is described by a plane wave in spinor space as well as by a plane wave in vector space. For the momentum spinor of a fermionic type particle having in the rest frame the following form

$$\mathbf{p}^T = (p_0, p_1, \overline{p_1}, -\overline{p_0})$$

quantity

$$m = p_1p_2 - p_0p_3 = p_1\overline{p_1} + p_0\overline{p_0}$$

is real and not equal to zero, and for the bosonic-type momentum spinor having in the rest frame the following form

$$\mathbf{p}^T = (p_0, p_1, p_0, p_1)$$

it is zero

$$m = p_1p_2 - p_0p_3 = p_1p_0 - p_0p_1 = 0$$

i.e., the boson satisfies the plane wave equation in spinor space with zero eigenvalue.

For the momentum spinor of a fermion-type particle we can consider another form in the rest system

$$\mathbf{p}^T = (p_0, p_1, -\overline{p_1}, \overline{p_0})$$

then the mass will be real and negative

$$m = p_1p_2 - p_0p_3 = -p_1\overline{p_1} - p_0\overline{p_0}$$

This particle with negative mass can be treated as an antiparticle, and in the rest frame its energy is equal to its mass modulo, but it is always positive

$$P_0 = \frac{1}{2}\mathbf{p}^\dagger S_0 \mathbf{p} = \frac{1}{2}(\overline{p_0}p_0 + \overline{p_1}p_1 + (-p_1)(-\overline{p_1}) + p_0\overline{p_0})$$
$$= \frac{1}{2}(\overline{p_0}p_0 + \overline{p_1}p_1 + p_1\overline{p_1} + p_0\overline{p_0})$$

To describe the behavior of an electron in the presence of an external electromagnetic field, it is common practice to add the electromagnetic potential vector to its momentum vector. We use the

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same approach at the spinor level and to each component of the momentum spinor of the electron we add the corresponding component of the electromagnetic potential spinor. For simplicity, the electron charge is equal to unity.

Further we need an expression for the commutation relation between the components of the momentum spinor, to which is added the corresponding component of the electromagnetic potential spinor, which is a function of the spinor coordinates

$$\big(p_0 + a_0(x_1,x_2)\big)\big(p_1 + a_1(x_1,x_2)\big) - \big(p_1 + a_1(x_1,x_2)\big)\big(p_0 + a_0(x_1,x_2)\big)$$

Let us replace the momenta by differential operators

$$p_0 \to \frac{\partial}{\partial x_1} \qquad p_1 \to -\frac{\partial}{\partial x_0} \qquad p_2 \to \frac{\partial}{\partial x_3} \qquad p_3 \to -\frac{\partial}{\partial x_2}$$

and find the commutation relation

$$\left\{\left(\frac{\partial}{\partial x_1} + a_0(x_0,x_1,x_2,x_3)\right)\left(-\frac{\partial}{\partial x_0} + a_1(x_0,x_1,x_2,x_3)\right) - \left(-\frac{\partial}{\partial x_0} + a_1(x_0,x_1,x_2,x_3)\right)\left(\frac{\partial}{\partial x_1} + a_0\big((x_0,x_1,x_2,x_3)\big)\right)\right\}\psi(x_0,x_1,x_2,x_3)$$
$$= \frac{\partial}{\partial x_1}(a_1\psi) - a_0\frac{\partial\psi}{\partial x_0} + \frac{\partial}{\partial x_0}(a_0\psi) - a_1\frac{\partial\psi}{\partial x_1}$$
$$= \frac{\partial a_1}{\partial x_1}\psi + a_1\frac{\partial\psi}{\partial x_1} - a_0\frac{\partial\psi}{\partial x_0} + \frac{\partial a_0}{\partial x_0}\psi + a_0\frac{\partial\psi}{\partial x_0} - a_1\frac{\partial\psi}{\partial x_1} = \frac{\partial a_1}{\partial x_1}\psi + \frac{\partial a_0}{\partial x_0}\psi$$
$$= \left\{\frac{\partial a_1(x_0,x_1,x_2,x_3)}{\partial x_1} + \frac{\partial a_0(x_0,x_1,x_2,x_3)}{\partial x_0}\right\}\psi(x_0,x_1,x_2,x_3)$$

Thus

$$(p_0 + a_0)(p_1 + a_1) - (p_1 + a_1)(p_0 + a_0) = \frac{\partial a_1}{\partial x_1} + \frac{\partial a_0}{\partial x_0}$$

Let us apply the proposed equation to analyze the wave function of the electron in a centrally symmetric electric field, this model is used to describe the hydrogen-like atom. For the components of the vector potential of a centrally symmetric electric field it is true that

$$A_0 = \frac{1}{2}\mathbf{a}^\dagger S_0\mathbf{a} = \frac{1}{2}(\overline{a_0}a_0 + \overline{a_1}a_1 + \overline{a_2}a_2 + \overline{a_3}a_3) = \frac{1}{R}$$

$$A_1 = \frac{1}{2}\mathbf{a}^\dagger S_1\mathbf{a} = \frac{1}{2}(\overline{a_0}a_1 + \overline{a_1}a_0 + \overline{a_2}a_3 + \overline{a_3}a_2) = 0$$

$$A_2 = \frac{1}{2}\mathbf{a}^\dagger S_2\mathbf{a} = \frac{1}{2}(-i\overline{a_0}a_1 + i\overline{a_1}a_0 - i\overline{a_2}a_3 + i\overline{a_3}a_2) = 0$$

$$A_3 = \frac{1}{2}\mathbf{a}^\dagger S_3\mathbf{a} = \frac{1}{2}(\overline{a_0}a_0 - \overline{a_1}a_1 + \overline{a_2}a_2 - \overline{a_3}a_3) = 0$$

$$\overline{a_0}a_0 + \overline{a_2}a_2 = \overline{a_1}a_1 + \overline{a_3}a_3$$

$$\overline{a_0}a_0 + \overline{a_2}a_2 = \frac{1}{R}$$

$$\overline{a_0}a_1 + \overline{a_2}a_3 = \overline{a_1}a_0 + \overline{a_3}a_2$$

$$\frac{1}{2}(\overline{a_0}a_1 + \overline{a_1}a_0 + \overline{a_2}a_3 + \overline{a_3}a_2) = \overline{a_0}a_1 + \overline{a_2}a_3 = 0$$

$$\overline{a_0}a_1 = -\overline{a_2}a_3$$
$$\overline{a_0} = i\overline{a_2}$$
$$a_0 = -ia_2$$

$$\overline{a_0}a_0 + \overline{a_2}a_2 = i\overline{a_2} * (-ia_2) + \overline{a_2}a_2 = 2\overline{a_2}a_2 = 2{a_2}^2 = \frac{1}{R}$$

As a result, it is possible to accept

$$a_0 = -\frac{i}{\sqrt{2R}} \qquad a_1 = \frac{1}{\sqrt{2R}} \qquad a_2 = \frac{1}{\sqrt{2R}} \qquad a_3 = -\frac{i}{\sqrt{2R}}$$

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$$\overline{a_0}a_1 = i\frac{1}{\sqrt{2R}}\frac{1}{\sqrt{2R}} = \frac{i}{2R}$$

$$\overline{a_2}a_3 = \frac{1}{\sqrt{2R}}\left(-i\frac{1}{\sqrt{2R}}\right) = -\frac{i}{2R}$$

$$R = \sqrt{X_1{}^2 + X_2{}^2 + X_3{}^2} =$$

$$\sqrt{\left(\frac{1}{2}(\overline{x_0}x_1 + \overline{x_1}x_0 + \overline{x_2}x_3 + \overline{x_3}x_2)\right)^2 + \left(\frac{1}{2}(-i\overline{x_0}x_1 + i\overline{x_1}x_0 - i\overline{x_2}x_3 + i\overline{x_3}x_2)\right)^2 + \left(\frac{1}{2}(\overline{x_0}x_0 - \overline{x_1}x_1 + \overline{x_2}x_2 - \overline{x_3}x_3)\right)^2}$$

$$=$$

$$\sqrt{\left(\frac{1}{2}(\overline{x_0}x_1 + \overline{x_1}x_0 + \overline{x_2}x_3 + \overline{x_3}x_2)\right)^2 - \left(\frac{1}{2}(-\overline{x_0}x_1 + \overline{x_1}x_0 - \overline{x_2}x_3 + \overline{x_3}x_2)\right)^2 + \left(\frac{1}{2}(\overline{x_0}x_0 - \overline{x_1}x_1 + \overline{x_2}x_2 - \overline{x_3}x_3)\right)^2}$$

We are looking for a solution of the spinor equation; we do not consider the electron's spin yet

$$\left(\frac{\partial}{\partial x_1}\frac{\partial}{\partial x_2} - \frac{\partial}{\partial x_0}\frac{\partial}{\partial x_3}\right)\varphi(x_0, x_1, x_2, x_3) = m\,\varphi(x_0, x_1, x_2, x_3)$$

This equation can be interpreted in another way. Let us take the invariant expression

$$(p_1 p_2 - p_0 p_3) = m$$

And let's do the substitution

$$p_0 \to \frac{\partial}{\partial x_1} + a_0(x_0, x_1, x_2, x_3) \qquad p_1 \to -\frac{\partial}{\partial x_0} + a_1(x_0, x_1, x_2, x_3)$$

$$p_2 \to \frac{\partial}{\partial x_3} + a_2(x_0, x_1, x_2, x_3) \qquad p_3 \to -\frac{\partial}{\partial x_2} + a_3(x_0, x_1, x_2, x_3)$$

$$\left\{\left(-\frac{\partial}{\partial x_0} + a_1\right)\left(\frac{\partial}{\partial x_3} + a_2\right) - \left(\frac{\partial}{\partial x_1} + a_0\right)\left(-\frac{\partial}{\partial x_2} + a_3\right)\right\}\varphi = m\varphi$$

We will consider this equation as an equation for determining the eigenvalues of m and the corresponding eigenfunctions

$$-\frac{\partial}{\partial x_0}\frac{\partial}{\partial x_3}\varphi + \frac{\partial}{\partial x_1}\frac{\partial}{\partial x_2}\varphi + \left(-\frac{\partial a_2}{\partial x_0} - \frac{\partial a_3}{\partial x_1}\right)\varphi - a_2\frac{\partial \varphi}{\partial x_0} + a_1\frac{\partial \varphi}{\partial x_3} - a_3\frac{\partial \varphi}{\partial x_1} + a_0\frac{\partial \varphi}{\partial x_2} + (a_1 a_2 - a_0 a_3)\varphi$$
$$= m\varphi$$

$$a_0 = -\frac{i}{\sqrt{2R}} \qquad a_1 = \frac{1}{\sqrt{2R}} \qquad a_2 = \frac{1}{\sqrt{2R}} \qquad a_3 = -\frac{i}{\sqrt{2R}}$$

$$a_1 a_2 - a_0 a_3 = \frac{1}{2R} + \frac{1}{2R} = \frac{1}{R}$$

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$$-\frac{\partial a_2}{\partial x_0}-\frac{\partial a_3}{\partial x_1}=-\frac{1}{\sqrt{2}}\frac{\partial}{\partial x_0}\left(\frac{1}{\sqrt{R}}\right)+i\frac{1}{\sqrt{2}}\frac{\partial}{\partial x_1}\left(\frac{1}{\sqrt{R}}\right)=-\frac{1}{\sqrt{2}}\frac{\partial}{\partial x_0}\left(\frac{1}{\sqrt[4]{R^2}}\right)+i\frac{1}{\sqrt{2}}\frac{\partial}{\partial x_1}\left(\frac{1}{\sqrt[4]{R^2}}\right)$$

$$=-\frac{1}{\sqrt{2}}\left(-\frac{1}{4}\frac{1}{(R^2)^{\frac{5}{4}}}\right)\frac{\partial}{\partial x_0}(R^2)+i\frac{1}{\sqrt{2}}\left(-\frac{1}{4}\frac{1}{(R^2)^{\frac{5}{4}}}\right)\frac{\partial}{\partial x_1}(R^2)=$$

$$=\frac{1}{\sqrt{2}}\left(\frac{1}{4}\frac{1}{(R^2)^{\frac{5}{4}}}\right)\left[\frac{\partial}{\partial x_0}(R^2)-i\frac{\partial}{\partial x_1}(R^2)\right]=\frac{1}{\left(\sqrt{2}R\right)^5}\left[\frac{\partial}{\partial x_0}(R^2)-i\frac{\partial}{\partial x_1}(R^2)\right]$$

$$R=\sqrt{{X_1}^2+{X_2}^2+{X_3}^2}=$$

$$\sqrt{\left(\frac{1}{2}(\overline{x_0}x_1+\overline{x_1}x_0+\overline{x_2}x_3+\overline{x_3}x_2)\right)^2-\left(\frac{1}{2}(-\overline{x_0}x_1+\overline{x_1}x_0-\overline{x_2}x_3+\overline{x_3}x_2)\right)^2+\left(\frac{1}{2}(\overline{x_0}x_0-\overline{x_1}x_1+\overline{x_2}x_2-\overline{x_3}x_3)\right)^2}$$

$$\frac{\partial}{\partial x_0}(R^2)=\frac{\partial}{\partial x_0}\left(\left(\frac{1}{2}(\overline{x_0}x_1+\overline{x_1}x_0+\overline{x_2}x_3+\overline{x_3}x_2)\right)^2-\left(\frac{1}{2}(-\overline{x_0}x_1+\overline{x_1}x_0-\overline{x_2}x_3+\overline{x_3}x_2)\right)^2\right.$$
$$\left.+\left(\frac{1}{2}(\overline{x_0}x_0-\overline{x_1}x_1+\overline{x_2}x_2-\overline{x_3}x_3)\right)^2\right)$$
$$=\frac{1}{4}\left(2(\overline{x_0}x_1+\overline{x_1}x_0+\overline{x_2}x_3+\overline{x_3}x_2)\frac{\partial}{\partial x_0}(\overline{x_0}x_1+\overline{x_1}x_0+\overline{x_2}x_3+\overline{x_3}x_2)\right.$$
$$-2(-\overline{x_0}x_1+\overline{x_1}x_0-\overline{x_2}x_3+\overline{x_3}x_2)\frac{\partial}{\partial x_0}(-\overline{x_0}x_1+\overline{x_1}x_0-\overline{x_2}x_3+\overline{x_3}x_2)$$
$$\left.+2(\overline{x_0}x_0-\overline{x_1}x_1+\overline{x_2}x_2-\overline{x_3}x_3)\frac{\partial}{\partial x_0}(\overline{x_0}x_0-\overline{x_1}x_1+\overline{x_2}x_2-\overline{x_3}x_3)\right)$$
$$=\frac{1}{4}(2(\overline{x_0}x_1+\overline{x_1}x_0+\overline{x_2}x_3+\overline{x_3}x_2)\overline{x_1}-2(-\overline{x_0}x_1+\overline{x_1}x_0-\overline{x_2}x_3+\overline{x_3}x_2)\overline{x_1}$$
$$+2(\overline{x_0}x_0-\overline{x_1}x_1+\overline{x_2}x_2-\overline{x_3}x_3)\overline{x_0})$$
$$=\frac{1}{2}\left((\overline{x_0}x_1+\overline{x_1}x_0+\overline{x_2}x_3+\overline{x_3}x_2)\overline{x_1}-(-\overline{x_0}x_1+\overline{x_1}x_0-\overline{x_2}x_3+\overline{x_3}x_2)\overline{x_1}\right.$$
$$\left.+(\overline{x_0}x_0-\overline{x_1}x_1+\overline{x_2}x_2-\overline{x_3}x_3)\overline{x_0}\right)$$
$$=\frac{1}{2}\left((\overline{x_0}x_1+\overline{x_2}x_3)\overline{x_1}-(-\overline{x_0}x_1-\overline{x_2}x_3)\overline{x_1}+(\overline{x_0}x_0-\overline{x_1}x_1+\overline{x_2}x_2-\overline{x_3}x_3)\overline{x_0}\right)$$
$$=\frac{1}{2}\left((\overline{x_0}x_1+\overline{x_2}x_3)\overline{x_1}+(\overline{x_0}x_1+\overline{x_2}x_3)\overline{x_1}+(\overline{x_0}x_0-\overline{x_1}x_1+\overline{x_2}x_2-\overline{x_3}x_3)\overline{x_0}\right)$$
$$=\frac{1}{2}\left((\overline{x_0}x_1+\overline{x_2}x_3)\overline{x_1}+(\overline{x_2}x_3)\overline{x_1}+(\overline{x_0}x_0+\overline{x_2}x_2-\overline{x_3}x_3)\overline{x_0}\right)$$
$$=\frac{1}{2}(\overline{x_0}x_1\overline{x_1}+2\overline{x_2}x_3\overline{x_1}+(\overline{x_0}x_0+\overline{x_2}x_2-\overline{x_3}x_3)\overline{x_0})$$
$$=\frac{1}{2}(2\overline{x_2}x_3\overline{x_1}-2\overline{x_3}x_3\overline{x_0}+(\overline{x_0}x_0+x_1\overline{x_1}+\overline{x_2}x_2+\overline{x_3}x_3)\overline{x_0})$$
$$=\frac{1}{2}(2x_3(\overline{x_2}\overline{x_1}-\overline{x_3}\overline{x_0})+(\overline{x_0}x_0+x_1\overline{x_1}+\overline{x_2}x_2+\overline{x_3}x_3)\overline{x_0})$$
$$=\frac{1}{2}(2x_3(\overline{x_2}\overline{x_1}-\overline{x_3}\overline{x_0})+(\overline{x_0}x_0+x_1\overline{x_1}+\overline{x_2}x_2+\overline{x_3}x_3)\overline{x_0})$$

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$$\frac{\partial}{\partial x_1}(R^2)=\frac{1}{2}\Big((\overline{x_0}x_1+\overline{x_1}x_0+\overline{x_2}x_3+\overline{x_3}x_2)\overline{x_0}+(-\overline{x_0}x_1+\overline{x_1}x_0-\overline{x_2}x_3+\overline{x_3}x_2)\overline{x_0}$$
$$-(\overline{x_0}x_0-\overline{x_1}x_1+\overline{x_2}x_2-\overline{x_3}x_3)\overline{x_1}\Big)$$
$$=\frac{1}{2}\Big((\overline{x_1}x_0+\overline{x_3}x_2)\overline{x_0}+(\overline{x_1}x_0+\overline{x_3}x_2)\overline{x_0}-(\overline{x_0}x_0-\overline{x_1}x_1+\overline{x_2}x_2-\overline{x_3}x_3)\overline{x_1}\Big)$$
$$=\frac{1}{2}(\overline{x_1}x_0\overline{x_0}+2\overline{x_3}x_2\overline{x_0}+(\overline{x_1}x_1-\overline{x_2}x_2+\overline{x_3}x_3)\overline{x_1})$$
$$=\frac{1}{2}(2x_2(\overline{x_3x_0}-\overline{x_2x_1})+(x_0\overline{x_0}+\overline{x_1}x_1+\overline{x_2}x_2+\overline{x_3}x_3)\overline{x_1})$$

Let's introduce the notations

$$\overline{x_2}\overline{x_1}-\overline{x_3}\overline{x_0}\equiv l$$

this quantity does not change under rotations and boosts and is some analog of the interval defined for Minkowski space and

$$\frac{1}{2}(x_0\overline{x_0}+\overline{x_1}x_1+\overline{x_2}x_2+\overline{x_3}x_3)\equiv t$$

this quantity represents time in four-dimensional vector space.

An interesting fact is that time is always a positive quantity. As an assumption it can be noted that since we observe that time value goes forward, i.e. the value of t grows, and it is possible only due to scaling of all components of spinor space, such scaling leads to increase of distance between any two points of Minkowski space. As a result, with the passage of time the Minkowski space should expand, herewith at first relatively quickly, and then more and more slowly.

$$\left[\frac{\partial}{\partial x_0}(R^2)-i\frac{\partial}{\partial x_1}(R^2)\right]$$
$$=\frac{1}{2}(2x_3(\overline{x_2x_1}-\overline{x_3x_0})+(\overline{x_0}x_0+x_1\overline{x_1}+\overline{x_2}x_2+\overline{x_3}x_3)\overline{x_0})$$
$$-i\frac{1}{2}(2x_2(\overline{x_3x_0}-\overline{x_2x_1})+(x_0\overline{x_0}+\overline{x_1}x_1+\overline{x_2}x_2+\overline{x_3}x_3)\overline{x_1})$$
$$=x_3(\overline{x_2x_1}-\overline{x_3x_0})+\frac{1}{2}(\overline{x_0}x_0+x_1\overline{x_1}+\overline{x_2}x_2+\overline{x_3}x_3)\overline{x_0}-ix_2(\overline{x_3x_0}-\overline{x_2x_1})$$
$$-i\frac{1}{2}(x_0\overline{x_0}+\overline{x_1}x_1+\overline{x_2}x_2+\overline{x_3}x_3)\overline{x_1}=x_3l+t\overline{x_0}+ix_2l-it\overline{x_1}$$
$$=l(x_3+ix_2)+t(\overline{x_0}-i\overline{x_1})$$

As a result, we have an equation for determining the eigenvalues of m and their corresponding eigenfunctions $\varphi(x_0,x_1,x_2,x_3)$

$$\left(-\frac{\partial}{\partial x_0}\frac{\partial}{\partial x_3}+\frac{\partial}{\partial x_1}\frac{\partial}{\partial x_2}\right)\varphi+\frac{1}{\sqrt{2R}}\left(-\frac{\partial\varphi}{\partial x_0}+\frac{\partial\varphi}{\partial x_3}+i\frac{\partial\varphi}{\partial x_1}-i\frac{\partial\varphi}{\partial x_2}\right)+\frac{1}{\left(\sqrt{2R}\right)^5}\big(l(x_3+ix_2)+t(\overline{x_0}-i\overline{x_1})\big)\varphi$$
$$+\frac{1}{R}\varphi=m\varphi$$

Instead of looking for solutions to this equation directly, we can first try substituting already known solutions to the Schrödinger equation for the hydrogen-like atom. If $\varphi(X_0,X_1,X_2,X_3)$ is one of these solutions, we need to find its derivatives over all spinor components

$$\frac{\partial\varphi}{\partial x_\mu}=\frac{\partial\varphi}{\partial X_\nu}\frac{\partial X_\nu}{\partial x_\mu}$$

$$X_0=\frac{1}{2}(\overline{x_0}x_0+\overline{x_1}x_1+\overline{x_2}x_2+\overline{x_3}x_3)$$

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$$X_1 = \frac{1}{2}(\overline{x_0}x_1 + \overline{x_1}x_0 + \overline{x_2}x_3 + \overline{x_3}x_2)$$

$$X_2 = \frac{1}{2}(-i\overline{x_0}x_1 + i\overline{x_1}x_0 - i\overline{x_2}x_3 + i\overline{x_3}x_2)$$

$$X_3 = \frac{1}{2}(\overline{x_0}x_0 - \overline{x_1}x_1 + \overline{x_2}x_2 - \overline{x_3}x_3)$$

For example

$$\frac{\partial\varphi}{\partial x_0} = \frac{\partial\varphi}{\partial X_0}\frac{\overline{x_0}}{2} + \frac{\partial\varphi}{\partial X_1}\frac{\overline{x_1}}{2} + \frac{\partial\varphi}{\partial X_2}\frac{i\overline{x_1}}{2} + \frac{\partial\varphi}{\partial X_3}\frac{\overline{x_0}}{2}$$

Let's pay attention to the shift in priorities. In the Schrödinger equation one looks for energy eigenvalues, while here it is proposed to look for mass eigenvalues, it seem more natural to us. The mass of a free particle is an invariant of the Lorentz transformations, and in the bound state the mass of the particle has a discrete series of allowed values, each of which corresponds to an energy eigenvalue, and the eigenfunction of these eigenvalues is the same. But these energy eigenvalues are not the same as the energy eigenvalues of the Schrödinger equation, because the equations are different. When an electron absorbs a photon, their spinors sum up and the mass of the electron changes. If the new mass coincides with some allowed value, the electron enters a new state. The kay idea here is the assumption that the interaction of spinors occurs simply by summing them.

The advantages of considering physical processes in spinor coordinate space may not be limited to electrodynamics. It may turn out, for example, that the spinor space is not subject to curvature under the influence of matter, as it takes place in the general theory of relativity for the vector coordinate space. On the contrary, it can be assumed that it is when the components of vector coordinate space are computed from the coordinate spinor that the momentum spinor with a multiplier of the order of the gravitational constant is added to this spinor. This results in a warp that affects other massive bodies.

To account for the electron spin, we will further represent the electron wave function as a four-component spinor function of four-component spinor coordinates

$$\boldsymbol{\psi}(x_0,x_1,x_2,x_3) = \begin{pmatrix}\psi_0(x_0,x_1,x_2,x_3)\\ \psi_1(x_0,x_1,x_2,x_3)\\ \psi_2(x_0,x_1,x_2,x_3)\\ \psi_3(x_0,x_1,x_2,x_3)\end{pmatrix} = \begin{pmatrix}u_0\\ u_1\\ u_2\\ u_3\end{pmatrix}\varphi(x_0,x_1,x_2,x_3)$$

where the coefficients u_μ are complex quantities independent of coordinates. In fact, as shown at the end of the paper, the wave function is a linear combination of such right-hand sides with operator coefficients.

We will search for the solution of the wave equation considered in the first part of this paper

$$(S_0P_0 - S_1P_1 - S_2P_2 - S_3P_3)(S_0P_0 + S_1P_1 + S_2P_2 + S_3P_3)\boldsymbol{\psi} = M^2\boldsymbol{\psi}$$

Let's express the left part through the components of the momentum spinor

$$P_\mu = \frac{1}{2}\mathbf{p}^\dagger S_\mu \mathbf{p}$$

$$P_0 = \frac{1}{2}\mathbf{p}^\dagger S_0 \mathbf{p} = \frac{1}{2}(\overline{p_0},\overline{p_1},\overline{p_2},\overline{p_3})\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}\begin{pmatrix}p_0\\p_1\\p_2\\p_3\end{pmatrix} = \frac{1}{2}(\overline{p_0},\overline{p_1},\overline{p_2},\overline{p_3})\begin{pmatrix}p_0\\p_1\\p_2\\p_3\end{pmatrix}$$

$$= \frac{1}{2}(\overline{p_0}p_0 + \overline{p_1}p_1 + \overline{p_2}p_2 + \overline{p_3}p_3)$$

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$$P_1 = \frac{1}{2}\mathbf{p}^\dagger S_1 \mathbf{p} = \frac{1}{2}(\overline{p_0}, \overline{p_1}, \overline{p_2}, \overline{p_3})\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} p_0 \\ p_1 \\ p_2 \\ p_3 \end{pmatrix} = \frac{1}{2}(\overline{p_0}, \overline{p_1}, \overline{p_2}, \overline{p_3})\begin{pmatrix} p_1 \\ p_0 \\ p_3 \\ p_2 \end{pmatrix}$$

$$= \frac{1}{2}(\overline{p_0}p_1 + \overline{p_1}p_0 + \overline{p_2}p_3 + \overline{p_3}p_2)$$

$$P_2 = \frac{1}{2}\mathbf{p}^\dagger S_2 \mathbf{p} = \frac{1}{2}(\overline{p_0}, \overline{p_1}, \overline{p_2}, \overline{p_3})\begin{pmatrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{pmatrix}\begin{pmatrix} p_0 \\ p_1 \\ p_2 \\ p_3 \end{pmatrix} = \frac{1}{2}(\overline{p_0}, \overline{p_1}, \overline{p_2}, \overline{p_3})\begin{pmatrix} -ip_1 \\ ip_0 \\ -ip_3 \\ ip_2 \end{pmatrix}$$

$$= \frac{1}{2}(-i\overline{p_0}p_1 + i\overline{p_1}p_0 - i\overline{p_2}p_3 + i\overline{p_3}p_2)$$

$$P_3 = \frac{1}{2}\mathbf{p}^\dagger S_3 \mathbf{p} = \frac{1}{2}(\overline{p_0}, \overline{p_1}, \overline{p_2}, \overline{p_3})\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}\begin{pmatrix} p_0 \\ p_1 \\ p_2 \\ p_3 \end{pmatrix} = \frac{1}{2}(\overline{p_0}, \overline{p_1}, \overline{p_2}, \overline{p_3})\begin{pmatrix} p_0 \\ -p_1 \\ p_2 \\ -p_3 \end{pmatrix}$$

$$= \frac{1}{2}(\overline{p_0}p_0 - \overline{p_1}p_1 + \overline{p_2}p_2 - \overline{p_3}p_3)$$

$$P_0 - P_3 = \frac{1}{2}(\overline{p_0}p_0 + \overline{p_1}p_1 + \overline{p_2}p_2 + \overline{p_3}p_3) - \frac{1}{2}(\overline{p_0}p_0 - \overline{p_1}p_1 + \overline{p_2}p_2 - \overline{p_3}p_3)$$

$$= \frac{1}{2}(\overline{p_0}p_0 + \overline{p_1}p_1 + \overline{p_2}p_2 + \overline{p_3}p_3 - \overline{p_0}p_0 + \overline{p_1}p_1 - \overline{p_2}p_2 + \overline{p_3}p_3)$$

$$= \frac{1}{2}(\overline{p_1}p_1 + \overline{p_3}p_3 + \overline{p_1}p_1 + \overline{p_3}p_3) = \overline{p_1}p_1 + \overline{p_3}p_3$$

$$P_0 + P_3 = \frac{1}{2}(\overline{p_0}p_0 + \overline{p_1}p_1 + \overline{p_2}p_2 + \overline{p_3}p_3) + \frac{1}{2}(\overline{p_0}p_0 - \overline{p_1}p_1 + \overline{p_2}p_2 - \overline{p_3}p_3) = \overline{p_0}p_0 + \overline{p_2}p_2$$

$$-P_1 + iP_2 = -\frac{1}{2}(\overline{p_0}p_1 + \overline{p_1}p_0 + \overline{p_2}p_3 + \overline{p_3}p_2) + i\frac{1}{2}(-i\overline{p_0}p_1 + i\overline{p_1}p_0 - i\overline{p_2}p_3 + i\overline{p_3}p_2)$$

$$= \frac{1}{2}(\overline{p_0}p_1 + \overline{p_1}p_0 + \overline{p_2}p_3 + \overline{p_3}p_2 + \overline{p_0}p_1 - \overline{p_1}p_0 + \overline{p_2}p_3 - \overline{p_3}p_2)$$

$$= \frac{1}{2}(\overline{p_0}p_1 + \overline{p_2}p_3 + \overline{p_0}p_1 + \overline{p_2}p_3) = \overline{p_0}p_1 + \overline{p_2}p_3$$

$$-P_1 - iP_2 = -\frac{1}{2}(\overline{p_0}p_1 + \overline{p_1}p_0 + \overline{p_2}p_3 + \overline{p_3}p_2) - i\frac{1}{2}(-i\overline{p_0}p_1 + i\overline{p_1}p_0 - i\overline{p_2}p_3 + i\overline{p_3}p_2)$$

$$= \frac{1}{2}(\overline{p_0}p_1 + \overline{p_1}p_0 + \overline{p_2}p_3 + \overline{p_3}p_2 - \overline{p_0}p_1 + \overline{p_1}p_0 - \overline{p_2}p_3 + \overline{p_3}p_2)$$

$$= \frac{1}{2}(\overline{p_1}p_0 + \overline{p_3}p_2 + \overline{p_1}p_0 + \overline{p_3}p_2) = \overline{p_1}p_0 + \overline{p_3}p_2$$

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$$S_0P_0 - S_1P_1 - S_2P_2 - S_3P_3$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} P_0 - \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} P_1 - \begin{pmatrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{pmatrix} P_2$$

$$- \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} P_3 = \begin{pmatrix} P_0 - P_3 & -P_1 + iP_2 & 0 & 0 \\ -P_1 - iP_2 & P_0 + P_3 & 0 & 0 \\ 0 & 0 & P_0 - P_3 & -P_1 + iP_2 \\ 0 & 0 & -P_1 - iP_2 & P_0 + P_3 \end{pmatrix} =$$

$$= \begin{pmatrix} \overline{p_1}p_1 + \overline{p_3}p_3 & \overline{p_0}p_1 + \overline{p_2}p_3 & 0 & 0 \\ \overline{p_1}p_0 + \overline{p_3}p_2 & \overline{p_0}p_0 + \overline{p_2}p_2 & 0 & 0 \\ 0 & 0 & \overline{p_1}p_1 + \overline{p_3}p_3 & \overline{p_0}p_1 + \overline{p_2}p_3 \\ 0 & 0 & \overline{p_1}p_0 + \overline{p_3}p_2 & \overline{p_0}p_0 + \overline{p_2}p_2 \end{pmatrix}$$

$$S_0P_0 + S_1P_1 + S_2P_2 + S_3P_3$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} P_0 + \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} P_1 + \begin{pmatrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{pmatrix} P_2$$

$$+ \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} P_3 = \begin{pmatrix} P_0 + P_3 & P_1 - iP_2 & 0 & 0 \\ P_1 + iP_2 & P_0 - P_3 & 0 & 0 \\ 0 & 0 & P_0 + P_3 & P_1 - iP_2 \\ 0 & 0 & P_1 + iP_2 & P_0 - P_3 \end{pmatrix}$$

$$= \begin{pmatrix} \overline{p_0}p_0 + \overline{p_2}p_2 & -\overline{p_0}p_1 - \overline{p_2}p_3 & 0 & 0 \\ -\overline{p_1}p_0 - \overline{p_3}p_2 & \overline{p_1}p_1 + \overline{p_3}p_3 & 0 & 0 \\ 0 & 0 & \overline{p_0}p_0 + \overline{p_2}p_2 & -\overline{p_0}p_1 - \overline{p_2}p_3 \\ 0 & 0 & -\overline{p_1}p_0 - \overline{p_3}p_2 & \overline{p_1}p_1 + \overline{p_3}p_3 \end{pmatrix}$$

Let's distinguish the direct products of vectors in these matrices

$$S_0P_0 + S_1P_1 + S_2P_2 + S_3P_3 = \begin{pmatrix} \overline{p_0}p_0 + \overline{p_2}p_2 & -\overline{p_0}p_1 - \overline{p_2}p_3 & 0 & 0 \\ -\overline{p_1}p_0 - \overline{p_3}p_2 & \overline{p_1}p_1 + \overline{p_3}p_3 & 0 & 0 \\ 0 & 0 & P_0 + P_3 & P_1 - iP_2 \\ 0 & 0 & P_1 + iP_2 & P_0 - P_3 \end{pmatrix}$$

$$= \begin{pmatrix} \overline{p_0}p_0 & -\overline{p_0}p_1 & 0 & 0 \\ -\overline{p_1}p_0 & \overline{p_1}p_1 & 0 & 0 \\ 0 & 0 & \overline{p_0}p_0 & -\overline{p_0}p_1 \\ 0 & 0 & -\overline{p_1}p_0 & \overline{p_1}p_1 \end{pmatrix} + \begin{pmatrix} \overline{p_2}p_2 & -\overline{p_2}p_3 & 0 & 0 \\ -\overline{p_3}p_2 & \overline{p_3}p_3 & 0 & 0 \\ 0 & 0 & \overline{p_2}p_2 & -\overline{p_2}p_3 \\ 0 & 0 & -\overline{p_3}p_2 & \overline{p_3}p_3 \end{pmatrix}$$

$$= \begin{pmatrix} -\overline{p_0} \\ \overline{p_1} \\ 0 \\ 0 \end{pmatrix} (-p_0, p_1, 0,0) + \begin{pmatrix} 0 \\ 0 \\ -\overline{p_0} \\ \overline{p_1} \end{pmatrix} (0,0,-p_0,p_1) + \begin{pmatrix} -\overline{p_2} \\ \overline{p_3} \\ 0 \\ 0 \end{pmatrix} (-p_2, p_3, 0,0) + \begin{pmatrix} 0 \\ 0 \\ -\overline{p_2} \\ \overline{p_3} \end{pmatrix} (0,0,-p_2,p_3)$$

$$S_0P_0 - S_1P_1 - S_2P_2 - S_3P_3 = \begin{pmatrix} \overline{p_1}p_1 + \overline{p_3}p_3 & \overline{p_0}p_1 + \overline{p_2}p_3 & 0 & 0 \\ \overline{p_1}p_0 + \overline{p_3}p_2 & \overline{p_0}p_0 + \overline{p_2}p_2 & 0 & 0 \\ 0 & 0 & \overline{p_1}p_1 + \overline{p_3}p_3 & \overline{p_0}p_1 + \overline{p_2}p_3 \\ 0 & 0 & \overline{p_1}p_0 + \overline{p_3}p_2 & \overline{p_0}p_0 + \overline{p_2}p_2 \end{pmatrix}$$

$$= \begin{pmatrix} \overline{p_1}p_1 & \overline{p_0}p_1 & 0 & 0 \\ \overline{p_1}p_0 & \overline{p_0}p_0 & 0 & 0 \\ 0 & 0 & \overline{p_1}p_1 & \overline{p_0}p_1 \\ 0 & 0 & \overline{p_1}p_0 & \overline{p_0}p_0 \end{pmatrix} + \begin{pmatrix} \overline{p_3}p_3 & \overline{p_2}p_3 & 0 & 0 \\ \overline{p_3}p_2 & \overline{p_2}p_2 & 0 & 0 \\ 0 & 0 & \overline{p_3}p_3 & \overline{p_2}p_3 \\ 0 & 0 & \overline{p_3}p_2 & \overline{p_2}p_2 \end{pmatrix}$$

$$= \begin{pmatrix} p_1\overline{p_1} - [p_1\overline{p_1} - \overline{p_1}p_1] & p_1\overline{p_0} - [p_1\overline{p_0} - \overline{p_0}p_1] & 0 & 0 \\ p_0\overline{p_1} - [p_0\overline{p_1} - \overline{p_1}p_0] & p_0\overline{p_0} - [p_0\overline{p_0} - \overline{p_0}p_0] & 0 & 0 \\ 0 & 0 & p_1\overline{p_1} - [p_1\overline{p_1} - \overline{p_1}p_1] & p_1\overline{p_0} - [p_1\overline{p_0} - \overline{p_0}p_1] \\ 0 & 0 & p_0\overline{p_1} - [p_0\overline{p_1} - \overline{p_1}p_0] & p_0\overline{p_0} - [p_0\overline{p_0} - \overline{p_0}p_0] \end{pmatrix}$$

$$+ \begin{pmatrix} p_3\overline{p_3} - [p_3\overline{p_3} - \overline{p_3}p_3] & p_3\overline{p_2} - [p_3\overline{p_2} - \overline{p_2}p_3] & 0 & 0 \\ p_2\overline{p_3} - [p_2\overline{p_3} - \overline{p_3}p_2] & p_2\overline{p_2} - [p_2\overline{p_2} - \overline{p_2}p_2] & 0 & 0 \\ 0 & 0 & p_3\overline{p_3} - [p_3\overline{p_3} - \overline{p_3}p_3] & p_3\overline{p_2} - [p_3\overline{p_2} - \overline{p_2}p_3] \\ 0 & 0 & p_2\overline{p_3} - [p_2\overline{p_3} - \overline{p_3}p_2] & p_2\overline{p_2} - [p_2\overline{p_2} - \overline{p_2}p_2] \end{pmatrix}$$

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$$=\begin{pmatrix}p_1\\p_0\\0\\0\end{pmatrix}(\overline{p_1},\overline{p_0},0,0)+\begin{pmatrix}0\\0\\p_1\\p_0\end{pmatrix}(0,0,\overline{p_1},\overline{p_0})$$

$$-\begin{pmatrix}[p_1\overline{p_1}-\overline{p_1}p_1] & [p_1\overline{p_0}-\overline{p_0}p_1] & 0 & 0\\ [p_0\overline{p_1}-\overline{p_1}p_0] & [p_0\overline{p_0}-\overline{p_0}p_0] & 0 & 0\\ 0 & 0 & [p_1\overline{p_1}-\overline{p_1}p_1] & [p_1\overline{p_0}-\overline{p_0}p_1]\\ 0 & 0 & [p_0\overline{p_1}-\overline{p_1}p_0] & [p_0\overline{p_0}-\overline{p_0}p_0]\end{pmatrix}$$

$$+\begin{pmatrix}p_3\\p_2\\0\\0\end{pmatrix}(\overline{p_3},\overline{p_2},0,0)+\begin{pmatrix}0\\0\\p_3\\p_2\end{pmatrix}(0,0,\overline{p_3},\overline{p_2})$$

$$-\begin{pmatrix}[p_3\overline{p_3}-\overline{p_3}p_3] & [p_3\overline{p_2}-\overline{p_2}p_3] & 0 & 0\\ [p_2\overline{p_3}-\overline{p_3}p_2] & [p_2\overline{p_2}-\overline{p_2}p_2] & 0 & 0\\ 0 & 0 & [p_3\overline{p_3}-\overline{p_3}p_3] & [p_3\overline{p_2}-\overline{p_2}p_3]\\ 0 & 0 & [p_2\overline{p_3}-\overline{p_3}p_2] & [p_2\overline{p_2}-\overline{p_2}p_2]\end{pmatrix}$$

Let's introduce the notations

$$\begin{pmatrix}-\overline{p_0}\\\overline{p_1}\\0\\0\end{pmatrix}(-p_0,p_1,0,0)+\begin{pmatrix}0\\0\\-\overline{p_0}\\\overline{p_1}\end{pmatrix}(0,0,-p_0,p_1)+\begin{pmatrix}-\overline{p_2}\\\overline{p_3}\\0\\0\end{pmatrix}(-p_2,p_3,0,0)+\begin{pmatrix}0\\0\\-\overline{p_2}\\\overline{p_3}\end{pmatrix}(0,0,-p_2,p_3)\equiv S^+$$

$$\begin{pmatrix}p_1\\p_0\\0\\0\end{pmatrix}(\overline{p_1},\overline{p_0},0,0)+\begin{pmatrix}0\\0\\p_1\\p_0\end{pmatrix}(0,0,\overline{p_1},\overline{p_0})+\begin{pmatrix}p_3\\p_2\\0\\0\end{pmatrix}(\overline{p_3},\overline{p_2},0,0)+\begin{pmatrix}0\\0\\p_3\\p_2\end{pmatrix}(0,0,\overline{p_3},\overline{p_2})\equiv S^-$$

$$\begin{pmatrix}[p_1\overline{p_1}-\overline{p_1}p_1] & [p_1\overline{p_0}-\overline{p_0}p_1] & 0 & 0\\ [p_0\overline{p_1}-\overline{p_1}p_0] & [p_0\overline{p_0}-\overline{p_0}p_0] & 0 & 0\\ 0 & 0 & [p_1\overline{p_1}-\overline{p_1}p_1] & [p_1\overline{p_0}-\overline{p_0}p_1]\\ 0 & 0 & [p_0\overline{p_1}-\overline{p_1}p_0] & [p_0\overline{p_0}-\overline{p_0}p_0]\end{pmatrix}$$

$$+\begin{pmatrix}[p_3\overline{p_3}-\overline{p_3}p_3] & [p_3\overline{p_2}-\overline{p_2}p_3] & 0 & 0\\ [p_2\overline{p_3}-\overline{p_3}p_2] & [p_2\overline{p_2}-\overline{p_2}p_2] & 0 & 0\\ 0 & 0 & [p_3\overline{p_3}-\overline{p_3}p_3] & [p_3\overline{p_2}-\overline{p_2}p_3]\\ 0 & 0 & [p_2\overline{p_3}-\overline{p_3}p_2] & [p_2\overline{p_2}-\overline{p_2}p_2]\end{pmatrix}\equiv K$$

Let us substitute differential operators instead of spinor components

$$p_0\to\frac{\partial}{\partial x_1}\equiv\partial_1\quad p_1\to-\frac{\partial}{\partial x_0}\equiv-\partial_0\quad p_2\to\frac{\partial}{\partial x_3}\equiv\partial_3\quad p_3\to-\frac{\partial}{\partial x_2}\equiv-\partial_2$$

$$\overline{p_0}\to\frac{\partial[\,]}{\partial x_1}\equiv\overline{\partial_1}\quad \overline{p_1}\to-\frac{\partial[\,]}{\partial x_0}\equiv-\overline{\partial_0}\quad \overline{p_2}\to\frac{\partial[\,]}{\partial x_3}\equiv\overline{\partial_3}\quad \overline{p_3}\to-\frac{\partial[\,]}{\partial x_2}\equiv-\overline{\partial_2}$$

Then the quantities included in the wave equation

$$(S^--K)S^+\boldsymbol{\psi}(x_0,x_1,x_2,x_3)=M^2\boldsymbol{\psi}(x_0,x_1,x_2,x_3)$$

will have the form

$$S^-=\begin{pmatrix}-\partial_0\\\partial_1\\0\\0\end{pmatrix}(-\overline{\partial_0},\overline{\partial_1},0,0)+\begin{pmatrix}0\\0\\-\partial_0\\\partial_1\end{pmatrix}(0,0,-\overline{\partial_0},\overline{\partial_1})+\begin{pmatrix}-\partial_2\\\partial_3\\0\\0\end{pmatrix}(-\overline{\partial_2},\overline{\partial_3},0,0)+\begin{pmatrix}0\\0\\-\partial_2\\\partial_3\end{pmatrix}(0,0,-\overline{\partial_2},\overline{\partial_3})$$

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$$S^{+}=\begin{pmatrix}-\overline{\partial_1}\\-\overline{\partial_0}\\0\\0\end{pmatrix}(-\partial_1,-\partial_0,0,0)+\begin{pmatrix}0\\0\\-\overline{\partial_1}\\-\overline{\partial_0}\end{pmatrix}(0,0,-\partial_1,-\partial_0)+\begin{pmatrix}-\overline{\partial_3}\\-\overline{\partial_2}\\0\\0\end{pmatrix}(-\partial_3,-\partial_2,0,0)$$
$$+\begin{pmatrix}0\\0\\-\overline{\partial_3}\\-\overline{\partial_2}\end{pmatrix}(0,0,-\partial_3,-\partial_2)$$

$$K=$$
$$=\begin{pmatrix}\partial_0\overline{\partial_0}-\overline{\partial_0}\partial_0 & (-\partial_0)\overline{\partial_1}-\overline{\partial_1}(-\partial_0) & 0 & 0\\ \partial_1(-\overline{\partial_0})-(-\overline{\partial_0})\partial_1 & \partial_1\overline{\partial_1}-\overline{\partial_1}\partial_1 & 0 & 0\\ 0 & 0 & \partial_0\overline{\partial_0}-\overline{\partial_0}\partial_0 & (-\partial_0)\overline{\partial_1}-\overline{\partial_1}(-\partial_0)\\ 0 & 0 & \partial_1(-\overline{\partial_0})-(-\overline{\partial_0})\partial_1 & \partial_1\overline{\partial_1}-\overline{\partial_1}\partial_1\end{pmatrix}$$
$$+\begin{pmatrix}\partial_2\overline{\partial_2}-\overline{\partial_2}\partial_2 & (-\partial_2)\overline{\partial_3}-\overline{\partial_3}(-\partial_2) & 0 & 0\\ \partial_3(-\overline{\partial_2})-(-\overline{\partial_2})\partial_3 & \partial_3\overline{\partial_3}-\overline{\partial_3}\partial_3 & 0 & 0\\ 0 & 0 & \partial_2\overline{\partial_2}-\overline{\partial_2}\partial_2 & (-\partial_2)\overline{\partial_3}-\overline{\partial_3}(-\partial_2)\\ 0 & 0 & \partial_3(-\overline{\partial_2})-(-\overline{\partial_2})\partial_3 & \partial_3\overline{\partial_3}-\overline{\partial_3}\partial_3\end{pmatrix}$$

Let us consider the case of a free particle and represent the electron field as a four-component spinor function of four-component spinor coordinates

$$\boldsymbol{\psi}(x_0,x_1,x_2,x_3)=\begin{pmatrix}\psi_0(x_0,x_1,x_2,x_3)\\\psi_1(x_0,x_1,x_2,x_3)\\\psi_2(x_0,x_1,x_2,x_3)\\\psi_3(x_0,x_1,x_2,x_3)\end{pmatrix}=\begin{pmatrix}u_0\\u_1\\u_2\\u_3\end{pmatrix}\varphi(x_0,x_1,x_2,x_3)$$

For a free particle, the components of the momentum spinor commute with each other, so all components of the matrix K are zero.

Let us use the model of a plane wave in spinor space

$$\varphi(x_0,x_1,x_2,x_3)=exp(p_0x_1-p_1x_0+p_2x_3-p_3x_2)$$

Substituting the plane wave solution into the differential equation, we obtain the algebraic equation

$$S^{-}S^{+}\begin{pmatrix}u_0\\u_1\\u_2\\u_3\end{pmatrix}\varphi(x_0,x_1,x_2,x_3)=M^2\begin{pmatrix}u_0\\u_1\\u_2\\u_3\end{pmatrix}\varphi(x_0,x_1,x_2,x_3)$$

$$S^{-}\left\{\begin{pmatrix}-\overline{p_0}\\\overline{p_1}\\0\\0\end{pmatrix}(-p_0u_0+p_1u_1)+\begin{pmatrix}0\\0\\-\overline{p_0}\\\overline{p_1}\end{pmatrix}(-p_0u_2+p_1u_3)+\begin{pmatrix}-\overline{p_2}\\\overline{p_3}\\0\\0\end{pmatrix}(-p_2u_0+p_3u_1)\right.$$
$$\left.+\begin{pmatrix}0\\0\\-\overline{p_2}\\\overline{p_3}\end{pmatrix}(-p_2u_2+p_3u_3)\right\}\varphi(x_0,x_1,x_2,x_3)=m^2\begin{pmatrix}u_0\\u_1\\u_2\\u_3\end{pmatrix}\varphi(x_0,x_1,x_2,x_3)$$

$$\left\{\begin{pmatrix}p_1\\p_0\\0\\0\end{pmatrix}(\overline{p_1},\overline{p_0},0,0)+\begin{pmatrix}0\\0\\p_1\\p_0\end{pmatrix}(0,0,\overline{p_1},\overline{p_0})+\begin{pmatrix}p_3\\p_2\\0\\0\end{pmatrix}(\overline{p_3},\overline{p_2},0,0)+\begin{pmatrix}0\\0\\p_3\\p_2\end{pmatrix}(0,0,\overline{p_3},\overline{p_2})\right\}$$

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$$\left\{\begin{pmatrix}-\overline{p_0}\\ \overline{p_1}\\ 0\\ 0\end{pmatrix}(-p_0u_0+p_1u_1)+\begin{pmatrix}0\\ 0\\ -\overline{p_0}\\ \overline{p_1}\end{pmatrix}(-p_0u_2+p_1u_3)+\begin{pmatrix}-\overline{p_2}\\ \overline{p_3}\\ 0\\ 0\end{pmatrix}(-p_2u_0+p_3u_1)\right.$$

$$\left.+\begin{pmatrix}0\\ 0\\ -\overline{p_2}\\ \overline{p_3}\end{pmatrix}(-p_2u_2+p_3u_3)\right\}\varphi(x_0,x_1,x_2,x_3)=M^2\begin{pmatrix}u_0\\ u_1\\ u_2\\ u_3\end{pmatrix}\varphi(x_0,x_1,x_2,x_3)$$

$$\left\{\begin{pmatrix}p_1\\ p_0\\ 0\\ 0\end{pmatrix}(\overline{p_1},\overline{p_0},0,0)+\begin{pmatrix}0\\ 0\\ p_1\\ p_0\end{pmatrix}(0,0,\overline{p_1},\overline{p_0})+\begin{pmatrix}p_3\\ p_2\\ 0\\ 0\end{pmatrix}(\overline{p_3},\overline{p_2},0,0)+\begin{pmatrix}0\\ 0\\ p_3\\ p_2\end{pmatrix}(0,0,\overline{p_3},\overline{p_2})\right\}$$

$$\left\{\begin{pmatrix}-\overline{p_0}\\ \overline{p_1}\\ 0\\ 0\end{pmatrix}(-p_0u_0+p_1u_1)+\begin{pmatrix}0\\ 0\\ -\overline{p_0}\\ \overline{p_1}\end{pmatrix}(-p_0u_2+p_1u_3)+\begin{pmatrix}-\overline{p_2}\\ \overline{p_3}\\ 0\\ 0\end{pmatrix}(-p_2u_0+p_3u_1)\right.$$

$$\left.+\begin{pmatrix}0\\ 0\\ -\overline{p_2}\\ \overline{p_3}\end{pmatrix}(-p_2u_2+p_3u_3)\right\}=M^2\begin{pmatrix}u_0\\ u_1\\ u_2\\ u_3\end{pmatrix}$$

$$\begin{pmatrix}p_1\\ p_0\\ 0\\ 0\end{pmatrix}(-\overline{p_1}\overline{p_0}+\overline{p_0}\overline{p_1})(-p_0u_0+p_1u_1)+\begin{pmatrix}p_3\\ p_2\\ 0\\ 0\end{pmatrix}(-\overline{p_3}\overline{p_0}+\overline{p_2}\overline{p_1})(-p_0u_0+p_1u_1)+$$

$$\begin{pmatrix}0\\ 0\\ p_1\\ p_0\end{pmatrix}(-\overline{p_1}\overline{p_0}+\overline{p_0}\overline{p_1})(-p_0u_2+p_1u_3)+\begin{pmatrix}0\\ 0\\ p_3\\ p_2\end{pmatrix}(-\overline{p_3}\overline{p_0}+\overline{p_2}\overline{p_1})(-p_0u_2+p_1u_3)+$$

$$\begin{pmatrix}p_1\\ p_0\\ 0\\ 0\end{pmatrix}(-\overline{p_1}\overline{p_2}+\overline{p_0}\overline{p_3})(-p_2u_0+p_3u_1)+\begin{pmatrix}p_3\\ p_2\\ 0\\ 0\end{pmatrix}(-\overline{p_3}\overline{p_2}+\overline{p_2}\overline{p_3})(-p_2u_0+p_3u_1)+$$

$$\begin{pmatrix}0\\ 0\\ p_1\\ p_0\end{pmatrix}(-\overline{p_1}\overline{p_2}+\overline{p_0}\overline{p_3})(-p_2u_2+p_3u_3)+\begin{pmatrix}0\\ 0\\ p_3\\ p_2\end{pmatrix}(-\overline{p_3}\overline{p_2}+\overline{p_2}\overline{p_3})(-p_2u_2+p_3u_3)=$$

$$=M^2\begin{pmatrix}u_0\\ u_1\\ u_2\\ u_3\end{pmatrix}$$

Let us take into account the commutativity of the momentum components, besides, let us introduce the notations

$$-\overline{p_3}\overline{p_0}+\overline{p_2}\overline{p_1}\equiv\overline{m}\qquad-\overline{p_1}\overline{p_2}+\overline{p_0}\overline{p_3}\equiv-\overline{m}$$

for the quantities which are invariant under any rotations and boosts, then we obtain

$$\left\{\begin{pmatrix}p_3\\ p_2\\ 0\\ 0\end{pmatrix}\overline{m}(-p_0u_0+p_1u_1)+\begin{pmatrix}0\\ 0\\ p_3\\ p_2\end{pmatrix}\overline{m}(-p_0u_2+p_1u_3)+\begin{pmatrix}p_1\\ p_0\\ 0\\ 0\end{pmatrix}(-\overline{m})(-p_2u_0+p_3u_1)\right.$$

$$\left.+\begin{pmatrix}0\\ 0\\ p_1\\ p_0\end{pmatrix}(-\overline{m})(-p_2u_2+p_3u_3)\right\}=M^2\begin{pmatrix}u_0\\ u_1\\ u_2\\ u_3\end{pmatrix}$$

$$\left\{u_0\left(-\begin{pmatrix}p_3\\p_2\\0\\0\end{pmatrix}\overline{m}p_0+\begin{pmatrix}p_1\\p_0\\0\\0\end{pmatrix}\overline{m}p_2\right)+u_1\left(\begin{pmatrix}p_3\\p_2\\0\\0\end{pmatrix}\overline{m}p_1-\begin{pmatrix}p_1\\p_0\\0\\0\end{pmatrix}\overline{m}p_3\right)+u_2\left(-\begin{pmatrix}0\\0\\p_3\\p_2\end{pmatrix}\overline{m}p_0+\begin{pmatrix}0\\0\\p_1\\p_0\end{pmatrix}\overline{m}p_2\right)\right.$$
$$\left.+u_3\left(\begin{pmatrix}0\\0\\p_3\\p_2\end{pmatrix}\overline{m}p_1-\begin{pmatrix}0\\0\\p_1\\p_0\end{pmatrix}\overline{m}p_3\right)\right\}=M^2\begin{pmatrix}u_0\\u_1\\u_2\\u_3\end{pmatrix}$$

$$\left\{u_0\overline{m}\begin{pmatrix}p_1p_2-p_3p_0\\p_0p_2-p_2p_0\\0\\0\end{pmatrix}+u_1\overline{m}\begin{pmatrix}p_3p_1-p_1p_3\\p_2p_1-p_0p_3\\0\\0\end{pmatrix}+u_2\overline{m}\begin{pmatrix}0\\0\\p_1p_2-p_3p_0\\p_0p_2-p_2p_0\end{pmatrix}+u_3\overline{m}\begin{pmatrix}0\\0\\p_3p_1-p_1p_3\\p_2p_1-p_0p_3\end{pmatrix}\right\}$$
$$=M^2\begin{pmatrix}u_0\\u_1\\u_2\\u_3\end{pmatrix}$$

Additionally, introducing notation for Lorentz invariant quantities

$$p_1p_2-p_3p_0\equiv m \qquad p_2p_1-p_0p_3\equiv m$$

we obtain

$$\left\{u_0\overline{m}\begin{pmatrix}m\\0\\0\\0\end{pmatrix}+u_1\overline{m}\begin{pmatrix}0\\m\\0\\0\end{pmatrix}+u_2\overline{m}\begin{pmatrix}0\\0\\m\\0\end{pmatrix}+u_3\overline{m}\begin{pmatrix}0\\0\\0\\m\end{pmatrix}\right\}=m^2\begin{pmatrix}u_0\\u_1\\u_2\\u_3\end{pmatrix}$$

$$\left\{u_0\begin{pmatrix}m^2\\0\\0\\0\end{pmatrix}+u_1\begin{pmatrix}0\\m^2\\0\\0\end{pmatrix}+u_2\begin{pmatrix}0\\0\\m^2\\0\end{pmatrix}+u_3\begin{pmatrix}0\\0\\0\\m^2\end{pmatrix}\right\}=m^2\begin{pmatrix}u_0\\u_1\\u_2\\u_3\end{pmatrix}$$

$$\begin{pmatrix}m^2&0&0&0\\0&m^2&0&0\\0&0&m^2&0\\0&0&0&m^2\end{pmatrix}\begin{pmatrix}u_0\\u_1\\u_2\\u_3\end{pmatrix}=M^2\begin{pmatrix}u_0\\u_1\\u_2\\u_3\end{pmatrix}$$

We see that in the case of a plane wave in spinor space, the matrix in the left part of the equation is diagonal and remains so at any rotations and boosts, the diagonal element also does not change.

In this case we can consider the matrix M^2 in the right part to be diagonal with the same elements on the diagonal m^2, then the equation can be rewritten as an equation for the problem of finding eigenvalues and eigenfunctions

$$S^-S^+\boldsymbol{\psi}(x_0,x_1,x_2,x_3)=m^2I\boldsymbol{\psi}(x_0,x_1,x_2,x_3)$$
$$S^-S^+\boldsymbol{\psi}(x_0,x_1,x_2,x_3)=m^2\boldsymbol{\psi}(x_0,x_1,x_2,x_3)$$

Let us compare our equation with the Dirac equation [6, formula (43.16)]

$$\begin{pmatrix}P_0+M&0&P_3&P_1-iP_2\\0&P_0+M&P_1+iP_2&-P_3\\P_3&P_1-iP_2&P_0-M&0\\P_1+iP_2&-P_3&0&P_0-M\end{pmatrix}\begin{pmatrix}u_0\\u_1\\u_2\\u_3\end{pmatrix}=0$$

In the rest frame of reference, the three components of momentum are zero and the equation is simplified

$$\begin{pmatrix}P_0+M&0&0&0\\0&P_0+M&0&0\\0&0&P_0-M&0\\0&0&0&P_0-M\end{pmatrix}\begin{pmatrix}u_0\\u_1\\u_2\\u_3\end{pmatrix}=0$$

That is, in the rest frame the Dirac equation and the spinor equation analyzed by us look identically and contain a diagonal matrix. The corresponding problem on eigenvalues and eigenvectors of these matrices has degenerate eigenvalues, which correspond to the linear space of

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eigenfunctions. In this space, one can choose an orthogonal basis of linearly independent functions, and this choice is quite arbitrary. For example, in [[8], formula (2.127)], solutions in the form of plane waves in the vector space have been proposed for the Dirac equation in the rest frame

$$u^i(0)\,exp(-iMt)$$
$$v^i(0)\,exp(+iMt)$$

and the following spinors are chosen as basis vectors

$$u^1(0)=\begin{pmatrix}1\\0\\0\\0\end{pmatrix}\quad u^2(0)=\begin{pmatrix}0\\1\\0\\0\end{pmatrix}\quad v^1(0)=\begin{pmatrix}0\\0\\1\\0\end{pmatrix}\quad v^2(0)=\begin{pmatrix}0\\0\\0\\1\end{pmatrix}$$

For transformation to a moving coordinate system in [[8], formula (2.133)] the following formula is used

$$\psi^i(X)=u^i(P)\,exp(-iPX)$$
$$\psi^i(X)=v^i(P)\,exp(+iPX)$$

where

$$u^1(P)=\sqrt{\frac{P_0+M}{2M}}\begin{pmatrix}1\\0\\\frac{P_3}{P_0+M}\\\frac{P_1+iP_2}{P_0+M}\end{pmatrix}\quad u^2(P)=\sqrt{\frac{P_0+M}{2M}}\begin{pmatrix}1\\0\\\frac{P_1-iP_2}{P_0+M}\\\frac{-P_3}{P_0+M}\end{pmatrix}\quad v^1(P)=\sqrt{\frac{P_0+M}{2M}}\begin{pmatrix}\frac{P_3}{P_0+M}\\\frac{P_1+iP_2}{P_0+M}\\1\\0\end{pmatrix}$$

$$v^2(P)=\sqrt{\frac{P_0+M}{2M}}\begin{pmatrix}\frac{P_1-iP_2}{P_0+M}\\\frac{-P_3}{P_0+M}\\0\\1\end{pmatrix}$$

The basis spinors form a complete system, that is, any four-component complex spinor can be represented as their linear combination and this arbitrary spinor will be a solution to the problem on eigenvalues and eigenfunctions in a resting coordinate system. The choice of the given particular basis has disadvantages, because if to find a four-dimensional current vector from any of these basis functions

$$j_\mu=\frac{1}{2}\left(u^1(0)\right)^\dagger S_\mu u^1(0)$$

then this current in the rest frame of reference

$$\mathbf{j}^{\mathbf{T}}=\left(\frac{1}{2},0,0,\frac{1}{2}\right)$$

has non-zero components, and the square of the length of the current vector is zero. It turns out that a resting electron creates a current, which contradicts physical common sense.

Since we have freedom of choice of the basis, it is reasonable to choose the spinor for the wave function as some set of momentum spinor components, for example

$$u(0)=\sqrt{\frac{e}{m}}\begin{pmatrix}p_2\\-p_3\\p_0\\-p_1\end{pmatrix}$$

An exhaustive list of 16 spinors of this kind, each corresponding to some particle of the fermionic field, is given in the last section of the paper. The proportionality factor is chosen so that in the rest frame the zero component of the current is equal to the charge of, for example, an electron or a positron.

The mass of electron $m=p_1p_2-p_3p_0$ and the phase of the plane spinor wave

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$$exp(\mathbf{p}^T \Sigma_{MM}\mathbf{x}) = exp(p_0x_1 - p_1x_0 + p_2x_3 - p_3x_2)$$

do not change at rotations and boosts. The matrix on the left side of the equation does not change either, remaining diagonal with m^2 on the diagonal.

For a fermion, which can be an electron or a positron in the rest frame takes place $\mathrm{p}^{\mathrm{T}} = (\mathrm{p}_0, \mathrm{p}_1, \overline{\mathrm{p}_1}, -\overline{\mathrm{p}_0})$, so the quantity

$$m = p_1p_2 - p_3p_0 = p_1\overline{p_1} + \overline{p_0}p_0$$

which, unlike the mass M in the Dirac equation, is complex in the general case, is also real for the fermion and can be positive or negative. For simplicity it is possible to consider the mass of the electron as negative and that of the positron as positive.

For the momentum spinor of a boson, such as a photon, it is true that $\mathbf{p}^{\mathrm{T}} = (p_0, p_1, p_0, p_1)$, so its mass is zero

$$m = p_1p_2 - p_3p_0 = p_1p_0 - p_1p_0 = 0$$

The given constructions are not abstract, but describe the physical reality, since the results of the processes occurring in the spinor space are displayed in the Minkowski vector space. In particular, the momentum vector corresponding to the momentum spinor has the following parameters

$$P_\mu = \frac{1}{2}Tr[\mathbf{p}\mathbf{p}^\dagger S_\mu]$$

the square of the length is equal to the square of the mass of the electron or positron

$$P_0{}^2 - P_1{}^2 - P_2{}^2 - P_3{}^2 = m^2$$

A spinor wave function $\boldsymbol{\psi}(x_0, x_1, x_2, x_3)$ at some point in spinor space can be given a probabilistic interpretation by establishing its correspondence with the vector wave function $\boldsymbol{\Psi}(X_0, X_1, X_2, X_3)$

$$\Psi_\mu = \frac{1}{2}Tr[\boldsymbol{\psi}\boldsymbol{\psi}^\dagger S_\mu]$$

taking its values in the corresponding point of physical space with coordinates

$$X_\mu = \frac{1}{2}Tr[\mathbf{x}\mathbf{x}^\dagger S_\mu]$$

We act within the classical concepts of quantum mechanics, simply to describe the state of a physical system we use spinor coordinate and momentum representations along with vector coordinate and momentum representations. Both types of representations equally have the right to be more substantial and in principle there is no need to express the wave function in one representation through the wave function in the other, both wave functions equally describe the same physical state. Moreover, since vector coordinates and momenta are simply expressed through spinor analogues, we would prioritize the spinor representations as the more fundamental ones.

Let us summarize the relations between quantum-mechanical quantities for the spinor space

$$\mathbf{x}^{\mathbf{T}} \equiv (x_0, x_1, x_2, x_3) \qquad \hat{\mathbf{x}}^{\mathbf{T}} \equiv (\hat{x}_0, \hat{x}_1, \hat{x}_2, \hat{x}_3)$$

$$\mathbf{p}^{\mathbf{T}} \equiv (p_0, p_1, p_2, p_3) \qquad \hat{\mathbf{p}}^{\mathbf{T}} \equiv (\hat{p}_0, \hat{p}_1, \hat{p}_2, \hat{p}_3)$$

$$(\mathbf{p}, \mathbf{x}) = p_0x_1 - p_1x_0 + p_2x_3 - p_3x_2$$

$$\overline{(\mathbf{p}, \mathbf{x})} = \overline{p_0x_1} - \overline{p_1x_0} + \overline{p_2x_3} - \overline{p_3x_2}$$

The complete orthonormalzed system of eigenvectors of the momentum operator

$$\hat{\mathbf{p}}|\mathbf{p}\rangle = \mathbf{p}|\mathbf{p}\rangle$$

$$\hat{p}_\alpha|\mathbf{p}\rangle = p_\alpha|\mathbf{p}\rangle$$

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$$\langle \mathbf{p}|\mathbf{p}'\rangle = (2\pi)^4\delta(\mathbf{p}-\mathbf{p}')$$

$$\int \frac{d^4p}{(2\pi)^4}|\mathbf{p}\rangle_{(\mathbf{x})}\,\langle\mathbf{p}|_{(\mathbf{x}')} = \mathbb{1}_{(\mathbf{x})(\mathbf{x}')}$$

$$\hat{\mathbf{p}}_{(\mathbf{x})(\mathbf{x}')} = \int \frac{d^4p}{(2\pi)^4}|\mathbf{p}\rangle_{(\mathbf{x})}\mathbf{p}\langle\mathbf{p}|_{(\mathbf{x}')}$$

$$\hat{p}_{\alpha(\mathbf{x})(\mathbf{x}')} = \int \frac{d^4p}{(2\pi)^4}|\mathbf{p}\rangle_{(\mathbf{x})}p_\alpha\langle\mathbf{p}|_{(\mathbf{x}')}$$

$$|\boldsymbol{\varphi}\rangle = \int \frac{d^4p}{(2\pi)^4}\,\boldsymbol{\varphi}(\mathbf{p})|\mathbf{p}\rangle$$

$$\boldsymbol{\varphi}(\mathbf{p}) = \langle\mathbf{p}|\boldsymbol{\varphi}\rangle$$

The complete orthonormalzed system of eigenvectors of the coordinate operator

$$\hat{\mathbf{x}}|\mathbf{x}\rangle = \mathbf{x}|\mathbf{x}\rangle$$

$$\hat{x}_\alpha|\mathbf{x}\rangle = x_\alpha|\mathbf{x}\rangle$$

$$\langle\mathbf{x}|\mathbf{x}'\rangle = \delta(\mathbf{x}-\mathbf{x}')$$

$$\int d^4x\,|\mathbf{x}\rangle_{(\mathbf{p})}\,\langle\mathbf{x}|_{(\mathbf{p}')} = \mathbb{1}_{(\mathbf{p})(\mathbf{p}')}$$

$$\hat{\mathbf{x}}_{(\mathbf{p})(\mathbf{p}')} = \int d^4x\,|\mathbf{x}\rangle_{(\mathbf{p})}\mathbf{x}\langle\mathbf{x}|_{(\mathbf{p}')}$$

$$\hat{x}_{\alpha(\mathbf{p})(\mathbf{p}')} = \int d^4x\,|\mathbf{x}\rangle_{(\mathbf{p})}x_\alpha\langle\mathbf{x}|_{(\mathbf{p}')}$$

$$|\boldsymbol{\varphi}\rangle = \int d^4x\;\boldsymbol{\varphi}(\mathbf{x})|\mathbf{x}\rangle$$

$$\boldsymbol{\varphi}(\mathbf{x}) = \langle\mathbf{x}|\boldsymbol{\varphi}\rangle$$

The relation between wave function in momentum and coordinate representations and the relation between eigenvectors of the coordinate operator and the momentum operator

$$\boldsymbol{\varphi}(\mathbf{x}) = \int \frac{d^4p}{(2\pi)^4}\;\boldsymbol{\varphi}(\mathbf{p})e^{i\left((\mathbf{p},\mathbf{x})+\overline{(\mathbf{p},\mathbf{x})}\right)}$$

$$|\boldsymbol{\varphi}\rangle = \int d^4x\;\boldsymbol{\varphi}(\mathbf{x})|\mathbf{x}\rangle = \int d^4x\left(\int \frac{d^4p}{(2\pi)^4}\;\boldsymbol{\varphi}(\mathbf{p})e^{i\left((\mathbf{p},\mathbf{x})+\overline{(\mathbf{p},\mathbf{x})}\right)}\right)|\mathbf{x}\rangle$$
$$= \int \frac{d^4p}{(2\pi)^4}\;\boldsymbol{\varphi}(\mathbf{p})\left(\int d^4x\;e^{i\left((\mathbf{p},\mathbf{x})+\overline{(\mathbf{p},\mathbf{x})}\right)}|\mathbf{x}\rangle\right)$$

$$|\boldsymbol{\varphi}\rangle = \int \frac{d^4p}{(2\pi)^4}\;\boldsymbol{\varphi}(\mathbf{p})|\mathbf{p}\rangle$$

$$|\mathbf{p}\rangle = \int d^4x\;e^{i\left((\mathbf{p},\mathbf{x})+\overline{(\mathbf{p},\mathbf{x})}\right)}|\mathbf{x}\rangle$$

$$\langle\mathbf{x}|\mathbf{p}\rangle = e^{i\left((\mathbf{p},\mathbf{x})+\overline{(\mathbf{p},\mathbf{x})}\right)}$$

The arbitrary choice of the basis of the linear space of the eigenvectors of the matrix takes place only for a free particle. In the general case the matrix K is not zero, the wave equation has no solution in the form of plane waves in spinor space and ceases to be invariant with respect to Lorentz transformations, and the eigenvalues become nondegenerate.

We propose to extend the scope of applicability of the presented equation consisting of differential operators in the form of partial derivatives on the components of coordinate spinors to case of a nonzero matrix K

$$(S^- - K)S^+\boldsymbol{\psi}(x_0,x_1,x_2,x_3) = M^2\boldsymbol{\psi}(x_0,x_1,x_2,x_3)$$

that is not only to the case of a plane wave, but to any situation in general. This transition is analogous to the transition from the application of the Schrödinger equation to a plane wave in vector space to its application in a general situation. The legitimacy of such transitions should be confirmed by the results of experiments.

This equation will be called the equation for the spinor wave function defined on the spinor coordinate space. Here the matrix M^2 is, generally speaking, neither diagonal nor real, but it does not depend on the coordinates and is determined solely by the parameters of the electromagnetic field. Only in the case of a plane wave it is diagonal and has on the diagonal the square of the mass of the free particle. We can try to simplify the problem and require that the matrix M^2 is diagonal with the same elements on the diagonal m^2, then the equation can be rewritten in the form of the equation for the problem of search of eigenvalues and eigenfunctions for any quantum states

$$(S^- - K)S^+\boldsymbol{\psi}(x_0,x_1,x_2,x_3) = m^2\boldsymbol{\psi}(x_0,x_1,x_2,x_3)$$

This approach is pleasant in the Dirac equation, where the mass is fixed and equated to the mass of a free particle, and at the same time results giving good agreement with experiment are obtained.

We are of the opinion that the spinor equation is more fundamental than the relativistic Schrödinger and Dirac equations, it is not a generalization of them, it is a refinement of them, because it describes nature at the spinor level, and hence is more precise and detailed than the equations for the wave function defined on the vector space.

Let us consider the proposed equation for the special case when the particle is in an external electromagnetic field, which we will also represent by a four-component spinor function at a point of the spinor coordinate space

$$\mathbf{a}(x_0,x_1,x_2,x_3) = \begin{pmatrix} a_0(x_0,x_1,x_2,x_3) \\ a_1(x_0,x_1,x_2,x_3) \\ a_2(x_0,x_1,x_2,x_3) \\ a_3(x_0,x_1,x_2,x_3) \end{pmatrix}$$

We will apply to the wave function of the electron the operators corresponding to the components of the momentum spinor, putting for simplicity the electron charge equal to unity

$$p_0 \to \frac{\partial}{\partial x_1} + a_0(x_0,x_1,x_2,x_3) \qquad p_1 \to -\frac{\partial}{\partial x_0} + a_1(x_0,x_1,x_2,x_3)$$

$$p_2 \to \frac{\partial}{\partial x_3} + a_2(x_0,x_1,x_2,x_3) \qquad p_3 \to -\frac{\partial}{\partial x_2} + a_3(x_0,x_1,x_2,x_3)$$

$$\overline{p_0} \to \frac{\partial\overline{[\,]}}{\partial \overline{x_1}} + \overline{a_0(x_0,x_1,x_2,x_3)} \qquad \overline{p_1} \to -\frac{\partial\overline{[\,]}}{\partial \overline{x_0}} + \overline{a_1(x_0,x_1,x_2,x_3)}$$

$$\overline{p_2} \to \frac{\partial\overline{[\,]}}{\partial \overline{x_3}} + \overline{a_2(x_0,x_1,x_2,x_3)} \qquad \overline{p_3} \to -\frac{\partial\overline{[\,]}}{\partial \overline{x_2}} + \overline{a_3(x_0,x_1,x_2,x_3)}$$

Note that the electromagnetic potential vector can be calculated from the electromagnetic potential spinor by the standard formula

$$A_\mu = \frac{1}{2}\mathbf{a}^\dagger S_\mu \mathbf{a}$$

The advantage of the spinor description over the vector description is that instead of summing up the components of the momentum and electromagnetic potential vectors as is usually done

$$P_\mu + A_\mu = \frac{1}{2}\mathbf{p}^\dagger S_\mu \mathbf{p} + \frac{1}{2}\mathbf{a}^\dagger S_\mu \mathbf{a}$$

now we sum the spinor components and then the resulting vector is

$$\frac{1}{2}(\mathbf{p}+\mathbf{a})^\dagger S_\mu (\mathbf{p}+\mathbf{a}) = \frac{1}{2}\mathbf{p}^\dagger S_\mu \mathbf{p} + \frac{1}{2}\mathbf{p}^\dagger S_\mu \mathbf{a} + \frac{1}{2}\mathbf{a}^\dagger S_\mu \mathbf{p} + \frac{1}{2}\mathbf{a}^\dagger S_\mu \mathbf{a}$$

in addition to the usual momentum and field vectors, contains an additional term

$$\frac{1}{2}\mathbf{p}^\dagger S_\mu \mathbf{a} + \frac{1}{2}\mathbf{a}^\dagger S_\mu \mathbf{p}$$

taking real values and describing the mutual influence of the fields of the electron and photon.

After the addition of the electromagnetic field the components of the momentum spinor do not commute, the corresponding commutators are found above

$$\left\{\left(\frac{\partial}{\partial x_1} + a_0\right)\left(-\frac{\partial}{\partial x_0} + a_1\right) - \left(-\frac{\partial}{\partial x_0} + a_1\right)\left(\frac{\partial}{\partial x_1} + a_0\right)\right\}\varphi = \left\{\frac{\partial a_1}{\partial x_1} + \frac{\partial a_0}{\partial x_0}\right\}\varphi$$

$$\left\{\left(\frac{\partial}{\partial x_3} + a_2\right)\left(-\frac{\partial}{\partial x_2} + a_3\right) - \left(-\frac{\partial}{\partial x_2} + a_3\right)\left(\frac{\partial}{\partial x_3} + a_2\right)\right\}\varphi = \left\{\frac{\partial a_3}{\partial x_3} + \frac{\partial a_2}{\partial x_2}\right\}\varphi$$

Let's find commutators for other operators

$$\left\{\left(\frac{\partial\overline{[\,]}}{\partial\overline{x_1}} + \overline{a_0}\right)\left(-\frac{\partial\overline{[\,]}}{\partial\overline{x_0}} + \overline{a_1}\right) - \left(-\frac{\partial\overline{[\,]}}{\partial\overline{x_0}} + \overline{a_1}\right)\left(\frac{\partial\overline{[\,]}}{\partial\overline{x_1}} + \overline{a_0}\right)\right\}\varphi =$$

$$\left(\frac{\partial\overline{[\,]}}{\partial\overline{x_1}} + \overline{a_0}\right)\left(-\frac{\partial\overline{[\,]}}{\partial\overline{x_0}} + \overline{a_1}\right)\varphi - \left(-\frac{\partial\overline{[\,]}}{\partial\overline{x_0}} + \overline{a_1}\right)\left(\frac{\partial\overline{[\,]}}{\partial\overline{x_1}} + \overline{a_0}\right)\varphi =$$

$$\left(\frac{\partial\overline{[\,]}}{\partial\overline{x_1}} + \overline{a_0}\right)\left(-\frac{\partial\bar{\varphi}}{\partial\overline{x_0}} + \overline{a_1}\varphi\right) - \left(-\frac{\partial\overline{[\,]}}{\partial\overline{x_0}} + \overline{a_1}\right)\left(\frac{\partial\bar{\varphi}}{\partial\overline{x_1}} + \overline{a_0}\varphi\right) =$$

$$\frac{\partial\overline{[\,]}}{\partial\overline{x_1}}\left(-\frac{\partial\bar{\varphi}}{\partial\overline{x_0}}\right) + \overline{a_0 a_1}\varphi + \frac{\partial\overline{[\,]}}{\partial\overline{x_1}}(\overline{a_1}\varphi) + \overline{a_0}\left(-\frac{\partial\bar{\varphi}}{\partial\overline{x_0}}\right) + \frac{\partial\overline{[\,]}}{\partial\overline{x_0}}\left(\frac{\partial\bar{\varphi}}{\partial\overline{x_1}}\right) - \overline{a_1 a_0}\varphi + \frac{\partial\overline{[\,]}}{\partial\overline{x_0}}(\overline{a_0}\varphi) - \overline{a_1}\frac{\partial\bar{\varphi}}{\partial\overline{x_1}} =$$

$$\frac{\partial\overline{[\,]}}{\partial\overline{x_1}}(\overline{a_1}\varphi) + \overline{a_0}\left(-\frac{\partial\bar{\varphi}}{\partial\overline{x_0}}\right) + \frac{\partial\overline{[\,]}}{\partial\overline{x_0}}(\overline{a_0}\varphi) - \overline{a_1}\frac{\partial\bar{\varphi}}{\partial\overline{x_1}} =$$

$$\frac{\partial\bar{\varphi}}{\partial\overline{x_1}}\overline{a_1} + \frac{\partial\overline{\overline{a_1}}}{\partial\overline{x_1}}\varphi + \overline{a_0}\left(-\frac{\partial\bar{\varphi}}{\partial\overline{x_0}}\right) + \frac{\partial\bar{\varphi}}{\partial\overline{x_0}}\overline{a_0} + \frac{\partial\overline{\overline{a_0}}}{\partial\overline{x_0}}\varphi - \overline{a_1}\frac{\partial\bar{\varphi}}{\partial\overline{x_1}} =$$

$$\frac{\partial\overline{\overline{a_1}}}{\partial\overline{x_1}}\varphi + \frac{\partial\overline{\overline{a_0}}}{\partial\overline{x_0}}\varphi = \left\{\frac{\partial a_1}{\partial\overline{x_1}} + \frac{\partial a_0}{\partial\overline{x_0}}\right\}\varphi$$

$$\left\{\left(\frac{\partial}{\partial x_1} + a_0\right)\left(-\frac{\partial\overline{[\,]}}{\partial\overline{x_0}} + \overline{a_1}\right) - \left(-\frac{\partial\overline{[\,]}}{\partial\overline{x_0}} + \overline{a_1}\right)\left(\frac{\partial}{\partial x_1} + a_0\right)\right\}\varphi =$$

$$\left(\frac{\partial}{\partial x_1} + a_0\right)\left(-\frac{\partial\overline{[\,]}}{\partial\overline{x_0}} + \overline{a_1}\right)\varphi - \left(-\frac{\partial\overline{[\,]}}{\partial\overline{x_0}} + \overline{a_1}\right)\left(\frac{\partial}{\partial x_1} + a_0\right)\varphi =$$

$$\left(\frac{\partial}{\partial x_1} + a_0\right)\left(-\frac{\partial\bar{\varphi}}{\partial\overline{x_0}} + \overline{a_1}\varphi\right) - \left(-\frac{\partial\overline{[\,]}}{\partial\overline{x_0}} + \overline{a_1}\right)\left(\frac{\partial\varphi}{\partial x_1} + a_0\varphi\right) =$$

$$\frac{\partial}{\partial x_1}\left(-\frac{\partial\bar{\varphi}}{\partial\overline{x_0}}\right) + a_0\overline{a_1}\varphi + \frac{\partial}{\partial x_1}(\overline{a_1}\varphi) + a_0\left(-\frac{\partial\bar{\varphi}}{\partial\overline{x_0}}\right) + \frac{\partial\overline{[\,]}}{\partial\overline{x_0}}\left(\frac{\partial\varphi}{\partial x_1}\right) - \overline{a_1}a_0\varphi + \frac{\partial\overline{[\,]}}{\partial\overline{x_0}}(a_0\varphi) - \overline{a_1}\frac{\partial\varphi}{\partial x_1} =$$

$$\frac{\partial}{\partial x_1}(\overline{a_1}\varphi) + a_0\left(-\frac{\partial\bar{\varphi}}{\partial\overline{x_0}}\right) + \frac{\partial\overline{[\,]}}{\partial\overline{x_0}}(a_0\varphi) - \overline{a_1}\frac{\partial\varphi}{\partial x_1} =$$

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$$\frac{\partial\varphi}{\partial x_1}\overline{a_1}+\frac{\partial\overline{a_1}}{\partial x_1}\varphi+a_0\left(-\frac{\partial\bar{\varphi}}{\partial\overline{x_0}}\right)+\frac{\partial\bar{\varphi}}{\partial\overline{x_0}}a_0+\frac{\partial\overline{a_0}}{\partial\overline{x_0}}\varphi-\overline{a_1}\frac{\partial\bar{\varphi}}{\partial\overline{x_1}}=$$

$$\frac{\partial\overline{a_1}}{\partial x_1}\varphi+\frac{\partial\overline{a_0}}{\partial\overline{x_0}}\varphi=\left\{\frac{\partial\overline{a_1}}{\partial x_1}+\frac{\partial\overline{a_0}}{\partial\overline{x_0}}\right\}\varphi$$

Further we will use these and analogous relations

$$\left\{\left(\frac{\partial}{\partial x_1}+a_0\right)\left(-\frac{\partial}{\partial x_0}+a_1\right)-\left(-\frac{\partial}{\partial x_0}+a_1\right)\left(\frac{\partial}{\partial x_1}+a_0\right)\right\}\varphi=\left\{\frac{\partial a_1}{\partial x_1}+\frac{\partial a_0}{\partial x_0}\right\}\varphi$$

$$\left\{\left(\frac{\partial}{\partial x_1}+a_0\right)\left(-\frac{\partial\overline{[\,]}}{\partial\overline{x_0}}+\overline{a_1}\right)-\left(-\frac{\partial\overline{[\,]}}{\partial\overline{x_0}}+\overline{a_1}\right)\left(\frac{\partial}{\partial x_1}+a_0\right)\right\}\varphi=\left\{\frac{\partial\overline{a_1}}{\partial x_1}+\frac{\partial\overline{a_0}}{\partial\overline{x_0}}\right\}\varphi$$

$$\left\{\left(-\frac{\partial}{\partial x_0}+a_1\right)\left(\frac{\partial\overline{[\,]}}{\partial\overline{x_1}}+\overline{a_0}\right)-\left(\frac{\partial\overline{[\,]}}{\partial\overline{x_1}}+\overline{a_0}\right)\left(-\frac{\partial}{\partial x_0}+a_1\right)\right\}\varphi=\left\{-\frac{\partial\overline{a_0}}{\partial x_0}-\frac{\partial\overline{a_1}}{\partial\overline{x_1}}\right\}\varphi$$

$$\left\{\left(-\frac{\partial}{\partial x_0}+a_1\right)\left(-\frac{\partial\overline{[\,]}}{\partial\overline{x_0}}+\overline{a_1}\right)-\left(-\frac{\partial\overline{[\,]}}{\partial\overline{x_0}}+\overline{a_1}\right)\left(-\frac{\partial}{\partial x_0}+a_1\right)\right\}\varphi=\left\{\left(-\frac{\partial\overline{a_1}}{\partial x_0}\right)+\frac{\partial\overline{a_1}}{\partial\overline{x_0}}\right\}\varphi$$

$$\left\{\left(\frac{\partial}{\partial x_1}+a_0\right)\left(\frac{\partial\overline{[\,]}}{\partial\overline{x_1}}+\overline{a_0}\right)-\left(\frac{\partial\overline{[\,]}}{\partial\overline{x_1}}+\overline{a_0}\right)\left(\frac{\partial}{\partial x_1}+a_0\right)\right\}\varphi=\left\{\left(\frac{\partial\overline{a_0}}{\partial x_1}\right)-\frac{\partial\overline{a_0}}{\partial\overline{x_1}}\right\}\varphi$$

$$\left\{\left(\frac{\partial\overline{[\,]}}{\partial\overline{x_1}}+\overline{a_0}\right)\left(-\frac{\partial\overline{[\,]}}{\partial\overline{x_0}}+\overline{a_1}\right)-\left(-\frac{\partial\overline{[\,]}}{\partial\overline{x_0}}+\overline{a_1}\right)\left(\frac{\partial\overline{[\,]}}{\partial\overline{x_1}}+\overline{a_0}\right)\right\}\varphi=\left\{\frac{\partial a_1}{\partial\overline{x_1}}+\frac{\partial a_0}{\partial\overline{x_0}}\right\}\varphi$$

$$\left(-\left(-\frac{\partial\overline{[\,]}}{\partial\overline{x_0}}+\overline{a_1}\right)\left(\frac{\partial\overline{[\,]}}{\partial\overline{x_1}}+\overline{a_0}\right)+\left(\frac{\partial\overline{[\,]}}{\partial\overline{x_1}}+\overline{a_0}\right)\left(-\frac{\partial\overline{[\,]}}{\partial\overline{x_0}}+\overline{a_1}\right)\right)\varphi=\left\{\frac{\overline{\partial a_1}}{\partial\overline{x_1}}+\frac{\overline{\partial a_0}}{\partial\overline{x_0}}\right\}\varphi$$

Earlier, by giving absoluteness to the requirement of invariance of the mass squared to the Lorentz transformations, we obtained a system of equations for interacting fields in electrodynamics in the case when these fields exist in vector space. But we can apply this approach to interacting fields in spinor space as well. Let us analyze again the formula

$$(p_0+a_0)(p_1+a_1)-(p_1+a_1)(p_0+a_0)=\frac{\partial a_1}{\partial x_1}+\frac{\partial a_0}{\partial x_0}$$

If we deal with the field of a free particle, then

$$p_0p_1-p_1p_0=0$$

But since we want to make the invariance principle absolute, we require commutability also in the presence of the electromagnetic field

$$(p_0+a_0)(p_1+a_1)-(p_1+a_1)(p_0+a_0)=0$$

This can be achieved if we take into account the dependence of the momentum spinor components on the coordinates and impose the condition

$$\frac{\partial(p_1+a_1)}{\partial x_1}+\frac{\partial(p_0+a_0)}{\partial x_0}=\left(\frac{\partial p_1}{\partial x_1}+\frac{\partial p_0}{\partial x_0}\right)+\left(\frac{\partial a_1}{\partial x_1}+\frac{\partial a_0}{\partial x_0}\right)=0$$

As in the case of vector space, we can treat in the spirit of Newton's law equations of the form

$$\left(\frac{\partial p_1}{\partial x_1}+\frac{\partial p_0}{\partial x_0}\right)=-\left(\frac{\partial a_1}{\partial x_1}+\frac{\partial a_0}{\partial x_0}\right)$$

If an external field is applied, the momentum of the electron field changes, if the momentum of the electron changes for some reason, the electromagnetic potential is perturbed and an electromagnetic field is generated.

Earlier we defined a matrix of switches

$$K \equiv \begin{pmatrix} [p_1\overline{p_1}-\overline{p_1}p_1] & [p_1\overline{p_0}-\overline{p_0}p_1] & 0 & 0 \\ [p_0\overline{p_1}-\overline{p_1}p_0] & [p_0\overline{p_0}-\overline{p_0}p_0] & 0 & 0 \\ 0 & 0 & [p_1\overline{p_1}-\overline{p_1}p_1] & [p_1\overline{p_0}-\overline{p_0}p_1] \\ 0 & 0 & [p_0\overline{p_1}-\overline{p_1}p_0] & [p_0\overline{p_0}-\overline{p_0}p_0] \end{pmatrix}$$

$$+\begin{pmatrix} [p_3\overline{p_3}-\overline{p_3}p_3] & [p_3\overline{p_2}-\overline{p_2}p_3] & 0 & 0 \\ [p_2\overline{p_3}-\overline{p_3}p_2] & [p_2\overline{p_2}-\overline{p_2}p_2] & 0 & 0 \\ 0 & 0 & [p_3\overline{p_3}-\overline{p_3}p_3] & [p_3\overline{p_2}-\overline{p_2}p_3] \\ 0 & 0 & [p_2\overline{p_3}-\overline{p_3}p_2] & [p_2\overline{p_2}-\overline{p_2}p_2] \end{pmatrix}$$

and noted that for a free particle it is equal to the zero-point matrix. We can require that this matrix is zero also in the presence of an arbitrary field. Absolutization of this requirement gives us an additional set of equations, besides the main one (for example, the Dirac equation), to describe the interaction between the field and the charged particle in the presence of spin. It is guaranteed that the basic equation remains true both for a free particle and for a particle in an external field.

We will not use the given considerations further in the paper, leaving them as an idea requiring a separate consideration.

Let's solve the equation

$$(S^- - K)S^+\boldsymbol{\psi}(x_0,x_1,x_2,x_3) = M^2\boldsymbol{\psi}(x_0,x_1,x_2,x_3)$$

$$(S^- - K)S^+\begin{pmatrix}\psi_0(x_0,x_1,x_2,x_3)\\ \psi_1(x_0,x_1,x_2,x_3)\\ \psi_2(x_0,x_1,x_2,x_3)\\ \psi_3(x_0,x_1,x_2,x_3)\end{pmatrix} = M^2\begin{pmatrix}\psi_0(x_0,x_1,x_2,x_3)\\ \psi_1(x_0,x_1,x_2,x_3)\\ \psi_2(x_0,x_1,x_2,x_3)\\ \psi_3(x_0,x_1,x_2,x_3)\end{pmatrix}$$

$$S^- = \begin{pmatrix}-\partial_0+a_1\\ \partial_1+a_0\\ 0\\ 0\end{pmatrix}\left((-\overline{\partial_0}+\overline{a_1}),(\overline{\partial_1}+\overline{a_0}),0,0\right) + \begin{pmatrix}0\\ 0\\ -\partial_0+a_1\\ \partial_1+a_0\end{pmatrix}\left(0,0,(-\overline{\partial_0}+\overline{a_1}),(\overline{\partial_1}+\overline{a_0})\right)$$

$$+\begin{pmatrix}-\partial_2+a_3\\ \partial_3+a_2\\ 0\\ 0\end{pmatrix}\left((-\overline{\partial_2}+\overline{a_3}),(\overline{\partial_3}+\overline{a_2}),0,0\right) + \begin{pmatrix}0\\ 0\\ -\partial_2+a_3\\ \partial_3+a_2\end{pmatrix}\left(0,0(-\overline{\partial_2}+\overline{a_3})(\overline{\partial_3}+\overline{a_2})\right)$$

$$S^+ = \begin{pmatrix}-(\overline{\partial_1}+\overline{a_0})\\ (-\overline{\partial_0}+\overline{a_1})\\ 0\\ 0\end{pmatrix}(-(\partial_1+a_0),(-\partial_0+a_1),0,0) + \begin{pmatrix}0\\ 0\\ -(\overline{\partial_1}+\overline{a_0})\\ (-\overline{\partial_0}+\overline{a_1})\end{pmatrix}(0,0,-(\partial_1+a_0)_0,(-\partial_0+a_1))$$

$$+\begin{pmatrix}-(\overline{\partial_3}+\overline{a_2})\\ (-\overline{\partial_2}+\overline{a_3})\\ 0\\ 0\end{pmatrix}(-(\partial_3+a_2),(-\partial_2+a_3),0,0) + \begin{pmatrix}0\\ 0\\ -(\overline{\partial_3}+\overline{a_2})\\ (-\overline{\partial_2}+\overline{a_3})\end{pmatrix}(0,0,-(\partial_3+a_2),(-\partial_2+a_3))$$

$$K =$$

$$= \begin{pmatrix} (\partial_0-a_1)(\overline{\partial_0}-\overline{a_1})-(\overline{\partial_0}-\overline{a_1})(\partial_0-a_1) & (-\partial_0+a_1)(\overline{\partial_1}+\overline{a_0})-(\overline{\partial_1}+\overline{a_0})(-\partial_0+a_1) & 0 & 0 \\ (\partial_1+a_0)(-\overline{\partial_0}+\overline{a_1})-(-\overline{\partial_0}+\overline{a_1})(\partial_1+a_0) & (\partial_1+a_0)(\overline{\partial_1}+\overline{a_0})-(\overline{\partial_1}+\overline{a_0})(\partial_1+a_0) & 0 & 0 \\ 0 & 0 & [p_1\overline{p_1}-\overline{p_1}p_1] & [p_1\overline{p_0}-\overline{p_0}p_1] \\ 0 & 0 & [p_0\overline{p_1}-\overline{p_1}p_0] & [p_0\overline{p_0}-\overline{p_0}p_0] \end{pmatrix}$$

$$+\begin{pmatrix} (\partial_2-a_3)(\overline{\partial_2}-\overline{a_3})-(\overline{\partial_2}-\overline{a_3})(\partial_2-a_3) & (-\partial_2+a_3)(\overline{\partial_3}+\overline{a_2})-(\overline{\partial_3}+\overline{a_2})(-\partial_2+a_3) & 0 & 0 \\ (\partial_3+a_2)(-\overline{\partial_2}+\overline{a_3})-(-\overline{\partial_2}+\overline{a_3})(\partial_3+a_2) & (\partial_3+a_2)(\overline{\partial_3}+\overline{a_2})-(\overline{\partial_3}+\overline{a_2})(\partial_3+a_2) & 0 & 0 \\ 0 & 0 & [p_3\overline{p_3}-\overline{p_3}p_3] & [p_3\overline{p_2}-\overline{p_2}p_3] \\ 0 & 0 & [p_2\overline{p_3}-\overline{p_3}p_2] & [p_2\overline{p_2}-\overline{p_2}p_2] \end{pmatrix}$$

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$$= \begin{pmatrix} -\frac{\partial \overline{a_1}}{\partial x_0}+\frac{\partial \overline{a_1}}{\partial \overline{x_0}} & -\frac{\partial \overline{a_0}}{\partial x_0}-\frac{\partial \overline{a_1}}{\partial \overline{x_1}} & 0 & 0 \\ \frac{\partial \overline{a_1}}{\partial x_1}+\frac{\partial \overline{a_0}}{\partial \overline{x_0}} & \frac{\partial \overline{a_0}}{\partial x_1}-\frac{\partial \overline{a_0}}{\partial \overline{x_1}} & 0 & 0 \\ 0 & 0 & -\frac{\partial \overline{a_1}}{\partial x_0}+\frac{\partial \overline{a_1}}{\partial \overline{x_0}} & -\frac{\partial \overline{a_0}}{\partial x_0}-\frac{\partial \overline{a_1}}{\partial \overline{x_1}} \\ 0 & 0 & \frac{\partial \overline{a_1}}{\partial x_1}+\frac{\partial \overline{a_0}}{\partial \overline{x_0}} & \frac{\partial \overline{a_0}}{\partial x_1}-\frac{\partial \overline{a_0}}{\partial \overline{x_1}} \end{pmatrix}$$

$$+ \begin{pmatrix} -\frac{\partial \overline{a_3}}{\partial x_2}+\frac{\partial \overline{a_3}}{\partial \overline{x_2}} & -\frac{\partial \overline{a_2}}{\partial x_2}-\frac{\partial \overline{a_3}}{\partial \overline{x_3}} & 0 & 0 \\ \frac{\partial \overline{a_3}}{\partial x_3}+\frac{\partial \overline{a_2}}{\partial \overline{x_2}} & \frac{\partial \overline{a_2}}{\partial x_3}-\frac{\partial \overline{a_2}}{\partial \overline{x_3}} & 0 & 0 \\ 0 & 0 & -\frac{\partial \overline{a_3}}{\partial x_2}+\frac{\partial \overline{a_3}}{\partial \overline{x_2}} & -\frac{\partial \overline{a_2}}{\partial x_2}-\frac{\partial \overline{a_3}}{\partial \overline{x_3}} \\ 0 & 0 & \frac{\partial \overline{a_3}}{\partial x_3}+\frac{\partial \overline{a_2}}{\partial \overline{x_2}} & \frac{\partial \overline{a_2}}{\partial x_3}-\frac{\partial \overline{a_2}}{\partial \overline{x_3}} \end{pmatrix}$$

Since the second factor S^+ in the left-hand side of the equation has a simpler structure than the first factor, perhaps as a first step we should find the eigenvalues and eigenfunctions of the equation

$$S^+\boldsymbol{\psi}(x_0,x_1,x_2,x_3) = M^2\boldsymbol{\psi}(x_0,x_1,x_2,x_3)$$

and use the when solving the equation as a whole.

$$S^-S^+\boldsymbol{\psi} = \left\{ \begin{array}{l} \begin{pmatrix} -\partial_0 + a_1 \\ \partial_1 + a_0 \\ 0 \\ 0 \end{pmatrix}\left((-\overline{\partial_0}+\overline{a_1}),(\overline{\partial_1}+\overline{a_0}),0,0\right) + \begin{pmatrix} 0 \\ 0 \\ -\partial_0 + a_1 \\ \partial_1 + a_0 \end{pmatrix}\left(0,0,(-\overline{\partial_0}+\overline{a_1}),(\overline{\partial_1}+\overline{a_0})\right) \\ + \begin{pmatrix} -\partial_2 + a_3 \\ \partial_3 + a_2 \\ 0 \\ 0 \end{pmatrix}\left((-\overline{\partial_2}+\overline{a_3}),(\overline{\partial_3}+\overline{a_2}),0,0\right) + \begin{pmatrix} 0 \\ 0 \\ -\partial_2 + a_3 \\ \partial_3 + a_2 \end{pmatrix}\left(0,0,(-\overline{\partial_2}+\overline{a_3})(\overline{\partial_3}+\overline{a_2})\right) \end{array} \right\}$$

$$\left\{ \begin{array}{l} \begin{pmatrix} -(\overline{\partial_1}+\overline{a_0}) \\ (-\overline{\partial_0}+\overline{a_1}) \\ 0 \\ 0 \end{pmatrix}\left(-\left(\frac{\partial\psi_0}{\partial x_1}+a_0\psi_0\right)+\left(-\frac{\partial\psi_1}{\partial x_0}+a_1\psi_1\right)\right) + \begin{pmatrix} 0 \\ 0 \\ -(\overline{\partial_1}+\overline{a_0}) \\ (-\overline{\partial_0}+\overline{a_1}) \end{pmatrix}\left(-\left(\frac{\partial\psi_2}{\partial x_1}+a_0\psi_2\right)+\left(-\frac{\partial\psi_3}{\partial x_0}+a_1\psi_3\right)\right) \\ + \begin{pmatrix} -(\overline{\partial_3}+\overline{a_2}) \\ (-\overline{\partial_2}+\overline{a_3}) \\ 0 \\ 0 \end{pmatrix}\left(-\left(\frac{\partial\psi_0}{\partial x_3}+a_2\psi_0\right)+\left(-\frac{\partial\psi_1}{\partial x_2}+a_3\psi_1\right)\right) + \begin{pmatrix} 0 \\ 0 \\ -(\overline{\partial_3}+\overline{a_2}) \\ (-\overline{\partial_2}+\overline{a_3}) \end{pmatrix}\left(-\left(\frac{\partial\psi_2}{\partial x_3}+a_2\psi_2\right)+\left(-\frac{\partial\psi_3}{\partial x_2}+a_3\psi_3\right)\right) \end{array} \right\}$$

$$S^-S^+\boldsymbol{\psi} = \begin{pmatrix} -\partial_0 + a_1 \\ \partial_1 + a_0 \\ 0 \\ 0 \end{pmatrix}\left(\frac{\partial \overline{a_1}}{\partial \overline{x_1}}+\frac{\partial \overline{a_0}}{\partial \overline{x_0}}\right)\left(-\left(\frac{\partial\psi_0}{\partial x_1}+a_0\psi_0\right)+\left(-\frac{\partial\psi_1}{\partial x_0}+a_1\psi_1\right)\right)$$

$$+\begin{pmatrix} -\partial_2 + a_3 \\ \partial_3 + a_2 \\ 0 \\ 0 \end{pmatrix}\left(-(-\overline{\partial_2}+\overline{a_3})(\overline{\partial_1}+\overline{a_0})+(\overline{\partial_3}+\overline{a_2})(-\overline{\partial_0}+\overline{a_1})\right)\left(-\left(\frac{\partial\psi_0}{\partial x_1}+a_0\psi_0\right)+\left(-\frac{\partial\psi_1}{\partial x_0}+a_1\psi_1\right)\right)$$

$$+\begin{pmatrix} 0 \\ 0 \\ -\partial_0 + a_1 \\ \partial_1 + a_0 \end{pmatrix}\left(\frac{\partial \overline{a_1}}{\partial \overline{x_1}}+\frac{\partial \overline{a_0}}{\partial \overline{x_0}}\right)\left(-\left(\frac{\partial\psi_2}{\partial x_1}+a_0\psi_2\right)+\left(-\frac{\partial\psi_3}{\partial x_0}+a_1\psi_3\right)\right)$$

$$+\begin{pmatrix} 0 \\ 0 \\ -\partial_2 + a_3 \\ \partial_3 + a_2 \end{pmatrix}\left(-(-\overline{\partial_2}+\overline{a_3})(\overline{\partial_1}+\overline{a_0})+(\overline{\partial_3}+\overline{a_2})(-\overline{\partial_0}+\overline{a_1})\right)\left(-\left(\frac{\partial\psi_2}{\partial x_1}+a_0\psi_2\right)+\left(-\frac{\partial\psi_3}{\partial x_0}+a_1\psi_3\right)\right)$$

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$$+\begin{pmatrix}-\partial_0+a_1\\ \partial_1+a_0\\ 0\\ 0\end{pmatrix}\left(-(-\overline{\partial_0}+\overline{a_1})(\overline{\partial_3}+\overline{a_2})+(\overline{\partial_1}+\overline{a_0})(-\overline{\partial_2}+\overline{a_3})\right)\left(-\left(\frac{\partial\psi_0}{\partial x_3}+a_2\psi_0\right)+\left(-\frac{\partial\psi_1}{\partial x_2}+a_3\psi_1\right)\right)$$

$$+\begin{pmatrix}-\partial_2+a_3\\ \partial_3+a_2\\ 0\\ 0\end{pmatrix}\left(\frac{\partial a_3}{\partial\overline{x_3}}+\frac{\partial a_2}{\partial\overline{x_2}}\right)\left(-\left(\frac{\partial\psi_0}{\partial x_3}+a_2\psi_0\right)+\left(-\frac{\partial\psi_1}{\partial x_2}+a_3\psi_1\right)\right)$$

$$+\begin{pmatrix}0\\ 0\\ -\partial_0+a_1\\ \partial_1+a_0\end{pmatrix}\left(-(-\overline{\partial_0}+\overline{a_1})(\overline{\partial_3}+\overline{a_2})+(\overline{\partial_1}+\overline{a_0})(-\overline{\partial_2}+\overline{a_3})\right)\left(-\left(\frac{\partial\psi_2}{\partial x_3}+a_2\psi_2\right)+\left(-\frac{\partial\psi_3}{\partial x_2}+a_3\psi_3\right)\right)$$

$$+\begin{pmatrix}0\\ 0\\ -\partial_2+a_3\\ \partial_3+a_2\end{pmatrix}\left(\frac{\partial a_3}{\partial\overline{x_3}}+\frac{\partial a_2}{\partial\overline{x_2}}\right)\left(-\left(\frac{\partial\psi_2}{\partial x_3}+a_2\psi_2\right)+\left(-\frac{\partial\psi_3}{\partial x_2}+a_3\psi_3\right)\right)$$

Let's calculate the expressions included in the equation

$$\left(-(-\overline{\partial_2}+\overline{a_3})(\overline{\partial_1}+\overline{a_0})+(\overline{\partial_3}+\overline{a_2})(-\overline{\partial_0}+\overline{a_1})\right)\varphi=$$

$$(\overline{\partial_3}+\overline{a_2})(-\overline{\partial_0}+\overline{a_1})\varphi-(-\overline{\partial_2}+\overline{a_3})(\overline{\partial_1}+\overline{a_0})\varphi=$$

$$(\overline{\partial_3}+\overline{a_2})\left(-\frac{\partial\overline{\varphi}}{\partial\overline{x_0}}+\overline{a_1}\varphi\right)-(-\overline{\partial_2}+\overline{a_3})\left(\frac{\partial\overline{\varphi}}{\partial\overline{x_1}}+\overline{a_0}\varphi\right)=$$

$$\overline{\partial_3}\left(-\frac{\partial\overline{\varphi}}{\partial\overline{x_0}}\right)+\overline{\partial_3}(\overline{a_1}\varphi)+\overline{a_2}\left(-\frac{\partial\overline{\varphi}}{\partial\overline{x_0}}\right)+\overline{a_2a_1}\varphi+\overline{\partial_2}\left(\frac{\partial\overline{\varphi}}{\partial\overline{x_1}}\right)-(-\overline{\partial_2})(\overline{a_0}\varphi)-\overline{a_3}\frac{\partial\overline{\varphi}}{\partial\overline{x_1}}-\overline{a_3a_0}\varphi=$$

$$\overline{\partial_3}\left(-\frac{\partial\overline{\varphi}}{\partial\overline{x_0}}\right)+\frac{\partial a_1}{\partial\overline{x_3}}\varphi-\overline{a_2}\frac{\partial\overline{\varphi}}{\partial\overline{x_0}}+\overline{a_2a_1}\varphi+\overline{\partial_2}\left(\frac{\partial\overline{\varphi}}{\partial\overline{x_1}}\right)+\overline{a_0}\frac{\partial\overline{\varphi}}{\partial\overline{x_2}}+\frac{\partial a_0}{\partial\overline{x_2}}\varphi-\overline{a_3}\frac{\partial\overline{\varphi}}{\partial\overline{x_1}}-\overline{a_3a_0}\varphi=$$

$$\overline{\partial_2}\left(\frac{\partial\overline{\varphi}}{\partial\overline{x_1}}\right)-\overline{\partial_3}\left(\frac{\partial\overline{\varphi}}{\partial\overline{x_0}}\right)+\left[\frac{\partial a_1}{\partial\overline{x_3}}+\frac{\partial a_0}{\partial\overline{x_2}}\right]\varphi+\overline{a_1}\frac{\partial\overline{\varphi}}{\partial\overline{x_3}}-\overline{a_2}\frac{\partial\overline{\varphi}}{\partial\overline{x_0}}+\overline{a_0}\frac{\partial\overline{\varphi}}{\partial\overline{x_2}}-\overline{a_3}\frac{\partial\overline{\varphi}}{\partial\overline{x_1}}+(\overline{a_2a_1}-\overline{a_3a_0})\varphi$$

It would be interesting in this context to consider for the presented spinor model the case of a centrally symmetric electric field and to find solutions of the spinor wave equation for the hydrogen-like atom, taking into account the presence of spin at the electron. For such a model we can take

$$a_0=-i\frac{1}{\sqrt{2R}}\qquad a_1=\frac{1}{\sqrt{2R}}\qquad a_2=\frac{1}{\sqrt{2R}}\qquad a_3=-i\frac{1}{\sqrt{2R}}$$

$$R=\sqrt{\left(\frac{1}{2}(\overline{x_0}x_1+\overline{x_1}x_0+\overline{x_2}x_3+\overline{x_3}x_2)\right)^2-\left(\frac{1}{2}(-\overline{x_0}x_1+\overline{x_1}x_0-\overline{x_2}x_3+\overline{x_3}x_2)\right)^2+\left(\frac{1}{2}(\overline{x_0}x_0-\overline{x_1}x_1+\overline{x_2}x_2-\overline{x_3}x_3)\right)^2}$$

As mentioned above, we can substitute into the equation the already known exact solutions of the Dirac equation for the hydrogen-like atom by expressing the components of the coordinate vector and derivatives on them through the components of the coordinate spinor and derivatives on them. It is likely that the solution of the Dirac equation would not make the spinor equation an identity; it would be evidence that more arbitrary assumptions are made in the Dirac equation than in the spinor equation, and that the latter claims to be a better description of nature.

We can also consider the case of a constant magnetic field directed along the z-axis

$$A_0=0\qquad A_1=-\frac{1}{2}B_3X_2\qquad A_2=\frac{1}{2}B_3X_1\qquad A_3=0$$

$$X_1=\frac{1}{2}(\overline{x_0}x_1+\overline{x_1}x_0+\overline{x_2}x_3+\overline{x_3}x_2)$$

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$$X_2 = \frac{1}{2}(-i\overline{x_0}x_1 + i\overline{x_1}x_0 - i\overline{x_2}x_3 + i\overline{x_3}x_2)$$

$$A_1 = \frac{1}{2}(\overline{a_0}a_1 + \overline{a_1}a_0 + \overline{a_2}a_3 + \overline{a_3}a_2)$$

$$A_2 = \frac{1}{2}(-i\overline{a_0}a_1 + i\overline{a_1}a_0 - i\overline{a_2}a_3 + i\overline{a_3}a_2)$$

$$A_0 = \frac{1}{2}(\overline{a_0}a_0 + \overline{a_1}a_1 + \overline{a_2}a_2 + \overline{a_3}a_3)$$

$$A_3 = \frac{1}{2}(\overline{a_0}a_0 - \overline{a_1}a_1 + \overline{a_2}a_2 - \overline{a_3}a_3)$$

Let's say

$$a_0 = i\overline{x_1}\sqrt{B_3/2} \qquad a_1 = -\overline{x_0}\sqrt{B_3/2}$$

$$a_2 = i\overline{x_3}\sqrt{B_3/2} \qquad a_3 = -\overline{x_2}\sqrt{B_3/2}$$

$$A_1 = \frac{1}{4}B_3(ix_1\overline{x_0} - i\overline{x_0}x_1 + i\overline{x_3}x_2 - i\overline{x_2}x_3) = -\frac{1}{2}B_3X_2$$

$$A_2 = \frac{1}{4}B_3(\overline{x_1}x_0 + \overline{x_0}x_1 + \overline{x_3}x_2 + \overline{x_2}x_3) = \frac{1}{2}B_3X_1$$

$$A_0 = \frac{1}{4}B_3(x_1\overline{x_1} + x_0\overline{x_0} + x_3\overline{x_3} + x_2\overline{x_2}) = \frac{1}{2}B_3t$$

$$A_3 = \frac{1}{4}B_3(x_1\overline{x_1} - x_0\overline{x_0} + x_3\overline{x_3} - x_2\overline{x_2}) = \frac{1}{2}B_3X_3$$

We see that the scalar potential A_0 grows with time, but does not depend on spatial coordinates, and the vector potential does not depend on time, so that there is no electric field. In this case

$$K = \begin{pmatrix} -\frac{\partial\overline{a_1}}{\partial x_0} + \frac{\partial\overline{a_1}}{\partial\overline{x_0}} & -\frac{\partial\overline{a_0}}{\partial x_0} - \frac{\partial\overline{a_1}}{\partial\overline{x_1}} & 0 & 0 \\ \frac{\partial\overline{a_1}}{\partial x_1} + \frac{\partial\overline{a_0}}{\partial\overline{x_0}} & \frac{\partial\overline{a_0}}{\partial x_1} - \frac{\partial\overline{a_0}}{\partial\overline{x_1}} & 0 & 0 \\ 0 & 0 & -\frac{\partial\overline{a_1}}{\partial x_0} + \frac{\partial\overline{a_1}}{\partial\overline{x_0}} & -\frac{\partial\overline{a_0}}{\partial x_0} - \frac{\partial\overline{a_1}}{\partial\overline{x_1}} \\ 0 & 0 & \frac{\partial\overline{a_1}}{\partial x_1} + \frac{\partial\overline{a_0}}{\partial\overline{x_0}} & \frac{\partial\overline{a_0}}{\partial x_1} - \frac{\partial\overline{a_0}}{\partial\overline{x_1}} \end{pmatrix}$$

$$+ \begin{pmatrix} -\frac{\partial\overline{a_3}}{\partial x_2} + \frac{\partial\overline{a_3}}{\partial\overline{x_2}} & -\frac{\partial\overline{a_2}}{\partial x_2} - \frac{\partial\overline{a_3}}{\partial\overline{x_3}} & 0 & 0 \\ \frac{\partial\overline{a_3}}{\partial x_3} + \frac{\partial\overline{a_2}}{\partial\overline{x_2}} & \frac{\partial\overline{a_2}}{\partial x_3} - \frac{\partial\overline{a_2}}{\partial\overline{x_3}} & 0 & 0 \\ 0 & 0 & -\frac{\partial\overline{a_3}}{\partial x_2} + \frac{\partial\overline{a_3}}{\partial\overline{x_2}} & -\frac{\partial\overline{a_2}}{\partial x_2} - \frac{\partial\overline{a_3}}{\partial\overline{x_3}} \\ 0 & 0 & \frac{\partial\overline{a_3}}{\partial x_3} + \frac{\partial\overline{a_2}}{\partial\overline{x_2}} & \frac{\partial\overline{a_2}}{\partial x_3} - \frac{\partial\overline{a_2}}{\partial\overline{x_3}} \end{pmatrix} =$$

$$= \sqrt{\frac{B_3}{2}}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -i \end{pmatrix} + \sqrt{B_3/2}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -i \end{pmatrix} = \sqrt{2B_3}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -i \end{pmatrix}$$

The equation considered up to now is rather cumbersome, therefore we would like to have a simpler and compact relativistic invariant equation for the fermion, taking into account the presence of a half-integer spin. Such equation really exists; its derivation is given in section 4 of the paper. Here we will give its form for the electron in the presence of the electromagnetic field

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$$\left(S^R + \overline{S^R} + S_R + \overline{S_R} - 4(m + \overline{m})I\right)\boldsymbol{\varphi}(\mathbf{x}) = 0$$

where

$$S^R = \begin{pmatrix} -(-\partial_2 + a_3) \\ -(\partial_3 + a_2) \\ (-\partial_0 + a_1) \\ (\partial_1 + a_0) \end{pmatrix} \left((\partial_1 + a_0), -(-\partial_0 + a_1), (\partial_3 + a_2), -(-\partial_2 + a_3)\right)$$

$$- \begin{pmatrix} -(-\partial_0 + a_1) \\ -(\partial_1 + a_0) \\ (-\partial_2 + a_3) \\ (\partial_3 + a_2) \end{pmatrix} \left((\partial_3 + a_2), -(-\partial_2 + a_3), (\partial_1 + a_0), -(-\partial_0 + a_1)\right)$$

$$+ \begin{pmatrix} (-\partial_0 + a_1) \\ (\partial_1 + a_0) \\ (-\partial_2 + a_3) \\ (\partial_3 + a_2) \end{pmatrix} \left((\partial_3 + a_2), -(-\partial_2 + a_3), -(\partial_1 + a_0), (-\partial_0 + a_1)\right)$$

$$- \begin{pmatrix} (-\partial_2 + a_3) \\ (\partial_3 + a_2) \\ (-\partial_0 + a_1) \\ (\partial_1 + a_0) \end{pmatrix} \left((\partial_1 + a_0), -(-\partial_0 + a_1), -(\partial_3 + a_2), (-\partial_2 + a_3)\right)$$

$$S_R = \begin{pmatrix} (\partial_1 + a_0) \\ -(-\partial_0 + a_1) \\ (\partial_3 + a_2) \\ -(-\partial_2 + a_3) \end{pmatrix} \left(-(-\partial_2 + a_3), -(\partial_3 + a_2), (-\partial_0 + a_1), (\partial_1 + a_0)\right)$$

$$- \begin{pmatrix} (\partial_3 + a_2) \\ -(-\partial_2 + a_3) \\ (\partial_1 + a_0) \\ -(-\partial_0 + a_1) \end{pmatrix} \left(-(-\partial_0 + a_1), -(\partial_1 + a_0), (-\partial_2 + a_3), (\partial_3 + a_2)\right)$$

$$+ \begin{pmatrix} (\partial_3 + a_2) \\ -(-\partial_2 + a_3) \\ -(\partial_1 + a_0) \\ (-\partial_0 + a_1) \end{pmatrix} \left((-\partial_0 + a_1), (\partial_1 + a_0), (-\partial_2 + a_3), (\partial_3 + a_2)\right)$$

$$- \begin{pmatrix} (\partial_1 + a_0) \\ -(-\partial_0 + a_1) \\ -(\partial_3 + a_2) \\ (-\partial_2 + a_3) \end{pmatrix} \left((-\partial_2 + a_3), (\partial_3 + a_2), (-\partial_0 + a_1), (\partial_1 + a_0)\right)$$

In general case electric and magnetic fields are expressed through partial derivatives of components of the vector potential by components of the space vector. We also can find the expression through these fields for the derivatives of the spinor components of the electromagnetic potential by the components of the coordinate spinor. To do this, we first find all derivatives

$$\frac{\partial A_\gamma}{\partial x_\mu} = \frac{\partial A_\gamma}{\partial X_\nu}\frac{\partial X_\nu}{\partial x_\mu}$$

then express the components of the vector potential through the components of the spinor potential, substitute the components of the electric and magnetic fields instead of the derivatives of the components of the vector potential by the components of the coordinate vector, and then find the required derivatives from the resulting system of linear equations.

From general considerations taking into account the substitutions

$$\overline{p_0} \to \frac{\partial[\,]}{\partial \overline{x_1}} \qquad \overline{p_1} \to -\frac{\partial[\,]}{\partial \overline{x_0}}$$

it is possible to write the commutation relations for the components of the momentum spinor and functions from the components of the coordinate spinor

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$$\frac{\partial[\varphi]}{\partial\overline{x_1}}=\frac{1}{c}[\varphi,\overline{p_0}]=\frac{1}{c}(\varphi\overline{p_0}-\overline{p_0}\varphi)$$

$$[x_1,\overline{p_0}]=(x_1\overline{p_0}-\overline{p_0}x_1)=c\frac{\partial\overline{x_1}}{\partial\overline{x_1}}=c$$

$$[\overline{x_1},p_0]=(\overline{x_1}p_0-p_0\overline{x_1})=\bar{c}$$

$$\frac{\partial[\varphi]}{\partial\overline{x_0}}=-\frac{1}{d}[\varphi,\overline{p_1}]=-\frac{1}{d}(\varphi\overline{p_1}-\overline{p_1}\varphi)$$

$$[x_0,\overline{p_1}]=(x_0\overline{p_1}-\overline{p_1}x_0)=-d\frac{\partial\overline{x_0}}{\partial\overline{x_0}}=-d$$

$$[\overline{x_0},p_1]=(\overline{x_0}p_1-p_1\overline{x_0})=-\bar{d}$$

All other combinations commute with each other. The constant coefficients c and d possibly include a minus sign, an imaginary unit and some degree of the rationalized Planck's constant.

Let's return to the relations

$$P_0{}^2-P_1{}^2-P_2{}^2-P_3{}^2=\overline{m}m=m^2$$

$$p_1p_2-p_0p_3=m \qquad \overline{p_1p_2}-\overline{p_0p_3}=\overline{m}$$

$$(\overline{p_1p_2}-\overline{p_0p_3})(p_1p_2-p_0p_3)=P_0{}^2-P_1{}^2-P_2{}^2-P_3{}^2=\overline{m}m=m^2$$

In this form they are equivalent, but if an external field is added, a difference arises, since in one case the field is added at the vector level and in the other at the spinor level

$$(P_0-A_0)^2-(P_1-A_1)^2-(P_2-A_2)^2-(P_3-A_3)^2=m^2$$

$$\left(\overline{(p_1-a_1)(p_2-a_2)}-\overline{(p_0-a_0)(p_3-a_3)}\right)\left((p_1-a_1)(p_2-a_2)-(p_0-a_0)(p_3-a_3)\right)=m^2$$

These relations correspond to differential equations including the relativistic Schrödinger equation

$$\left(\frac{\partial^2}{\partial X_0{}^2}-\frac{\partial^2}{\partial X_1{}^2}-\frac{\partial^2}{\partial X_2{}^2}-\frac{\partial^2}{\partial X_3{}^2}\right)\varphi(X_0,X_1,X_2,X_3)=m^2\varphi(X_0,X_1,X_2,X_3)$$

$$\left(\frac{\partial}{\partial x_1}\frac{\partial}{\partial x_2}-\frac{\partial}{\partial x_0}\frac{\partial}{\partial x_3}\right)\varphi(x_0,x_1,x_2,x_3)=m\,\varphi(x_0,x_1,x_2,x_3)$$

$$\left(\frac{\partial[\,]}{\partial\overline{x_1}}\frac{\partial[\,]}{\partial\overline{x_2}}-\frac{\partial[\,]}{\partial\overline{x_0}}\frac{\partial[\,]}{\partial\overline{x_3}}\right)\varphi(x_0,x_1,x_2,x_3)=\overline{m}\,\varphi(x_0,x_1,x_2,x_3)$$

$$\left(\frac{\partial[\,]}{\partial\overline{x_1}}\frac{\partial[\,]}{\partial\overline{x_2}}-\frac{\partial[\,]}{\partial\overline{x_0}}\frac{\partial[\,]}{\partial\overline{x_3}}\right)\left(\frac{\partial}{\partial x_1}\frac{\partial}{\partial x_2}-\frac{\partial}{\partial x_0}\frac{\partial}{\partial x_3}\right)\varphi(x_0,x_1,x_2,x_3)=m^2\varphi(x_0,x_1,x_2,x_3)$$

The corresponding inhomogeneous equation is

$$\left(\left(\frac{\partial[\,]}{\partial\overline{x_1}}\frac{\partial[\,]}{\partial\overline{x_2}}-\frac{\partial[\,]}{\partial\overline{x_0}}\frac{\partial[\,]}{\partial\overline{x_3}}\right)\left(\frac{\partial}{\partial x_1}\frac{\partial}{\partial x_2}-\frac{\partial}{\partial x_0}\frac{\partial}{\partial x_3}\right)-m^2\right)\varphi(\mathbf{x})=\delta(\mathbf{x})$$

where the delta function can be represented as

$$\delta(\mathbf{x})=\int\frac{d^4p}{(2\pi)^4}\,e^{i\left(p_0x_1-p_1x_0+p_2x_3-p_3x_2+\overline{(p_0x_1-p_1x_0+p_2x_3-p_3x_2)}\right)}$$

has a solution

$$\varphi(\mathbf{x})=\int\frac{d^4p}{(2\pi)^4}\frac{e^{i\left(p_0x_1-p_1x_0+p_2x_3-p_3x_2+\overline{(p_0x_1-p_1x_0+p_2x_3-p_3x_2)}\right)}}{(\overline{p_1p_2}-\overline{p_0p_3})(p_1p_2-p_0p_3)-m^2}$$

For a free particle the eigenfunctions and eigenvalues solving these equations should coincide, but in the presence of an external field the eigenvalues and the corresponding eigenfunctions will differ because of the above mentioned difference in summation in one case of vector components and in the other case of spinor components.

While the Dirac equation is sometimes referred to as extracting the square root of the Klein-Gordon equation, here we see a different way of doing it.

Let us check the truth of the relation

$$(\overline{p_1 p_2} - \overline{p_0 p_3})(p_1 p_2 - p_0 p_3) = {P_0}^2 - {P_1}^2 - {P_2}^2 - {P_3}^2$$

$$\begin{aligned}
&4(P_0 P_0 - P_1 P_1 - P_2 P_2 - P_3 P_3) = \\
&= (\overline{p_0}p_0 + \overline{p_1}p_1 + \overline{p_2}p_2 + \overline{p_3}p_3)(\overline{p_0}p_0 + \overline{p_1}p_1 + \overline{p_2}p_2 + \overline{p_3}p_3) \\
&\quad - (\overline{p_0}p_1 + \overline{p_1}p_0 + \overline{p_2}p_3 + \overline{p_3}p_2)(\overline{p_0}p_1 + \overline{p_1}p_0 + \overline{p_2}p_3 + \overline{p_3}p_2) \\
&\quad + (-\overline{p_0}p_1 + \overline{p_1}p_0 - \overline{p_2}p_3 + \overline{p_3}p_2)(-\overline{p_0}p_1 + \overline{p_1}p_0 - \overline{p_2}p_3 + \overline{p_3}p_2) \\
&\quad - (\overline{p_0}p_0 - \overline{p_1}p_1 + \overline{p_2}p_2 - \overline{p_3}p_3)(\overline{p_0}p_0 - \overline{p_1}p_1 + \overline{p_2}p_2 - \overline{p_3}p_3)
\end{aligned}$$

$$\begin{aligned}
&(\overline{p_0}p_0 + \overline{p_1}p_1 + \overline{p_2}p_2 + \overline{p_3}p_3)(\overline{p_0}p_0 + \overline{p_1}p_1 + \overline{p_2}p_2 + \overline{p_3}p_3) \\
&\quad - (\overline{p_0}p_0 - \overline{p_1}p_1 + \overline{p_2}p_2 - \overline{p_3}p_3)(\overline{p_0}p_0 - \overline{p_1}p_1 + \overline{p_2}p_2 - \overline{p_3}p_3) = \\
&= \overline{p_0}p_0(\overline{p_1}p_1 + \overline{p_2}p_2 + \overline{p_3}p_3) + \overline{p_1}p_1(\overline{p_0}p_0 + \overline{p_2}p_2 + \overline{p_3}p_3) + \overline{p_2}p_2(\overline{p_0}p_0 + \overline{p_1}p_1 + \overline{p_3}p_3) \\
&\quad + \overline{p_3}p_3(\overline{p_0}p_0 + \overline{p_1}p_1 + \overline{p_2}p_2) - \overline{p_0}p_0(-\overline{p_1}p_1 + \overline{p_2}p_2 - \overline{p_3}p_3) + \overline{p_1}p_1(\overline{p_0}p_0 + \overline{p_2}p_2 - \overline{p_3}p_3) \\
&\quad - \overline{p_2}p_2(\overline{p_0}p_0 - \overline{p_1}p_1 - \overline{p_3}p_3) + \overline{p_3}p_3(\overline{p_0}p_0 - \overline{p_1}p_1 + \overline{p_2}p_2) \\
&= \overline{p_0}p_0(\overline{p_1}p_1 + \overline{p_3}p_3) + \overline{p_1}p_1(\overline{p_0}p_0 + \overline{p_2}p_2) + \overline{p_2}p_2(\overline{p_1}p_1 + \overline{p_3}p_3) + \overline{p_3}p_3(\overline{p_0}p_0 + \overline{p_2}p_2) \\
&\quad - \overline{p_0}p_0(-\overline{p_1}p_1 - \overline{p_3}p_3) + \overline{p_1}p_1(\overline{p_0}p_0 + \overline{p_2}p_2) - \overline{p_2}p_2(-\overline{p_1}p_1 - \overline{p_3}p_3) \\
&\quad + \overline{p_3}p_3(\overline{p_0}p_0 + \overline{p_2}p_2) \\
&= \overline{p_0}p_0(\overline{p_1}p_1 + \overline{p_3}p_3) + \overline{p_1}p_1(\overline{p_0}p_0 + \overline{p_2}p_2) + \overline{p_2}p_2(\overline{p_1}p_1 + \overline{p_3}p_3) + \overline{p_3}p_3(\overline{p_0}p_0 + \overline{p_2}p_2) \\
&\quad + \overline{p_0}p_0(\overline{p_1}p_1 + \overline{p_3}p_3) + \overline{p_1}p_1(\overline{p_0}p_0 + \overline{p_2}p_2) + \overline{p_2}p_2(\overline{p_1}p_1 + \overline{p_3}p_3) + \overline{p_3}p_3(\overline{p_0}p_0 + \overline{p_2}p_2)
\end{aligned}$$

$$\begin{aligned}
&-(\overline{p_0}p_1 + \overline{p_1}p_0 + \overline{p_2}p_3 + \overline{p_3}p_2)(\overline{p_0}p_1 + \overline{p_1}p_0 + \overline{p_2}p_3 + \overline{p_3}p_2) \\
&\quad + (-\overline{p_0}p_1 + \overline{p_1}p_0 - \overline{p_2}p_3 + \overline{p_3}p_2)(-\overline{p_0}p_1 + \overline{p_1}p_0 - \overline{p_2}p_3 + \overline{p_3}p_2) = \\
&= -\overline{p_0}p_1(\overline{p_1}p_0 + \overline{p_2}p_3 + \overline{p_3}p_2) - \overline{p_1}p_0(\overline{p_0}p_1 + \overline{p_2}p_3 + \overline{p_3}p_2) - \overline{p_2}p_3(\overline{p_0}p_1 + \overline{p_1}p_0 + \overline{p_3}p_2) \\
&\quad - \overline{p_3}p_2(\overline{p_0}p_1 + \overline{p_1}p_0 + \overline{p_2}p_3) - \overline{p_0}p_1(\overline{p_1}p_0 - \overline{p_2}p_3 + \overline{p_3}p_2) + \overline{p_1}p_0(-\overline{p_0}p_1 - \overline{p_2}p_3 + \overline{p_3}p_2) \\
&\quad - \overline{p_2}p_3(-\overline{p_0}p_1 + \overline{p_1}p_0 + \overline{p_3}p_2) + \overline{p_3}p_2(-\overline{p_0}p_1 + \overline{p_1}p_0 - \overline{p_2}p_3) \\
&= -\overline{p_0}p_1(\overline{p_1}p_0 + \overline{p_3}p_2) - \overline{p_1}p_0(\overline{p_0}p_1 + \overline{p_2}p_3) - \overline{p_2}p_3(\overline{p_1}p_0 + \overline{p_3}p_2) - \overline{p_3}p_2(\overline{p_0}p_1 + \overline{p_2}p_3) \\
&\quad - \overline{p_0}p_1(\overline{p_1}p_0 + \overline{p_3}p_2) + \overline{p_1}p_0(-\overline{p_0}p_1 - \overline{p_2}p_3) - \overline{p_2}p_3(+\overline{p_1}p_0 + \overline{p_3}p_2) \\
&\quad + \overline{p_3}p_2(-\overline{p_0}p_1 - \overline{p_2}p_3)
\end{aligned}$$

$$\begin{aligned}
&4(P_0 P_0 - P_1 P_1 - P_2 P_2 - P_3 P_3) = \\
&= \overline{p_0}p_0(\overline{p_1}p_1 + \overline{p_3}p_3) + \overline{p_1}p_1(\overline{p_0}p_0 + \overline{p_2}p_2) + \overline{p_2}p_2(\overline{p_1}p_1 + \overline{p_3}p_3) + \overline{p_3}p_3(\overline{p_0}p_0 + \overline{p_2}p_2) \\
&\quad + \overline{p_0}p_0(\overline{p_1}p_1 + \overline{p_3}p_3) + \overline{p_1}p_1(\overline{p_0}p_0 + \overline{p_2}p_2) + \overline{p_2}p_2(\overline{p_1}p_1 + \overline{p_3}p_3) + \overline{p_3}p_3(\overline{p_0}p_0 + \overline{p_2}p_2) \\
&\quad - \overline{p_0}p_1(\overline{p_1}p_0 + \overline{p_3}p_2) - \overline{p_1}p_0(\overline{p_0}p_1 + \overline{p_2}p_3) - \overline{p_2}p_3(\overline{p_1}p_0 + \overline{p_3}p_2) - \overline{p_3}p_2(\overline{p_0}p_1 + \overline{p_2}p_3) \\
&\quad - \overline{p_0}p_1(\overline{p_1}p_0 + \overline{p_3}p_2) + \overline{p_1}p_0(-\overline{p_0}p_1 - \overline{p_2}p_3) - \overline{p_2}p_3(\overline{p_1}p_0 + \overline{p_3}p_2) \\
&\quad + \overline{p_3}p_2(-\overline{p_0}p_1 - \overline{p_2}p_3)
\end{aligned}$$

To obtain this result, we did not have to make assumptions about commutability of the spinor components among themselves. Accordingly, a similar expression takes place for the phase of a plane wave in vector space

$$\begin{aligned}
&4(P_0 X_0 - P_1 X_1 - P_2 X_2 - P_3 X_3) = \\
&= \overline{p_0}p_0(\overline{x_1}x_1 + \overline{x_3}x_3) + \overline{p_1}p_1(\overline{x_0}x_0 + \overline{x_2}x_2) + \overline{p_2}p_2(\overline{x_1}x_1 + \overline{x_3}x_3) + \overline{p_3}p_3(\overline{x_0}x_0 + \overline{x_2}x_2) \\
&\quad + \overline{p_0}p_0(\overline{x_1}x_1 + \overline{x_3}x_3) + \overline{p_1}p_1(\overline{x_0}x_0 + \overline{x_2}x_2) + \overline{p_2}p_2(\overline{x_1}x_1 + \overline{x_3}x_3) + \overline{p_3}p_3(\overline{x_0}x_0 + \overline{x_2}x_2) \\
&\quad - \overline{p_0}p_1(\overline{x_1}x_0 + \overline{x_3}x_2) - \overline{p_1}p_0(\overline{x_0}x_1 + \overline{x_2}x_3) - \overline{p_2}p_3(\overline{x_1}x_0 + \overline{x_3}x_2) - \overline{p_3}p_2(\overline{x_0}x_1 + \overline{x_2}x_3) \\
&\quad - \overline{p_0}p_1(\overline{x_1}x_0 + \overline{x_3}x_2) + \overline{p_1}p_0(-\overline{x_0}x_1 - \overline{x_2}x_3) - \overline{p_2}p_3(\overline{x_1}x_0 + \overline{x_3}x_2) \\
&\quad + \overline{p_3}p_2(-\overline{x_0}x_1 - \overline{x_2}x_3)
\end{aligned}$$

Further we assume that the components of the momentum spinor commute, which takes place for a free particle, then we obtain

$$4(P_0P_0 - P_1P_1 - P_2P_2 - P_3P_3) =$$

$$= \bar{p}_0p_0(\bar{p}_1p_1 + \bar{p}_3p_3) + \bar{p}_1p_1(\bar{p}_0p_0 + \bar{p}_2p_2) + \bar{p}_2p_2(\bar{p}_1p_1 + \bar{p}_3p_3) + \bar{p}_3p_3(\bar{p}_0p_0 + \bar{p}_2p_2)$$
$$+ \bar{p}_0p_0(\bar{p}_1p_1 + \bar{p}_3p_3) + \bar{p}_1p_1(\bar{p}_0p_0 + \bar{p}_2p_2) + \bar{p}_2p_2(\bar{p}_1p_1 + \bar{p}_3p_3) + \bar{p}_3p_3(\bar{p}_0p_0 + \bar{p}_2p_2)$$
$$- \bar{p}_0p_1(\bar{p}_1p_0 + \bar{p}_3p_2) - \bar{p}_1p_0(\bar{p}_0p_1 + \bar{p}_2p_3) - \bar{p}_2p_3(\bar{p}_1p_0 + \bar{p}_3p_2) - \bar{p}_3p_2(\bar{p}_0p_1 + \bar{p}_2p_3)$$
$$- \bar{p}_0p_1(\bar{p}_1p_0 + \bar{p}_3p_2) + \bar{p}_1p_0(-\bar{p}_0p_1 - \bar{p}_2p_3) - \bar{p}_2p_3(\bar{p}_1p_0 + \bar{p}_3p_2)$$
$$+ \bar{p}_3p_2(-\bar{p}_0p_1 - \bar{p}_2p_3)$$
$$= 2\bar{p}_0p_0(\bar{p}_1p_1 + \bar{p}_3p_3) + 2\bar{p}_1p_1(\bar{p}_0p_0 + \bar{p}_2p_2) + 2\bar{p}_2p_2(\bar{p}_1p_1 + \bar{p}_3p_3) + 2\bar{p}_3p_3(\bar{p}_0p_0 + \bar{p}_2p_2)$$
$$- \bar{p}_0p_1(\bar{p}_1p_0 + \bar{p}_3p_2) - \bar{p}_1p_0(\bar{p}_0p_1 + \bar{p}_2p_3) - \bar{p}_2p_3(\bar{p}_1p_0 + \bar{p}_3p_2) - \bar{p}_3p_2(\bar{p}_0p_1 + \bar{p}_2p_3)$$
$$- \bar{p}_0p_1(\bar{p}_1p_0 + \bar{p}_3p_2) + \bar{p}_1p_0(-\bar{p}_0p_1 - \bar{p}_2p_3) - \bar{p}_2p_3(\bar{p}_1p_0 + \bar{p}_3p_2)$$
$$+ \bar{p}_3p_2(-\bar{p}_0p_1 - \bar{p}_2p_3)$$
$$= 2\bar{p}_0p_0(\bar{p}_1p_1 + \bar{p}_3p_3) + 2\bar{p}_1p_1(\bar{p}_0p_0 + \bar{p}_2p_2) + 2\bar{p}_2p_2(\bar{p}_1p_1 + \bar{p}_3p_3) + 2\bar{p}_3p_3(\bar{p}_0p_0 + \bar{p}_2p_2)$$
$$- \overline{2p_0}p_1(\bar{p}_1p_0 + \bar{p}_3p_2) - 2\bar{p}_2p_3(\bar{p}_1p_0 + \bar{p}_3p_2) - 2\bar{p}_0p_1(\bar{p}_1p_0 + \bar{p}_3p_2)$$
$$- 2\bar{p}_2p_3(\bar{p}_1p_0 + \bar{p}_3p_2)$$
$$= 2\bar{p}_0p_0(\bar{p}_1p_1 + \bar{p}_3p_3) + 2\bar{p}_1p_1(\bar{p}_0p_0 + \bar{p}_2p_2) + 2\bar{p}_2p_2(\bar{p}_1p_1 + \bar{p}_3p_3) + 2\bar{p}_3p_3(\bar{p}_0p_0 + \bar{p}_2p_2)$$
$$- \overline{2p_0}p_1(\bar{p}_1p_0 + \bar{p}_3p_2) - 2\bar{p}_2p_3(\bar{p}_1p_0 + \bar{p}_3p_2) - 2\bar{p}_0p_1(\bar{p}_1p_0 + \bar{p}_3p_2)$$
$$- 2\bar{p}_2p_3(\bar{p}_1p_0 + \bar{p}_3p_2)$$
$$= 2\bar{p}_0p_0(\bar{p}_3p_3) + 2\bar{p}_1p_1(\bar{p}_2p_2) + 2\bar{p}_2p_2(\bar{p}_1p_1) + 2\bar{p}_3p_3(\bar{p}_0p_0) - \overline{2p_0}p_1(\bar{p}_3p_2) - 2\bar{p}_2p_3(\bar{p}_1p_0)$$
$$- 2\bar{p}_0p_1(\bar{p}_3p_2) - 2\bar{p}_2p_3(\bar{p}_1p_0)$$
$$= 4\bar{p}_0p_0(\bar{p}_3p_3) + 4\bar{p}_1p_1(\bar{p}_2p_2) - 4\bar{p}_0p_1(\bar{p}_3p_2) - 4\bar{p}_2p_3(\bar{p}_1p_0)$$

On the other hand, we can write

$$\bar{m}m = \overline{(p_1p_2 - p_0p_3)}(p_1p_2 - p_0p_3) = \overline{p_1p_2}p_1p_2 - \overline{p_1p_2}p_0p_3 - \overline{p_0p_3}p_1p_2 + \overline{p_0p_3}p_0p_3$$

Thus, the results of calculations coincide.

Let us compare the phases of plane waves in vector and spinor spaces. Let us hypothesize that the plane wave in spinor space has a more complicated form than it was supposed earlier in the paper, namely, it contains an additional conjugate multiplier

$$exp\left(-i\,\overline{(p_0x_1 - p_1x_0 + p_2x_3 - p_3x_2)}(p_0x_1 - p_1x_0 + p_2x_3 - p_3x_2)\right)$$

The phase of the wave in this form is closer to the generally accepted phase of a plane wave in vector space. But the phases calculated by two methods do not coincide with each other, although both of them are invariant under Lorentz transformations

$$\overline{(p_0x_1 - p_1x_0 + p_2x_3 - p_3x_2)}(p_0x_1 - p_1x_0 + p_2x_3 - p_3x_2) \neq P_0X_0 - P_1X_1 - P_2X_2 - P_3X_3$$

Let us slightly modify the expression for the phase of the plane wave

$$\left(\frac{\partial}{\partial x_1}\frac{\partial}{\partial x_2} - \frac{\partial}{\partial x_0}\frac{\partial}{\partial x_3}\right) exp[(p_0x_1 - p_1x_0 + \overline{p_2x_3} - \overline{p_3x_2})(\overline{p_0x_1} - \overline{p_1x_0} + p_2x_3 - p_3x_2)] =$$
$$\Big((-p_3)p_0(p_0x_1 - p_1x_0 + \overline{p_2x_3} - \overline{p_3x_2})(\overline{p_0x_1} - \overline{p_1x_0} + p_2x_3 - p_3x_2) + p_0$$
$$- p_2(-p_1)(\overline{p_0x_1} - \overline{p_1x_0} + p_2x_3 - p_3x_2)(p_0x_1 - p_1x_0 + \overline{p_2x_3} - \overline{p_3x_2}) - p_1\Big)$$
$$exp[(p_0x_1 - p_1x_0 + \overline{p_2x_3} - \overline{p_3x_2})(\overline{p_0x_1} - \overline{p_1x_0} + p_2x_3 - p_3x_2)]$$
$$= \left((-p_3)p_0f(\mathbf{x})\overline{f(\mathbf{x})} + p_0 - p_2(-p_1)\overline{f(\mathbf{x})}f(\mathbf{x}) - p_1\right)$$
$$exp[(p_0x_1 - p_1x_0 + \overline{p_2x_3} - \overline{p_3x_2})(\overline{p_0x_1} - \overline{p_1x_0} + p_2x_3 - p_3x_2)]$$
$$= \left((p_2p_1 - p_3p_0)f(\mathbf{x})\overline{f(\mathbf{x})} + p_0 - p_1\right)$$

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$$exp[(p_0x_1 - p_1x_0 + \overline{p_2x_3} - \overline{p_3x_2})(\overline{p_0x_1} - \overline{p_1x_0} + p_2x_3 - p_3x_2)]$$

where

$$f(\mathbf{x}) \equiv (p_0x_1 - p_1x_0 + \overline{p_2x_3} - \overline{p_3x_2})$$

Let's change the order of derivatives

$$\left(\frac{\partial}{\partial x_2}\frac{\partial}{\partial x_1} - \frac{\partial}{\partial x_3}\frac{\partial}{\partial x_0}\right) exp[(p_0x_1 - p_1x_0 + \overline{p_2x_3} - \overline{p_3x_2})(\overline{p_0x_1} - \overline{p_1x_0} + p_2x_3 - p_3x_2)] =$$

$$(p_0(-p_3)(\overline{p_0x_1} - \overline{p_1x_0} + p_2x_3 - p_3x_2)(p_0x_1 - p_1x_0 + \overline{p_2x_3} - \overline{p_3x_2}) - p_3 - (-p_1)p_2(p_0x_1 - p_1x_0 + \overline{p_2x_3} - \overline{p_3x_2})(\overline{p_0x_1} - \overline{p_1x_0} + p_2x_3 - p_3x_2) + p_2)$$

$$exp[(p_0x_1 - p_1x_0 + \overline{p_2x_3} - \overline{p_3x_2})(\overline{p_0x_1} - \overline{p_1x_0} + p_2x_3 - p_3x_2)]$$

$$= \left(p_0(-p_3)\overline{f(\mathbf{x})}f(\mathbf{x}) - p_3 - (-p_1)p_2 f(\mathbf{x})\overline{f(\mathbf{x})} + p_2\right)$$

$$exp[(p_0x_1 - p_1x_0 + \overline{p_2x_3} - \overline{p_3x_2})(\overline{p_0x_1} - \overline{p_1x_0} + p_2x_3 - p_3x_2)] =$$

$$= \left((p_1p_2 - p_0p_3)f(\mathbf{x})\overline{f(\mathbf{x})} - p_3 + p_2\right)$$

$$exp[(p_0x_1 - p_1x_0 + \overline{p_2x_3} - \overline{p_3x_2})(\overline{p_0x_1} - \overline{p_1x_0} + p_2x_3 - p_3x_2)]$$

and write the difference of the two equations

$$\left(\left(\frac{\partial}{\partial x_1}\frac{\partial}{\partial x_2} - \frac{\partial}{\partial x_0}\frac{\partial}{\partial x_3}\right) - \left(\frac{\partial}{\partial x_2}\frac{\partial}{\partial x_1} - \frac{\partial}{\partial x_3}\frac{\partial}{\partial x_0}\right)\right)$$

$$exp[(p_0x_1 - p_1x_0 + \overline{p_2x_3} - \overline{p_3x_2})(\overline{p_0x_1} - \overline{p_1x_0} + p_2x_3 - p_3x_2)] =$$

$$\left[\left((p_2p_1 - p_3p_0)f(\mathbf{x})\overline{f(\mathbf{x})} + p_0 - p_1\right) - \left((p_1p_2 - p_0p_3)f(\mathbf{x})\overline{f(\mathbf{x})} - p_3 + p_2\right)\right]$$

$$exp[(p_0x_1 - p_1x_0 + \overline{p_2x_3} - \overline{p_3x_2})(\overline{p_0x_1} - \overline{p_1x_0} + p_2x_3 - p_3x_2)]$$

$$= [p_0 - p_2 + p_3 - p_1]$$

$$exp[(p_0x_1 - p_1x_0 + \overline{p_2x_3} - \overline{p_3x_2})(\overline{p_0x_1} - \overline{p_1x_0} + p_2x_3 - p_3x_2)]$$

Add an imaginary unit to the phase

$$\left(\left(\frac{\partial}{\partial x_1}\frac{\partial}{\partial x_2} - \frac{\partial}{\partial x_0}\frac{\partial}{\partial x_3}\right) - \left(\frac{\partial}{\partial x_2}\frac{\partial}{\partial x_1} - \frac{\partial}{\partial x_3}\frac{\partial}{\partial x_0}\right)\right)$$

$$exp[-i(p_0x_1 - p_1x_0 + \overline{p_2x_3} - \overline{p_3x_2})(\overline{p_0x_1} - \overline{p_1x_0} + p_2x_3 - p_3x_2)] =$$

$$= \left[\left(-(p_2p_1 - p_3p_0)f(\mathbf{x})\overline{f(\mathbf{x})} - ip_0 + ip_1\right) - \left(-(p_1p_2 - p_0p_3)f(\mathbf{x})\overline{f(\mathbf{x})} + ip_3 - ip_2\right)\right]$$

$$exp[-i(p_0x_1 - p_1x_0 + \overline{p_2x_3} - \overline{p_3x_2})(\overline{p_0x_1} - \overline{p_1x_0} + p_2x_3 - p_3x_2)] =$$

$$= i[p_2 - p_0 + p_1 - p_3]$$

$$exp[-i(p_0x_1 - p_1x_0 + \overline{p_2x_3} - \overline{p_3x_2})(\overline{p_0x_1} - \overline{p_1x_0} + p_2x_3 - p_3x_2)]$$

Thus, we obtained a differential equation with an eigenvalue independent of coordinates

$$i[p_2 - p_0 + p_1 - p_3]$$

to which corresponds the eigenfunction

$$exp[-i(p_0x_1 - p_1x_0 + \overline{p_2x_3} - \overline{p_3x_2})(\overline{p_0x_1} - \overline{p_1x_0} + p_2x_3 - p_3x_2)]$$

which is a plane wave with imaginary phase and bounded amplitude.

Now we can define the function

$$D(\mathbf{x}) = \int \frac{d^4p}{(2\pi)^4} \frac{exp[-i(p_0x_1 - p_1x_0 + \overline{p_2x_3} - \overline{p_3x_2})(\overline{p_0x_1} - \overline{p_1x_0} + p_2x_3 - p_3x_2)]}{i[p_2 - p_0 + p_1 - p_3]}$$

which satisfies to equation

$$\left(\left(\frac{\partial}{\partial x_1}\frac{\partial}{\partial x_2} - \frac{\partial}{\partial x_0}\frac{\partial}{\partial x_3}\right) - \left(\frac{\partial}{\partial x_2}\frac{\partial}{\partial x_1} - \frac{\partial}{\partial x_3}\frac{\partial}{\partial x_0}\right)\right) D(\mathbf{x}) = \delta(\mathbf{x})$$

where

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$$\delta(\mathbf{x}) = \int \frac{d^4p}{(2\pi)^4} exp[-i(p_0x_1 - p_1x_0 + \overline{p_2x_3} - \overline{p_3x_2})(\overline{p_0x_1} - \overline{p_1x_0} + p_2x_3 - p_3x_2)]$$

thus, $D(\mathbf{x})$ has the properties of the Green's function.

4. Path Integral and Second Quantization in Spinor Coordinate Space

Based on the above, we can modify the theory of the path integral. We will consider it in the notations in which it is presented in [9]. For a free scalar field with sources J(X) the path integral has the form

$$Z(J) = \int D\varphi(\mathbf{X})\, exp(i\mathcal{S}(\varphi(\mathbf{X}))) = \int D\varphi(\mathbf{X})\, exp\left(i\int d^4X\{\mathcal{L}(\varphi(\mathbf{X})) + J(\mathbf{X})\varphi(\mathbf{X})\}\right)$$

$$= \int D\varphi(\mathbf{X})\, exp\left(i\int d^4X\left\{\frac{1}{2}\left(\left(\frac{\partial\varphi}{\partial X_0}\right)^2 - \left(\frac{\partial\varphi}{\partial X_1}\right)^2 - \left(\frac{\partial\varphi}{\partial X_2}\right)^2 - \left(\frac{\partial\varphi}{\partial X_3}\right)^2 - m^2\varphi(\mathbf{X})^2\right)\right.\right.$$

$$\left.\left. + J(\mathbf{X})\varphi(\mathbf{X})\right\}\right)$$

It includes the action of

$$\mathcal{S}(\varphi(\mathbf{X}) = \int d^4X\{\mathcal{L}(\varphi(\mathbf{X})) + J(\mathbf{X})\varphi(\mathbf{X})\}$$

and the Lagrangian density for the free field

$$\mathcal{L}(\varphi(\mathbf{X}) = \frac{1}{2}\left(\left(\frac{\partial\varphi}{\partial X_0}\right)^2 - \left(\frac{\partial\varphi}{\partial X_1}\right)^2 - \left(\frac{\partial\varphi}{\partial X_2}\right)^2 - \left(\frac{\partial\varphi}{\partial X_3}\right)^2 - m^2\varphi(\mathbf{X})^2\right)$$

For convenience and clarity, the following notations are introduced

$$(\partial\varphi)^2 = \partial_\mu\varphi\partial^\mu\varphi = \eta^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi = (\partial_0\varphi)^2 - (\partial_1\varphi)^2 - (\partial_2\varphi)^2 - (\partial_3\varphi)^2$$

$$= \left(\frac{\partial\varphi}{\partial X_0}\right)^2 - \left(\frac{\partial\varphi}{\partial X_1}\right)^2 - \left(\frac{\partial\varphi}{\partial X_2}\right)^2 - \left(\frac{\partial\varphi}{\partial X_3}\right)^2$$

$$\partial_\mu \equiv \frac{\partial}{\partial X_\mu}$$

For the general case the Lagrangian density has the form

$$\mathcal{L}(\varphi(\mathbf{X})) = \frac{1}{2}(\partial\varphi(\mathbf{X}))^2 - V(\varphi(\mathbf{X}))$$

where $V(\varphi(\mathbf{X}))$-polynomial over the field $\varphi(\mathbf{X})$.

Substituting the Lagrangian into the Euler equation

$$\partial_\mu\frac{\delta\mathcal{L}}{\delta(\partial_\mu\varphi)} - \frac{\delta\mathcal{L}}{\delta\varphi} = 0$$

the field equation of motion is obtained.

The free field theory is developed for a special kind of polynomial

$$V(\varphi(X)) = \frac{1}{2}m^2\varphi^2$$

$$\mathcal{L}(\varphi) = \frac{1}{2}[(\partial\varphi)^2 - m^2\varphi^2]$$

$$\frac{\delta\mathcal{L}}{\delta(\partial_\mu\varphi)} = \frac{1}{2}\frac{\delta(\partial\varphi)^2}{\delta(\partial_\mu\varphi)} = \frac{1}{2}\frac{\delta[(\partial_0\varphi)^2 - (\partial_1\varphi)^2 - (\partial_2\varphi)^2 - (\partial_3\varphi)^2]}{\delta(\partial_\mu\varphi)} = \pm\frac{1}{2}\frac{\delta(\partial_\mu\varphi)^2}{\delta(\partial_\mu\varphi)} = \pm\partial_\mu\varphi$$

$$\frac{\delta\mathcal{L}}{\delta\varphi} = \frac{1}{2}\left[-m^2\frac{\delta\varphi^2}{\delta\varphi}\right] = -m^2\varphi$$

In summary, Euler's equation defines the equation of motion

$$\partial_0(\partial_0\varphi) - \partial_0(\partial_0\varphi) - \partial_0(\partial_0\varphi) - \partial_0(\partial_0\varphi) + m^2\varphi = 0$$

$$\partial_0^2\varphi - \partial_1^2\varphi - \partial_2^2\varphi - \partial_3^2\varphi + m^2\varphi = 0$$
$$\partial^2\varphi \equiv \partial_0^2\varphi - \partial_1^2\varphi - \partial_2^2\varphi - \partial_3^2\varphi$$
$$\partial^2\varphi + m^2\varphi = 0$$
$$(\partial^2 + m^2)\varphi = 0$$

The notations used here are

$$\partial^2\varphi \equiv \partial_0^2\varphi - \partial_1^2\varphi - \partial_2^2\varphi - \partial_3^2\varphi$$
$$\partial^2 \equiv \partial_0^2 - \partial_1^2 - \partial_2^2 - \partial_3^2$$

Thus, there is a correspondence of the Lagrangian and the equation of motion for the free field

$$\mathcal{L}(\varphi(\mathbf{X})) = \frac{1}{2}\left[(\partial_0\varphi(\mathbf{X}))^2 - (\partial_1\varphi(\mathbf{X}))^2 - (\partial_2\varphi(\mathbf{X}))^2 - (\partial_3\varphi(\mathbf{X}))^2 - m^2\varphi(\mathbf{X})^2\right]$$

$$\mathcal{L}(\varphi) = \frac{1}{2}[(\partial\varphi)^2 - m^2\varphi^2]$$

$$\mathcal{L}(\varphi) = \frac{1}{2}[(\partial_0\varphi)^2 - (\partial_1\varphi)^2 - (\partial_2\varphi)^2 - (\partial_3\varphi)^2 - m^2\varphi^2]$$

$$\partial_0^2\varphi(\mathbf{X}) - \partial_1^2\varphi(\mathbf{X}) - \partial_2^2\varphi(\mathbf{X}) - \partial_3^2\varphi(\mathbf{X}) + m^2\varphi(\mathbf{X}) = 0$$

Our proposal is to replace the Lagrangian in vector coordinate space by the Lagrangian in spinor coordinate space. For this purpose, we use the equation of motion in spinor coordinate space and we want to find the Lagrangian for which the Euler equation defines this equation of motion

$$\left(\frac{\partial}{\partial x_1}\frac{\partial}{\partial x_2} - \frac{\partial}{\partial x_0}\frac{\partial}{\partial x_3}\right)\varphi(\mathbf{x}) + m\varphi(\mathbf{x}) = 0$$
$$(\partial_1\partial_2 - \partial_0\partial_3)\varphi(\mathbf{x}) + m\varphi(\mathbf{x}) = 0$$
$$\partial_\mu\frac{\delta\mathcal{L}}{\delta(\partial_\mu\varphi(\mathbf{x}))} - \frac{\delta\mathcal{L}}{\delta\varphi(\mathbf{x})} = 0$$

For the sake of clarity, we use the same notation for the spinor coordinate derivative as for the vector coordinate derivative; the context allows us to distinguish between them

$$\partial_\mu \equiv \frac{\partial}{\partial x_\mu}$$

et us write the Lagrangian plus sources in the form

$$\mathcal{L}(\varphi(\mathbf{x})) = \frac{1}{2}[\partial_1\varphi(\mathbf{x})\partial_2\varphi(\mathbf{x}) - \partial_0\varphi(\mathbf{x})\partial_3\varphi(\mathbf{x}))] - V(\varphi(\mathbf{x})) + j(\mathbf{x})\varphi(\mathbf{x})$$

And let's substitute the Lagrangian into the Euler equation

$$\partial_0\frac{\delta\mathcal{L}}{\delta(\partial_0)} + \partial_1\frac{\delta\mathcal{L}}{\delta(\partial_1)} + \partial_2\frac{\delta\mathcal{L}}{\delta(\partial_2)} + \partial_3\frac{\delta\mathcal{L}}{\delta(\partial_3)} - \frac{\delta\mathcal{L}}{\delta\varphi} = 0$$

$$\frac{1}{2}[-\partial_0(\partial_3\varphi(\mathbf{x})) + \partial_1(\partial_2\varphi(\mathbf{x})) + \partial_2(\partial_1\varphi(\mathbf{x})) - \partial_3(\partial_0\varphi(\mathbf{x}))] - \frac{\delta\mathcal{L}}{\delta\varphi} = 0$$

For the case of a free field the derivative operators commute, so we can write

$$\partial_1\partial_2\varphi(\mathbf{x}) - \partial_0\partial_3\varphi(\mathbf{x}) - \left(\frac{\delta\mathcal{L}}{\delta\varphi}\right) = 0$$

$$\left(\frac{\partial}{\partial x_1}\frac{\partial}{\partial x_2} - \frac{\partial}{\partial x_0}\frac{\partial}{\partial x_3}\right)\varphi(\mathbf{x}) - \left(\frac{\delta\mathcal{L}}{\delta\varphi}\right) = 0$$

$$\left(\frac{\partial}{\partial x_1}\frac{\partial}{\partial x_2} - \frac{\partial}{\partial x_0}\frac{\partial}{\partial x_3}\right)\varphi(\mathbf{x}) - \left(\frac{\delta V(\varphi)}{\delta\varphi}\right) = 0$$

It is pleasant that the Euler equation in invariant form works also in this situation, so that we obtain the desired form of the equation of motion in the spinor coordinate space. It is important that

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the proposed Lagrangian has a relativistically invariant form, even in the general case, and not only at commuting derivatives. The polynomial has the form

$$V(\varphi) = \frac{1}{2} m\varphi(\mathbf{x})^2 + \frac{g}{3!}\varphi(\mathbf{x})^3 + \frac{\lambda}{4!}\varphi(\mathbf{x})^4 + \cdots$$

In the case of a free field we restrict ourselves to the first term of the polynomial

$$V(\varphi) = \frac{1}{2} m\varphi(\mathbf{x})^2$$

Then the Lagrangian density and the equation of motion for the scalar field in spinor coordinate space have the form

$$\mathcal{L}\big(\varphi(\mathbf{x})\big) = \frac{1}{2}[\partial_1\varphi(\mathbf{x})\partial_2\varphi(\mathbf{x}) - \partial_0\varphi(\mathbf{x})\partial_3\varphi(\mathbf{x}))] - \frac{1}{2} m\varphi(\mathbf{x})^2$$

$$\frac{1}{2}(\partial_1\partial_2 + \partial_2\partial_1 - \partial_0\partial_3 - \partial_3\partial_0)\varphi(\mathbf{x}) + m\varphi(\mathbf{x}) = 0$$

For a free field when the derivative operators commute, we obtain

$$(\partial_1\partial_2 - \partial_0\partial_3)\varphi(\mathbf{x}) + m\varphi(\mathbf{x}) = 0$$

In the spinor equation of motion there is a plus sign before the mass, although in the rest of the paper there was a minus sign. To return to the minus sign it is enough to put a plus sign in front of the polynomial $V(\varphi)$ in the Lagrangian.

Now we have to find the path integral, which, along with the Lagrangian, includes the sources

$$Z(j) = \int D\varphi(\mathbf{x})\, exp\left(i\int d^4x\{\mathcal{L}(\varphi(\mathbf{x})) + j(\mathbf{x})\varphi(\mathbf{x})\}\right)$$

$$= \int D\varphi(\mathbf{x})\, exp\left(i\int d^4x\left\{\frac{1}{2}[\partial_1\varphi(\mathbf{x})\partial_2\varphi(\mathbf{x}) - \partial_0\varphi(\mathbf{x})\partial_3\varphi(\mathbf{x}))] - \frac{1}{2}m\varphi(\mathbf{x})^2 + j(\mathbf{x})\varphi(\mathbf{x})\right\}\right)$$

The components of spinors are complex, and we have already noted that the derivatives on complex variables are applied to the degree functions, which, most likely, can describe physical fields, respectively, the finding of an indefinite integral for the function of a complex variable can be treated similarly, i.e. as an indefinite integral from the degree function.

It is possible to recover Planck's constant, which provides a transition to the classical limit

$$Z(j) = \int D\varphi(\mathbf{x})\, exp\left(\frac{i}{\hbar}\int d^4x\mathcal{L}(\varphi(\mathbf{x})\right)$$

One of the steps in computing the path integral in [9] is to find the free propagator from Eq.

$$-(\partial^2 + m^2)D(\mathbf{X} - \mathbf{Y}) = \delta(\mathbf{X} - \mathbf{Y})$$

the solution of which has the form

$$D(\mathbf{X} - \mathbf{Y}) = \int \frac{d^4P}{(2\pi)^4}\frac{e^{i\mathbf{P}(\mathbf{X}-\mathbf{Y})}}{P^2 - m^2 + i\varepsilon}$$

herewith

$$\delta(\mathbf{X} - \mathbf{Y}) = \int \frac{d^4P}{(2\pi)^4} e^{i\mathbf{P}(\mathbf{X}-\mathbf{Y})}$$

In our case, we want to find

$$Z(j) = \int D\varphi(\mathbf{x})\, exp\left(i\int d^4x\left\{\frac{1}{2}[\partial_1\varphi(\mathbf{x})\partial_2\varphi(\mathbf{x}) - \partial_0\varphi(\mathbf{x})\partial_3\varphi(\mathbf{x}))] - \frac{1}{2}m\varphi(\mathbf{x})^2 + j(\mathbf{x})\varphi(\mathbf{x})\right\}\right)$$

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After integration by parts by analogy with [[9], Chapter 1.3] we obtain for the special case of a free field

$$Z(j) = \int D\varphi(\mathbf{x})\, exp\left(i\int d^4x\left\{-\frac{1}{2}\varphi(\mathbf{x})[(\partial_1\partial_2 - \partial_0\partial_3) + m]\varphi(\mathbf{x}) + j(\mathbf{x})\varphi(\mathbf{x})\right\}\right)$$

In the process of calculation, it is necessary to find the solution of the equation

$$-(\partial_1\partial_2 - \partial_0\partial_3 + m)D(\mathbf{x}-\mathbf{y}) = \delta(\mathbf{x}-\mathbf{y})$$

For this purpose, we pass to the momentum space by means of the integral transformation

$$\varphi(\mathbf{x}) = \int \frac{d^4p}{(2\pi)^4}\,\varphi(\mathbf{p})e^{i(p_0x_1 - p_1x_0 + p_2x_3 - p_3x_2 + \overline{(\mathbf{p},\mathbf{x})})}$$

The assumed propagator has the form

$$D(\mathbf{x}-\mathbf{y}) = \int \frac{d^4p}{(2\pi)^4}\,\frac{e^{i(p_0(x_1-y_1) - p_1(x_0-y_0) + p_2(x_3-y_3) - p_3(x_2-y_2) + \overline{(\mathbf{p},\mathbf{x}-\mathbf{y})})}}{(p_1p_2 - p_0p_3) - m}$$

which is verified by substitution into Eq. Here it is assumed that the representation of the delta function

$$\delta(\mathbf{x}-\mathbf{y}) = \int \frac{d^4p}{(2\pi)^4}\,e^{i(p_0(x_1-y_1) - p_1(x_0-y_0) + p_2(x_3-y_3) - p_3(x_2-y_2) + \overline{(\mathbf{p},\mathbf{x}-\mathbf{y})})}$$

We added a conjugate phase to the exponent

$$\overline{(\mathbf{p},\mathbf{x})} = \overline{p_0x_1} - \overline{p_1x_0} + \overline{p_2x_3} - \overline{p_3x_2}$$

which, on the one hand, provides convergence of the integral, and on the other hand, it does not affect the result of differentiation on variables x_μ.

We note at once that there is no simple correspondence between the so defined phase of a plane wave in spinor space and the phase of a plane wave in vector space, e.g.

$$\left(p_0(x_1-y_1) - p_1(x_0-y_0) + p_2(x_3-y_3) - p_3(x_2-y_2) + \overline{(\mathbf{p},\mathbf{x}-\mathbf{y})}\right)^2 \neq P_0X_0 - P_1X_1 - P_2X_2 - P_3X_3$$

but both parts of the inequality are invariant under Lorentz transformations.

One can see the difference between the propagators, since in one case m^2 is real and positive, while in spinor space m is complex in general. We can use the relation

$$\frac{1}{(p_1p_2 - p_0p_3) - m} = \frac{\overline{(p_1p_2 - p_0p_3)} + \overline{m}}{\left(\overline{(p_1p_2 - p_0p_3)} + \overline{m}\ \right)\left((p_1p_2 - p_0p_3) - m\ \right)} = \frac{\overline{(p_1p_2 - p_0p_3)} + \overline{m}}{P^2 - m^2 + (\overline{m} - m)(p_1p_2 - p_0p_3)}$$

$$= \frac{\overline{(p_1p_2 - p_0p_3)} + \overline{m}}{P^2 - m^2}$$

where

$$P^2 \equiv P_0^2 - P_1^2 - P_2^2 - P_3^2$$

in which it is taken into account that the fermion mass is real. Now the propagator has the form

$$D(\mathbf{x}) = \int \frac{d^4p}{(2\pi)^4}\frac{\overline{(p_1p_2 - p_0p_3)} + \overline{m}}{P^2 - m^2}e^{i(p_0x_1 - p_1x_0 + p_2x_3 - p_3x_2 + \overline{(\mathbf{p},\mathbf{x})})}$$

The derivatives of the scalar field on spinor coordinates can be expressed through the derivatives on vector coordinates

$$\partial_0\varphi(\mathbf{x}) = \frac{\partial\varphi(\mathbf{x})}{\partial x_0} = \frac{\partial\varphi\big(\mathbf{X}(\mathbf{x})\big)}{\partial x_0}$$

$$= \frac{\partial\varphi\big(\mathbf{X}(\mathbf{x})\big)}{\partial X_0}\frac{\partial X_0(\mathbf{x})}{\partial x_0} + \frac{\partial\varphi\big(\mathbf{X}(\mathbf{x})\big)}{\partial X_1}\frac{\partial X_1(\mathbf{x})}{\partial x_0} + \frac{\partial\varphi\big(\mathbf{X}(\mathbf{x})\big)}{\partial X_2}\frac{\partial X_2(\mathbf{x})}{\partial x_0} + \frac{\partial\varphi\big(\mathbf{X}(\mathbf{x})\big)}{\partial X_3}\frac{\partial X_3(\mathbf{x})}{\partial x_0}$$

$$= \frac{\partial\varphi}{\partial X_0}\frac{\overline{x_0}}{2} + \frac{\partial\varphi}{\partial X_1}\frac{\overline{x_1}}{2} + \frac{\partial\varphi}{\partial X_2}\frac{i\overline{x_1}}{2} + \frac{\partial\varphi}{\partial X_3}\frac{\overline{x_0}}{2}$$

$$\partial_0\varphi(\mathbf{x}) = \frac{\partial\varphi}{\partial X_0}\frac{\overline{x_0}}{2} + \frac{\partial\varphi}{\partial X_1}\frac{\overline{x_1}}{2} + \frac{\partial\varphi}{\partial X_2}\frac{i\overline{x_1}}{2} + \frac{\partial\varphi}{\partial X_3}\frac{\overline{x_0}}{2}$$

$$\partial_1 \varphi(\mathbf{x}) = \frac{\partial \varphi}{\partial X_0}\frac{\overline{x_1}}{2} + \frac{\partial \varphi}{\partial X_1}\frac{\overline{x_0}}{2} - \frac{\partial \varphi}{\partial X_2}\frac{i\overline{x_0}}{2} - \frac{\partial \varphi}{\partial X_3}\frac{\overline{x_1}}{2}$$

$$\partial_2 \varphi(\mathbf{x}) = \frac{\partial \varphi}{\partial X_0}\frac{\overline{x_2}}{2} + \frac{\partial \varphi}{\partial X_1}\frac{\overline{x_3}}{2} + \frac{\partial \varphi}{\partial X_2}\frac{i\overline{x_3}}{2} + \frac{\partial \varphi}{\partial X_3}\frac{\overline{x_2}}{2}$$

$$\partial_3 \varphi(\mathbf{x}) = \frac{\partial \varphi}{\partial X_0}\frac{\overline{x_3}}{2} + \frac{\partial \varphi}{\partial X_1}\frac{\overline{x_2}}{2} - \frac{\partial \varphi}{\partial X_2}\frac{i\overline{x_2}}{2} - \frac{\partial \varphi}{\partial X_3}\frac{\overline{x_3}}{2}$$

If in the right part to represent the wave function as a plane wave in vector space

$$\varphi(\mathbf{X}) = exp(P_0X_0 - P_1X_1 - P_2X_2 - P_3X_3)$$

then in the left part it should be represented as a plane wave of a special form in spinor space

$$\varphi(\mathbf{x}) = exp\left(\overline{(p_0x_1 - p_1x_0 + p_2x_3 - p_3x_2)}(p_0x_1 - p_1x_0 + p_2x_3 - p_3x_2)\right)$$

Only in this case the left and right parts will be dimensionally consistent, e.g.

$$\partial_1 \varphi(\mathbf{x}) = (\overline{p_0x_1} - \overline{p_1x_0} + \overline{p_2x_3} - \overline{p_3x_2})p_0$$

$$\frac{\partial \varphi}{\partial X_0}\frac{\overline{x_0}}{2} = P_0\frac{\overline{x_0}}{2} = \frac{1}{4}(\overline{p_0}p_0 + \overline{p_1}p_1 + \overline{p_2}p_2 + \overline{p_3}p_3)\overline{x_0}$$

In any case, a complete coincidence will not be obtained due to the mismatch of dimensionless exponents of the exponents

$$\overline{(p_0x_1 - p_1x_0 + p_2x_3 - p_3x_2)}(p_0x_1 - p_1x_0 + p_2x_3 - p_3x_2) \neq P_0X_0 - P_1X_1 - P_2X_2 - P_3X_3$$

Since we call the field under consideration a scalar field, we expect its value to be invariant to Lorentz transformations. But how to formalize this statement and to what exactly does this transformation apply? We propose to consider that the value of a scalar field is the scalar product of the representatives of a spinor field, which is the most fundamental field in nature, and vectors, tensors and, among others, scalars are formed from the spinors representing it. The scalar product is defined by means of the metric tensor of the spinor space. From any two spinors we can obtain a scalar, in general the complex case. But if we want to obtain a scalar with real values, we must impose some restrictions on the original spinors. For example, to any spinor u we can correspond a scalar U taking real values, whose value does not change under the action of the Lorentz transformation on the spinor and the action of the same transformation on the conjugate spinor

$$U = -i(\mathbf{u}^T \Sigma_{MM}\overline{\mathbf{u}}) = \mathbf{u}^T S_2 \overline{\mathbf{u}} = (N * \mathbf{u})^T S_2 (N * \overline{\mathbf{u}})$$

$$U = -i(u_0 * \overline{u_1} - u_1 * \overline{u_0} + u_2 * \overline{u_3} - u_3 * \overline{u_2})$$

When a spinor and its conjugate spinor are simultaneously rotated or boosted by some angle, the scalar undergoes a rotation or boost by zero angle.

We can find the derivatives of the scalar by the components of the coordinate spinor

$$\frac{\partial U(\mathbf{x})}{\partial x_\mu} = \left(\frac{\partial \mathbf{u}(\mathbf{x})}{\partial x_\mu}\right)^T S_2\, \overline{\mathbf{u}} + \mathbf{u}(\mathbf{x})^T S_2 \left(\frac{\partial \overline{\mathbf{u}(\mathbf{x})}}{\partial x_\mu}\right)$$

The components of the coordinate spinor are complex quantities, the derivative on them is taken formally, since physical fields can be represented by power functions of the components of the coordinate spinor and its conjugate.

What are the advantages of the transition from path integral in vector space to path integral in spinor space? A possible answer is that there are new conditions for working with divergent integrals. Now integration is performed over spinor space, so that in the numerator there is a four-dimensional

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differential element d^4p instead of element d^4P in the case of vector space. The spinor element has the order of magnitude P^2 instead of P^4 for the vector element, whish decreases the order of magnitude of the numerator, while the order of magnitude of the denominator does not change.

If the spinor coordinate space is indeed more fundamental, and the vector coordinate space is an offspring of it, then we ma y benefit from this transition in any case.

Now let us move from the scalar field to the field of an electron, that is, the field of a particle with half-integer spin. We will use gamma matrices in the Weyl basis

$$\gamma_0^V=\begin{pmatrix}0&0&1&0\\0&0&0&1\\1&0&0&0\\0&1&0&0\end{pmatrix}\qquad \gamma_1^V=\begin{pmatrix}0&0&0&1\\0&0&1&0\\0&-1&0&0\\-1&0&0&0\end{pmatrix}$$

$$\gamma_2^V=\begin{pmatrix}0&0&0&-i\\0&0&i&0\\0&i&0&0\\-i&0&0&0\end{pmatrix}\qquad \gamma_3^V=\begin{pmatrix}0&0&1&0\\0&0&0&-1\\-1&0&0&0\\0&1&0&0\end{pmatrix}$$

Let us consider the linear combination of these matrices with components of the momentum vector as coefficients, substituting the expressions of the vector components through the components of the momentum spinor

$$\gamma_0^V P_0+\gamma_1^V P_1+\gamma_2^V P_2+\gamma_3^V P_3=$$

$$\begin{pmatrix}0&0&1&0\\0&0&0&1\\1&0&0&0\\0&1&0&0\end{pmatrix}P_0+\begin{pmatrix}0&0&0&1\\0&0&1&0\\0&-1&0&0\\-1&0&0&0\end{pmatrix}P_1+\begin{pmatrix}0&0&0&-i\\0&0&i&0\\0&i&0&0\\-i&0&0&0\end{pmatrix}P_2+\begin{pmatrix}0&0&1&0\\0&0&0&-1\\-1&0&0&0\\0&1&0&0\end{pmatrix}P_3=$$

$$\begin{pmatrix}0&0&P_0+P_3&P_1-iP_2\\0&0&P_1+iP_2&P_0-P_3\\P_0-P_3&-P_1+iP_2&0&0\\-P_1-iP_2&P_0+P_3&0&0\end{pmatrix}=$$

$$\begin{pmatrix}0&0&\bar p_0p_0+\bar p_2p_2&-\bar p_0p_1-\bar p_2p_3\\0&0&-\bar p_1p_0-\bar p_3p_2&\bar p_1p_1+\bar p_3p_3\\\bar p_1p_1+\bar p_3p_3&\bar p_0p_1+\bar p_2p_3&0&0\\\bar p_1p_0+\bar p_3p_2&\bar p_0p_0+\bar p_2p_2&0&0\end{pmatrix}=$$

$$\begin{pmatrix}0&0&\bar p_0p_0&-\bar p_0p_1\\0&0&-\bar p_1p_0&\bar p_1p_1\\\bar p_1p_1&\bar p_0p_1&0&0\\\bar p_1p_0&\bar p_0p_0&0&0\end{pmatrix}+\begin{pmatrix}0&0&\bar p_2p_2&-\bar p_2p_3\\0&0&-\bar p_3p_2&\bar p_3p_3\\\bar p_3p_3&\bar p_2p_3&0&0\\\bar p_3p_2&\bar p_2p_2&0&0\end{pmatrix}=$$

$$\begin{pmatrix}0&0&\bar p_0p_0&-\bar p_0p_1\\0&0&-\bar p_1p_0&\bar p_1p_1\\p_1\bar p_1-[p_1\bar p_1-\bar p_1p_1]&p_1\bar p_0-[p_1\bar p_0-\bar p_0p_1]&0&0\\p_0\bar p_1-[p_0\bar p_1-\bar p_1p_0]&p_0\bar p_0-[p_0\bar p_0-\bar p_0p_0]&0&0\end{pmatrix}+$$

$$\begin{pmatrix}0&0&\bar p_2p_2&-\bar p_2p_3\\0&0&-\bar p_3p_2&\bar p_3p_3\\p_3\bar p_3-[p_3\bar p_3-\bar p_3p_3]&p_3\bar p_2-[p_3\bar p_2-\bar p_2p_3]&0&0\\p_2\bar p_3-[p_2\bar p_3-\bar p_3p_2]&p_2\bar p_2-[p_2\bar p_2-\bar p_2p_2]&0&0\end{pmatrix}=$$

$$\begin{pmatrix}0&0&\bar p_0p_0&-\bar p_0p_1\\0&0&-\bar p_1p_0&\bar p_1p_1\\p_1\bar p_1&p_1\bar p_0&0&0\\p_0\bar p_1&p_0\bar p_0&0&0\end{pmatrix}+\begin{pmatrix}0&0&\bar p_2p_2&-\bar p_2p_3\\0&0&-\bar p_3p_2&\bar p_3p_3\\p_3\bar p_3&p_3\bar p_2&0&0\\p_2\bar p_3&p_2\bar p_2&0&0\end{pmatrix}=$$

$$-\begin{pmatrix}0&0&0&0\\0&0&0&0\\[p_1\bar p_1-\bar p_1p_1]&[p_1\bar p_0-\bar p_0p_1]&0&0\\[p_0\bar p_1-\bar p_1p_0]&[p_0\bar p_0-\bar p_0p_0]&0&0\end{pmatrix}-\begin{pmatrix}0&0&0&0\\0&0&0&0\\[p_3\bar p_3-\bar p_3p_3]&[p_3\bar p_2-\bar p_2p_3]&0&0\\[p_2\bar p_3-\bar p_3p_2]&[p_2\bar p_2-\bar p_2p_2]&0&0\end{pmatrix}$$

$$\equiv S^V(\mathbf{p})-K^V(\mathbf{p})$$

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Let us represent the matrix $S^V(\mathbf{p})$ as a sum of direct products of spinors

$$S^V(\mathbf{p}) = \begin{pmatrix} 0 \\ 0 \\ p_1 \\ p_0 \end{pmatrix} (\overline{p_1}, \overline{p_0}, 0,0) + \begin{pmatrix} \overline{p_0} \\ -\overline{p_1} \\ 0 \\ 0 \end{pmatrix} (0,0,p_0,-p_1) + \begin{pmatrix} 0 \\ 0 \\ p_3 \\ p_2 \end{pmatrix} (\overline{p_3}, \overline{p_2}, 0,0) + \begin{pmatrix} \overline{p_2} \\ -\overline{p_3} \\ 0 \\ 0 \end{pmatrix} (0,0,p_2,-p_3)$$

For a free field the components of the momentum spinor commute, therefore

$$\gamma_0^V P_0 + \gamma_1^V P_1 + \gamma_2^V P_2 + \gamma_3^V P_3 = S^V(\mathbf{p})$$

Complex mass

$$m = p_1 p_2 - p_0 p_3$$

does not change at rotations and boosts for an arbitrary complex spinor. Moreover, by a direct check it is possible to check that for an arbitrary spinor

$$S^V(\mathbf{p}) S^V(\mathbf{p}) = \overline{m} m I = m^2 I$$

For a free field, when all components of the momentum spinor commute, we can write the relativistic equation of motion of the fermionic field

$$S^V S^V \boldsymbol{\varphi}(\mathbf{x}) = \overline{m} m I \boldsymbol{\varphi}(\mathbf{x})$$

Where the matrix of derivatives S^V is obtained from the matrix $S^V(\mathbf{p})$ by substitutions

$$p_1 \to -\partial_0 \qquad p_0 \to \partial_1 \qquad p_3 \to -\partial_2 \qquad p_2 \to \partial_3$$

$$\overline{p_1} \to -\overline{\partial_0} \qquad \overline{p_0} \to \overline{\partial_1} \qquad \overline{p_3} \to -\overline{\partial_2} \qquad \overline{p_2} \to \overline{\partial_3}$$

$$\overline{\partial_\mu} \varphi(\mathbf{x}) \equiv \frac{\partial \overline{\varphi(\mathbf{x})}}{\partial \overline{x_\mu}}$$

$$S^V = \begin{pmatrix} 0 \\ 0 \\ -\partial_0 \\ \partial_1 \end{pmatrix} (-\overline{\partial_0}, \overline{\partial_1}, 0,0) + \begin{pmatrix} \overline{\partial_1} \\ \overline{\partial_0} \\ 0 \\ 0 \end{pmatrix} (0,0,\partial_1,\partial_0) + \begin{pmatrix} 0 \\ 0 \\ -\partial_2 \\ \partial_3 \end{pmatrix} (-\overline{\partial_2}, \overline{\partial_3}, 0,0) + \begin{pmatrix} \overline{\partial_3} \\ \overline{\partial_2} \\ 0 \\ 0 \end{pmatrix} (0,0,\partial_3,\partial_2)$$

However, it is generally accepted to write for this field another equation, the Dirac equation, which does not possess the invariance property anymore

$$(S^V - mI)\boldsymbol{\varphi}(\mathbf{x}) = 0$$

And for the more general case, when the momentum components do not commute, we need to write the equation

$$(S^V - K^V - mI)\boldsymbol{\varphi}(\mathbf{x}) = 0$$

$$K^V(\mathbf{p}) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ [p_1\overline{p_1} - \overline{p_1}p_1] & [p_1\overline{p_0} - \overline{p_0}p_1] & 0 & 0 \\ [p_0\overline{p_1} - \overline{p_1}p_0] & [p_0\overline{p_0} - \overline{p_0}p_0] & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ [p_3\overline{p_3} - \overline{p_3}p_3] & [p_3\overline{p_2} - \overline{p_2}p_3] & 0 & 0 \\ [p_2\overline{p_3} - \overline{p_3}p_2] & [p_2\overline{p_2} - \overline{p_2}p_2] & 0 & 0 \end{pmatrix}$$

$$K^V = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ [\partial_0\overline{\partial_0} - \overline{\partial_0}\partial_0] & [-\partial_0\overline{\partial_1} + \overline{\partial_1}\partial_0] & 0 & 0 \\ [-\partial_1\overline{\partial_0} + \overline{\partial_0}\partial_1] & [\partial_1\overline{\partial_1} - \overline{\partial_1}\partial_1] & 0 & 0 \end{pmatrix}$$

$$+ \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ [\partial_2\overline{\partial_2} - \overline{\partial_2}\partial_2] & [-\partial_2\overline{\partial_3} + \overline{\partial_3}\partial_2] & 0 & 0 \\ [-\partial_3\overline{\partial_2} + \overline{\partial_2}\partial_3] & [\partial_3\overline{\partial_3} - \overline{\partial_3}\partial_3] & 0 & 0 \end{pmatrix}$$

Further we will consider the equation of motion for a free field

$$(S^V - mI)\boldsymbol{\varphi}(\mathbf{x}) = 0$$

We again want to find the path integral

$$Z(j) = \int D\varphi(\mathbf{x})\, exp\left(i \int d^4x\{\mathcal{L}(\varphi(\mathbf{x})) + j(\mathbf{x})\varphi(\mathbf{x})\}\right)$$

for which we need the Lagrangian, from which the Euler equation is derived equation of motion

$$(S^V - mI)\boldsymbol{\varphi}(\mathbf{x}) = 0$$

It is suggested to use the Lagrangian

$$\mathcal{L} = \frac{1}{2}\boldsymbol{\varphi}(\mathbf{x})^T S^V \boldsymbol{\varphi}(\mathbf{x}) - \frac{1}{2} m \boldsymbol{\varphi}(\mathbf{x})^T \boldsymbol{\varphi}(\mathbf{x})$$

Let us substitute the Lagrangian into the Euler equation and obtain the equation of motion

$$\partial_0 \frac{\delta \mathcal{L}}{\delta(\partial_0)} + \partial_1 \frac{\delta \mathcal{L}}{\delta(\partial_1)} + \partial_2 \frac{\delta \mathcal{L}}{\delta(\partial_2)} + \partial_3 \frac{\delta \mathcal{L}}{\delta(\partial_3)} - \frac{\delta \mathcal{L}}{\delta \varphi} = 0$$

$$\frac{1}{2} S^V \boldsymbol{\varphi}(\mathbf{x}) + m\boldsymbol{\varphi}(\mathbf{x}) = 0$$

Since the Lagrangian includes, along with the derivatives of ∂_μ, the derivatives of $\overline{\partial_\mu}$, it is logical to use a different definition of Euler's equation

$$\partial_0 \frac{\delta \mathcal{L}}{\delta(\partial_0)} + \overline{\partial_0} \frac{\delta \mathcal{L}}{\delta(\overline{\partial_0})} + \partial_1 \frac{\delta \mathcal{L}}{\delta(\partial_1)} + \overline{\partial_1} \frac{\delta \mathcal{L}}{\delta(\overline{\partial_1})} + \partial_2 \frac{\delta \mathcal{L}}{\delta(\partial_2)} + \overline{\partial_2} \frac{\delta \mathcal{L}}{\delta(\overline{\partial_2})} + \partial_3 \frac{\delta \mathcal{L}}{\delta(\partial_3)} + \overline{\partial_3} \frac{\delta \mathcal{L}}{\delta(\overline{\partial_3})} - \frac{\delta \mathcal{L}}{\delta \varphi} = 0$$

Then for the free field case when the derivative operators commute with each other, we obtain the equation of motion

$$S^V \boldsymbol{\varphi}(\mathbf{x}) + m\boldsymbol{\varphi}(\mathbf{x}) = 0$$

If the derivative operators do not commute, additional terms will appear in the equation of motion in the form of matrices similar to the K^V matrix, and these additional terms will not necessarily coincide with K^V. In this connection it is necessary to consider the Lagrangian as more fundamental notion than the equation of motion and to derive the equation of motion from the Lagrangian, i.e. to take as a basis not the derivation of the equation of motion in momentum space, with what we started, but to take as an axiom the form of the Lagrangian in the form of field derivatives in the relativistically invariant form. Then, if to follow the invariance principle quite strictly, we should start from the product of two matrices, i.e. to use the Lagrangian

$$\mathcal{L} = \frac{1}{2}[\boldsymbol{\varphi}(\mathbf{x})^T S^V S^V \boldsymbol{\varphi}(\mathbf{x}) - m^2 \boldsymbol{\varphi}(\mathbf{x})^T \boldsymbol{\varphi}(\mathbf{x})]$$

Or, not limited to fermions,

$$\mathcal{L} = \frac{1}{2}[\boldsymbol{\varphi}(\mathbf{x})^T S^V S^V \boldsymbol{\varphi}(\mathbf{x}) - m\overline{m} \boldsymbol{\varphi}(\mathbf{x})^T \boldsymbol{\varphi}(\mathbf{x})]$$

Nevertheless, further we will search for the path integral in the simplest case with the originally proposed Lagrangian and in addition assume commutativity of all derivative operators

$$Z(j) = \int D\varphi(\mathbf{x})\, exp\left(i \int d^4x \left\{\frac{1}{2}\boldsymbol{\varphi}(\mathbf{x})^T S^V \boldsymbol{\varphi}(\mathbf{x}) - \frac{1}{2} m \boldsymbol{\varphi}(\mathbf{x})^T \boldsymbol{\varphi}(\mathbf{x}) + \mathbf{j}(\mathbf{x})^T \boldsymbol{\varphi}(\mathbf{x})\right\}\right)$$

After integration by parts, we presumably obtain

$$Z(j) = \int D\varphi(\mathbf{x})\, exp\left(i \int d^4x \left\{- \frac{1}{2}\boldsymbol{\varphi}(\mathbf{x})^T [S^V + mI] \boldsymbol{\varphi}(\mathbf{x}) + j(\mathbf{x})\boldsymbol{\varphi}(\mathbf{x})\right\}\right)$$

Then it is necessary to find the solution of the equation

$$-(S^V + mI)\mathbf{D}(\mathbf{x}) = I\delta(\mathbf{x})$$

For this purpose, we pass to the momentum space by means of the integral transformation

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$$\boldsymbol{\varphi}(\mathbf{x}) = \int \frac{d^4p}{(2\pi)^4}\, \boldsymbol{\varphi}(\mathbf{p}) e^{i(p_0x_1 - p_1x_0 + p_2x_3 - p_3x_2 + \overline{(\mathbf{p},\mathbf{x})})}$$

We get the equation

$$(S^V(\mathbf{p}) - mI)D^V(\mathbf{p}) = I$$

with the decision

$$D^V(\mathbf{p}) = \frac{S^V(\mathbf{p}) + \overline{m}I}{P^2 - \overline{m}m}$$

Indeed

$$\frac{(S^V(\mathbf{p}) - mI)(S^V(\mathbf{p}) + \overline{m}I)}{P^2 - \overline{m}m} = \frac{(P^2 - \overline{m}m)I}{P^2 - \overline{m}m} = I$$

Here we use the equality, which is valid for an arbitrary complex spinor $\mathbf{p}$

$$(S^V(\mathbf{p}) - mI)(S^V(\mathbf{p}) + \overline{m}I) = P^2I - (m - \overline{m})S^V(\mathbf{p}) - \overline{m}mI = (P^2 - m^2)I$$

$$P_\mu = \frac{1}{2}\mathbf{p}^\dagger S_\mu \mathbf{p}$$

$$P^2 = P_0^2 - P_1^2 - P_2^2 - P_3^2$$

It is based on the correlation verified earlier in our work

$$(p_1p_2 - p_0p_3)(\overline{p_1}\overline{p_2} - \overline{p_0}\overline{p_3}) = P_0^2 - P_1^2 - P_2^2 - P_3^2$$

it is also taken into account that we consider fermions whose mass is real.

As a result, the propagator has the form

$$D^V(\mathbf{x}) = \int \frac{d^4p}{(2\pi)^4} \frac{S^V(\mathbf{p}) + \overline{m}I}{P^2 - \overline{m}m} e^{i(p_0x_1 - p_1x_0 + p_2x_3 - p_3x_2 + \overline{(\mathbf{p},\mathbf{x})})}$$

here we assume the validity of the relation

$$\delta(\mathbf{x}) = \int \frac{d^4p}{(2\pi)^4}\, e^{i(p_0x_1 - p_1x_0 + p_2x_3 - p_3x_2 + \overline{(\mathbf{p},\mathbf{x})})}$$

In the case of a fermion, the mass in integration is a fixed real quantity, and it can be considered negative for the electron and positive for the positron. Theoretically, the mass can be complex or purely imaginary. If we put mass equal to zero, it may be possible to apply this Lagrangian to describe massless particles. I wonder if there are particles with complex or purely imaginary mass. In the latter case, the square of the mass will still be positive and the particle will satisfy the Klein-Gordon equation. Such particles can interact among themselves, but not with particles whose mass is real.

Let's return to the question about the use of completely relativistically invariant Lagrangian

$$\mathcal{L} = \frac{1}{2}[\boldsymbol{\varphi}(\mathbf{x})^T S^V S^V \boldsymbol{\varphi}(\mathbf{x}) - m^2\boldsymbol{\varphi}(\mathbf{x})^T\boldsymbol{\varphi}(\mathbf{x})]$$

Let's find the product of matrices

$$S^V(\mathbf{p})S^V(\mathbf{p}) =$$

$$\left(\begin{pmatrix}0\\0\\p_1\\p_0\end{pmatrix}(\overline{p_1},\overline{p_0},0,0) + \begin{pmatrix}\overline{p_0}\\-\overline{p_1}\\0\\0\end{pmatrix}(0,0,p_0,-p_1) + \begin{pmatrix}0\\0\\p_3\\p_2\end{pmatrix}(\overline{p_3},\overline{p_2},0,0) + \begin{pmatrix}\overline{p_2}\\-\overline{p_3}\\0\\0\end{pmatrix}(0,0,p_2,-p_3)\right)$$

$$\left(\begin{pmatrix}0\\0\\p_1\\p_0\end{pmatrix}(\overline{p_1},\overline{p_0},0,0) + \begin{pmatrix}\overline{p_0}\\-\overline{p_1}\\0\\0\end{pmatrix}(0,0,p_0,-p_1) + \begin{pmatrix}0\\0\\p_3\\p_2\end{pmatrix}(\overline{p_3},\overline{p_2},0,0) + \begin{pmatrix}\overline{p_2}\\-\overline{p_3}\\0\\0\end{pmatrix}(0,0,p_2,-p_3)\right) =$$

$$(\overline{p_1}\overline{p_2} - \overline{p_0}\overline{p_3})\begin{pmatrix}0\\0\\p_1\\p_0\end{pmatrix}(0,0,p_2,-p_3) + (p_0p_3 - p_1p_2)\begin{pmatrix}\overline{p_0}\\-\overline{p_1}\\0\\0\end{pmatrix}(\overline{p_3},\overline{p_2},0,0) +$$

$$(\overline{p_3}\overline{p_0} - \overline{p_2}\overline{p_1})\begin{pmatrix}0\\0\\p_3\\p_2\end{pmatrix}(0,0,p_0,-p_1) + (p_2p_1 - p_3p_0)\begin{pmatrix}\overline{p_2}\\-\overline{p_3}\\0\\0\end{pmatrix}(\overline{p_1},\overline{p_0},0,0) =$$

$$\bar{m}\begin{pmatrix}0\\0\\p_1\\p_0\end{pmatrix}(0,0,p_2,-p_3) - m\begin{pmatrix}\overline{p_0}\\-\overline{p_1}\\0\\0\end{pmatrix}(\overline{p_3},\overline{p_2},0,0) - \bar{m}\begin{pmatrix}0\\0\\p_3\\p_2\end{pmatrix}(0,0,p_0,-p_1) + m\begin{pmatrix}\overline{p_2}\\-\overline{p_3}\\0\\0\end{pmatrix}(\overline{p_1},\overline{p_0},0,0) =$$

$$m\left\{\begin{pmatrix}0\\0\\p_1\\p_0\end{pmatrix}(0,0,p_2,-p_3) - \begin{pmatrix}\overline{p_0}\\-\overline{p_1}\\0\\0\end{pmatrix}(\overline{p_3},\overline{p_2},0,0) - \begin{pmatrix}0\\0\\p_3\\p_2\end{pmatrix}(0,0,p_0,-p_1) + \begin{pmatrix}\overline{p_2}\\-\overline{p_3}\\0\\0\end{pmatrix}(\overline{p_1},\overline{p_0},0,0)\right\}$$

$$\equiv mS^{VV}(\mathbf{p})$$

The assumption that the following equalities hold is used

$$p_1p_2 - p_0p_3 = p_2p_1 - p_3p_0 = m$$

$$\overline{p_1p_2} - \overline{p_0p_3} = \overline{p_2p_1} - \overline{p_3p_0} = \bar{m}$$

$$\bar{m} = m$$

Further we find the product of matrices

$$S^{VV}(\mathbf{p})S^{VV}(\mathbf{p}) =$$

$$\left\{\begin{pmatrix}0\\0\\p_1\\p_0\end{pmatrix}(0,0,p_2,-p_3) - \begin{pmatrix}\overline{p_0}\\-\overline{p_1}\\0\\0\end{pmatrix}(\overline{p_3},\overline{p_2},0,0) - \begin{pmatrix}0\\0\\p_3\\p_2\end{pmatrix}(0,0,p_0,-p_1) + \begin{pmatrix}\overline{p_2}\\-\overline{p_3}\\0\\0\end{pmatrix}(\overline{p_1},\overline{p_0},0,0)\right\}$$

$$\left\{\begin{pmatrix}0\\0\\p_1\\p_0\end{pmatrix}(0,0,p_2,-p_3) - \begin{pmatrix}\overline{p_0}\\-\overline{p_1}\\0\\0\end{pmatrix}(\overline{p_3},\overline{p_2},0,0) - \begin{pmatrix}0\\0\\p_3\\p_2\end{pmatrix}(0,0,p_0,-p_1) + \begin{pmatrix}\overline{p_2}\\-\overline{p_3}\\0\\0\end{pmatrix}(\overline{p_1},\overline{p_0},0,0)\right\}$$

$$= (p_2p_1 - p_3p_0)\begin{pmatrix}0\\0\\p_1\\p_0\end{pmatrix}(0,0,p_2,-p_3) + (\overline{p_3p_0} - \overline{p_2p_1})\begin{pmatrix}\overline{p_0}\\-\overline{p_1}\\0\\0\end{pmatrix}(\overline{p_3},\overline{p_2},0,0)$$

$$+(p_0p_3 - p_1p_2)\begin{pmatrix}0\\0\\p_3\\p_2\end{pmatrix}(0,0,p_0,-p_1) + (\overline{p_2p_1} - \overline{p_3p_0})\begin{pmatrix}\overline{p_2}\\-\overline{p_3}\\0\\0\end{pmatrix}(\overline{p_1},\overline{p_0},0,0)$$

$$= m\left\{\begin{pmatrix}0\\0\\p_1\\p_0\end{pmatrix}(0,0,p_2,-p_3) - \begin{pmatrix}0\\0\\p_3\\p_2\end{pmatrix}(0,0,p_0,-p_1)\right\}$$

$$+\bar{m}\left\{\begin{pmatrix}\overline{p_2}\\-\overline{p_3}\\0\\0\end{pmatrix}(\overline{p_1},\overline{p_0},0,0) - \begin{pmatrix}\overline{p_0}\\-\overline{p_1}\\0\\0\end{pmatrix}(\overline{p_3},\overline{p_2},0,0)\right\}$$

$$= m\left\{\begin{pmatrix}0 & 0 & 0 & 0\\0 & 0 & 0 & 0\\0 & 0 & p_1p_2 & -p_1p_3\\0 & 0 & p_0p_2 & -p_0p_3\end{pmatrix} - \begin{pmatrix}0 & 0 & 0 & 0\\0 & 0 & 0 & 0\\0 & 0 & p_3p_0 & -p_3p_1\\0 & 0 & p_2p_0 & -p_2p_1\end{pmatrix}\right\}$$

$$+\bar{m}\left\{\begin{pmatrix}\overline{p_2p_1} & \overline{p_2p_0} & 0 & 0\\-\overline{p_3p_1} & -\overline{p_3p_0} & 0 & 0\\0 & 0 & 0 & 0\\0 & 0 & 0 & 0\end{pmatrix} - \begin{pmatrix}\overline{p_0p_3} & \overline{p_0p_2} & 0 & 0\\-\overline{p_1p_3} & -\overline{p_1p_2} & 0 & 0\\0 & 0 & 0 & 0\\0 & 0 & 0 & 0\end{pmatrix}\right\}$$

$$= m\left\{\begin{pmatrix}0 & 0 & 0 & 0\\0 & 0 & 0 & 0\\0 & 0 & p_1p_2 - p_3p_0 & 0\\0 & 0 & 0 & -p_0p_3 + p_2p_1\end{pmatrix}\right\} + \bar{m}\left\{\begin{pmatrix}\overline{p_2p_1} - \overline{p_0p_3} & 0 & 0 & 0\\0 & -\overline{p_3p_0} + \overline{p_1p_2} & 0 & 0\\0 & 0 & 0 & 0\\0 & 0 & 0 & 0\end{pmatrix}\right\}$$

$$= \begin{pmatrix}\bar{m}\bar{m} & 0 & 0 & 0\\0 & \bar{m}\bar{m} & 0 & 0\\0 & 0 & mm & 0\\0 & 0 & 0 & mm\end{pmatrix}$$

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Again we use the equality

$$(p_1p_2 - p_0p_3)(\overline{p_1p_2} - \overline{p_0p_3}) = P_0^2 - P_1^2 - P_2^2 - P_3^2 = P^2$$

and consider that the mass of the fermion is real, i.e.

$$p_1p_2 - p_0p_3 = \overline{p_1p_2} - \overline{p_0p_3}$$

$$(p_1p_2 - p_0p_3)(p_1p_2 - p_0p_3) = (\overline{p_1p_2} - \overline{p_0p_3})(\overline{p_1p_2} - \overline{p_0p_3}) = P^2$$

therefore, the relations are valid

$$S^{VV}(\mathbf{p})S^{VV}(\mathbf{p}) = \begin{pmatrix} P^2 & 0 & 0 & 0 \\ 0 & P^2 & 0 & 0 \\ 0 & 0 & P^2 & 0 \\ 0 & 0 & 0 & P^2 \end{pmatrix} = P^2 I$$

$$(S^{VV}(\mathbf{p}) - mI)(S^{VV}(\mathbf{p}) + mI) = P^2 I - m^2 I = (P^2 - m^2)I$$

$$\frac{(S^{VV}(\mathbf{p}) - mI)(S^{VV}(\mathbf{p}) + mI)}{P^2 - m^2} = I$$

But the main advantage of the obtained matrix is the following

$$S^{VV}(\mathbf{p}) = \begin{pmatrix} 0 \\ 0 \\ p_1 \\ p_0 \end{pmatrix}(0,0,p_2,-p_3) - \begin{pmatrix} \overline{p}_0 \\ -\overline{p}_1 \\ 0 \\ 0 \end{pmatrix}(\overline{p}_3,\overline{p}_2,0,0) - \begin{pmatrix} 0 \\ 0 \\ p_3 \\ p_2 \end{pmatrix}(0,0,p_0,-p_1) + \begin{pmatrix} \overline{p}_2 \\ -\overline{p}_3 \\ 0 \\ 0 \end{pmatrix}(\overline{p}_1,\overline{p}_0,0,0) =$$

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & p_1p_2 & -p_1p_3 \\ 0 & 0 & p_0p_2 & -p_0p_3 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & p_3p_0 & -p_3p_1 \\ 0 & 0 & p_2p_0 & -p_2p_1 \end{pmatrix}$$

$$+ \begin{pmatrix} \overline{p}_2\overline{p}_1 & \overline{p}_2\overline{p}_0 & 0 & 0 \\ -\overline{p}_3\overline{p}_1 & -\overline{p}_3\overline{p}_0 & 0 & 0 \\ 0 \quad 0 & & 0 & 0 \\ 0 \quad 0 & & 0 & 0 \end{pmatrix} - \begin{pmatrix} \overline{p}_0\overline{p}_3 & \overline{p}_0\overline{p}_2 & 0 & 0 \\ -\overline{p}_1\overline{p}_3 & -\overline{p}_1\overline{p}_2 & 0 & 0 \\ 0 \quad 0 & & 0 & 0 \\ 0 \quad 0 & & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & p_1p_2 - p_3p_0 & 0 \\ 0 & 0 & 0 & -p_0p_3 + p_2p_1 \end{pmatrix} + \begin{pmatrix} \overline{p}_2\overline{p}_1 - \overline{p}_0\overline{p}_3 & 0 & 0 & 0 \\ 0 & -\overline{p}_3\overline{p}_0 + \overline{p}_1\overline{p}_2 & 0 & 0 \\ 0 \quad 0 & & 0 & 0 \\ 0 \quad 0 & & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} \overline{m} & 0 & 0 & 0 \\ 0 & \overline{m} & 0 & 0 \\ 0 & 0 & m & 0 \\ 0 & 0 & 0 & m \end{pmatrix}$$

This matrix does not change at rotations and boosts, so it can be stated that the equation of motion, e.g., in the form of

$$\left(S^{VV} - \begin{pmatrix} \overline{m} & 0 & 0 & 0 \\ 0 & \overline{m} & 0 & 0 \\ 0 & 0 & m & 0 \\ 0 & 0 & 0 & m \end{pmatrix}\right)\boldsymbol{\varphi}(\mathbf{x}) = 0$$

where

$$S^{VV} = \begin{pmatrix} 0 \\ 0 \\ -\partial_0 \\ \partial_1 \end{pmatrix}(0,0,\partial_3,\partial_2) - \begin{pmatrix} \overline{\partial}_1 \\ \overline{\partial}_0 \\ 0 \\ 0 \end{pmatrix}(-\overline{\partial}_2,\overline{\partial}_3,0,0) - \begin{pmatrix} 0 \\ 0 \\ -\partial_2 \\ \partial_3 \end{pmatrix}(0,0,\partial_1,\partial_0) + \begin{pmatrix} \overline{\partial}_3 \\ \overline{\partial}_2 \\ 0 \\ 0 \end{pmatrix}(-\overline{\partial}_0,\overline{\partial}_1,0,0)$$

is truly relativistically invariant, respectively we can use the invariant Lagrangian

$$\mathcal{L} = \frac{1}{2}[\boldsymbol{\varphi}(\mathbf{x})^T S^{VV}\boldsymbol{\varphi}(\mathbf{x}) - m\boldsymbol{\varphi}(\mathbf{x})^T\boldsymbol{\varphi}(\mathbf{x})]$$

to which corresponds the relativistically invariant propagator of the boson having a real mass, which is negative for the electron and positive for the positron

$$D^{VV}(\mathbf{x}) = \int \frac{d^4p}{(2\pi)^4} \frac{S^{VV}(\mathbf{p}) + mI}{P^2 - m^2} e^{i(p_0x_1 - p_1x_0 + p_2x_3 - p_3x_2 + \overline{(\mathbf{p},\mathbf{x})})}$$

Let us compare the propagator in spinor space with the propagator of the fermion given in [[9], formula II.2.22 and formula II.5.18]

$$D(\mathbf{X}) = \int \frac{d^4P}{(2\pi)^4} \frac{e^{-i\mathbf{PX}}}{\gamma^\mu P_\mu - mI} = \int \frac{d^4P}{(2\pi)^4} \frac{\gamma^\mu P_\mu + mI}{P^2 - m^2} e^{-i\mathbf{PX}}$$

In [9] this formula is obtained by applying the second quantization procedure or using Grassmann integrals. The results are similar, but the integration here is performed in the vector momentum space. The Dirac equation and the corresponding Lagrangian are not relativistically invariant. Besides, here the mass is considered always real and positive, but then it is not clear how electron and positron differ from the point of view of this formula.

Let us consider in detail the derivation of the expression for the fermion propagator in [[9], Sec. II.2]. It is based on the assumption of relativistic invariance of the Dirac equation and therefore the calculations are carried out in the rest frame, and then the result is extended to an arbitrary frame of reference. Thus for the field spinor u the spinor $u\_\equiv\ \boldsymbol{u}^\dagger\gamma^0$ is defined and it is asserted that the value of

$$\boldsymbol{u}^\dagger\gamma^0 u = \boldsymbol{u}^\dagger \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} u$$

is a Lorentz scalar. But it is not so, since in the spinor space the scalar is formed exclusively by the scalar product of two spinors, where the metric tensor of the spinor space is included

$$\boldsymbol{u}^\dagger \Sigma_{MM} u = \boldsymbol{u}^\dagger \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} u$$

there are no other ways to construct a scalar in the spinor space.

Nevertheless, this fact and the fact of non-invariance of the Dirac equation itself do not cancel the value of the second quantization procedure and the final form of the fermion propagator, which allows to make accurate predictions of the experimental results.

We hope that the proposed Lagrangian for the spinor coordinate space can find application in the calculation of the path integral, but already in the spinor space. Whether such a calculation in spinor space has an advantage over the calculation of the path integral in vector space can be shown by their real comparison.

By analogy with the propagator of a photon, more precisely of a massive vector meson, given in [[9], formula I.5.3]

$$D_{\nu\lambda}(\mathbf{X}) = \int \frac{d^4P}{(2\pi)^4} \frac{-\eta_{\nu\lambda} + P_\nu P_\lambda/m^2}{P^2 - m^2} e^{i\mathbf{PX}}$$

we can assume the propagator form in the spinor space without revealing for compactness the expression of the momentum vector components through the momentum spinor components

$$D_{\nu\lambda}(\mathbf{x}) = \int \frac{d^4p}{(2\pi)^4} \frac{-\eta_{\nu\lambda} + P_\nu P_\lambda/m^2}{P^2 - \overline{m}m} e^{i(p_0x_1 - p_1x_0 + p_2x_3 - p_3x_2 + \overline{(\mathbf{p},\mathbf{x})})}$$

Among other things, the equation

$$\left(S^{VV} - \begin{pmatrix} \overline{m} & 0 & 0 & 0 \\ 0 & \overline{m} & 0 & 0 \\ 0 & 0 & m & 0 \\ 0 & 0 & 0 & m \end{pmatrix} \right) \boldsymbol{\varphi}(\mathbf{x}) = 0$$

can be modified to take into account the electromagnetic potential, the electron charge is taken as a unit

$$p_0 \to \partial_1 + a_0 \qquad p_1 \to -\partial_0 + a_1 \qquad p_2 \to \partial_3 + a_2 \qquad p_3 \to -\partial_2 + a_3$$
$$\overline{p_0} \to \overline{\partial_1} + \overline{a_0} \qquad \overline{p_1} \to -\overline{\partial_0} + \overline{a_1} \qquad \overline{p_2} \to \overline{\partial_3} + \overline{a_2} \qquad \overline{p_3} \to -\overline{\partial_2} + \overline{a_3}$$

$$S^{VV} = \begin{pmatrix} 0 \\ 0 \\ -\partial_0 + a_1 \\ \partial_1 + a_0 \end{pmatrix} (0,0,\partial_3 + a_2, \partial_2 - a_3) - \begin{pmatrix} \overline{\partial_1} + \overline{a_0} \\ \overline{\partial_0} - \overline{a_1} \\ 0 \\ 0 \end{pmatrix} (-\overline{\partial_2} + \overline{a_3}, \overline{\partial_3} + \overline{a_2}, 0,0)$$
$$- \begin{pmatrix} 0 \\ 0 \\ -\partial_2 + a_3 \\ \partial_3 + a_2 \end{pmatrix} (0,0,\partial_1 + a_0, \partial_0 - a_1) + \begin{pmatrix} \overline{\partial_3} + \overline{a_2} \\ \overline{\partial_2} - \overline{a_3} \\ 0 \\ 0 \end{pmatrix} (-\overline{\partial_0} + \overline{a_1}, \overline{\partial_1} + \overline{a_0}, 0,0)$$

and apply, in particular, to analyze the radiation spectrum of a hydrogen-like atom.

Let us formulate again the difference between the equations, the second of which is derived from the Dirac equation with gamma matrices in the Weyl basis

$$\left(S^{VV} - \begin{pmatrix} \overline{m} & 0 & 0 & 0 \\ 0 & \overline{m} & 0 & 0 \\ 0 & 0 & m & 0 \\ 0 & 0 & 0 & m \end{pmatrix} \right) \boldsymbol{\varphi}(\mathbf{x}) = 0$$

$$(S^V - mI)\boldsymbol{\varphi}(\mathbf{x}) = 0$$

The difference is, the matrix $S^{VV}(\mathbf{p})$ (p) remains unchanged under any rotations and boosts applied to the spinor $\mathbf{p}$, while the matrix $S^V(\mathbf{p})$ (p) changes under any rotations and boosts.

$$S^V = \begin{pmatrix} 0 \\ 0 \\ -\partial_0 \\ \partial_1 \end{pmatrix} (-\overline{\partial_0}, \overline{\partial_1}, 0,0) + \begin{pmatrix} \overline{\partial_1} \\ \overline{\partial_0} \\ 0 \\ 0 \end{pmatrix} (0,0,\partial_1, \partial_0) + \begin{pmatrix} 0 \\ 0 \\ -\partial_2 \\ \partial_3 \end{pmatrix} (-\overline{\partial_2}, \overline{\partial_3}, 0,0) + \begin{pmatrix} \overline{\partial_3} \\ \overline{\partial_2} \\ 0 \\ 0 \end{pmatrix} (0,0,\partial_3, \partial_2)$$

$$S^{VV} = \begin{pmatrix} 0 \\ 0 \\ -\partial_0 \\ \partial_1 \end{pmatrix} (0,0,\partial_3, \partial_2) - \begin{pmatrix} \overline{\partial_1} \\ \overline{\partial_0} \\ 0 \\ 0 \end{pmatrix} (-\overline{\partial_2}, \overline{\partial_3}, 0,0) - \begin{pmatrix} 0 \\ 0 \\ -\partial_2 \\ \partial_3 \end{pmatrix} (0,0,\partial_1, \partial_0) + \begin{pmatrix} \overline{\partial_3} \\ \overline{\partial_2} \\ 0 \\ 0 \end{pmatrix} (-\overline{\partial_0}, \overline{\partial_1}, 0,0)$$

Equally radically different are the corresponding Lagrangians and propagators.

By analogy with [9, Chapter II.2] we will carry out the procedure of second quantization of the fermion field. Let us write the equation

$$\left(S^{VV} - \begin{pmatrix} \overline{m} & 0 & 0 & 0 \\ 0 & \overline{m} & 0 & 0 \\ 0 & 0 & m & 0 \\ 0 & 0 & 0 & m \end{pmatrix} \right) \boldsymbol{\varphi}(\mathbf{x}) = 0$$

in the momentum space, for which we apply the integral transformation

$$\boldsymbol{\varphi}(\mathbf{x}) = \int \frac{d^4p}{(2\pi)^4} \boldsymbol{\varphi}(\mathbf{p}) e^{i(p_0x_1 - p_1x_0 + p_2x_3 - p_3x_2)}$$

Let's substitute the wave function into the equation and obtain

$$\left(S^{VV}(\mathbf{p}) - \begin{pmatrix} \overline{m} & 0 & 0 & 0 \\ 0 & \overline{m} & 0 & 0 \\ 0 & 0 & m & 0 \\ 0 & 0 & 0 & m \end{pmatrix} \right) \boldsymbol{\varphi}(\mathbf{p}) = 0$$

$$S^{VV}(\mathbf{p}) = \begin{pmatrix} 0 \\ 0 \\ p_1 \\ p_0 \end{pmatrix} (0,0,p_2,-p_3) - \begin{pmatrix} \overline{p_0} \\ -\overline{p_1} \\ 0 \\ 0 \end{pmatrix} (\overline{p_3}, \overline{p_2}, 0,0) - \begin{pmatrix} 0 \\ 0 \\ p_3 \\ p_2 \end{pmatrix} (0,0,p_0,-p_1) + \begin{pmatrix} \overline{p_2} \\ -\overline{p_3} \\ 0 \\ 0 \end{pmatrix} (\overline{p_1}, \overline{p_0}, 0,0)$$

Let us define two sets of four reference spinors

$$\mathbf{u1} = \begin{pmatrix} 0 \\ 0 \\ p_1 \\ p_0 \end{pmatrix} \quad \mathbf{u2} = \begin{pmatrix} \overline{p_0} \\ -\overline{p_1} \\ 0 \\ 0 \end{pmatrix} \quad \mathbf{u3} = \begin{pmatrix} 0 \\ 0 \\ p_3 \\ p_2 \end{pmatrix} \quad \mathbf{u4} = \begin{pmatrix} \overline{p_2} \\ -\overline{p_3} \\ 0 \\ 0 \end{pmatrix}$$

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$$\mathbf{v1} = \begin{pmatrix} p_1 \\ p_0 \\ 0 \\ 0 \end{pmatrix} \quad \mathbf{v2} = \begin{pmatrix} 0 \\ 0 \\ \overline{p_0} \\ -\overline{p_1} \end{pmatrix} \quad \mathbf{v3} = \begin{pmatrix} p_3 \\ p_2 \\ 0 \\ 0 \end{pmatrix} \quad \mathbf{v4} = \begin{pmatrix} 0 \\ 0 \\ \overline{p_2} \\ -\overline{p_3} \end{pmatrix}$$

$$\mathbf{v1} = \gamma_0^V \mathbf{u1} \quad \mathbf{v2} = \gamma_0^V \mathbf{u2} \quad \mathbf{v3} = \gamma_0^V \mathbf{u3} \quad \mathbf{v4} = \gamma_0^V \mathbf{u4}$$

where

$$\gamma_0^V = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

And let's express the matrix through them

$$S^{VV}(\mathbf{p}) = \begin{pmatrix} 0 \\ 0 \\ p_1 \\ p_0 \end{pmatrix} (0,0,p_2,-p_3) - \begin{pmatrix} \overline{p_0} \\ -\overline{p_1} \\ 0 \\ 0 \end{pmatrix} (\overline{p_3},\overline{p_2},0,0) - \begin{pmatrix} 0 \\ 0 \\ p_3 \\ p_2 \end{pmatrix} (0,0,p_0,-p_1) + \begin{pmatrix} \overline{p_2} \\ -\overline{p_3} \\ 0 \\ 0 \end{pmatrix} (\overline{p_1},\overline{p_0},0,0)$$

$$= \mathbf{u1(p)v4^+(p) - u2(p)v3^+(p) - u3(p)v2^+(p) + u4(p)v1^+(p)}$$

Developing the idea of invariance, we pass to the set of reference spinors with wider filling, but continuing to form matrices possessing the invariance property

$$\mathbf{u1} = \begin{pmatrix} -p_3 \\ -p_2 \\ p_1 \\ p_0 \end{pmatrix} \quad \mathbf{u2} = \begin{pmatrix} p_2 \\ -p_3 \\ p_0 \\ -p_1 \end{pmatrix} \quad \mathbf{u3} = \begin{pmatrix} -p_1 \\ -p_0 \\ p_3 \\ p_2 \end{pmatrix} \quad \mathbf{u4} = \begin{pmatrix} p_0 \\ -p_1 \\ p_2 \\ -p_3 \end{pmatrix}$$

$$\mathbf{v1} = \begin{pmatrix} p_1 \\ p_0 \\ p_3 \\ p_2 \end{pmatrix} \quad \mathbf{v2} = \begin{pmatrix} p_0 \\ -p_1 \\ -p_2 \\ p_3 \end{pmatrix} \quad \mathbf{v3} = \begin{pmatrix} p_3 \\ p_2 \\ p_1 \\ p_0 \end{pmatrix} \quad \mathbf{v4} = \begin{pmatrix} p_2 \\ -p_3 \\ -p_0 \\ p_1 \end{pmatrix}$$

Let's express through the reference spinors the matrix

$$S^R(\mathbf{p}) = \begin{pmatrix} -p_3 \\ -p_2 \\ p_1 \\ p_0 \end{pmatrix} (p_0,-p_1,p_2,-p_3) - \begin{pmatrix} -p_1 \\ -p_0 \\ p_3 \\ p_2 \end{pmatrix} (p_2,-p_3,p_0,-p_1)$$

$$+ \begin{pmatrix} p_1 \\ p_0 \\ p_3 \\ p_2 \end{pmatrix} (p_2,-p_3,-p_0,p_1) - \begin{pmatrix} p_3 \\ p_2 \\ p_1 \\ p_0 \end{pmatrix} (p_0,-p_1,-p_2,p_3)$$

$$= \mathbf{u1(p)u4^T(p) - u3(p)u2^T(p) + v1(p)v4^T(p) - v3(p)v2^T(p)}$$

$$S^R(\mathbf{p}) = \begin{pmatrix} -p_3 \\ -p_2 \\ p_1 \\ p_0 \end{pmatrix} (p_0,-p_1,p_2,-p_3) - \begin{pmatrix} -p_1 \\ -p_0 \\ p_3 \\ p_2 \end{pmatrix} (p_2,-p_3,p_0,-p_1)$$

$$+ \begin{pmatrix} p_1 \\ p_0 \\ p_3 \\ p_2 \end{pmatrix} (p_2,-p_3,-p_0,p_1) - \begin{pmatrix} p_3 \\ p_2 \\ p_1 \\ p_0 \end{pmatrix} (p_0,-p_1,-p_2,p_3)$$

$$= \begin{pmatrix} -p_3p_0 & p_3p_1 & -p_3p_2 & p_3p_3 \\ -p_2p_0 & p_2p_1 & -p_2p_2 & p_2p_3 \\ p_1p_0 & -p_1p_1 & p_1p_2 & -p_1p_3 \\ p_0p_0 & -p_0p_1 & p_0p_2 & -p_0p_3 \end{pmatrix} - \begin{pmatrix} -p_1p_2 & p_1p_3 & -p_1p_0 & p_1p_1 \\ -p_0p_2 & p_0p_3 & -p_0p_0 & p_0p_1 \\ p_3p_2 & -p_3p_3 & p_3p_0 & -p_3p_1 \\ p_2p_2 & -p_2p_3 & p_2p_0 & -p_2p_1 \end{pmatrix}$$

$$+ \begin{pmatrix} p_1p_2 & -p_1p_3 & -p_1p_0 & p_1p_1 \\ p_0p_2 & -p_0p_3 & -p_0p_0 & p_0p_1 \\ p_3p_2 & -p_3p_3 & -p_3p_0 & p_3p_1 \\ p_2p_2 & -p_2p_3 & -p_2p_0 & p_2p_1 \end{pmatrix} - \begin{pmatrix} p_3p_0 & -p_3p_1 & -p_3p_2 & p_3p_3 \\ p_2p_0 & -p_2p_1 & -p_2p_2 & p_2p_3 \\ p_1p_0 & -p_1p_1 & -p_1p_2 & p_1p_3 \\ p_0p_0 & -p_0p_1 & -p_0p_2 & p_0p_3 \end{pmatrix}$$

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$$= \begin{pmatrix} -p_3p_0 + p_1p_2 & 0 & 0 & 0 \\ 0 & p_2p_1 - p_0p_3 & 0 & 0 \\ 0 & 0 & p_1p_2 - p_3p_0 & 0 \\ 0 & 0 & 0 & -p_0p_3 + p_2p_1 \end{pmatrix}$$
$$+ \begin{pmatrix} p_1p_2 - p_3p_0 & 0 & 0 & 0 \\ 0 & -p_0p_3 + p_2p_1 & 0 & 0 \\ 0 & 0 & -p_3p_0 + p_1p_2 & 0 \\ 0 & 0 & 0 & p_2p_1 - p_0p_3 \end{pmatrix}$$
$$= \begin{pmatrix} m+m & 0 & 0 & 0 \\ 0 & m+m & 0 & 0 \\ 0 & 0 & m+m & 0 \\ 0 & 0 & 0 & m+m \end{pmatrix}$$

and matrix

$$S_R(\mathbf{p}) = \begin{pmatrix} p_0 \\ -p_1 \\ p_2 \\ -p_3 \end{pmatrix} (-p_3, -p_2, p_1, p_0) - \begin{pmatrix} p_2 \\ -p_3 \\ p_0 \\ -p_1 \end{pmatrix} (-p_1, -p_0, p_3, p_2)$$
$$+ \begin{pmatrix} p_2 \\ -p_3 \\ -p_0 \\ p_1 \end{pmatrix} (p_1, p_0, p_3, p_2) - \begin{pmatrix} p_0 \\ -p_1 \\ -p_2 \\ p_3 \end{pmatrix} (p_3, p_2, p_1, p_0)$$
$$= \mathbf{u4}(\mathbf{p})\mathbf{u1}^T(\mathbf{p}) - \mathbf{u2}(\mathbf{p})\mathbf{u3}^T(\mathbf{p}) + \mathbf{v4}(\mathbf{p})\mathbf{v1}^T(\mathbf{p}) - \mathbf{v2}(\mathbf{p})\mathbf{v3}^T(\mathbf{p})$$

$$S_R(\mathbf{p}) = \begin{pmatrix} p_0 \\ -p_1 \\ p_2 \\ -p_3 \end{pmatrix} (-p_3, -p_2, p_1, p_0) - \begin{pmatrix} p_2 \\ -p_3 \\ p_0 \\ -p_1 \end{pmatrix} (-p_1, -p_0, p_3, p_2)$$
$$+ \begin{pmatrix} p_2 \\ -p_3 \\ -p_0 \\ p_1 \end{pmatrix} (p_1, p_0, p_3, p_2) - \begin{pmatrix} p_0 \\ -p_1 \\ -p_2 \\ p_3 \end{pmatrix} (p_3, p_2, p_1, p_0) =$$
$$= \begin{pmatrix} -p_0p_3 & -p_0p_2 & p_0p_1 & p_0p_0 \\ p_1p_3 & p_1p_2 & -p_1p_1 & -p_1p_0 \\ -p_2p_3 & -p_2p_2 & p_2p_1 & p_2p_0 \\ p_3p_3 & p_3p_2 & -p_3p_1 & -p_3p_0 \end{pmatrix} - \begin{pmatrix} -p_2p_1 & -p_2p_0 & p_2p_3 & p_2p_2 \\ p_3p_1 & p_3p_0 & -p_3p_3 & -p_3p_2 \\ -p_0p_1 & -p_0p_0 & p_0p_3 & p_0p_2 \\ p_1p_1 & p_1p_0 & -p_1p_3 & -p_1p_2 \end{pmatrix}$$
$$+ \begin{pmatrix} p_2p_1 & p_2p_0 & p_2p_3 & p_2p_2 \\ -p_3p_1 & -p_3p_0 & -p_3p_3 & -p_3p_2 \\ -p_0p_1 & -p_0p_0 & -p_0p_3 & -p_0p_2 \\ p_1p_1 & p_1p_0 & p_1p_3 & p_1p_2 \end{pmatrix} - \begin{pmatrix} p_0p_3 & p_0p_2 & p_0p_1 & p_0p_0 \\ -p_1p_3 & -p_1p_2 & -p_1p_1 & -p_1p_0 \\ -p_2p_3 & -p_2p_2 & -p_2p_1 & -p_2p_0 \\ p_3p_3 & p_3p_2 & p_3p_1 & p_3p_0 \end{pmatrix}$$
$$= \begin{pmatrix} -p_0p_3 + p_2p_1 & 0 & 0 & 0 \\ 0 & p_1p_2 - p_3p_0 & 0 & 0 \\ 0 & 0 & p_2p_1 - p_0p_3 & 0 \\ 0 & 0 & 0 & -p_3p_0 + p_1p_2 \end{pmatrix}$$
$$+ \begin{pmatrix} p_2p_1 - p_0p_3 & 0 & 0 & 0 \\ 0 & -p_3p_0 + p_1p_2 & 0 & 0 \\ 0 & 0 & -p_0p_3 + p_2p_1 & 0 \\ 0 & 0 & 0 & p_1p_2 - p_3p_0 \end{pmatrix}$$
$$= \begin{pmatrix} m+m & 0 & 0 & 0 \\ 0 & m+m & 0 & 0 \\ 0 & 0 & m+m & 0 \\ 0 & 0 & 0 & m+m \end{pmatrix}$$

here

$$m = p_1p_2 - p_0p_3$$

Let us decompose the fermion field into plane waves with operator coefficients

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$$\boldsymbol{\varphi}(\mathbf{x}) = \int \frac{d^4p}{(2\pi)^4}$$

$$\begin{bmatrix} d_1(\mathbf{p})\mathbf{u1}(\mathbf{p}) + id_2(\mathbf{p})\mathbf{u3}(\mathbf{p}) + ib_2(\mathbf{p})\overline{\mathbf{u2}}(\mathbf{p}) + b_1(\mathbf{p})\overline{\mathbf{u4}}(\mathbf{p}) \\ +d_4(\mathbf{p})\mathbf{v1}(\mathbf{p}) + id_3(\mathbf{p})\mathbf{v3}(\mathbf{p}) + ib_3(\mathbf{p})\overline{\mathbf{v2}}(\mathbf{p}) + b_4(\mathbf{p})\overline{\mathbf{v4}}(\mathbf{p}) \end{bmatrix} e^{i(p_0x_1 - p_1x_0 + p_2x_3 - p_3x_2 + \overline{(\mathbf{p},\mathbf{x})})}$$

$$+\begin{bmatrix} b_1^*(\mathbf{p})\overline{\mathbf{u1}}(\mathbf{p}) + ib_2^*(\mathbf{p})\overline{\mathbf{u3}}(\mathbf{p}) + id_2^*(\mathbf{p})\mathbf{u2}(\mathbf{p}) + d_1^*(\mathbf{p})\mathbf{u4}(\mathbf{p}) \\ +b_4^*(\mathbf{p})\overline{\mathbf{v1}}(\mathbf{p}) + ib_3^*(\mathbf{p})\overline{\mathbf{v3}}(\mathbf{p}) + id_3^*(\mathbf{p})\mathbf{v2}(\mathbf{p}) + d_4^*(\mathbf{p})\mathbf{v4}(\mathbf{p}) \end{bmatrix} e^{-i(p_0x_1 - p_1x_0 + p_2x_3 - p_3x_2 + \overline{(\mathbf{p},\mathbf{x})})}$$

Let's impose the anticommutation conditions on the operator coefficients

$$b_1(\mathbf{p})b_1^*(\mathbf{p}') + b_1^*(\mathbf{p}')b_1(\mathbf{p}) = \delta(\mathbf{p} - \mathbf{p}') \qquad b_1^*(\mathbf{p}')b_1(\mathbf{p}) + b_1(\mathbf{p})b_1^*(\mathbf{p}') = \delta(\mathbf{p}' - \mathbf{p})$$
$$d_1(\mathbf{p})d_1^*(\mathbf{p}') + d_1^*(\mathbf{p}')d_1(\mathbf{p}) = \delta(\mathbf{p} - \mathbf{p}') \qquad d_1^*(\mathbf{p})d_1(\mathbf{p}') + d_1(\mathbf{p}')d_1^*(\mathbf{p}) = \delta(\mathbf{p}' - \mathbf{p})$$
$$d_2(\mathbf{p})d_2^*(\mathbf{p}') + d_2^*(\mathbf{p}')d_2(\mathbf{p}) = \delta(\mathbf{p} - \mathbf{p}') \qquad d_2^*(\mathbf{p})d_2(\mathbf{p}') + d_2(\mathbf{p}')d_2^*(\mathbf{p}) = \delta(\mathbf{p}' - \mathbf{p})$$
$$b_2(\mathbf{p})b_2^*(\mathbf{p}') + b_2^*(\mathbf{p}')b_2(\mathbf{p}) = \delta(\mathbf{p} - \mathbf{p}') \qquad b_2^*(\mathbf{p})b_2(\mathbf{p}') + b_2(\mathbf{p}')b_2^*(\mathbf{p}) = \delta(\mathbf{p}' - \mathbf{p})$$
$$d_3(\mathbf{p})d_3^*(\mathbf{p}') + d_3^*(\mathbf{p}')d_3(\mathbf{p}) = \delta(\mathbf{p} - \mathbf{p}') \qquad d_3^*(\mathbf{p})d_3(\mathbf{p}') + d_3(\mathbf{p}')d_3^*(\mathbf{p}) = \delta(\mathbf{p} - \mathbf{p}')$$
$$b_3(\mathbf{p})b_3^*(\mathbf{p}') + b_3^*(\mathbf{p}')b_3(\mathbf{p}) = \delta(\mathbf{p} - \mathbf{p}') \qquad b_3^*(\mathbf{p})b_3(\mathbf{p}') + b_3(\mathbf{p}')b_3^*(\mathbf{p}) = \delta(\mathbf{p}' - \mathbf{p})$$
$$b_4(\mathbf{p})b_4^*(\mathbf{p}') + b_4^*(\mathbf{p}')b_4(\mathbf{p}) = \delta(\mathbf{p} - \mathbf{p}') \qquad b_4^*(\mathbf{p})b_4(\mathbf{p}') + b_4(\mathbf{p}')b_4^*(\mathbf{p}) = \delta(\mathbf{p}' - \mathbf{p})$$
$$d_4(\mathbf{p})d_4^*(\mathbf{p}') + d_4^*(\mathbf{p}')d_4(\mathbf{p}) = \delta(\mathbf{p} - \mathbf{p}') \qquad d_4^*(\mathbf{p})d_4(\mathbf{p}') + d_4(\mathbf{p}')d_4^*(\mathbf{p}) = \delta(\mathbf{p}' - \mathbf{p})$$

We consider the rest anticommutators to be equal to zero. Then we can write the expression for the anticommutator of the field

$$\{\varphi_i(\mathbf{x}), \varphi_j(\mathbf{x}')\} = \varphi_i(\mathbf{x})\varphi_j(\mathbf{x}') + \varphi_j(\mathbf{x}')\varphi_i(\mathbf{x}) = \left(\boldsymbol{\varphi}(\mathbf{x})\boldsymbol{\varphi}^T(\mathbf{x}') + \left(\boldsymbol{\varphi}(\mathbf{x}')\boldsymbol{\varphi}^T(\mathbf{x})\right)^T\right)_{ij}$$

$$\boldsymbol{\varphi}(\mathbf{x})\boldsymbol{\varphi}^T(\mathbf{x}') + \left(\boldsymbol{\varphi}(\mathbf{x}')\boldsymbol{\varphi}^T(\mathbf{x})\right)^T =$$
$$\iint \frac{d^4p}{(2\pi)^4}\frac{d^4p'}{(2\pi)^4} =$$
$$\begin{bmatrix} d_1(\mathbf{p})\mathbf{u1}(\mathbf{p}) + id_2(\mathbf{p})\mathbf{u3}(\mathbf{p}) + ib_2(\mathbf{p})\overline{\mathbf{u2}}(\mathbf{p}) + b_1(\mathbf{p})\overline{\mathbf{u4}}(\mathbf{p}) \\ +d_4(\mathbf{p})\mathbf{v1}(\mathbf{p}) + id_3(\mathbf{p})\mathbf{v3}(\mathbf{p}) + ib_3(\mathbf{p})\overline{\mathbf{v2}}(\mathbf{p}) + b_4(\mathbf{p})\overline{\mathbf{v4}}(\mathbf{p}) \end{bmatrix}$$
$$\begin{bmatrix} b_1^*(\mathbf{p}')\mathbf{u1}^+(\mathbf{p}') + ib_2^*(\mathbf{p}')\mathbf{u3}^+(\mathbf{p}') + id_2^*(\mathbf{p}')\mathbf{u2}^{\boldsymbol{T}}(\mathbf{p}') + d_1^*(\mathbf{p}')\mathbf{u4}^{\boldsymbol{T}}(\mathbf{p}') \\ +b_4^*(\mathbf{p}')\mathbf{v1}^+(\mathbf{p}') + ib_3^*(\mathbf{p}')\mathbf{v3}^+(\mathbf{p}') + id_3^*(\mathbf{p}')\mathbf{v2}^{\boldsymbol{T}}(\mathbf{p}') + d_4^*(\mathbf{p}')\mathbf{v4}^{\boldsymbol{T}}(\mathbf{p}') \end{bmatrix}$$
$$e^{i(p_0x_1 - p_1x_0 + p_2x_3 - p_3x_2 + \overline{(\mathbf{p},\mathbf{x})})} e^{-i(p_0'x_1' - p_1'x_0' + p_2'x_3' - p_3'x_2' + \overline{(\mathbf{p}',\mathbf{x}')})}$$
$$+$$
$$\left(\begin{matrix} \begin{bmatrix} d_1(\mathbf{p}')\mathbf{u1}(\mathbf{p}') + id_2(\mathbf{p}')\mathbf{u3}(\mathbf{p}') + ib_2(\mathbf{p}')\overline{\mathbf{u2}}(\mathbf{p}') + b_1(\mathbf{p}')\overline{\mathbf{u4}}(\mathbf{p}') \\ +d_4(\mathbf{p}')\mathbf{v1}(\mathbf{p}') + id_3(\mathbf{p}')\mathbf{v3}(\mathbf{p}') + ib_3(\mathbf{p}')\overline{\mathbf{v2}}(\mathbf{p}') + b_4(\mathbf{p}')\overline{\mathbf{v4}}(\mathbf{p}') \end{bmatrix} \\ \begin{bmatrix} b_1^*(\mathbf{p})\mathbf{u1}^+(\mathbf{p}) + ib_2^*(\mathbf{p})\mathbf{u3}^+(\mathbf{p}) + id_2^*(\mathbf{p})\mathbf{u2}^{\boldsymbol{T}}(\mathbf{p}) + d_1^*(\mathbf{p})\mathbf{u4}^{\boldsymbol{T}}(\mathbf{p}) \\ +d_4^*(\mathbf{p})\mathbf{v1}^+(\mathbf{p}) + id_3^*(\mathbf{p})\mathbf{v3}^+(\mathbf{p}) + id_3^*(\mathbf{p})\mathbf{v2}^{\boldsymbol{T}}(\mathbf{p}) + d_4^*(\mathbf{p})\mathbf{v4}^{\boldsymbol{T}}(\mathbf{p}) \end{bmatrix} \\ e^{i(p_0'x_1' - p_1'x_0' + p_2'x_3' - p_3'x_2' + \overline{(\mathbf{p}',\mathbf{x}')})} e^{-i(p_0x_1 - p_1x_0 + p_2x_3 - p_3x_2 + \overline{(\mathbf{p},\mathbf{x})})} \end{matrix}\right)^T$$
$$+$$
$$\begin{bmatrix} b_1^*(\mathbf{p})\overline{\mathbf{u1}}(\mathbf{p}) + ib_2^*(\mathbf{p})\overline{\mathbf{u3}}(\mathbf{p}) + id_2^*(\mathbf{p})\mathbf{u2}(\mathbf{p}) + d_1^*(\mathbf{p})\mathbf{u4}(\mathbf{p}) \\ +b_4^*(\mathbf{p})\overline{\mathbf{v1}}(\mathbf{p}) + ib_3^*(\mathbf{p})\overline{\mathbf{v3}}(\mathbf{p}) + id_3^*(\mathbf{p})\mathbf{v2}(\mathbf{p}) + d_4^*(\mathbf{p})\mathbf{v4}(\mathbf{p}) \end{bmatrix}$$
$$\begin{bmatrix} d_1(\mathbf{p}')\mathbf{u1}^{\boldsymbol{T}}(\mathbf{p}') + id_2(\mathbf{p}')\mathbf{u3}^{\boldsymbol{T}}(\mathbf{p}') + ib_2(\mathbf{p}')\mathbf{u2}^+(\mathbf{p}') + b_1(\mathbf{p}')\mathbf{u4}^+(\mathbf{p}') \\ +d_4(\mathbf{p}')\mathbf{v1}^{\boldsymbol{T}}(\mathbf{p}') + id_3(\mathbf{p}')\mathbf{v3}^{\boldsymbol{T}}(\mathbf{p}') + ib_3(\mathbf{p}')\mathbf{v2}^+(\mathbf{p}') + b_4(\mathbf{p}')\mathbf{v4}^+(\mathbf{p}') \end{bmatrix}$$
$$e^{-i(p_0x_1 - p_1x_0 + p_2x_3 - p_3x_2 + \overline{(\mathbf{p},\mathbf{x})})} e^{i(p_0'x_1' - p_1'x_0' + p_2'x_3' - p_3'x_2' + \overline{(\mathbf{p}',\mathbf{x}')})}$$
$$+$$

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$$\begin{pmatrix} \begin{bmatrix} b_1^*(\mathbf{p}')\overline{\mathbf{u1}}(\mathbf{p}') + ib_2^*(\mathbf{p}')\overline{\mathbf{u3}}(\mathbf{p}') + id_2^*(\mathbf{p}')\mathbf{u2}(\mathbf{p}') + d_1^*(\mathbf{p}')\mathbf{u4}(\mathbf{p}') \\ +b_4^*(\mathbf{p}')\overline{\mathbf{v1}}(\mathbf{p}') + ib_3^*(\mathbf{p}')\overline{\mathbf{v3}}(\mathbf{p}') + id_3^*(\mathbf{p}')\mathbf{v2}(\mathbf{p}') + d_4^*(\mathbf{p}')\mathbf{v4}(\mathbf{p}') \end{bmatrix} \\ \begin{bmatrix} d_1(\mathbf{p})\mathbf{u1}^{\boldsymbol{T}}(\mathbf{p}) + id_2(\mathbf{p})\mathbf{u3}^{\boldsymbol{T}}(\mathbf{p}) + ib_2(\mathbf{p})\mathbf{u2}^{+}(\mathbf{p}) + b_1(\mathbf{p})\mathbf{u4}^{+}(\mathbf{p}) \\ +d_4(\mathbf{p})\mathbf{v1}^{\boldsymbol{T}}(\mathbf{p}) + id_3(\mathbf{p})\mathbf{v3}^{\boldsymbol{T}}(\mathbf{p}) + ib_3(\mathbf{p})\mathbf{v2}^{+}(\mathbf{p}) + b_4(\mathbf{p})\mathbf{v4}^{+}(\mathbf{p}) \end{bmatrix} \\ e^{-i(p_0{}'x_1{}'-p_1{}'x_0{}'+p_2{}'x_3{}'-p_3{}'x_2{}'+\overline{(\mathbf{p}',\mathbf{x}')})}e^{i(p_0x_1-p_1x_0+p_2x_3-p_3x_2+\overline{(\mathbf{p},\mathbf{x})})} \end{pmatrix}^T$$

$$= \iint \frac{d^4p}{(2\pi)^4}\frac{d^4p'}{(2\pi)^4} \begin{bmatrix} \begin{bmatrix} d_1(\mathbf{p})d_1^*(\mathbf{p}')\mathbf{u1}(\mathbf{p})\mathbf{u4}^{\boldsymbol{T}}(\mathbf{p}') + d_1(\mathbf{p}')d_1^*(\mathbf{p})\left(\mathbf{u1}(\mathbf{p}')\mathbf{u4}^{\boldsymbol{T}}(\mathbf{p})\right)^{\boldsymbol{T}} \\ -d_2(\mathbf{p})d_2^*(\mathbf{p}')\mathbf{u3}(\mathbf{p})\mathbf{u2}^{\boldsymbol{T}}(\mathbf{p}') - d_2(\mathbf{p}')d_2^*(\mathbf{p})\left(\mathbf{u3}(\mathbf{p}')\mathbf{u2}^{\boldsymbol{T}}(\mathbf{p})\right)^{\boldsymbol{T}} + \cdots \end{bmatrix} \\ e^{i(p_0x_1-p_1x_0+p_2x_3-p_3x_2+\overline{(\mathbf{p},\mathbf{x})})}e^{-i(p_0{}'x_1{}'-p_1{}'x_0{}'+p_2{}'x_3{}'-p_3{}'x_2{}'+\overline{(\mathbf{p}',\mathbf{x}')})} \\ + \\ \begin{bmatrix} b_1(\mathbf{p})\overline{b_1^*(\mathbf{p}')\mathbf{u4}}(\mathbf{p})\mathbf{u1}^{+}(\mathbf{p}') + \ b_1\left(\mathbf{p}'b_1^*(\mathbf{p})\right)\left(\overline{\mathbf{u4}}(\mathbf{p}')\mathbf{u1}^{+}(\mathbf{p})\right)^{\boldsymbol{T}} \\ -b_2(\mathbf{p})b_2^*(\mathbf{p}')\overline{\mathbf{u2}}(\mathbf{p})\mathbf{u3}^{+}(\mathbf{p}') - b_2(\mathbf{p}')b_2^*(\mathbf{p})\left(\overline{\mathbf{u2}}(\mathbf{p}')\mathbf{u3}^{+}(\mathbf{p})\right)^{\boldsymbol{T}} + \cdots \end{bmatrix} \\ e^{i(p_0{}'x_1{}'-p_1{}'x_0{}'+p_2{}'x_3{}'-p_3{}'x_2{}'+\overline{(\mathbf{p}',\mathbf{x}')})}e^{-i(p_0x_1-p_1x_0+p_2x_3-p_3x_2+\overline{(\mathbf{p},\mathbf{x})})} \end{bmatrix}$$

$$+ \iint \frac{d^4p}{(2\pi)^4}\frac{d^4p'}{(2\pi)^4} \begin{bmatrix} \begin{bmatrix} b_1^*(\mathbf{p})b_1(\mathbf{p}')\overline{\mathbf{u1}}(\mathbf{p})\mathbf{u4}^{+}(\mathbf{p}') + b_1^*(\mathbf{p}')b_1(\mathbf{p})\left(\overline{\mathbf{u1}}(\mathbf{p}')\mathbf{u4}^{+}(\mathbf{p})\right)^{\boldsymbol{T}} \\ -b_2^*(\mathbf{p})b_2(\mathbf{p}')\overline{\mathbf{u3}}(\mathbf{p})\mathbf{u2}^{+}(\mathbf{p}') - b_2^*(\mathbf{p}')b_2(\mathbf{p})\left(\overline{\mathbf{u3}}(\mathbf{p}')\mathbf{u2}^{+}(\mathbf{p})\right)^{\boldsymbol{T}} + \cdots \end{bmatrix} \\ e^{-i(p_0x_1-p_1x_0+p_2x_3-p_3x_2+\overline{(\mathbf{p},\mathbf{x})})}e^{i(p_0{}'x_1{}'-p_1{}'x_0{}'+p_2{}'x_3{}'-p_3{}'x_2{}'+\overline{(\mathbf{p}',\mathbf{x}')})} \\ + \\ \begin{bmatrix} d_1^*(\mathbf{p})d_1(\mathbf{p}')\mathbf{u4}(\mathbf{p})\mathbf{u1}^{\boldsymbol{T}}(\mathbf{p}') + d_1^*(\mathbf{p}')d_1(\mathbf{p})\left(\mathbf{u4}(\mathbf{p}')\mathbf{u1}^{\boldsymbol{T}}(\mathbf{p})\right)^{\boldsymbol{T}} \\ -d_2^*(\mathbf{p})d_2(\mathbf{p}')\mathbf{u2}(\mathbf{p})\mathbf{u3}^{\boldsymbol{T}}(\mathbf{p}') - d_2^*(\mathbf{p}')d_2(\mathbf{p})\left(\mathbf{u2}(\mathbf{p}')\mathbf{u3}^{\boldsymbol{T}}(\mathbf{p})\right)^{\boldsymbol{T}} + \cdots \end{bmatrix} \\ e^{-i(p_0{}'x_1{}'-p_1{}'x_0{}'+p_2{}'x_3{}'-p_3{}'x_2{}'+\overline{(\mathbf{p}',\mathbf{x}')})}e^{i(p_0x_1-p_1x_0+p_2x_3-p_3x_2+\overline{(\mathbf{p},\mathbf{x})})} \end{bmatrix}$$

$$= \iint \frac{d^4p}{(2\pi)^4}\frac{d^4p'}{(2\pi)^4} \begin{bmatrix} \begin{bmatrix} d_1(\mathbf{p})d_1^*(\mathbf{p}')\mathbf{u1}(\mathbf{p})\mathbf{u4}^{\boldsymbol{T}}(\mathbf{p}') + d_1(\mathbf{p}')d_1^*(\mathbf{p})\left(\mathbf{u4}(\mathbf{p})\mathbf{u1}^{\boldsymbol{T}}(\mathbf{p}')\right) \\ -d_2(\mathbf{p})d_2^*(\mathbf{p}')\mathbf{u3}(\mathbf{p})\mathbf{u2}^{\boldsymbol{T}}(\mathbf{p}') - d_2(\mathbf{p}')d_2^*(\mathbf{p})\left(\mathbf{u2}(\mathbf{p})\mathbf{u3}^{\boldsymbol{T}}(\mathbf{p}')\right) + \cdots \end{bmatrix} \\ e^{i(p_0x_1-p_1x_0+p_2x_3-p_3x_2+\overline{(\mathbf{p},\mathbf{x})})}e^{-i(p_0{}'x_1{}'-p_1{}'x_0{}'+p_2{}'x_3{}'-p_3{}'x_2{}'+\overline{(\mathbf{p}',\mathbf{x}')})} \\ + \\ \begin{bmatrix} b_1(\mathbf{p})b_1^*(\mathbf{p}')\overline{\mathbf{u4}}(\mathbf{p})\mathbf{u1}^{+}(\mathbf{p}') + \ b_1\left(\mathbf{p}'b_1^*(\mathbf{p})\right)\left(\overline{\mathbf{u1}}(\mathbf{p})\mathbf{u4}^{+}(\mathbf{p}')\right) \\ -b_2(\mathbf{p})b_2^*(\mathbf{p}')\overline{\mathbf{u2}}(\mathbf{p})\mathbf{u3}^{+}(\mathbf{p}') - b_2(\mathbf{p}')b_2^*(\mathbf{p})\left(\overline{\mathbf{u3}}(\mathbf{p})\mathbf{u2}^{+}(\mathbf{p}')\right) + \cdots \end{bmatrix} \\ e^{i(p_0{}'x_1{}'-p_1{}'x_0{}'+p_2{}'x_3{}'-p_3{}'x_2{}'+\overline{(\mathbf{p}',\mathbf{x}')})}e^{-i(p_0x_1-p_1x_0+p_2x_3-p_3x_2+\overline{(\mathbf{p},\mathbf{x})})} \end{bmatrix}$$

$$+ \iint \frac{d^4p}{(2\pi)^4}\frac{d^4p'}{(2\pi)^4} \begin{bmatrix} \begin{bmatrix} b_1^*(\mathbf{p})b_1(\mathbf{p}')\overline{\mathbf{u1}}(\mathbf{p})\mathbf{u4}^{+}(\mathbf{p}') + b_1^*(\mathbf{p}')b_1(\mathbf{p})\left(\overline{\mathbf{u4}}(\mathbf{p})\mathbf{u1}^{+}(\mathbf{p}')\right) \\ -b_2^*(\mathbf{p})b_2(\mathbf{p}')\overline{\mathbf{u3}}(\mathbf{p})\mathbf{u2}^{+}(\mathbf{p}') - b_2^*(\mathbf{p}')b_2(\mathbf{p})\left(\overline{\mathbf{u2}}(\mathbf{p})\mathbf{u3}^{+}(\mathbf{p}')\right) + \cdots \end{bmatrix} \\ e^{-i(p_0x_1-p_1x_0+p_2x_3-p_3x_2+\overline{(\mathbf{p},\mathbf{x})})}e^{i(p_0{}'x_1{}'-p_1{}'x_0{}'+p_2{}'x_3{}'-p_3{}'x_2{}'+\overline{(\mathbf{p}',\mathbf{x}')})} \\ + \\ \begin{bmatrix} d_1^*(\mathbf{p})d_1(\mathbf{p}')\mathbf{u4}(\mathbf{p})\mathbf{u1}^{\boldsymbol{T}}(\mathbf{p}') + d_1^*(\mathbf{p}')d_1(\mathbf{p})\left(\mathbf{u1}(\mathbf{p})\mathbf{u4}^{\boldsymbol{T}}(\mathbf{p}')\right) \\ -d_2^*(\mathbf{p})d_2(\mathbf{p}')\mathbf{u2}(\mathbf{p})\mathbf{u3}^{\boldsymbol{T}}(\mathbf{p}') - d_2^*(\mathbf{p}')d_2(\mathbf{p})\left(\mathbf{u3}(\mathbf{p})\mathbf{u2}^{\boldsymbol{T}}(\mathbf{p}')\right) + \cdots \end{bmatrix} \\ e^{-i(p_0{}'x_1{}'-p_1{}'x_0{}'+p_2{}'x_3{}'-p_3{}'x_2{}'+\overline{(\mathbf{p}',\mathbf{x}')})}e^{i(p_0x_1-p_1x_0+p_2x_3-p_3x_2+\overline{(\mathbf{p},\mathbf{x})})} \end{bmatrix}$$

$$= \int \frac{d^4p}{(2\pi)^4} \begin{bmatrix} \begin{bmatrix} \mathbf{u1}(\mathbf{p})\mathbf{u4}^{\boldsymbol{T}}(\mathbf{p}) + \cdots \\ -\mathbf{u3}(\mathbf{p})\mathbf{u2}^{\boldsymbol{T}}(\mathbf{p}) + \cdots \end{bmatrix} \\ e^{i(p_0(x_1-x_1{}')-p_1(x_0-x_0{}')+p_2(x_3-x_3{}')-p_3(x_2-x_2{}')+\overline{(\mathbf{p},\mathbf{x}-\mathbf{x}')})} \\ + \\ \begin{bmatrix} \overline{\mathbf{u4}}(\mathbf{p})\mathbf{u1}^{+}(\mathbf{p}) + \cdots \\ -\overline{\mathbf{u2}}(\mathbf{p})\mathbf{u3}^{+}(\mathbf{p}) + \cdots \end{bmatrix} \\ e^{-i(p_0(x_1-x_1{}')-p_1(x_0-x_0{}')+p_2(x_3-x_3{}')-p_3(x_2-x_2{}')+\overline{(\mathbf{p},\mathbf{x}-\mathbf{x}')})} \end{bmatrix}$$

$$+\int \frac{d^4p}{(2\pi)^4}\left[\begin{matrix}\begin{bmatrix}\overline{\mathbf{u1}}(\mathbf{p})\mathbf{u4}^+(\mathbf{p})+\cdots\\ -\overline{\mathbf{u3}}(\mathbf{p})\mathbf{u2}^+(\mathbf{p})+\cdots\end{bmatrix}\\ e^{-i\left(p_0(x_1-x_1{}')-p_1(x_0-x_0{}')+p_2(x_3-x_3{}')-p_3(x_2-x_2{}')+\overline{(\mathbf{p},\mathbf{x}-\mathbf{x}')}\right)}\\ +\\ \begin{bmatrix}\mathbf{u4}(\mathbf{p})\mathbf{u1}^T(\mathbf{p})+\cdots\\ -\mathbf{u2}(\mathbf{p})\mathbf{u3}^T(\mathbf{p})+\cdots\end{bmatrix}\\ e^{i\left(p_0(x_1-x_1{}')-p_1(x_0-x_0{}')+p_2(x_3-x_3{}')-p_3(x_2-x_2{}')+\overline{(\mathbf{p},\mathbf{x}-\mathbf{x}')}\right)}\end{matrix}\right]$$

$$=\int \frac{d^4p}{(2\pi)^4}\left[\begin{matrix}\begin{bmatrix}\mathbf{u1}(\mathbf{p})\mathbf{u4}^T(\mathbf{p})-\mathbf{u3}(\mathbf{p})\mathbf{u2}^T(\mathbf{p})+\cdots+\\ \mathbf{u4}(\mathbf{p})\mathbf{u1}^T(\mathbf{p})-\mathbf{u2}(\mathbf{p})\mathbf{u3}^T(\mathbf{p})+\cdots+\end{bmatrix}\\ e^{i\left(p_0(x_1-x_1{}')-p_1(x_0-x_0{}')+p_2(x_3-x_3{}')-p_3(x_2-x_2{}')+\overline{(\mathbf{p},\mathbf{x}-\mathbf{x}')}\right)}\\ +\\ \begin{bmatrix}\overline{\mathbf{u4}}(\mathbf{p})\mathbf{u1}^+(\mathbf{p})-\overline{\mathbf{u2}}(\mathbf{p})\mathbf{u3}^+(\mathbf{p})+\cdots+\\ \overline{\mathbf{u1}}(\mathbf{p})\mathbf{u4}^+(\mathbf{p})-\overline{\mathbf{u3}}(\mathbf{p})\mathbf{u2}^+(\mathbf{p})+\cdots+\end{bmatrix}\\ e^{-i\left(p_0(x_1-x_1{}')-p_1(x_0-x_0{}')+p_2(x_3-x_3{}')-p_3(x_2-x_2{}')+\overline{(\mathbf{p},\mathbf{x}-\mathbf{x}')}\right)}\end{matrix}\right]$$

$$=\int \frac{d^4p}{(2\pi)^4}\left[\begin{matrix}\begin{bmatrix}\mathbf{u1}(\mathbf{p})\mathbf{u4}^T(\mathbf{p})-\mathbf{u3}(\mathbf{p})\mathbf{u2}^T(\mathbf{p})+\mathbf{v1}(\mathbf{p})\mathbf{v4}^T(\mathbf{p})-\mathbf{v3}(\mathbf{p})\mathbf{v2}^T(\mathbf{p})+\\ \mathbf{u4}(\mathbf{p})\mathbf{u1}^T(\mathbf{p})-\mathbf{u2}(\mathbf{p})\mathbf{u3}^T(\mathbf{p})+\mathbf{v4}(\mathbf{p})\mathbf{v1}^T(\mathbf{p})-\mathbf{v2}(\mathbf{p})\mathbf{v3}^T(\mathbf{p})\end{bmatrix}\\ e^{i\left(p_0(x_1-x_1{}')-p_1(x_0-x_0{}')+p_2(x_3-x_3{}')-p_3(x_2-x_2{}')+\overline{(\mathbf{p},\mathbf{x}-\mathbf{x}')}\right)}\\ +\\ \begin{bmatrix}\overline{\mathbf{u4}}(\mathbf{p})\mathbf{u1}^+(\mathbf{p})-\overline{\mathbf{u2}}(\mathbf{p})\mathbf{u3}^+(\mathbf{p})+\overline{\mathbf{v4}}(\mathbf{p})\mathbf{v1}^+(\mathbf{p})-\overline{\mathbf{v2}}(\mathbf{p})\mathbf{v3}^+(\mathbf{p})+\\ \overline{\mathbf{u1}}(\mathbf{p})\mathbf{u4}^+(\mathbf{p})-\overline{\mathbf{u3}}(\mathbf{p})\mathbf{u2}^+(\mathbf{p})+\overline{\mathbf{v1}}(\mathbf{p})\mathbf{v4}^+(\mathbf{p})-\overline{\mathbf{v3}}(\mathbf{p})\mathbf{v2}^+(\mathbf{p})\end{bmatrix}\\ e^{-i\left(p_0(x_1-x_1{}')-p_1(x_0-x_0{}')+p_2(x_3-x_3{}')-p_3(x_2-x_2{}')+\overline{(\mathbf{p},\mathbf{x}-\mathbf{x}')}\right)}\end{matrix}\right]$$

$$=\int \frac{d^4p}{(2\pi)^4}\left(S^R(\mathbf{p})+S_R(\mathbf{p})\right)e^{\left(i\left(p_0(x_1-x_1{}')-p_1(x_0-x_0{}')+p_2(x_3-x_3{}')-p_3(x_2-x_2{}')+\overline{(\mathbf{p},\mathbf{x}-\mathbf{x}')}\right)\right)}+$$

$$\int \frac{d^4p}{(2\pi)^4}\left(\overline{S_R}(\mathbf{p})+\overline{S^R}(\mathbf{p})\right)e^{-\left(i\left(p_0(x_1-x_1{}')-p_1(x_0-x_0{}')+p_2(x_3-x_3{}')-p_3(x_2-x_2{}')+\overline{(\mathbf{p},\mathbf{x}-\mathbf{x}')}\right)\right)}$$

$$=$$

$$\int \frac{d^4p}{(2\pi)^4}4\begin{pmatrix}m&0&0&0\\0&m&0&0\\0&0&m&0\\0&0&0&m\end{pmatrix}e^{\left(i\left(p_0(x_1-x_1{}')-p_1(x_0-x_0{}')+p_2(x_3-x_3{}')-p_3(x_2-x_2{}')+\overline{(\mathbf{p},\mathbf{x}-\mathbf{x}')}\right)\right)}+$$

$$\int \frac{d^4p}{(2\pi)^4}4\begin{pmatrix}\overline{m}&0&0&0\\0&\overline{m}&0&0\\0&0&\overline{m}&0\\0&0&0&\overline{m}\end{pmatrix}e^{-\left(i\left(p_0(x_1-x_1{}')-p_1(x_0-x_0{}')+p_2(x_3-x_3{}')-p_3(x_2-x_2{}')+\overline{(\mathbf{p},\mathbf{x}-\mathbf{x}')}\right)\right)}$$

$$=4mI\delta(\mathbf{x}'-\mathbf{x})+4\overline{m}I\delta(\mathbf{x}-\mathbf{x}')$$

We will consider this relation as a proof of the anti-symmetry of the fermion wave function under the stipulated anticommutation relations.

It is important that all the above deductions are valid in any frame of reference, while the proof of anticommutativity of the fermion field in [9] is carried out for the rest frame.

Let us calculate the total energy of the fermion field

$$E=P_0=\int d^4x\,\boldsymbol{\varphi}^+(\mathbf{x})S_0\boldsymbol{\varphi}(\mathbf{x})$$

$$=\int d^4x\iint \frac{d^4p}{(2\pi)^4}\frac{d^4p'}{(2\pi)^4}$$

$$\left[\begin{matrix}\begin{bmatrix}d_1^*(\mathbf{p}')\mathbf{u1}^+(\mathbf{p}')-id_2^*(\mathbf{p}')\mathbf{u3}^+(\mathbf{p}')-ib_2^*(\mathbf{p}')\mathbf{u2}^T(\mathbf{p}')+b_1^*(\mathbf{p}')\mathbf{u4}^T(\mathbf{p}')\\ +d_4^*(\mathbf{p}')\mathbf{v1}^+(\mathbf{p}')-id_3^*(\mathbf{p}')\mathbf{v3}^+(\mathbf{p}')-ib_3^*(\mathbf{p}')\mathbf{v2}^T(\mathbf{p}')+b_4^*(\mathbf{p}')\mathbf{v4}^T(\mathbf{p}')\end{bmatrix}e^{-i\left(p_0'x_1-p_1'x_0+p_2'x_3-p_3'x_2+\overline{(\mathbf{p}',\mathbf{x})}\right)}\\ +\begin{bmatrix}b_1(\mathbf{p}')\mathbf{u1}^T(\mathbf{p}')-ib_2(\mathbf{p}')\mathbf{u3}^T(\mathbf{p}')-id_2(\mathbf{p}')\mathbf{u2}^+(\mathbf{p}')+d_1(\mathbf{p}')\mathbf{u4}^+(\mathbf{p}')\\ +b_4(\mathbf{p}')\mathbf{v1}^T(\mathbf{p}')-ib_3(\mathbf{p}')\mathbf{v3}^T(\mathbf{p}')-id_3(\mathbf{p}')\mathbf{v2}^+(\mathbf{p}')+d_4(\mathbf{p}')\mathbf{v4}^+(\mathbf{p}')\end{bmatrix}e^{i\left(p_0'x_1-p_1'x_0+p_2'x_3-p_3'x_2+\overline{(\mathbf{p}',\mathbf{x})}\right)}\end{matrix}\right]$$

$$\left[\begin{matrix}\begin{bmatrix}d_1(\mathbf{p})\mathbf{u1}^T(\mathbf{p})+id_2(\mathbf{p})\mathbf{u3}^T(\mathbf{p})+ib_2(\mathbf{p})\mathbf{u2}^+(\mathbf{p})+b_1(\mathbf{p})\mathbf{u4}^+(\mathbf{p})\\ +d_4(\mathbf{p})\mathbf{v1}^T(\mathbf{p})+id_3(\mathbf{p})\mathbf{v3}^T(\mathbf{p})+ib_3(\mathbf{p})\mathbf{v2}^+(\mathbf{p})+b_4(\mathbf{p})\mathbf{v4}^+(\mathbf{p})\end{bmatrix}e^{i\left(p_0x_1-p_1x_0+p_2x_3-p_3x_2+\overline{(\mathbf{p},\mathbf{x})}\right)}\\ +\begin{bmatrix}b_1^*(\mathbf{p})\mathbf{u1}^+(\mathbf{p})+ib_2^*(\mathbf{p})\mathbf{u3}^+(\mathbf{p})+id_2^*(\mathbf{p})\mathbf{u2}^T(\mathbf{p})+d_1^*(\mathbf{p})\mathbf{u4}^T(\mathbf{p})\\ +b_4^*(\mathbf{p})\mathbf{v1}^+(\mathbf{p})+ib_3^*(\mathbf{p})\mathbf{v3}^+(\mathbf{p})+id_3^*(\mathbf{p})\mathbf{v2}^T(\mathbf{p})+d_4^*(\mathbf{p})\mathbf{v4}^T(\mathbf{p})\end{bmatrix}e^{-i\left(p_0x_1-p_1x_0+p_2x_3-p_3x_2+\overline{(\mathbf{p},\mathbf{x})}\right)}\end{matrix}\right]$$

$$= \int d^4x \iint \frac{d^4p}{(2\pi)^4}\frac{d^4p'}{(2\pi)^4}$$

$$\left[\begin{array}{c} \begin{bmatrix} d_1^*(\mathbf{p}')\mathbf{u1}^+(\mathbf{p}') - id_2^*(\mathbf{p}')\mathbf{u3}^+(\mathbf{p}') - ib_2^*(\mathbf{p}')\mathbf{u2}^T(\mathbf{p}') + b_1^*(\mathbf{p}')\mathbf{u4}^T(\mathbf{p}') \\ +d_4^*(\mathbf{p}')\mathbf{v1}^+(\mathbf{p}') - id_3^*(\mathbf{p}')\mathbf{v3}^+(\mathbf{p}') - ib_3^*(\mathbf{p}')\mathbf{v2}^T(\mathbf{p}') + b_4^*(\mathbf{p}')\mathbf{v4}^T(\mathbf{p}') \end{bmatrix} \\ \begin{bmatrix} d_1(\mathbf{p})\mathbf{u1}(\mathbf{p}) + id_2(\mathbf{p})\mathbf{u3}(\mathbf{p}) + ib_2(\mathbf{p})\overline{\mathbf{u2}}(\mathbf{p}) + b_1(\mathbf{p})\overline{\mathbf{u4}}(\mathbf{p}) \\ +d_4(\mathbf{p})\mathbf{v1}(\mathbf{p}) + id_3(\mathbf{p})\mathbf{v3}(\mathbf{p}) + ib_3(\mathbf{p})\overline{\mathbf{v2}}(\mathbf{p}) + b_4(\mathbf{p})\overline{\mathbf{v4}}(\mathbf{p}) \end{bmatrix} \\ e^{-i(p_0'x_1 - p_1'x_0 + p_2'x_3 - p_3'x_2 + \overline{(\mathbf{p}',\mathbf{x})})} e^{i(p_0x_1 - p_1x_0 + p_2x_3 - p_3x_2 + \overline{(\mathbf{p},\mathbf{x})})} \\ + \begin{bmatrix} b_1(\mathbf{p}')\mathbf{u1}^T(\mathbf{p}') - ib_2(\mathbf{p}')\mathbf{u3}^T(\mathbf{p}') - id_2(\mathbf{p}')\mathbf{u2}^+(\mathbf{p}') + d_1(\mathbf{p}')\mathbf{u4}^+(\mathbf{p}') \\ +b_4(\mathbf{p}')\mathbf{v1}^T(\mathbf{p}') - ib_3(\mathbf{p}')\mathbf{v3}^T(\mathbf{p}') - id_3(\mathbf{p}')\mathbf{v2}^+(\mathbf{p}') + d_4(\mathbf{p}')\mathbf{v4}^+(\mathbf{p}') \end{bmatrix} \\ \begin{bmatrix} b_1^*(\mathbf{p})\overline{\mathbf{u1}}(\mathbf{p}) + ib_2^*(\mathbf{p})\overline{\mathbf{u3}}(\mathbf{p}) + id_2^*(\mathbf{p})\mathbf{u2}(\mathbf{p}) + d_1^*(\mathbf{p})\mathbf{u4}(\mathbf{p}) \\ +b_4^*(\mathbf{p})\overline{\mathbf{v1}}(\mathbf{p}) + ib_3^*(\mathbf{p})\overline{\mathbf{v3}}(\mathbf{p}) + id_3^*(\mathbf{p})\mathbf{v2}(\mathbf{p}) + d_4^*(\mathbf{p})\mathbf{v4}(\mathbf{p}) \end{bmatrix} \\ e^{i(p_0'x_1 - p_1'x_0 + p_2'x_3 - p_3'x_2 + \overline{(\mathbf{p}',\mathbf{x})})} e^{-i(p_0x_1 - p_1x_0 + p_2x_3 - p_3x_2 + \overline{(\mathbf{p},\mathbf{x})})} \end{array}\right]$$

$$= \iint \frac{d^4p}{(2\pi)^4}\frac{d^4p'}{(2\pi)^4}$$

$$\left[\begin{array}{c} \begin{bmatrix} d_1^*(\mathbf{p}')\mathbf{u1}^+(\mathbf{p}') - id_2^*(\mathbf{p}')\mathbf{u3}^+(\mathbf{p}') - ib_2^*(\mathbf{p}')\mathbf{u2}^T(\mathbf{p}') + b_1^*(\mathbf{p}')\mathbf{u4}^T(\mathbf{p}') \\ +d_4^*(\mathbf{p}')\mathbf{v1}^+(\mathbf{p}') - id_3^*(\mathbf{p}')\mathbf{v3}^+(\mathbf{p}') - ib_3^*(\mathbf{p}')\mathbf{v2}^T(\mathbf{p}') + b_4^*(\mathbf{p}')\mathbf{v4}^T(\mathbf{p}') \end{bmatrix} \\ \begin{bmatrix} d_1(\mathbf{p})\mathbf{u1}(\mathbf{p}) + id_2(\mathbf{p})\mathbf{u3}(\mathbf{p}) + ib_2(\mathbf{p})\overline{\mathbf{u2}}(\mathbf{p}) + b_1(\mathbf{p})\overline{\mathbf{u4}}(\mathbf{p}) \\ +d_4(\mathbf{p})\mathbf{v1}(\mathbf{p}) + id_3(\mathbf{p})\mathbf{v3}(\mathbf{p}) + ib_3(\mathbf{p})\overline{\mathbf{v2}}(\mathbf{p}) + b_4(\mathbf{p})\overline{\mathbf{v4}}(\mathbf{p}) \end{bmatrix} \\ \delta(\mathbf{p}' - \mathbf{p}) \\ + \begin{bmatrix} b_1(\mathbf{p}')\mathbf{u1}^T(\mathbf{p}') - ib_2(\mathbf{p}')\mathbf{u3}^T(\mathbf{p}') - id_2(\mathbf{p}')\mathbf{u2}^+(\mathbf{p}') + d_1(\mathbf{p}')\mathbf{u4}^+(\mathbf{p}') \\ +b_4(\mathbf{p}')\mathbf{v1}^T(\mathbf{p}') - ib_3(\mathbf{p}')\mathbf{v3}^T(\mathbf{p}') - id_3(\mathbf{p}')\mathbf{v2}^+(\mathbf{p}') + d_4(\mathbf{p}')\mathbf{v4}^+(\mathbf{p}') \end{bmatrix} \\ \begin{bmatrix} b_1^*(\mathbf{p})\overline{\mathbf{u1}}(\mathbf{p}) + ib_2^*(\mathbf{p})\overline{\mathbf{u3}}(\mathbf{p}) + id_2^*(\mathbf{p})\mathbf{u2}(\mathbf{p}) + d_1^*(\mathbf{p})\mathbf{u4}(\mathbf{p}) \\ +b_4^*(\mathbf{p})\overline{\mathbf{v1}}(\mathbf{p}) + ib_3^*(\mathbf{p})\overline{\mathbf{v3}}(\mathbf{p}) + id_3^*(\mathbf{p})\mathbf{v2}(\mathbf{p}) + d_4^*(\mathbf{p})\mathbf{v4}(\mathbf{p}) \end{bmatrix} \\ \delta(\mathbf{p} - \mathbf{p}') \end{array}\right]$$

$$= \int \frac{d^4p}{(2\pi)^4} \begin{bmatrix} d_1^*(\mathbf{p})d_1(\mathbf{p})\mathbf{u1}^+(\mathbf{p})\mathbf{u1}(\mathbf{p}) + d_1(\mathbf{p})d_1^*(\mathbf{p})\mathbf{u4}^+(\mathbf{p})\mathbf{u4}(\mathbf{p}) \\ +b_1(\mathbf{p})b_1^*(\mathbf{p})\mathbf{u1}^T(\mathbf{p})\overline{\mathbf{u1}}(\mathbf{p}) + b_1^*(\mathbf{p})b_1(\mathbf{p})\mathbf{u4}^T(\mathbf{p})\overline{\mathbf{u4}}(\mathbf{p}) \\ +b_2(\mathbf{p})b_2^*(\mathbf{p})\mathbf{u3}^T(\mathbf{p})\overline{\mathbf{u3}}(\mathbf{p}) + b_2^*(\mathbf{p})b_2(\mathbf{p})\mathbf{u2}^T(\mathbf{p})\overline{\mathbf{u2}}(\mathbf{p}) \\ +d_2^*(\mathbf{p})d_2(\mathbf{p})\mathbf{u3}^+(\mathbf{p})\mathbf{u3}(\mathbf{p}) + d_2(\mathbf{p})d_2^*(\mathbf{p})\mathbf{u2}^+(\mathbf{p})\mathbf{u2}(\mathbf{p}) \\ +d_4^*(\mathbf{p})d_4(\mathbf{p})\mathbf{v1}^+(\mathbf{p})\mathbf{v1}(\mathbf{p}) + d_4(\mathbf{p})d_4^*(\mathbf{p})\mathbf{v4}^+(\mathbf{p})\mathbf{v4}(\mathbf{p}) \\ +b_4(\mathbf{p})b_4^*(\mathbf{p})\mathbf{v1}^T(\mathbf{p})\overline{\mathbf{v1}}(\mathbf{p}) + b_4^*(\mathbf{p})b_4(\mathbf{p})\mathbf{v4}^T(\mathbf{p})\overline{\mathbf{v4}}(\mathbf{p}) \\ +b_3(\mathbf{p})b_3^*(\mathbf{p})\mathbf{v3}^T(\mathbf{p})\overline{\mathbf{v3}}(\mathbf{p}) + b_3^*(\mathbf{p})b_3(\mathbf{p})\mathbf{v2}^T(\mathbf{p})\overline{\mathbf{v2}}(\mathbf{p}) \\ +d_3^*(\mathbf{p})d_3(\mathbf{p})\mathbf{v3}^+(\mathbf{p})\mathbf{v3}(\mathbf{p}) + d_3(\mathbf{p})d_3^*(\mathbf{p})\mathbf{v2}^+(\mathbf{p})\mathbf{v2}(\mathbf{p}) \end{bmatrix}$$

$$= \int \frac{d^4p}{(2\pi)^4}\, e_0(\mathbf{p}) \begin{bmatrix} b_1(\mathbf{p})b_1^*(\mathbf{p}) + b_1^*(\mathbf{p})b_1(\mathbf{p}) + d_1^*(\mathbf{p})d_1(\mathbf{p}) + d_1(\mathbf{p})d_1^*(\mathbf{p}) \\ +b_2(\mathbf{p})b_2^*(\mathbf{p}) + b_2^*(\mathbf{p})b_2(\mathbf{p}) + d_2^*(\mathbf{p})d_2(\mathbf{p}) + d_2(\mathbf{p})d_2^*(\mathbf{p}) \\ +b_4(\mathbf{p})b_4^*(\mathbf{p}) + b_4^*(\mathbf{p})b_4(\mathbf{p}) + d_4^*(\mathbf{p})d_4(\mathbf{p}) + d_4(\mathbf{p})d_4^*(\mathbf{p}) \\ +b_3(\mathbf{p})b_3^*(\mathbf{p}) + b_3^*(\mathbf{p})b_3(\mathbf{p}) + d_3^*(\mathbf{p})d_3(\mathbf{p}) + d_3(\mathbf{p})d_3^*(\mathbf{p}) \end{bmatrix}$$

$$= 8\int \frac{d^4p}{(2\pi)^4}\, e_0(\mathbf{p})\delta(\mathbf{0}) = 8\int \frac{d^4x}{(2\pi)^4}\int \frac{d^4p}{(2\pi)^4}\, e_0(\mathbf{p})$$

here

$$e_0(\mathbf{p}) = \overline{p_0}p_0 + \overline{p_1}p_1 + \overline{p_2}p_2 + \overline{p_3}p_3$$

Each summand in brackets represents the operator of the number of particles with a certain reference spinor. The operator's action consists of consecutive application of the annihilation operator and the operator of the birth of a particle. On initial examination, it would appear that the energy associated with zero-point fluctuations in the vacuum has been overlooked. However, an examination of the final expression reveals that the field always possesses a constant energy, regardless of the particles that contribute to it. This constant energy of the field can be interpreted as the energy of zero-point fluctuations of the vacuum.

The following relations were taken into account in the derivation

$$b_1(\mathbf{p})b_1^*(\mathbf{p}) + b_1^*(\mathbf{p})b_1(\mathbf{p}) = \delta(\mathbf{0}) \qquad b_1^*(\mathbf{p}')b_1(\mathbf{p}) + b_1(\mathbf{p})b_1^*(\mathbf{p}) = \delta(\mathbf{0})$$
$$d_1(\mathbf{p})d_1^*(\mathbf{p}) + d_1^*(\mathbf{p})d_1(\mathbf{p}) = \delta(\mathbf{0}) \qquad d_1^*(\mathbf{p})d_1(\mathbf{p}) + d_1(\mathbf{p})d_1^*(\mathbf{p}) = \delta(\mathbf{0})$$
$$d_2(\mathbf{p})d_2^*(\mathbf{p}) + d_2^*(\mathbf{p}')d_2(\mathbf{p}) = \delta(\mathbf{0}) \qquad d_2^*(\mathbf{p})d_2(\mathbf{p}) + d_2(\mathbf{p})d_2^*(\mathbf{p}) = \delta(\mathbf{0})$$
$$b_2(\mathbf{p})b_2^*(\mathbf{p}) + b_2^*(\mathbf{p})b_2(\mathbf{p}) = \delta(\mathbf{0}) \qquad b_2^*(\mathbf{p})b_2(\mathbf{p}) + b_2(\mathbf{p})b_2^*(\mathbf{p}) = \delta(\mathbf{0})$$
$$d_3(\mathbf{p})d_3^*(\mathbf{p}) + d_3^*(\mathbf{p})d_3(\mathbf{p}) = \delta(\mathbf{0}) \qquad d_3^*(\mathbf{p})d_3(\mathbf{p}) + d_3(\mathbf{p})d_3^*(\mathbf{p}) = \delta(\mathbf{0})$$

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$$b_3(\mathbf{p})b_3^*(\mathbf{p}) + b_3^*(\mathbf{p})b_3(\mathbf{p}) = \delta(\mathbf{0}) \qquad b_3^*(\mathbf{p})b_3(\mathbf{p}) + b_3(\mathbf{p})b_3^*(\mathbf{p}) = \delta(\mathbf{0})$$
$$b_4(\mathbf{p})b_4^*(\mathbf{p}) + b_4^*(\mathbf{p})b_4(\mathbf{p}) = \delta(\mathbf{0}) \qquad b_4^*(\mathbf{p})b_4(\mathbf{p}) + b_4(\mathbf{p})b_4^*(\mathbf{p}) = \delta(\mathbf{0})$$
$$d_4(\mathbf{p})d_4^*(\mathbf{p}) + d_4^*(\mathbf{p})d_4(\mathbf{p}) = \delta(\mathbf{0}) \qquad d_4^*(\mathbf{p})d_4(\mathbf{p}) + d_4(\mathbf{p})d_4^*(\mathbf{p}) = \delta(\mathbf{0})$$
$$\delta(\mathbf{0}) = \int \frac{d^4x}{(2\pi)^4}$$

Other components of the total field momentum are calculated by the formula

$$P_\mu = \int d^4x\, \boldsymbol{\varphi}^+(\mathbf{x})S_\mu\boldsymbol{\varphi}(\mathbf{x})$$

Total momentum

$$\mathbf{P}^\mathbf{T} \equiv (P_0, P_1, P_2, P_3)$$

is a vector in Minkowski space. The density of the current as a function of coordinates is

$$J_\mu = \pm\frac{e}{m_e}\boldsymbol{\varphi}^+(\mathbf{x})S_\mu\boldsymbol{\varphi}(\mathbf{x}) = \pm\frac{e}{m_e}F_\mu(\mathbf{x})$$

where

$$F_\mu(\mathbf{x}) = \boldsymbol{\varphi}^+(\mathbf{x})S_\mu\boldsymbol{\varphi}(\mathbf{x})$$

is a four-dimensional probability density current, which is transformed as a four-dimensional vector by Lorentz transformations. Multiplication by $\pm\frac{e}{m_e}$ transforms it into a four-dimensional current density.

Let us perform a series of transformations analogous to those presented by Dirac in [10, Lecture 11]

$$P_0 = \int \frac{d^4x}{(2\pi)^4}\boldsymbol{\varphi}^+(\mathbf{x})S_0\boldsymbol{\varphi}(\mathbf{x}) = \frac{1}{2m}\int \frac{d^4x}{(2\pi)^4}\boldsymbol{\varphi}^+(\mathbf{x})S_0[S^R\boldsymbol{\varphi}(\mathbf{x})]$$
$$= \frac{1}{2m}\int \frac{d^4x}{(2\pi)^4}\boldsymbol{\varphi}^+(\mathbf{x})S_0\left[\int \frac{d^4p}{(2\pi)^4}\, S^R(\mathbf{p})\boldsymbol{\varphi}(\mathbf{p})e^{i(p_0x_1 - p_1x_0 + p_2x_3 - p_3x_2 + \overline{(\mathbf{p},\mathbf{x})})}\right]$$
$$= \frac{1}{2m}\int \frac{d^4p}{(2\pi)^4}\left[\int \frac{d^4x}{(2\pi)^4}\boldsymbol{\varphi}^+(\mathbf{x})S_0e^{i(p_0x_1 - p_1x_0 + p_2x_3 - p_3x_2 + \overline{(\mathbf{p},\mathbf{x})})}\right]S^R(\mathbf{p})\boldsymbol{\varphi}(\mathbf{p})$$
$$= \frac{1}{2m}\int \frac{d^4p}{(2\pi)^4}\, \boldsymbol{\varphi}^+(\mathbf{p})S_0[S^R(\mathbf{p})\boldsymbol{\varphi}(\mathbf{p})] = \int \frac{d^4p}{(2\pi)^4}\, \boldsymbol{\varphi}^+(\mathbf{p})S_0\boldsymbol{\varphi}(\mathbf{p}) = \int \frac{d^4p}{(2\pi)^4}\, \boldsymbol{\varphi}^+(\mathbf{p})\boldsymbol{\varphi}(\mathbf{p})$$
$$= \int \frac{d^4p}{(2\pi)^4}\, [\varphi_0^+(\mathbf{p})\varphi_0(\mathbf{p}) + \varphi_1^+(\mathbf{p})\varphi_1(\mathbf{p}) + \varphi_2^+(\mathbf{p})\varphi_2(\mathbf{p}) + \varphi_3^+(\mathbf{p})\varphi_3(\mathbf{p})]$$

For an arbitrary component of the total momentum we have

$$P_\mu = \int \frac{d^4p}{(2\pi)^4}\, \boldsymbol{\varphi}^+(\mathbf{p})S_\mu\boldsymbol{\varphi}(\mathbf{p})$$

Following Dirac's argument in [10], the value of

$$P_0 = H = \int \frac{d^4p}{(2\pi)^4}\, [\varphi_0^+(\mathbf{p})\varphi_0(\mathbf{p}) + \varphi_1^+(\mathbf{p})\varphi_1(\mathbf{p}) + \varphi_2^+(\mathbf{p})\varphi_2(\mathbf{p}) + \varphi_3^+(\mathbf{p})\varphi_3(\mathbf{p})]$$

can be treated as either a Hamiltonian or a total energy operator, with $\varphi_\mu^+(\mathbf{p})$ representing the birth operator and $\varphi_\mu(\mathbf{p})$ representing the annihilation operator.

In [10] the quantization procedure includes the use of one definite Lorentzian reference frame, i.e. it is not invariant. In our case all deductions are valid in any reference frame in the spinor space, and it means invariance to change of reference frames in the Minkowski space also.

The following relations are used in the transformations

$$S^R\boldsymbol{\varphi}(\mathbf{x}) = 2m\boldsymbol{\varphi}(\mathbf{x})$$

$$\boldsymbol{\varphi}(\mathbf{x}) = \frac{1}{2m}S^R\boldsymbol{\varphi}(\mathbf{x})$$

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$$\boldsymbol{\varphi}(\mathbf{x}) = \int \frac{d^4p}{(2\pi)^4}\, \boldsymbol{\varphi}(\mathbf{p})e^{i(p_0x_1 - p_1x_0 + p_2x_3 - p_3x_2 + \overline{(\mathbf{p},\mathbf{x})})}$$

$$\boldsymbol{\varphi}(\mathbf{p}) = \int \frac{d^4x'}{(2\pi)^4}\, \boldsymbol{\varphi}(\mathbf{x}')e^{-i(p_0x_1' - p_1x_0' + p_2x_3' - p_3x_2' + \overline{(\mathbf{p},\mathbf{x}')})}$$

$$\delta(\mathbf{p}) = \int \frac{d^4x'}{(2\pi)^4}\, e^{-i(p_0x_1' - p_1x_0' + p_2x_3' - p_3x_2' + \overline{(\mathbf{p},\mathbf{x}')})}$$

$$\delta(\mathbf{x}) = \int \frac{d^4p}{(2\pi)^4}\, e^{i(p_0x_1 - p_1x_0 + p_2x_3 - p_3x_2 + \overline{(\mathbf{p},\mathbf{x})})}$$

$$\boldsymbol{\varphi}^{+}(\mathbf{p}) = \int \frac{d^4x}{(2\pi)^4}\, \boldsymbol{\varphi}^{+}(\mathbf{x})e^{i(p_0x_1 - p_1x_0 + p_2x_3 - p_3x_2 + \overline{(\mathbf{p},\mathbf{x})})}$$

$$S^R\boldsymbol{\varphi}(\mathbf{x}) = \int \frac{d^4p}{(2\pi)^4}\, S^R(\mathbf{p})\, \boldsymbol{\varphi}(\mathbf{p})e^{i(p_0x_1 - p_1x_0 + p_2x_3 - p_3x_2 + \overline{(\mathbf{p},\mathbf{x})})}$$

$$S^R(\mathbf{p}) = 2mI$$

$$S^R = \begin{pmatrix} \partial_2 \\ -\partial_3 \\ -\partial_0 \\ \partial_1 \end{pmatrix}(\partial_1, \partial_0, \partial_3, \partial_2) - \begin{pmatrix} \partial_0 \\ -\partial_1 \\ -\partial_2 \\ \partial_3 \end{pmatrix}(\partial_3, \partial_2, \partial_1, \partial_0)$$
$$+ \begin{pmatrix} -\partial_0 \\ \partial_1 \\ -\partial_2 \\ \partial_3 \end{pmatrix}(\partial_3, \partial_2, -\partial_1, -\partial_0) - \begin{pmatrix} -\partial_2 \\ \partial_3 \\ -\partial_0 \\ \partial_1 \end{pmatrix}(\partial_1, \partial_0, -\partial_3, -\partial_2)$$

$$S^R(\mathbf{p}) = \begin{pmatrix} -p_3 \\ -p_2 \\ p_1 \\ p_0 \end{pmatrix}(p_0, -p_1, p_2, -p_3) - \begin{pmatrix} -p_1 \\ -p_0 \\ p_3 \\ p_2 \end{pmatrix}(p_2, -p_3, p_0, -p_1)$$
$$+ \begin{pmatrix} p_1 \\ p_0 \\ p_3 \\ p_2 \end{pmatrix}(p_2, -p_3, -p_0, p_1) - \begin{pmatrix} p_3 \\ p_2 \\ p_1 \\ p_0 \end{pmatrix}(p_0, -p_1, -p_2, p_3)$$

The chain of reasoning can be organized in a slightly different way as well

$$P_0 = \int \frac{d^4x}{(2\pi)^4} \boldsymbol{\varphi}^{+}(\mathbf{x})S_0\boldsymbol{\varphi}(\mathbf{x}) = \frac{1}{2\overline{m}}\frac{1}{2m}\int \frac{d^4x}{(2\pi)^4}[S^R\boldsymbol{\varphi}(\mathbf{x})]^{+}[S^R\boldsymbol{\varphi}(\mathbf{x})]$$
$$= \frac{1}{2\overline{m}}\frac{1}{2m}\int \frac{d^4x}{(2\pi)^4}\left[\int \frac{d^4p'}{(2\pi)^4}\, S^R(\mathbf{p}')\boldsymbol{\varphi}(\mathbf{p}')e^{i(p_0'x_1 - p_1'x_0 + p_2'x_3 - p_3'x_2 + \overline{(\mathbf{p}',\mathbf{x})})}\right]^{+}$$
$$\left[\int \frac{d^4p}{(2\pi)^4}\, S^R(\mathbf{p})\boldsymbol{\varphi}(\mathbf{p})e^{i(p_0x_1 - p_1x_0 + p_2x_3 - p_3x_2 + \overline{(\mathbf{p},\mathbf{x})})}\right]$$
$$= \frac{1}{2\overline{m}}\frac{1}{2m}\int \frac{d^4x}{(2\pi)^4}\left[\int \frac{d^4p'}{(2\pi)^4}\, S^R(\mathbf{p}')\boldsymbol{\varphi}(\mathbf{p}')\right]^{+} e^{-i(p_0'x_1 - p_1'x_0 + p_2'x_3 - p_3'x_2 + \overline{(\mathbf{p}',\mathbf{x})})}$$
$$\left[\int \frac{d^4p}{(2\pi)^4}\, S^R(\mathbf{p})\boldsymbol{\varphi}(\mathbf{p})\right]e^{i(p_0x_1 - p_1x_0 + p_2x_3 - p_3x_2 + \overline{(\mathbf{p},\mathbf{x})})}$$
$$= \frac{1}{2\overline{m}}\frac{1}{2m}\left[\int \frac{d^4p'}{(2\pi)^4}\, S^R(\mathbf{p}')\boldsymbol{\varphi}(\mathbf{p}')\right]^{+}\left[\int \frac{d^4p}{(2\pi)^4}\, S^R(\mathbf{p})\boldsymbol{\varphi}(\mathbf{p})\right]\delta(\mathbf{p}' - \mathbf{p})$$
$$= \frac{1}{2\overline{m}}\frac{1}{2m}\int \frac{d^4p'}{(2\pi)^4}\int \frac{d^4p}{(2\pi)^4}\, [S^R(\mathbf{p}')\boldsymbol{\varphi}(\mathbf{p}')]^{+}[S^R(\mathbf{p})\boldsymbol{\varphi}(\mathbf{p})]\delta(\mathbf{p}' - \mathbf{p})$$
$$= \frac{1}{2\overline{m}}\frac{1}{2m}\int \frac{d^4p}{(2\pi)^4}\, [S^R(\mathbf{p})\boldsymbol{\varphi}(\mathbf{p})]^{+}[S^R(\mathbf{p})\boldsymbol{\varphi}(\mathbf{p})]$$

$$= \frac{1}{2\overline{m}}\frac{1}{2m}\int \frac{d^4p}{(2\pi)^4}\, \boldsymbol{\varphi}(\mathbf{p})^+[S^R(\mathbf{p})]^+[S^R(\mathbf{p})\boldsymbol{\varphi}(\mathbf{p})]$$

$$= \frac{1}{2\overline{m}}\frac{1}{2m}\int \frac{d^4p}{(2\pi)^4}\, \boldsymbol{\varphi}(\mathbf{p})^+\left[\overline{S^R}(\mathbf{p})\right]^T[S^R(\mathbf{p})]\boldsymbol{\varphi}(\mathbf{p})$$

$$= \frac{1}{2\overline{m}}\frac{1}{2m}\int \frac{d^4p}{(2\pi)^4}\, \boldsymbol{\varphi}(\mathbf{p})^+\left[2\overline{(p_1p_2 - p_3p_0)}I\right]^T[2(p_1p_2 - p_3p_0)I]\boldsymbol{\varphi}(\mathbf{p})$$

$$= \frac{1}{\overline{m}}\frac{1}{m}\int \frac{d^4p}{(2\pi)^4}\, \boldsymbol{\varphi}(\mathbf{p})^+\overline{(p_1p_2 - p_3p_0)}(p_1p_2 - p_3p_0)\boldsymbol{\varphi}(\mathbf{p})$$

$$= \frac{1}{\overline{m}}\frac{1}{m}\int \frac{d^4p}{(2\pi)^4}(P_0^2 - P_1^2 - P_2^2 - P_3^2)\, \boldsymbol{\varphi}(\mathbf{p})^+\boldsymbol{\varphi}(\mathbf{p})$$

$$= \int \frac{d^4p}{(2\pi)^4}\boldsymbol{\varphi}(\mathbf{p})^+\boldsymbol{\varphi}(\mathbf{p})$$

Here it is taken into account that

$$S^R(\mathbf{p}) = 2(p_1p_2 - p_3p_0)I$$

$$\overline{(p_1p_2 - p_0p_3)}(p_1p_2 - p_0p_3) = P_0^2 - P_1^2 - P_2^2 - P_3^2 = \overline{m}m =$$

$$= (S_0P_0 - S_1P_1 - S_2P_2 - S_3P_3)(S_0P_0 + S_1P_1 + S_2P_2 + S_3P_3)$$

Let us draw an analogy between our approach and the relations given in [[11]], Volume 1, Chapter 3, Section 3.3.1]. There it is noted that the birth and annihilation operators of the fermionic field must satisfy such commutation relations that the equality expressing translational invariance is satisfied

$$\boldsymbol{\varphi}(\mathbf{X}+\mathbf{A}) = e^{i\mathbf{P}^T\mathbf{A}}\,\boldsymbol{\varphi}(\mathbf{X})e^{-i\mathbf{P}^T\mathbf{A}}$$

which in differential form is written as

$$\partial_\mu\boldsymbol{\varphi}(\mathbf{X}) = i[P_\mu, \boldsymbol{\varphi}(\mathbf{X})]$$

The coordinates here are the components of the Minkowski vector space. On the basis of these relations the anticommutation relations between the birth and annihilation operators are derived.

At substantiation of the Schrödinger equation we have to assume that the zero component of momentum, i.e. energy, commutes with the rest of the momentum components, which allows us to represent the exponent of the sum as a product of exponents.

$$e^{i(P_0X_0 - P_1X_1 - P_2X_2 - P_3X_3)} = e^{i(P_0X_0)}e^{-i(P_1X_1 + P_2X_2 + P_3X_3)}$$

and consider time and energy separately from spatial coordinates and momenta. In this case it is possible to independently perform translation in time and space.

$$\boldsymbol{\varphi}(\mathbf{X}+\mathbf{A}) = e^{i(P_0A_0)}e^{-i(P_1A_1+P_2A_2+P_3A_3)}\,\boldsymbol{\varphi}(\mathbf{X})e^{i(P_1A_1+P_2A_2+P_3A_3)}$$

$$= e^{-i(P_1A_1+P_2A_2+P_3A_3)}\left[e^{i(P_0A_0)}\boldsymbol{\varphi}(\mathbf{X})e^{-i(P_0A_0)}\right]e^{-i(P_0A_0)}e^{i(P_1A_1+P_2A_2+P_3A_3)}$$

In the spinor coordinate space, we can express the translational invariance of the field operator by the relations

$$\boldsymbol{\varphi}(\mathbf{x}+\mathbf{a}) = e^{i(p_0a_1 - p_1a_0 + p_2a_3 - p_3a_2 + \overline{(\mathbf{p},\mathbf{a})})}\,\boldsymbol{\varphi}(\mathbf{x})e^{-i(p_0a_1 - p_1a_0 + p_2a_3 - p_3a_2 + \overline{(\mathbf{p},\mathbf{a})})}$$

$$\partial_0\boldsymbol{\varphi}(\mathbf{x}) = i[-p_1, \boldsymbol{\varphi}(\mathbf{x})] \qquad \partial_1\boldsymbol{\varphi}(\mathbf{x}) = i[p_0, \boldsymbol{\varphi}(\mathbf{x})]$$

$$\partial_2\boldsymbol{\varphi}(\mathbf{x}) = i[-p_3, \boldsymbol{\varphi}(\mathbf{x})] \qquad \partial_3\boldsymbol{\varphi}(\mathbf{x}) = i[p_2, \boldsymbol{\varphi}(\mathbf{x})]$$

$$[p_1, x_0] = i \qquad [p_0, x_1] = -i$$

$$[p_3, x_2] = i \qquad [p_2, x_3] = -i$$

It is interesting to find out in what relation these translational operators are - one operator acts in vector space, the other in spinor space. In both cases the following interpretation can be given. Suppose we know the result of an operator acting on an arbitrary state at a point in space 1, and we want to know the result of its action on a state at point 2. Then we translate the state from point 2 to point 1, act on it by the operator, and transfer the obtained result back to point 2.

Both operators act on the same state, but in one case the state is labeled by spinor coordinates and in the other by vector coordinates. The translation mechanism of the operators is essentially the same, but it is not possible to replace the action of one translation operator by some combination of actions of the other. Because of this, the question arises as to which of these operators better describes

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nature. Our point of view is that the translation operator in spinor space is primary, and the operator in vector space just successfully copies it, without being exact, but being some approximation. It attracted the attention of physicists first because vector space is more accessible for investigation. When integrating over a four-dimensional vector space in some cases there is a divergence, then use renormalization. When integrating over four-dimensional spinor space, the differential element has two orders of magnitude of the vector momentum component smaller, while the denominator in the integrand remains of the same order as when integrating over vector space. This difference possibly affects the convergence.

Let us calculate the total mass of the fermion field

$$M = \int d^4x\, \boldsymbol{\varphi}^T(\mathbf{x})\, \boldsymbol{\varphi}(\mathbf{x}) =$$

$$\int d^4x \iint \frac{d^4p}{(2\pi)^4}\frac{d^4p'}{(2\pi)^4}$$

$$\begin{bmatrix} d_1(\mathbf{p}')\mathbf{u1}^T(\mathbf{p}') + id_2(\mathbf{p}')\mathbf{u3}^T(\mathbf{p}') + ib_2(\mathbf{p}')\mathbf{u2}^+(\mathbf{p}') + b_1(\mathbf{p}')\mathbf{u4}^+(\mathbf{p}') \\ +d_4(\mathbf{p}')\mathbf{v1}^T(\mathbf{p}') + id_3(\mathbf{p}')\mathbf{v3}^T(\mathbf{p}') + ib_3(\mathbf{p}')\mathbf{v2}^+(\mathbf{p}') + b_4(\mathbf{p}')\mathbf{v4}^+(\mathbf{p}') \end{bmatrix}$$

$$\begin{bmatrix} b_1^*(\mathbf{p})\overline{\mathbf{u1}}(\mathbf{p}) + ib_2^*(\mathbf{p})\overline{\mathbf{u3}}(\mathbf{p}) + id_2^*(\mathbf{p})\mathbf{u2}(\mathbf{p}) + d_1^*(\mathbf{p})\mathbf{u4}(\mathbf{p}) \\ +b_4^*(\mathbf{p})\overline{\mathbf{v1}}(\mathbf{p}) + ib_3^*(\mathbf{p})\overline{\mathbf{v3}}(\mathbf{p}) + id_3^*(\mathbf{p})\mathbf{v2}(\mathbf{p}) + d_4^*(\mathbf{p})\mathbf{v4}(\mathbf{p}) \end{bmatrix}$$

$$e^{i(p'_0x_1 - p'_1x_0 + p'_2x_3 - p'_3x_2 + \overline{(\mathbf{p}',\mathbf{x})})} e^{-i(p_0x_1 - p_1x_0 + p_2x_3 - p_3x_2 + \overline{(\mathbf{p},\mathbf{x})})}$$

$$+ \int d^4x \iint \frac{d^4p}{(2\pi)^4}\frac{d^4p'}{(2\pi)^4}$$

$$\begin{bmatrix} b_1^*(\mathbf{p}')\mathbf{u1}^+(\mathbf{p}') + ib_2^*(\mathbf{p}')\mathbf{u3}^+(\mathbf{p}') + id_2^*(\mathbf{p}')\mathbf{u2}^T(\mathbf{p}') + d_1^*(\mathbf{p}')\mathbf{u4}^T(\mathbf{p}') \\ +b_4^*(\mathbf{p}')\mathbf{v1}^+(\mathbf{p}') + ib_3^*(\mathbf{p}')\mathbf{v3}^+(\mathbf{p}') + id_3^*(\mathbf{p}')\mathbf{v2}^T(\mathbf{p}') + d_4^*(\mathbf{p}')\mathbf{v4}^T(\mathbf{p}') \end{bmatrix}$$

$$\begin{bmatrix} d_1(\mathbf{p})\mathbf{u1}(\mathbf{p}) + id_2(\mathbf{p})\mathbf{u3}(\mathbf{p}) + ib_2(\mathbf{p})\overline{\mathbf{u2}}(\mathbf{p}) + b_1(\mathbf{p})\overline{\mathbf{u4}}(\mathbf{p}) \\ +d_4(\mathbf{p})\mathbf{v1}(\mathbf{p}) + id_3(\mathbf{p})\mathbf{v3}(\mathbf{p}) + ib_3(\mathbf{p})\overline{\mathbf{v2}}(\mathbf{p}) + b_4(\mathbf{p})\overline{\mathbf{v4}}(\mathbf{p}) \end{bmatrix}$$

$$e^{-i(p'_0x_1 - p'_1x_0 + p'_2x_3 - p'_3x_2 + \overline{(\mathbf{p}',\mathbf{x})})} e^{i(p_0x_1 - p_1x_0 + p_2x_3 - p_3x_2 + \overline{(\mathbf{p},\mathbf{x})})}$$

$$= \iint \frac{d^4p}{(2\pi)^4}\frac{d^4p'}{(2\pi)^4}$$

$$\begin{bmatrix} d_1(\mathbf{p}')\mathbf{u1}^T(\mathbf{p}') + id_2(\mathbf{p}')\mathbf{u3}^T(\mathbf{p}') + ib_2(\mathbf{p}')\mathbf{u2}^+(\mathbf{p}') + b_1(\mathbf{p}')\mathbf{u4}^+(\mathbf{p}') \\ +d_4(\mathbf{p}')\mathbf{v1}^T(\mathbf{p}') + id_3(\mathbf{p}')\mathbf{v3}^T(\mathbf{p}') + ib_3(\mathbf{p}')\mathbf{v2}^+(\mathbf{p}') + b_4(\mathbf{p}')\mathbf{v4}^+(\mathbf{p}') \end{bmatrix}$$

$$\begin{bmatrix} b_1^*(\mathbf{p})\overline{\mathbf{u1}}(\mathbf{p}) + ib_2^*(\mathbf{p})\overline{\mathbf{u3}}(\mathbf{p}) + id_2^*(\mathbf{p})\mathbf{u2}(\mathbf{p}) + d_1^*(\mathbf{p})\mathbf{u4}(\mathbf{p}) \\ +b_4^*(\mathbf{p})\overline{\mathbf{v1}}(\mathbf{p}) + ib_3^*(\mathbf{p})\overline{\mathbf{v3}}(\mathbf{p}) + id_3^*(\mathbf{p})\mathbf{v2}(\mathbf{p}) + d_4^*(\mathbf{p})\mathbf{v4}(\mathbf{p}) \end{bmatrix}$$

$$\delta(\mathbf{p} - \mathbf{p}')$$

$$+ \iint \frac{d^4p}{(2\pi)^4}\frac{d^4p'}{(2\pi)^4}$$

$$\begin{bmatrix} b_1^*(\mathbf{p}')\mathbf{u1}^+(\mathbf{p}') + ib_2^*(\mathbf{p}')\mathbf{u3}^+(\mathbf{p}') + id_2^*(\mathbf{p}')\mathbf{u2}^T(\mathbf{p}') + d_1^*(\mathbf{p}')\mathbf{u4}^T(\mathbf{p}') \\ +b_4^*(\mathbf{p}')\mathbf{v1}^+(\mathbf{p}') + ib_3^*(\mathbf{p}')\mathbf{v3}^+(\mathbf{p}') + id_3^*(\mathbf{p}')\mathbf{v2}^T(\mathbf{p}') + d_4^*(\mathbf{p}')\mathbf{v4}^T(\mathbf{p}') \end{bmatrix}$$

$$\begin{bmatrix} d_1(\mathbf{p})\mathbf{u1}(\mathbf{p}) + id_2(\mathbf{p})\mathbf{u3}(\mathbf{p}) + ib_2(\mathbf{p})\overline{\mathbf{u2}}(\mathbf{p}) + b_1(\mathbf{p})\overline{\mathbf{u4}}(\mathbf{p}) \\ +d_4(\mathbf{p})\mathbf{v1}(\mathbf{p}) + id_3(\mathbf{p})\mathbf{v3}(\mathbf{p}) + ib_3(\mathbf{p})\overline{\mathbf{v2}}(\mathbf{p}) + b_4(\mathbf{p})\overline{\mathbf{v4}}(\mathbf{p}) \end{bmatrix}$$

$$\delta(\mathbf{p}' - \mathbf{p})$$

$$= \int \frac{d^4p}{(2\pi)^4} \begin{bmatrix} d_1(\mathbf{p})d_1^*(\mathbf{p})\mathbf{u1}^T(\mathbf{p})\mathbf{u4}(\mathbf{p}) + b_1(\mathbf{p})b_1^*(\mathbf{p})\mathbf{u4}^+(\mathbf{p})\overline{\mathbf{u1}}(\mathbf{p}) \\ -d_2(\mathbf{p})d_2^*(\mathbf{p})\mathbf{u3}^T(\mathbf{p})\mathbf{u2}(\mathbf{p}) - b_2(\mathbf{p})b_2^*(\mathbf{p})\mathbf{u2}^+(\mathbf{p})\overline{\mathbf{u3}}(\mathbf{p}) \\ +d_4(\mathbf{p})d_4^*(\mathbf{p})\mathbf{v1}^T(\mathbf{p})\mathbf{v4}(\mathbf{p}) + b_4(\mathbf{p})b_4^*(\mathbf{p})\mathbf{v4}^+(\mathbf{p})\overline{\mathbf{v1}}(\mathbf{p}) \\ -d_3(\mathbf{p})d_3^*(\mathbf{p})\mathbf{v3}^T(\mathbf{p})\mathbf{v2}(\mathbf{p}) - b_3(\mathbf{p})b_3^*(\mathbf{p})\mathbf{v2}^+(\mathbf{p})\overline{\mathbf{v3}}(\mathbf{p}) \end{bmatrix}$$

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$$+\int \frac{d^4p}{(2\pi)^4}\begin{bmatrix} b_1^*(\mathbf{p})\, b_1(\mathbf{p})\mathbf{u1}^+(\mathbf{p})\overline{\mathbf{u4}}(\mathbf{p}) + d_1^*(\mathbf{p})d_1(\mathbf{p})\mathbf{u4}^T(\mathbf{p})\mathbf{u1}(\mathbf{p}) \\ -b_2^*(\mathbf{p})b_2(\mathbf{p})\mathbf{u3}^+(\mathbf{p})\overline{\mathbf{u2}}(\mathbf{p}) - d_2^*(\mathbf{p})d_2(\mathbf{p})\mathbf{u2}(\mathbf{p})\mathbf{u3}(\mathbf{p}) \\ +b_4^*(\mathbf{p})b_4(\mathbf{p})\mathbf{v1}^+(\mathbf{p})\overline{\mathbf{v4}}(\mathbf{p}) + d_4^*(\mathbf{p})d_4(\mathbf{p})\mathbf{v4}^T(\mathbf{p})\mathbf{v1}(\mathbf{p}) \\ -b_3^*(\mathbf{p})b_3(\mathbf{p})\mathbf{v3}^+(\mathbf{p})\overline{\mathbf{v2}}(\mathbf{p}) - d_3^*(\mathbf{p})d_3(\mathbf{p})\mathbf{v2}^T(\mathbf{p})\mathbf{v3}(\mathbf{p}) \end{bmatrix}$$

$$= \int \frac{d^4p}{(2\pi)^4}\,(m+\overline{m})\begin{bmatrix} d_1(\mathbf{p})d_1^*(\mathbf{p}) + b_1(\mathbf{p})b_1^*(\mathbf{p}) + d_4(\mathbf{p})d_4^*(\mathbf{p}) + b_4(\mathbf{p})b_4^*(\mathbf{p}) \\ +b_2(\mathbf{p})b_2^*(\mathbf{p}) + d_2(\mathbf{p})d_2^*(\mathbf{p}) + d_3(\mathbf{p})d_3^*(\mathbf{p}) + b_3(\mathbf{p})b_3^*(\mathbf{p}) \\ +b_1^*(\mathbf{p})\, b_1(\mathbf{p}) + d_1^*(\mathbf{p})d_1(\mathbf{p}) + b_4^*(\mathbf{p})b_4(\mathbf{p}) + d_4^*(\mathbf{p})d_4(\mathbf{p}) \\ +b_2^*(\mathbf{p})b_2(\mathbf{p}) + d_2^*(\mathbf{p})d_2(\mathbf{p}) + b_3^*(\mathbf{p})b_3(\mathbf{p}) + d_3^*(\mathbf{p})d_3(\mathbf{p}) \end{bmatrix}$$

$$= \int \frac{d^4p}{(2\pi)^4} 8(m+\overline{m})\,\delta(\mathbf{0}) = \int \frac{d^4x}{(2\pi)^4}\int \frac{d^4p}{(2\pi)^4}\, 8(m+\overline{m})$$

The ratios used in the derivation are

$$\mathbf{u1}^T(\mathbf{p})\mathbf{u4}(\mathbf{p}) = -p_3p_0 + p_2p_1 + p_1p_2 - p_0p_3 = 2m$$
$$\mathbf{u4}^T(\mathbf{p})\mathbf{u1}(\mathbf{p}) = -p_0p_3 + p_1p_2 + p_2p_1 - p_3p_0 = 2m$$
$$\mathbf{u3}^T(\mathbf{p})\mathbf{u2}(\mathbf{p}) = -p_1p_2 + p_0p_3 + p_3p_0 - p_2p_1 = -2m$$
$$\mathbf{u2}^T(\mathbf{p})\mathbf{u3}(\mathbf{p}) = -p_2p_1 + p_3p_0 + p_0p_3 - p_1p_2 = -2m$$
$$\mathbf{u1}^T(\mathbf{p})\mathbf{u4}(\mathbf{p}) = -p_3p_0 + p_2p_1 + p_1p_2 - p_0p_3 = 2m$$
$$\mathbf{v1}^T(\mathbf{p})\mathbf{v4}(\mathbf{p}) = p_1p_2 - p_3p_0 - p_0p_3 + p_2p_1 = 2m$$
$$\mathbf{u1}^+(\mathbf{p})\overline{\mathbf{u4}}(\mathbf{p}) = \overline{-p_3p_0 + p_2p_1 + p_1p_2 - p_0p_3} = 2\overline{m}$$

$$b_1(\mathbf{p})b_1^*(\mathbf{p}) + b_1^*(\mathbf{p})b_1(\mathbf{p}) = b_1^*(\mathbf{p})b_1(\mathbf{p}) + b_1(\mathbf{p})b_1^*(\mathbf{p}) = \delta(\mathbf{0})$$
$$d_1(\mathbf{p})d_1^*(\mathbf{p}) + d_1^*(\mathbf{p})d_1(\mathbf{p}) = d_1^*(\mathbf{p})d_1(\mathbf{p}) + d_1(\mathbf{p})d_1^*(\mathbf{p}) = \delta(\mathbf{0})$$
$$d_2(\mathbf{p})d_2^*(\mathbf{p}) + d_2^*(\mathbf{p})d_2(\mathbf{p}) = b_2^*(\mathbf{p})b_2(\mathbf{p}) + b_2(\mathbf{p})b_2^*(\mathbf{p}) = \delta(\mathbf{0})$$
$$b_2(\mathbf{p})b_2^*(\mathbf{p}) + b_2^*(\mathbf{p})b_2(\mathbf{p}) = d_2^*(\mathbf{p})d_2(\mathbf{p}) + d_2(\mathbf{p})d_2^*(\mathbf{p}) = \delta(\mathbf{0})$$
$$d_3(\mathbf{p})d_3^*(\mathbf{p}) + d_3^*(\mathbf{p})d_3(\mathbf{p}) = b_3^*(\mathbf{p})b_3(\mathbf{p}) + b_3(\mathbf{p})b_3^*(\mathbf{p}) = \delta(\mathbf{0})$$
$$b_3(\mathbf{p})b_3^*(\mathbf{p}) + b_3^*(\mathbf{p})b_3(\mathbf{p}) = d_3^*(\mathbf{p})d_3(\mathbf{p}) + d_3(\mathbf{p})d_3^*(\mathbf{p}) = \delta(\mathbf{0})$$
$$b_4(\mathbf{p})b_4^*(\mathbf{p}) + b_4^*(\mathbf{p})b_4(\mathbf{p}) = b_4^*(\mathbf{p})b_4(\mathbf{p}) + b_4(\mathbf{p})b_4^*(\mathbf{p}) = \delta(\mathbf{0})$$
$$d_4(\mathbf{p})d_4^*(\mathbf{p}) + d_4^*(\mathbf{p})d_4(\mathbf{p}) = d_4^*(\mathbf{p})d_4(\mathbf{p}) + d_4(\mathbf{p})d_4^*(\mathbf{p}) = \delta(\mathbf{0})$$

$$\delta(\mathbf{0}) = \int \frac{d^4x}{(2\pi)^4}$$

Let us give an interpretation of the operator coefficients for this approach

$$\mathbf{u1} = \begin{pmatrix} -p_3 \\ -p_2 \\ p_1 \\ p_0 \end{pmatrix} \quad \mathbf{u2} = \begin{pmatrix} p_2 \\ -p_3 \\ p_0 \\ -p_1 \end{pmatrix} \quad \mathbf{u3} = \begin{pmatrix} -p_1 \\ -p_0 \\ p_3 \\ p_2 \end{pmatrix} \quad \mathbf{u4} = \begin{pmatrix} p_0 \\ -p_1 \\ p_2 \\ -p_3 \end{pmatrix}$$

$$m_{\mathbf{u1}} = -p_2p_1 + p_3p_0 = -m$$
$$m_{\mathbf{u2}} = -p_3p_0 + p_2p_1 = m$$
$$m_{\mathbf{u3}} = -p_0p_3 + p_1p_2 = m$$
$$m_{\mathbf{u4}} = -p_1p_2 + p_0p_3 = -m$$

$$\mathbf{v1} = \begin{pmatrix} p_1 \\ p_0 \\ p_3 \\ p_2 \end{pmatrix} \quad \mathbf{v2} = \begin{pmatrix} p_0 \\ -p_1 \\ -p_2 \\ p_3 \end{pmatrix} \quad \mathbf{v3} = \begin{pmatrix} p_3 \\ p_2 \\ p_1 \\ p_0 \end{pmatrix} \quad \mathbf{v4} = \begin{pmatrix} p_2 \\ -p_3 \\ -p_0 \\ p_1 \end{pmatrix}$$

$$m_{\mathbf{v1}} = p_0p_3 - p_1p_2 = -m$$
$$m_{\mathbf{v2}} = p_1p_2 - p_0p_3 = m$$

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$$m_{\mathbf{v3}} = p_2 p_1 - p_3 p_0 = m$$
$$m_{\mathbf{v4}} = p_3 p_0 - p_2 p_1 = -m$$

$$\boldsymbol{\varphi}(\mathbf{x}) = \int \frac{d^4 p}{(2\pi)^4}$$
$$\begin{bmatrix} d_1(\mathbf{p})\mathbf{u1}(\mathbf{p}) + id_2(\mathbf{p})\mathbf{u3}(\mathbf{p}) + ib_2(\mathbf{p})\overline{\mathbf{u2}}(\mathbf{p}) + b_1(\mathbf{p})\overline{\mathbf{u4}}(\mathbf{p}) \\ +d_4(\mathbf{p})\mathbf{v1}(\mathbf{p}) + id_3(\mathbf{p})\mathbf{v3}(\mathbf{p}) + ib_3(\mathbf{p})\overline{\mathbf{v2}}(\mathbf{p}) + b_4(\mathbf{p})\overline{\mathbf{v4}}(\mathbf{p}) \end{bmatrix} e^{i(p_0x_1 - p_1x_0 + p_2x_3 - p_3x_2 + \overline{(\mathbf{p},\mathbf{x})})}$$
$$+ \begin{bmatrix} b_1^*(\mathbf{p})\overline{\mathbf{u1}}(\mathbf{p}) + ib_2^*(\mathbf{p})\overline{\mathbf{u3}}(\mathbf{p}) + id_2^*(\mathbf{p})\mathbf{u2}(\mathbf{p}) + d_1^*(\mathbf{p})\mathbf{u4}(\mathbf{p}) \\ +b_4^*(\mathbf{p})\overline{\mathbf{v1}}(\mathbf{p}) + ib_3^*(\mathbf{p})\overline{\mathbf{v3}}(\mathbf{p}) + id_3^*(\mathbf{p})\mathbf{v2}(\mathbf{p}) + d_4^*(\mathbf{p})\mathbf{v4}(\mathbf{p}) \end{bmatrix} e^{-i(p_0x_1 - p_1x_0 + p_2x_3 - p_3x_2 + \overline{(\mathbf{p},\mathbf{x})})}$$

$d_1^*(\mathbf{p})$ creates and $d_1(\mathbf{p})$ destroys a particle $\mathbf{u1}(\mathbf{p}) = \begin{pmatrix} -p_3 \\ -p_2 \\ p_1 \\ p_0 \end{pmatrix}$ with mass $-m$, spin up and momentum in the interval d^4p, $d_1^*(\mathbf{p})d_1(\mathbf{p})$ is the operator of the number of such particles

$b_1(\mathbf{p})$ creates and $b_1^*(\mathbf{p})$ destroys a particle $\overline{\mathbf{u1}}(\mathbf{p}) = \begin{pmatrix} -\overline{p_3} \\ -\overline{p_2} \\ \overline{p_1} \\ \overline{p_0} \end{pmatrix}$ with mass $-\overline{m}$, spin up and momentum in the interval d^4p, $b_1(\mathbf{p})b_1^*(\mathbf{p})$ is the operator of the number of such particles

$d_1(\mathbf{p})$ creates and $d_1^*(\mathbf{p})$ destroys a particle $\mathbf{u4}(\mathbf{p}) = \begin{pmatrix} p_0 \\ -p_1 \\ p_2 \\ -p_3 \end{pmatrix}$ with mass $-m$, spin up and momentum in the interval d^4p, $d_1(\mathbf{p})d_1^*(\mathbf{p})$ is the operator of the number of such particles

$b_1^*(\mathbf{p})$ creates and $b_1(\mathbf{p})$ destroys a particle $\overline{\mathbf{u4}}(\mathbf{p}) = \begin{pmatrix} \overline{p_0} \\ -\overline{p_1} \\ \overline{p_2} \\ -\overline{p_3} \end{pmatrix}$ with mass $-\overline{m}$, spin up and momentum in the interval d^4p, $b_1^*(\mathbf{p})b_1(\mathbf{p})$ is the operator of the number of such particles

Note that $\mathbf{u1}(\mathbf{p})$ and $\mathbf{u4}(\mathbf{p})$are translated into each other by a linear transformation, this is also true for other pairs of spinors

$$\mathbf{u4} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \mathbf{u1}$$
$$\mathbf{u1} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \mathbf{u4}$$

It is known [[9], formula II.1.30] that the charge conjugation operation transforms an electron into a positron with a change of the sign of the charge. Let us apply the charge conjugation to the reference spinor

$$\begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix} \mathbf{u1} = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} -p_3 \\ -p_2 \\ p_1 \\ p_0 \end{pmatrix} = -i \begin{pmatrix} p_0 \\ -p_1 \\ p_2 \\ -p_3 \end{pmatrix} = -i\mathbf{u4}$$

As a result $\mathbf{u1}$ not only transforms to $\mathbf{u4}$, but also changes a sign of mass due to the imaginary unit in the charge conjugation matrix. This confirms our thesis that the charge conjugation synchronously changes signs of charge and mass.

The properties of all particles and operators are summarized in a table

creates	**destroys**	**particle spinor**	**vector**	**number**	**mass**	**spin**	**wave sign**

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$d_1^*(\mathbf{p})$	$d_1(\mathbf{p})$	$\mathbf{u1}(\mathbf{p}) = \begin{pmatrix} -p_3 \\ -p_2 \\ p_1 \\ p_0 \end{pmatrix}$	$\begin{pmatrix} P_0 \\ P_1 \\ -P_2 \\ -P_3 \end{pmatrix}$	$d_1^*(\mathbf{p})d_1(\mathbf{p})$	$-m$	up	+
$d_1(\mathbf{p})$	$d_1^*(\mathbf{p})$	$\mathbf{u4}(\mathbf{p}) = \begin{pmatrix} p_0 \\ -p_1 \\ p_2 \\ -p_3 \end{pmatrix}$	$\begin{pmatrix} P_0 \\ -P_1 \\ -P_2 \\ P_3 \end{pmatrix}$	$d_1(\mathbf{p})d_1^*(\mathbf{p})$	$-m$	up	-
$b_1(\mathbf{p})$	$b_1^*(\mathbf{p})$	$\overline{\mathbf{u1}}(\mathbf{p}) = \begin{pmatrix} -\overline{p_3} \\ -\overline{p_2} \\ \overline{p_1} \\ \overline{p_0} \end{pmatrix}$	$\begin{pmatrix} P_0 \\ P_1 \\ P_2 \\ -P_3 \end{pmatrix}$	$b_1(\mathbf{p})b_1^*(\mathbf{p})$	$-\overline{m}$	up	-
$b_1^*(\mathbf{p})$	$b_1(\mathbf{p})$	$\overline{\mathbf{u4}}(\mathbf{p}) = \begin{pmatrix} \overline{p_0} \\ -\overline{p_1} \\ \overline{p_2} \\ -\overline{p_3} \end{pmatrix}$	$\begin{pmatrix} P_0 \\ -P_1 \\ P_2 \\ P_3 \end{pmatrix}$	$b_1^*(\mathbf{p})b_1(\mathbf{p})$	$-\overline{m}$	up	+
$d_4^*(\mathbf{p})$	$d_4(\mathbf{p})$	$\mathbf{v1}(\mathbf{p}) = \begin{pmatrix} p_1 \\ p_0 \\ p_3 \\ p_2 \end{pmatrix}$	$\begin{pmatrix} P_0 \\ P_1 \\ -P_2 \\ -P_3 \end{pmatrix}$	$d_4^*(\mathbf{p})d_4$	$-m$	down	+
$d_4(\mathbf{p})$	$d_4^*(\mathbf{p})$	$\mathbf{v4}(\mathbf{p}) = \begin{pmatrix} p_2 \\ -p_3 \\ -p_0 \\ p_1 \end{pmatrix}$	$\begin{pmatrix} P_0 \\ -P_1 \\ -P_2 \\ P_3 \end{pmatrix}$	$d_4(\mathbf{p})d_4^*(\mathbf{p})$	$-m$	down	-
$b_4(\mathbf{p})$	$b_4^*(\mathbf{p})$	$\overline{\mathbf{v1}}(\mathbf{p}) = \begin{pmatrix} \overline{p_1} \\ \overline{p_0} \\ \overline{p_3} \\ \overline{p_2} \end{pmatrix}$	$\begin{pmatrix} P_0 \\ P_1 \\ P_2 \\ -P_3 \end{pmatrix}$	$b_4(\mathbf{p})b_4^*(\mathbf{p})$	$-\overline{m}$	down	-
$b_4^*(\mathbf{p})$	$b_4(\mathbf{p})$	$\overline{\mathbf{v4}}(\mathbf{p}) = \begin{pmatrix} \overline{p_2} \\ -\overline{p_3} \\ -\overline{p_0} \\ \overline{p_1} \end{pmatrix}$	$\begin{pmatrix} P_0 \\ -P_1 \\ P_2 \\ P_3 \end{pmatrix}$	$b_4^*(\mathbf{p})b_4(\mathbf{p})$	$-\overline{m}$	down	+
$d_2^*(\mathbf{p})$	$d_2(\mathbf{p})$	$\mathbf{u3}(\mathbf{p}) = \begin{pmatrix} -p_1 \\ -p_0 \\ p_3 \\ p_2 \end{pmatrix}$	$\begin{pmatrix} P_0 \\ P_1 \\ -P_2 \\ -P_3 \end{pmatrix}$	$d_2^*(\mathbf{p})d_2(\mathbf{p})$	m	up	+
$d_2(\mathbf{p})$	$d_2^*(\mathbf{p})$	$\mathbf{u2}(\mathbf{p}) = \begin{pmatrix} p_2 \\ -p_3 \\ p_0 \\ -p_1 \end{pmatrix}$	$\begin{pmatrix} P_0 \\ -P_1 \\ -P_2 \\ P_3 \end{pmatrix}$	$d_2(\mathbf{p})d_2^*(\mathbf{p})$	m	up	-
$b_2(\mathbf{p})$	$b_2^*(\mathbf{p})$	$\overline{\mathbf{u3}}(\mathbf{p}) = \begin{pmatrix} -\overline{p_1} \\ -\overline{p_0} \\ \overline{p_3} \\ \overline{p_2} \end{pmatrix}$	$\begin{pmatrix} P_0 \\ P_1 \\ P_2 \\ -P_3 \end{pmatrix}$	$b_2(\mathbf{p})b_2^*(\mathbf{p})$	$\overline{m}$	up	-
$b_2^*(\mathbf{p})$	$b_2(\mathbf{p})$	$\overline{\mathbf{u2}}(\mathbf{p}) = \begin{pmatrix} \overline{p_2} \\ -\overline{p_3} \\ \overline{p_0} \\ -\overline{p_1} \end{pmatrix}$	$\begin{pmatrix} P_0 \\ -P_1 \\ P_2 \\ P_3 \end{pmatrix}$	$b_2^*(\mathbf{p})b_2(\mathbf{p})$	$\overline{m}$	up	+
$d_3^*(\mathbf{p})$	$d_3(\mathbf{p})$	$\mathbf{v3}(\mathbf{p}) = \begin{pmatrix} p_3 \\ p_2 \\ p_1 \\ p_0 \end{pmatrix}$	$\begin{pmatrix} P_0 \\ P_1 \\ -P_2 \\ -P_3 \end{pmatrix}$	$d_3^*(\mathbf{p})d_3(\mathbf{p})$	m	down	+
$d_3(\mathbf{p})$	$d_3^*(\mathbf{p})$	$\mathbf{v2}(\mathbf{p}) = \begin{pmatrix} p_0 \\ -p_1 \\ -p_2 \\ p_3 \end{pmatrix}$	$\begin{pmatrix} P_0 \\ -P_1 \\ -P_2 \\ P_3 \end{pmatrix}$	$d_3(\mathbf{p})d_3^*(\mathbf{p})$	m	down	-
$b_3(\mathbf{p})$	$b_3^*(\mathbf{p})$	$\overline{\mathbf{v3}}(\mathbf{p}) = \begin{pmatrix} \overline{p_3} \\ \overline{p_2} \\ \overline{p_1} \\ \overline{p_0} \end{pmatrix}$	$\begin{pmatrix} P_0 \\ P_1 \\ P_2 \\ -P_3 \end{pmatrix}$	$b_3(\mathbf{p})b_3^*(\mathbf{p})$	$\overline{m}$	down	-

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$b_3^*(\mathbf{p})$	$b_3(\mathbf{p})$	$\overline{\mathbf{v2}}(\mathbf{p}) = \begin{pmatrix} \overline{p_0} \\ -\overline{p_1} \\ -\overline{p_2} \\ \overline{p_3} \end{pmatrix}$	$\begin{pmatrix} P_0 \\ -P_1 \\ P_2 \\ P_3 \end{pmatrix}$	$b_3^*(\mathbf{p})b_3(\mathbf{p})$	$\overline{m}$	down	+

Here the column "vector" shows the vector obtained from the corresponding spinor by the formula of the form

$$U1_\mu = \frac{1}{2}\mathbf{u1}^\dagger S_\mu \mathbf{u1}$$

and

$$P_\mu = \frac{1}{2}\mathbf{p}^\dagger S_\mu \mathbf{p}$$

Although we have used the term vector for quantities like **U1**, they are not really vectors in the sense that if a Lorentz transformation is applied to a coordinate spinor and hence a coordinate vector, the true vector must undergo the same transformation. For a momentum vector this is the case, but if the sign of one or more components in the momentum vector is changed, it will no longer be transformed according to this law. For example, charge conjugation changes the signs of some components

$$C^T S_0 C = S_0 \qquad C^T S_1 C = -S_1 \qquad C^T S_2 C = S_2 \qquad C^T S_3 C = -S_3$$

so the electron current and the positron current cannot be vectors at the same time, and in fact, as can be seen from the table, neither is a vector.

By the words $d_1(\mathbf{p})$ destroys the particle $\mathbf{u1}(\mathbf{p})$ it should be understood that this operator transforms this particle into the particle $\mathbf{u4}(\mathbf{p})$, and the operator $d_1^*(\mathbf{p})$ performs the reverse transformation of $\mathbf{u4}(\mathbf{p})$ into $\mathbf{u1}(\mathbf{p})$. The action of the operator $d_1(\mathbf{p})$ on any other particle gives zero. Since both these particles have the same mass, the total mass of the fermionic field does not change from these transformations. The mass m itself can have any sign or even be complex.

Although we call the spinors presented in the table particles, they actually describe the same particle whose characteristic property is a mass with a certain sign. A particle with mass of opposite sign is described by the other sixteen spinors. Let's compare the momenta of two particles with different mass signs

$$\begin{pmatrix} p_0 \\ p_1 \\ p_2 \\ p_3 \end{pmatrix} \qquad p_1 p_2 - p_0 p_3 = m$$

$$\begin{pmatrix} p_2 \\ p_3 \\ p_0 \\ p_1 \end{pmatrix} \qquad p_3 p_0 - p_2 p_1 = -m$$

If we add sixteen spinors of the field of one particle with sixteen corresponding spinors of another particle, it will look the same as if the momenta of the particles were summed directly. It is clear that the momenta themselves cannot be directly summed, but summing the fields does not look impossible and leads to the same result as adding the momenta directly. The result can be represented as a sum of two other momenta, the mass of each of which is zero

$$\begin{pmatrix} p_0 \\ p_1 \\ p_2 \\ p_3 \end{pmatrix} + \begin{pmatrix} p_2 \\ p_3 \\ p_0 \\ p_1 \end{pmatrix} = \begin{pmatrix} p_0 + p_2 \\ p_1 + p_3 \\ p_2 + p_0 \\ p_3 + p_1 \end{pmatrix} = \begin{pmatrix} p_0 \\ p_1 \\ p_0 \\ p_1 \end{pmatrix} + \begin{pmatrix} p_2 \\ p_3 \\ p_2 \\ p_3 \end{pmatrix}$$

Perhaps, such summation is an adequate model for describing the phenomenon of annihilation of particles with different masses. As an illustrative example, consider the case of an electron and a positron at rest

$$\begin{pmatrix} p_0 \\ p_1 \\ \overline{p_1} \\ -\overline{p_0} \end{pmatrix} \qquad p_1 \overline{p_1} - p_0(-\overline{p_0}) = m$$

$$\begin{pmatrix} \overline{p_1} \\ -\overline{p_0} \\ p_0 \\ p_1 \end{pmatrix} \qquad -\overline{p_0}p_0 - \overline{p_1}p_1 = -m$$

$$\begin{pmatrix} p_0 \\ p_1 \\ \overline{p_1} \\ -\overline{p_0} \end{pmatrix} + \begin{pmatrix} \overline{p_1} \\ -\overline{p_0} \\ p_0 \\ p_1 \end{pmatrix} = \begin{pmatrix} p_0 + \overline{p_1} \\ p_1 - \overline{p_0} \\ \overline{p_1} + p_0 \\ -\overline{p_0} + p_1 \end{pmatrix} = \begin{pmatrix} p_0 \\ p_1 \\ p_0 \\ p_1 \end{pmatrix} + \begin{pmatrix} \overline{p_1} \\ -\overline{p_0} \\ \overline{p_1} \\ -\overline{p_0} \end{pmatrix}$$

As a result of addition and separation, two photons with zero mass are obtained, having oppositely directed spatial components of the momentum vector, i.e. flying apart. At this interaction the total energy, the total momentum and the total mass are conserved. We can also say that the total charge is conserved, although in our interpretation the charge is not a numerical characteristic that can be calculated, the sign of the charge is determined by the structure of the spinor. In turn, this structure is determined precisely by the sign of the mass. Thus, a change in the sign of the mass leads ultimately to a change in the sign of the charge.

In the table below two last columns with a set of spinors and vectors corresponding to the particle with opposite sign of mass are added. It is supposed that at annihilation the spinors of the particle and antiparticle, which are in the same row of the table, are summed. The set of 16 spinors remains the same, but the order of their arrangement changes when the sign of mass changes.

creates	destroys	particle spinor	vector	mass	spin	wave sign	antiparticle spinor	vector
$d_1^*(\mathbf{p})$	$d_1(\mathbf{p})$	$\mathbf{u1}(\mathbf{p}) = \begin{pmatrix} -p_3 \\ -p_2 \\ p_1 \\ p_0 \end{pmatrix}$	$\begin{pmatrix} P_0 \\ P_1 \\ -P_2 \\ -P_3 \end{pmatrix}$	$-m$	up	+	$\mathbf{u1}(\mathbf{p}) = \begin{pmatrix} -p_1 \\ -p_0 \\ p_3 \\ p_2 \end{pmatrix}$	$\begin{pmatrix} P_0 \\ P_1 \\ -P_2 \\ -P_3 \end{pmatrix}$
$d_1(\mathbf{p})$	$d_1^*(\mathbf{p})$	$\mathbf{u4}(\mathbf{p}) = \begin{pmatrix} p_0 \\ -p_1 \\ p_2 \\ -p_3 \end{pmatrix}$	$\begin{pmatrix} P_0 \\ -P_1 \\ -P_2 \\ P_3 \end{pmatrix}$	$-m$	up	-	$\mathbf{u4}(\mathbf{p}) = \begin{pmatrix} p_2 \\ -p_3 \\ p_0 \\ -p_1 \end{pmatrix}$	$\begin{pmatrix} P_0 \\ -P_1 \\ -P_2 \\ P_3 \end{pmatrix}$
$b_1(\mathbf{p})$	$b_1^*(\mathbf{p})$	$\overline{\mathbf{u1}}(\mathbf{p}) = \begin{pmatrix} -\overline{p_3} \\ -\overline{p_2} \\ \overline{p_1} \\ \overline{p_0} \end{pmatrix}$	$\begin{pmatrix} P_0 \\ P_1 \\ P_2 \\ -P_3 \end{pmatrix}$	$-\overline{m}$	up	-	$\overline{\mathbf{u1}}(\mathbf{p}) = \begin{pmatrix} -\overline{p_1} \\ -\overline{p_0} \\ \overline{p_3} \\ \overline{p_2} \end{pmatrix}$	$\begin{pmatrix} P_0 \\ P_1 \\ P_2 \\ -P_3 \end{pmatrix}$
$b_1^*(\mathbf{p})$	$b_1(\mathbf{p})$	$\overline{\mathbf{u4}}(\mathbf{p}) = \begin{pmatrix} \overline{p_0} \\ -\overline{p_1} \\ \overline{p_2} \\ -\overline{p_3} \end{pmatrix}$	$\begin{pmatrix} P_0 \\ -P_1 \\ P_2 \\ P_3 \end{pmatrix}$	$-\overline{m}$	up	+	$\overline{\mathbf{u4}}(\mathbf{p}) = \begin{pmatrix} \overline{p_2} \\ -\overline{p_3} \\ \overline{p_0} \\ -\overline{p_1} \end{pmatrix}$	$\begin{pmatrix} P_0 \\ -P_1 \\ P_2 \\ P_3 \end{pmatrix}$
$d_4^*(\mathbf{p})$	$d_4(\mathbf{p})$	$\mathbf{v1}(\mathbf{p}) = \begin{pmatrix} p_1 \\ p_0 \\ p_3 \\ p_2 \end{pmatrix}$	$\begin{pmatrix} P_0 \\ P_1 \\ -P_2 \\ -P_3 \end{pmatrix}$	$-m$	down	+	$\mathbf{v1}(\mathbf{p}) = \begin{pmatrix} p_3 \\ p_2 \\ p_1 \\ p_0 \end{pmatrix}$	$\begin{pmatrix} P_0 \\ P_1 \\ -P_2 \\ -P_3 \end{pmatrix}$
$d_4(\mathbf{p})$	$d_4^*(\mathbf{p})$	$\mathbf{v4}(\mathbf{p}) = \begin{pmatrix} p_2 \\ -p_3 \\ -p_0 \\ p_1 \end{pmatrix}$	$\begin{pmatrix} P_0 \\ -P_1 \\ -P_2 \\ P_3 \end{pmatrix}$	$-m$	down	-	$\mathbf{v4}(\mathbf{p}) = \begin{pmatrix} p_0 \\ -p_1 \\ -p_2 \\ p_3 \end{pmatrix}$	$\begin{pmatrix} P_0 \\ -P_1 \\ -P_2 \\ P_3 \end{pmatrix}$
$b_4(\mathbf{p})$	$b_4^*(\mathbf{p})$	$\overline{\mathbf{v1}}(\mathbf{p}) = \begin{pmatrix} \overline{p_1} \\ \overline{p_0} \\ \overline{p_3} \\ \overline{p_2} \end{pmatrix}$	$\begin{pmatrix} P_0 \\ P_1 \\ P_2 \\ -P_3 \end{pmatrix}$	$-\overline{m}$	down	-	$\overline{\mathbf{v1}}(\mathbf{p}) = \begin{pmatrix} \overline{p_3} \\ \overline{p_2} \\ \overline{p_1} \\ \overline{p_0} \end{pmatrix}$	$\begin{pmatrix} P_0 \\ P_1 \\ P_2 \\ -P_3 \end{pmatrix}$
$b_4^*(\mathbf{p})$	$b_4(\mathbf{p})$	$\overline{\mathbf{v4}}(\mathbf{p}) = \begin{pmatrix} \overline{p_2} \\ -\overline{p_3} \\ -\overline{p_0} \\ \overline{p_1} \end{pmatrix}$	$\begin{pmatrix} P_0 \\ -P_1 \\ P_2 \\ P_3 \end{pmatrix}$	$-\overline{m}$	down	+	$\overline{\mathbf{v4}}(\mathbf{p}) = \begin{pmatrix} \overline{p_0} \\ -\overline{p_1} \\ -\overline{p_2} \\ \overline{p_3} \end{pmatrix}$	$\begin{pmatrix} P_0 \\ -P_1 \\ P_2 \\ P_3 \end{pmatrix}$
$d_2^*(\mathbf{p})$	$d_2(\mathbf{p})$	$\mathbf{u3}(\mathbf{p}) = \begin{pmatrix} -p_1 \\ -p_0 \\ p_3 \\ p_2 \end{pmatrix}$	$\begin{pmatrix} P_0 \\ P_1 \\ -P_2 \\ -P_3 \end{pmatrix}$	m	up	+	$\mathbf{u3}(\mathbf{p}) = \begin{pmatrix} -p_3 \\ -p_2 \\ p_1 \\ p_0 \end{pmatrix}$	$\begin{pmatrix} P_0 \\ P_1 \\ -P_2 \\ -P_3 \end{pmatrix}$

$d_2(\mathbf{p})$	$d_2^*(\mathbf{p})$	$\mathbf{u2}(\mathbf{p})=\begin{pmatrix}p_2\\-p_3\\p_0\\-p_1\end{pmatrix}$	$\begin{pmatrix}P_0\\-P_1\\-P_2\\P_3\end{pmatrix}$	m	up	-	$\mathbf{u2}(\mathbf{p})=\begin{pmatrix}p_0\\-p_1\\p_2\\-p_3\end{pmatrix}$	$\begin{pmatrix}P_0\\-P_1\\-P_2\\P_3\end{pmatrix}$
$b_2(\mathbf{p})$	$b_2^*(\mathbf{p})$	$\overline{\mathbf{u3}}(\mathbf{p})=\begin{pmatrix}-\overline{p_1}\\-\overline{p_0}\\\overline{p_3}\\\overline{p_2}\end{pmatrix}$	$\begin{pmatrix}P_0\\P_1\\P_2\\-P_3\end{pmatrix}$	$\overline{m}$	up	-	$\overline{\mathbf{u3}}(\mathbf{p})=\begin{pmatrix}-\overline{p_3}\\-\overline{p_2}\\\overline{p_1}\\\overline{p_0}\end{pmatrix}$	$\begin{pmatrix}P_0\\P_1\\P_2\\-P_3\end{pmatrix}$
$b_2^*(\mathbf{p})$	$b_2(\mathbf{p})$	$\overline{\mathbf{u2}}(\mathbf{p})=\begin{pmatrix}\overline{p_2}\\-\overline{p_3}\\\overline{p_0}\\-\overline{p_1}\end{pmatrix}$	$\begin{pmatrix}P_0\\-P_1\\P_2\\P_3\end{pmatrix}$	$\overline{m}$	up	+	$\overline{\mathbf{u2}}(\mathbf{p})=\begin{pmatrix}\overline{p_0}\\-\overline{p_1}\\\overline{p_2}\\-\overline{p_3}\end{pmatrix}$	$\begin{pmatrix}P_0\\-P_1\\P_2\\P_3\end{pmatrix}$
$d_3^*(\mathbf{p})$	$d_3(\mathbf{p})$	$\mathbf{v3}(\mathbf{p})=\begin{pmatrix}p_3\\p_2\\p_1\\p_0\end{pmatrix}$	$\begin{pmatrix}P_0\\P_1\\-P_2\\-P_3\end{pmatrix}$	m	down	+	$\mathbf{v3}(\mathbf{p})=\begin{pmatrix}p_1\\p_0\\p_3\\p_2\end{pmatrix}$	$\begin{pmatrix}P_0\\P_1\\-P_2\\-P_3\end{pmatrix}$
$d_3(\mathbf{p})$	$d_3^*(\mathbf{p})$	$\mathbf{v2}(\mathbf{p})=\begin{pmatrix}p_0\\-p_1\\-p_2\\p_3\end{pmatrix}$	$\begin{pmatrix}P_0\\-P_1\\-P_2\\P_3\end{pmatrix}$	m	down	-	$\mathbf{v2}(\mathbf{p})=\begin{pmatrix}p_2\\-p_3\\-p_0\\p_1\end{pmatrix}$	$\begin{pmatrix}P_0\\-P_1\\-P_2\\P_3\end{pmatrix}$
$b_3(\mathbf{p})$	$b_3^*(\mathbf{p})$	$\overline{\mathbf{v3}}(\mathbf{p})=\begin{pmatrix}\overline{p_3}\\\overline{p_2}\\\overline{p_1}\\\overline{p_0}\end{pmatrix}$	$\begin{pmatrix}P_0\\P_1\\P_2\\-P_3\end{pmatrix}$	$\overline{m}$	down	-	$\overline{\mathbf{v3}}(\mathbf{p})=\begin{pmatrix}\overline{p_1}\\\overline{p_0}\\\overline{p_3}\\\overline{p_2}\end{pmatrix}$	$\begin{pmatrix}P_0\\P_1\\P_2\\-P_3\end{pmatrix}$
$b_3^*(\mathbf{p})$	$b_3(\mathbf{p})$	$\overline{\mathbf{v2}}(\mathbf{p})=\begin{pmatrix}\overline{p_0}\\-\overline{p_1}\\-\overline{p_2}\\\overline{p_3}\end{pmatrix}$	$\begin{pmatrix}P_0\\-P_1\\P_2\\P_3\end{pmatrix}$	$\overline{m}$	down	+	$\overline{\mathbf{v2}}(\mathbf{p})=\begin{pmatrix}\overline{p_2}\\-\overline{p_3}\\-\overline{p_0}\\\overline{p_1}\end{pmatrix}$	$\begin{pmatrix}P_0\\-P_1\\P_2\\P_3\end{pmatrix}$

It is possible to assume that the reason and condition of distinction between particles with different charge is the presence of their non-zero mass. If the mass is zero, then in the given table there are no differences between spinors of the particle and antiparticle, i.e. there is no mechanism for formation of the internal degree of freedom, which we treat as charge.

Let us see what result we get if we apply another definition of anticommutativity of the fermionic field.

$$\boldsymbol{\varphi}(\mathbf{x})=\int\frac{d^4p}{(2\pi)^4}$$

$$\begin{bmatrix}d_1(\mathbf{p})\mathbf{u1}(\mathbf{p})+id_2(\mathbf{p})\mathbf{u3}(\mathbf{p})+ib_2(\mathbf{p})\overline{\mathbf{u2}}(\mathbf{p})+\ b_1(\mathbf{p})\overline{\mathbf{u4}}(\mathbf{p})\\+d_4(\mathbf{p})\mathbf{v1}(\mathbf{p})+id_3(\mathbf{p})\mathbf{v3}(\mathbf{p})+ib_3(\mathbf{p})\overline{\mathbf{v2}}(\mathbf{p})+b_4(\mathbf{p})\overline{\mathbf{v4}}(\mathbf{p})\end{bmatrix}e^{i\left(p_0x_1-p_1x_0+p_2x_3-p_3x_2+\overline{(\mathbf{p},\mathbf{x})}\right)}$$

$$+\begin{bmatrix}b_1^*(\mathbf{p})\overline{\mathbf{u1}}(\mathbf{p})+ib_2^*(\mathbf{p})\overline{\mathbf{u3}}(\mathbf{p})+id_2^*(\mathbf{p})\mathbf{u2}(\mathbf{p})+d_1^*(\mathbf{p})\mathbf{u4}(\mathbf{p})\\+b_4^*(\mathbf{p})\overline{\mathbf{v1}}(\mathbf{p})+ib_3^*(\mathbf{p})\overline{\mathbf{v3}}(\mathbf{p})+id_3^*(\mathbf{p})\mathbf{v2}(\mathbf{p})+d_4^*(\mathbf{p})\mathbf{v4}(\mathbf{p})\end{bmatrix}e^{-i\left(p_0x_1-p_1x_0+p_2x_3-p_3x_2+\overline{(\mathbf{p},\mathbf{x})}\right)}$$

$$\{\varphi_i(\mathbf{x}),\overline{\varphi_j}(\mathbf{x}')\}=\varphi_i(\mathbf{x})\overline{\varphi_j}(\mathbf{x}')+\overline{\varphi_j}(\mathbf{x}')\varphi_i(\mathbf{x})=\left(\boldsymbol{\varphi}(\mathbf{x})\boldsymbol{\varphi}^+(\mathbf{x}')+\left(\overline{\boldsymbol{\varphi}}(\mathbf{x}')\boldsymbol{\varphi}^T(\mathbf{x})\right)^T\right)_{ij}$$

$$\boldsymbol{\varphi}(\mathbf{x})\boldsymbol{\varphi}^+(\mathbf{x}')+\left(\overline{\boldsymbol{\varphi}}(\mathbf{x}')\boldsymbol{\varphi}^T(\mathbf{x})\right)^T=$$

$$\iint\frac{d^4p}{(2\pi)^4}\frac{d^4p'}{(2\pi)^4}$$

$$\begin{bmatrix}d_1(\mathbf{p})\mathbf{u1}(\mathbf{p})+id_2(\mathbf{p})\mathbf{u3}(\mathbf{p})+ib_2(\mathbf{p})\overline{\mathbf{u2}}(\mathbf{p})+\ b_1(\mathbf{p})\overline{\mathbf{u4}}(\mathbf{p})\\+d_4(\mathbf{p})\mathbf{v1}(\mathbf{p})+id_3(\mathbf{p})\mathbf{v3}(\mathbf{p})+ib_3(\mathbf{p})\overline{\mathbf{v2}}(\mathbf{p})+b_4(\mathbf{p})\overline{\mathbf{v4}}(\mathbf{p})\end{bmatrix}$$

$$\begin{bmatrix}d_1^*(\mathbf{p}')\mathbf{u1}^+(\mathbf{p}')-id_2^*(\mathbf{p}')\mathbf{u3}^+(\mathbf{p}')-ib_2^*(\mathbf{p}')\mathbf{u2}^T(\mathbf{p}')+b_1^*(\mathbf{p}')\mathbf{u4}^T(\mathbf{p}')\\+d_4^*(\mathbf{p}')\mathbf{v1}^+(\mathbf{p}')-id_3^*(\mathbf{p}')\mathbf{v3}^+(\mathbf{p}')-ib_3^*(\mathbf{p}')\mathbf{v2}^T(\mathbf{p}')+b_4^*(\mathbf{p}')\mathbf{v4}^T(\mathbf{p}')\end{bmatrix}$$

$$e^{i\left(p_0x_1-p_1x_0+p_2x_3-p_3x_2+\overline{(\mathbf{p},\mathbf{x})}\right)}e^{-i\left(p_0'x_1'-p_1'x_0'+p_2'x_3'-p_3'x_2'+\overline{(\mathbf{p}',\mathbf{x}')}\right)}$$

$$+$$

$$\left(\begin{matrix}\begin{bmatrix} d_1^*(\mathbf{p}')\overline{\mathbf{u1}}(\mathbf{p}') - id_2^*(\mathbf{p}')\overline{\mathbf{u3}}(\mathbf{p}') - ib_2^*(\mathbf{p}')\mathbf{u2}(\mathbf{p}') + b_1(\mathbf{p}')\mathbf{u4}(\mathbf{p}') \\ +d_4^*(\mathbf{p}')\overline{\mathbf{v1}}(\mathbf{p}') - id_3^*(\mathbf{p}')\overline{\mathbf{v3}}(\mathbf{p}') - ib_3^*(\mathbf{p}')\mathbf{v2}(\mathbf{p}') + b_4^*(\mathbf{p}')\mathbf{v4}(\mathbf{p}') \end{bmatrix} \\ \begin{bmatrix} d_1(\mathbf{p})\mathbf{u1}^T(\mathbf{p}) + id_2(\mathbf{p})\mathbf{u3}^T(\mathbf{p}) + ib_2(\mathbf{p}')\mathbf{u2}^+(\mathbf{p}') + b_1^*(\mathbf{p})\mathbf{u4}^+(\mathbf{p}) \\ +d_4(\mathbf{p})\mathbf{v1}^T(\mathbf{p}) + id_3(\mathbf{p})\mathbf{v3}^T(\mathbf{p}) + ib_3(\mathbf{p})\mathbf{v2}^+(\mathbf{p}) + b_4(\mathbf{p})\mathbf{v4}^+(\mathbf{p}) \end{bmatrix} \\ e^{-i(p_0{}'x_1{}'-p_1{}'x_0{}'+p_2{}'x_3{}'-p_3{}'x_2{}'+\overline{(\mathbf{p}',\mathbf{x}')})} e^{i(p_0x_1-p_1x_0+p_2x_3-p_3x_2+\overline{(\mathbf{p},\mathbf{x})})} \end{matrix}\right)^T$$

$$+$$

$$\begin{bmatrix} b_1^*(\mathbf{p})\overline{\mathbf{u1}}(\mathbf{p}) + ib_2^*(\mathbf{p})\overline{\mathbf{u3}}(\mathbf{p}) + id_2^*(\mathbf{p})\mathbf{u2}(\mathbf{p}) + d_1^*(\mathbf{p})\mathbf{u4}(\mathbf{p}) \\ +b_4^*(\mathbf{p})\overline{\mathbf{v1}}(\mathbf{p}) + ib_3^*(\mathbf{p})\overline{\mathbf{v3}}(\mathbf{p}) + id_3^*(\mathbf{p})\mathbf{v2}(\mathbf{p}) + d_4^*(\mathbf{p})\mathbf{v4}(\mathbf{p}) \end{bmatrix}$$

$$\begin{bmatrix} b_1(\mathbf{p}')\mathbf{u1}^T(\mathbf{p}') - ib_2(\mathbf{p}')\mathbf{u3}^T(\mathbf{p}') - id_2(\mathbf{p}')\mathbf{u2}^+(\mathbf{p}') + d_1(\mathbf{p}')\mathbf{u4}^+(\mathbf{p}') \\ +b_4(\mathbf{p}')\mathbf{v1}^T(\mathbf{p}') - ib_3(\mathbf{p}')\mathbf{v3}^T(\mathbf{p}') - id_3(\mathbf{p}')\mathbf{v2}^+(\mathbf{p}') + d_4(\mathbf{p}')\mathbf{v4}^+(\mathbf{p}') \end{bmatrix}$$

$$e^{-i(p_0x_1-p_1x_0+p_2x_3-p_3x_2+\overline{(\mathbf{p},\mathbf{x})})} e^{i(p_0{}'x_1{}'-p_1{}'x_0{}'+p_2{}'x_3{}'-p_3{}'x_2{}'+\overline{(\mathbf{p}',\mathbf{x}')})}$$

$$+$$

$$\left(\begin{matrix}\begin{bmatrix} b_1(\mathbf{p}')\mathbf{u1}(\mathbf{p}') - ib_2(\mathbf{p}')\mathbf{u3}(\mathbf{p}') - id_2(\mathbf{p}')\overline{\mathbf{u2}}(\mathbf{p}') + d_1(\mathbf{p}')\overline{\mathbf{u4}}(\mathbf{p}') \\ +b_4(\mathbf{p}')\mathbf{v1}(\mathbf{p}') - ib_3(\mathbf{p}')\mathbf{v3}(\mathbf{p}') - id_3(\mathbf{p}')\overline{\mathbf{v2}}(\mathbf{p}') + d_4(\mathbf{p}')\overline{\mathbf{v4}}(\mathbf{p}') \end{bmatrix} \\ \begin{bmatrix} b_1^*(\mathbf{p})\mathbf{u1}^+(\mathbf{p}) + ib_2^*(\mathbf{p})\mathbf{u3}^+(\mathbf{p}) + id_2^*(\mathbf{p})\mathbf{u2}^T(\mathbf{p}) + d_1^*(\mathbf{p})\mathbf{u4}^T(\mathbf{p}) \\ +b_4^*(\mathbf{p})\mathbf{v1}^+(\mathbf{p}) + ibib_{3_3}^*(\mathbf{p})\mathbf{v3}^+(\mathbf{p}) + id_3^*(\mathbf{p})\mathbf{v2}^T(\mathbf{p}) + d_4^*(\mathbf{p})\mathbf{v4}^T(\mathbf{p}) \end{bmatrix} \\ e^{i(p_0{}'x_1{}'-p_1{}'x_0{}'+p_2{}'x_3{}'-p_3{}'x_2{}'+\overline{(\mathbf{p}',\mathbf{x}')})} e^{-i(p_0x_1-p_1x_0+p_2x_3-p_3x_2+\overline{(\mathbf{p},\mathbf{x})})} \end{matrix}\right)^T$$

$$= \iint \frac{d^4p}{(2\pi)^4}\frac{d^4p'}{(2\pi)^4} \begin{bmatrix} \begin{bmatrix} d_1(\mathbf{p})d_1^*(\mathbf{p}')\mathbf{u1}(\mathbf{p})\mathbf{u1}^+(\mathbf{p}') + \left(d_1^*(\mathbf{p}')d_1(\mathbf{p})\overline{\mathbf{u1}}(\mathbf{p}')\mathbf{u1}^T(\mathbf{p})\right)^T + \cdots \\ +d_2(\mathbf{p})d_2^*(\mathbf{p}')\mathbf{u3}(\mathbf{p})\mathbf{u3}^+(\mathbf{p}') + \left(d_2^*(\mathbf{p}')d_2(\mathbf{p})\overline{\mathbf{u3}}(\mathbf{p}')\mathbf{u3}^T(\mathbf{p})\right)^T + \cdots \end{bmatrix} \\ e^{i(p_0x_1-p_1x_0+p_2x_3-p_3x_2+\overline{(\mathbf{p},\mathbf{x})})} e^{-i(p_0{}'x_1{}'-p_1{}'x_0{}'+p_2{}'x_3{}'-p_3{}'x_2{}'+\overline{(\mathbf{p}',\mathbf{x}')})} \\ + \\ \begin{bmatrix} b_1(\mathbf{p})b_1^*(\mathbf{p}')\overline{\mathbf{u4}}(\mathbf{p})\mathbf{u4}^T(\mathbf{p}') + \left(b_1^*(\mathbf{p}')b_1(\mathbf{p})\mathbf{u4}(\mathbf{p}')\mathbf{u4}^+(\mathbf{p})\right)^T + \cdots \\ +b_2(\mathbf{p})b_2^*(\mathbf{p}')\overline{\mathbf{u2}}(\mathbf{p})\mathbf{u2}^T(\mathbf{p}') + \left(b_2^*(\mathbf{p}')b_2(\mathbf{p}')\mathbf{u2}(\mathbf{p}')\mathbf{u2}^+(\mathbf{p}')\right)^T + \cdots \end{bmatrix} \\ e^{i(p_0{}'x_1{}'-p_1{}'x_0{}'+p_2{}'x_3{}'-p_3{}'x_2{}'+\overline{(\mathbf{p}',\mathbf{x}')})} e^{-i(p_0x_1-p_1x_0+p_2x_3-p_3x_2+\overline{(\mathbf{p},\mathbf{x})})} \end{bmatrix}$$

$$+ \iint \frac{d^4p}{(2\pi)^4}\frac{d^4p'}{(2\pi)^4} \begin{bmatrix} \begin{bmatrix} b_1^*(\mathbf{p})b_1(\mathbf{p}')\overline{\mathbf{u1}}(\mathbf{p})\mathbf{u1}^T(\mathbf{p}') + \left(b_1(\mathbf{p}')b_1^*(\mathbf{p})\mathbf{u1}(\mathbf{p}')\mathbf{u1}^+(\mathbf{p})\right)^T + \cdots \\ +b_2^*(\mathbf{p})b_2(\mathbf{p}')\overline{\mathbf{u3}}(\mathbf{p})\mathbf{u3}^T(\mathbf{p}') + \left(b_2(\mathbf{p}')b_2^*(\mathbf{p})\mathbf{u3}(\mathbf{p}')\mathbf{u3}^+(\mathbf{p})\right)^T + \cdots \end{bmatrix} \\ e^{-i(p_0x_1-p_1x_0+p_2x_3-p_3x_2+\overline{(\mathbf{p},\mathbf{x})})} e^{i(p_0{}'x_1{}'-p_1{}'x_0{}'+p_2{}'x_3{}'-p_3{}'x_2{}'+\overline{(\mathbf{p}',\mathbf{x}')})} \\ + \\ \begin{bmatrix} d_1^*(\mathbf{p})d_1(\mathbf{p}')\mathbf{u4}(\mathbf{p})\mathbf{u4}^+(\mathbf{p}') + \left(d_1(\mathbf{p}')d_1^*(\mathbf{p})\overline{\mathbf{u4}}(\mathbf{p}')\mathbf{u4}^T(\mathbf{p})\right)^T + \cdots \\ +d_2^*(\mathbf{p})d_2(\mathbf{p}')\mathbf{u2}(\mathbf{p})\mathbf{u2}^+(\mathbf{p}') + \left(d_2(\mathbf{p}')d_2^*(\mathbf{p})\overline{\mathbf{u2}}(\mathbf{p}')\mathbf{u2}^T(\mathbf{p})\right)^T + \cdots \end{bmatrix} \\ e^{-i(p_0{}'x_1{}'-p_1{}'x_0{}'+p_2{}'x_3{}'-p_3{}'x_2{}'+\overline{(\mathbf{p}',\mathbf{x}')})} e^{i(p_0x_1-p_1x_0+p_2x_3-p_3x_2+\overline{(\mathbf{p},\mathbf{x})})} \end{bmatrix}$$

$$= \iint \frac{d^4p}{(2\pi)^4}\frac{d^4p'}{(2\pi)^4} \begin{bmatrix} \begin{bmatrix} d_1(\mathbf{p})d_1^*(\mathbf{p}')\mathbf{u1}(\mathbf{p})\mathbf{u1}^+(\mathbf{p}') + \left(d_1^*(\mathbf{p}')d_1(\mathbf{p})\mathbf{u1}(\mathbf{p})\mathbf{u1}^+(\mathbf{p}')\right) + \cdots \\ +d_2(\mathbf{p})d_2^*(\mathbf{p}')\mathbf{u3}(\mathbf{p})\mathbf{u3}^+(\mathbf{p}') + \left(d_2^*(\mathbf{p}')d_2(\mathbf{p})\mathbf{u3}(\mathbf{p})\mathbf{u3}^+(\mathbf{p}')\right) + \cdots \end{bmatrix} \\ e^{i(p_0x_1-p_1x_0+p_2x_3-p_3x_2+\overline{(\mathbf{p},\mathbf{x})})} e^{-i(p_0{}'x_1{}'-p_1{}'x_0{}'+p_2{}'x_3{}'-p_3{}'x_2{}'+\overline{(\mathbf{p}',\mathbf{x}')})} \\ + \\ \begin{bmatrix} b_1(\mathbf{p})b_1^*(\mathbf{p}')\overline{\mathbf{u4}}(\mathbf{p})\mathbf{u4}^T(\mathbf{p}') + \left(b_1^*(\mathbf{p}')b_1(\mathbf{p})\overline{\mathbf{u4}}(\mathbf{p})\mathbf{u4}^T(\mathbf{p}')\right) + \cdots \\ +b_2(\mathbf{p})b_2^*(\mathbf{p}')\overline{\mathbf{u2}}(\mathbf{p})\mathbf{u2}^T(\mathbf{p}') + \left(b_2^*(\mathbf{p}')b_2(\mathbf{p}')\overline{\mathbf{u2}}(\mathbf{p})\mathbf{u2}^T(\mathbf{p}')\right) + \cdots \end{bmatrix} \\ e^{i(p_0{}'x_1{}'-p_1{}'x_0{}'+p_2{}'x_3{}'-p_3{}'x_2{}'+\overline{(\mathbf{p}',\mathbf{x}')})} e^{-i(p_0x_1-p_1x_0+p_2x_3-p_3x_2+\overline{(\mathbf{p},\mathbf{x})})} \end{bmatrix}$$

$$+\iint \frac{d^4p}{(2\pi)^4}\frac{d^4p'}{(2\pi)^4}\left[\begin{array}{c}\left[\begin{array}{c}b_1^*(\mathbf{p})b_1(\mathbf{p}')\overline{\mathbf{u1}}(\mathbf{p})\mathbf{u1}^T(\mathbf{p}')+\left(b_1(\mathbf{p}')b_1^*(\mathbf{p})\overline{\mathbf{u1}}(\mathbf{p})\mathbf{u1}^T(\mathbf{p}')(\mathbf{p}')\right)+\cdots\\+b_2^*(\mathbf{p})b_2(\mathbf{p}')\overline{\mathbf{u3}}(\mathbf{p})\mathbf{u3}^T(\mathbf{p}')+\left(b_2(\mathbf{p}')b_2^*(\mathbf{p})\overline{\mathbf{u3}}(\mathbf{p})\mathbf{u3}^T(\mathbf{p}')\right)+\cdots\end{array}\right]\\ e^{-i\left(p_0x_1-p_1x_0+p_2x_3-p_3x_2+\overline{(\mathbf{p},\mathbf{x})}\right)}e^{i\left(p_0'x_1'-p_1'x_0'+p_2'x_3'-p_3'x_2'+\overline{(\mathbf{p}',\mathbf{x}')}\right)}\\+\\\left[\begin{array}{c}d_1^*(\mathbf{p})d_1(\mathbf{p}')\mathbf{u4}(\mathbf{p})\mathbf{u4}^+(\mathbf{p}')+\left(d_1(\mathbf{p}')d_1^*(\mathbf{p})\mathbf{u4}(\mathbf{p})\mathbf{u4}^+(\mathbf{p}')\right)+\cdots\\+d_2^*(\mathbf{p})d_2(\mathbf{p}')\mathbf{u2}(\mathbf{p})\mathbf{u2}^+(\mathbf{p}')+\left(d_2(\mathbf{p}')d_2^*(\mathbf{p})\mathbf{u2}(\mathbf{p})\mathbf{u2}^+(\mathbf{p}')\right)+\cdots\end{array}\right]\\ e^{-i\left(p_0'x_1'-p_1'x_0'+p_2'x_3'-p_3'x_2'+\overline{(\mathbf{p}',\mathbf{x}')}\right)}e^{i\left(p_0x_1-p_1x_0+p_2x_3-p_3x_2+\overline{(\mathbf{p},\mathbf{x})}\right)}\end{array}\right]$$

$$=\int\frac{d^4p}{(2\pi)^4}\left[\begin{array}{c}\left[\begin{array}{c}\mathbf{u1}(\mathbf{p})\mathbf{u1}^+(\mathbf{p})+\cdots\\+\mathbf{u3}(\mathbf{p})\mathbf{u3}^+(\mathbf{p})+\cdots\end{array}\right]\\ e^{i\left(p_0(x_1-x_1')-p_1(x_0-x_0')+p_2(x_3-x_3')-p_3(x_2-x_2')+\overline{(\mathbf{p},\mathbf{x}-\mathbf{x}')}\right)}\\+\\\left[\begin{array}{c}\overline{\mathbf{u4}}(\mathbf{p})\mathbf{u4}^T(\mathbf{p})+\cdots\\+\overline{\mathbf{u2}}(\mathbf{p})\mathbf{u2}^T(\mathbf{p})+\cdots\end{array}\right]\\ e^{-i\left(p_0(x_1-x_1')-p_1(x_0-x_0')+p_2(x_3-x_3')-p_3(x_2-x_2')+\overline{(\mathbf{p},\mathbf{x}-\mathbf{x}')}\right)}\end{array}\right]$$

$$+\int\frac{d^4p}{(2\pi)^4}\left[\begin{array}{c}\left[\begin{array}{c}\overline{\mathbf{u1}}(\mathbf{p})\mathbf{u1}^T(\mathbf{p})+\cdots\\+\overline{\mathbf{u3}}(\mathbf{p})\mathbf{u3}^T(\mathbf{p})+\cdots\end{array}\right]\\ e^{-i\left(p_0(x_1-x_1')-p_1(x_0-x_0')+p_2(x_3-x_3')-p_3(x_2-x_2')+\overline{(\mathbf{p},\mathbf{x}-\mathbf{x}')}\right)}\\+\\\left[\begin{array}{c}\mathbf{u4}(\mathbf{p})\mathbf{u4}^+(\mathbf{p})+\cdots\\+\mathbf{u2}(\mathbf{p})\mathbf{u2}^+(\mathbf{p})+\cdots\end{array}\right]\\ e^{i\left(p_0(x_1-x_1')-p_1(x_0-x_0')+p_2(x_3-x_3')-p_3(x_2-x_2')+\overline{(\mathbf{p},\mathbf{x}-\mathbf{x}')}\right)}\end{array}\right]$$

$$=\int\frac{d^4p}{(2\pi)^4}\left[\begin{array}{c}\left[\begin{array}{c}\mathbf{u1}(\mathbf{p})\mathbf{u1}^+(\mathbf{p})+\mathbf{u3}(\mathbf{p})\mathbf{u3}^+(\mathbf{p})+\\\mathbf{u4}(\mathbf{p})\mathbf{u4}^+(\mathbf{p})+\mathbf{u2}(\mathbf{p})\mathbf{u2}^+(\mathbf{p})+\cdots\end{array}\right]\\ e^{i\left(p_0(x_1-x_1')-p_1(x_0-x_0')+p_2(x_3-x_3')-p_3(x_2-x_2')+\overline{(\mathbf{p},\mathbf{x}-\mathbf{x}')}\right)}\\+\\\left[\begin{array}{c}\overline{\mathbf{u4}}(\mathbf{p})\mathbf{u4}^T(\mathbf{p})+\overline{\mathbf{u2}}(\mathbf{p})\mathbf{u2}^T(\mathbf{p})+\\\overline{\mathbf{u1}}(\mathbf{p})\mathbf{u1}^T(\mathbf{p})+\overline{\mathbf{u3}}(\mathbf{p})\mathbf{u3}^T(\mathbf{p})+\cdots\end{array}\right]\\ e^{-i\left(p_0(x_1-x_1')-p_1(x_0-x_0')+p_2(x_3-x_3')-p_3(x_2-x_2')+\overline{(\mathbf{p},\mathbf{x}-\mathbf{x}')}\right)}\end{array}\right]$$

$$=\int\frac{d^4p}{(2\pi)^4}\left[\begin{array}{c}\left[\begin{array}{c}\mathbf{u1}(\mathbf{p})\mathbf{u1}^+(\mathbf{p})+\mathbf{u2}(\mathbf{p})\mathbf{u2}^+(\mathbf{p})+\mathbf{u3}(\mathbf{p})\mathbf{u3}^+(\mathbf{p})+\mathbf{u4}(\mathbf{p})\mathbf{u4}^+(\mathbf{p})+\\\mathbf{v1}(\mathbf{p})\mathbf{v1}^+(\mathbf{p})+\mathbf{v2}(\mathbf{p})\mathbf{v2}^+(\mathbf{p})+\mathbf{v3}(\mathbf{p})\mathbf{v3}^+(\mathbf{p})+\mathbf{v4}(\mathbf{p})\mathbf{v4}^+(\mathbf{p})\end{array}\right]\\ e^{i\left(p_0(x_1-x_1')-p_1(x_0-x_0')+p_2(x_3-x_3')-p_3(x_2-x_2')+\overline{(\mathbf{p},\mathbf{x}-\mathbf{x}')}\right)}\\+\\\left[\begin{array}{c}\overline{\mathbf{u1}}(\mathbf{p})\mathbf{u1}^+(\mathbf{p})+\overline{\mathbf{u2}}(\mathbf{p})\mathbf{u2}^+(\mathbf{p})+\overline{\mathbf{u3}}(\mathbf{p})\mathbf{u3}^+(\mathbf{p})+\overline{\mathbf{u4}}(\mathbf{p})\mathbf{u4}^+(\mathbf{p})+\\\overline{\mathbf{v1}}(\mathbf{p})\mathbf{v1}^+(\mathbf{p})+\overline{\mathbf{v2}}(\mathbf{p})\mathbf{v2}^+(\mathbf{p})+\overline{\mathbf{v3}}(\mathbf{p})\mathbf{v3}^+(\mathbf{p})+\overline{\mathbf{v4}}(\mathbf{p})\mathbf{v4}^+(\mathbf{p})\end{array}\right]\\ e^{-i\left(p_0(x_1-x_1')-p_1(x_0-x_0')+p_2(x_3-x_3')-p_3(x_2-x_2')+\overline{(\mathbf{p},\mathbf{x}-\mathbf{x}')}\right)}\end{array}\right]$$

$$=\int\frac{d^4p}{(2\pi)^4}\left(T^R(\mathbf{p})+T_R(\mathbf{p})\right)e^{\left(i\left(p_0(x_1-x_1')-p_1(x_0-x_0')+p_2(x_3-x_3')-p_3(x_2-x_2')+\overline{(\mathbf{p},\mathbf{x}-\mathbf{x}')}\right)\right)}$$

$$+\int\frac{d^4p}{(2\pi)^4}\left(\overline{T_R}(\mathbf{p})+\overline{T^R}(\mathbf{p})\right)e^{-\left(i\left(p_0(x_1-x_1')-p_1(x_0-x_0')+p_2(x_3-x_3')-p_3(x_2-x_2')+\overline{(\mathbf{p},\mathbf{x}-\mathbf{x}')}\right)\right)}$$

$$=\int\frac{d^4p}{(2\pi)^4}4\begin{pmatrix}e(\mathbf{p})&0&0&0\\0&e(\mathbf{p})&0&0\\0&0&e(\mathbf{p})&0\\0&0&0&e(\mathbf{p})\end{pmatrix}e^{\left(i\left(p_0(x_1-x_1')-p_1(x_0-x_0')+p_2(x_3-x_3')-p_3(x_2-x_2')\right)+\overline{(\mathbf{p},\mathbf{x}-\mathbf{x}')}\right)}$$

$$+\int\frac{d^4p}{(2\pi)^4}4\begin{pmatrix}e(\mathbf{p})&0&0&0\\0&e(\mathbf{p})&0&0\\0&0&e(\mathbf{p})&0\\0&0&0&e(\mathbf{p})\end{pmatrix}e^{-\left(i\left(p_0(x_1-x_1')-p_1(x_0-x_0')+p_2(x_3-x_3')-p_3(x_2-x_2')+\overline{(\mathbf{p},\mathbf{x}-\mathbf{x}')}\right)\right)}$$

$$=4e(\mathbf{p})I\delta(\mathbf{x}'-\mathbf{x})+4e(\mathbf{p})I\delta(\mathbf{x}-\mathbf{x}')$$

where

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$$T^R(\mathbf{p}) = \mathbf{u1}(\mathbf{p})\mathbf{u1}^+(\mathbf{p}) + \mathbf{u2}(\mathbf{p})\mathbf{u2}^+(\mathbf{p}) + \mathbf{u3}(\mathbf{p})\mathbf{u3}^+(\mathbf{p}) + \mathbf{u4}(\mathbf{p})\mathbf{u4}^+(\mathbf{p})$$

$$T_R(\mathbf{p}) = \mathbf{v1}(\mathbf{p})\mathbf{v1}^+(\mathbf{p}) + \mathbf{v2}(\mathbf{p})\mathbf{v2}^+(\mathbf{p}) + \mathbf{v3}(\mathbf{p})\mathbf{v3}^+(\mathbf{p}) + \mathbf{v4}(\mathbf{p})\mathbf{v4}^+(\mathbf{p})$$

$$T^R(\mathbf{p}) + T_R(\mathbf{p}) + \overline{T^R}(\mathbf{p}) + \overline{T_R}(\mathbf{p}) =$$

$$4(p_0\overline{p_0} + p_1\overline{p_1} + p_2\overline{p_2} + p_3\overline{p_3})\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = 4e(\mathbf{p})I$$

In deriving this result, the following relations are taken into account

$$T^R(\mathbf{p}) + T_R(\mathbf{p}) = \mathbf{u1}(\mathbf{p})\mathbf{u1}^+(\mathbf{p}) + \mathbf{u2}(\mathbf{p})\mathbf{u2}^+(\mathbf{p}) + \mathbf{u3}(\mathbf{p})\mathbf{u3}^+(\mathbf{p}) + \mathbf{u4}(\mathbf{p})\mathbf{u4}^+(\mathbf{p})$$
$$+\mathbf{v1}(\mathbf{p})\mathbf{v1}^+(\mathbf{p}) + \mathbf{v2}(\mathbf{p})\mathbf{v2}^+(\mathbf{p}) + \mathbf{v3}(\mathbf{p})\mathbf{v3}^+(\mathbf{p}) + \mathbf{v4}(\mathbf{p})\mathbf{v4}^+(\mathbf{p}) =$$

$$\begin{pmatrix} -p_3 \\ -p_2 \\ p_1 \\ p_0 \end{pmatrix}(-\overline{p_3}, -\overline{p_2}, \overline{p_1}, \overline{p_0}) + \begin{pmatrix} p_2 \\ -p_3 \\ p_0 \\ -p_1 \end{pmatrix}(\overline{p_2}, -\overline{p_3}, \overline{p_0}, -\overline{p_1})$$

$$+\begin{pmatrix} -p_1 \\ -p_0 \\ p_3 \\ p_2 \end{pmatrix}(-\overline{p_1}, -\overline{p_0}, \overline{p_3}, \overline{p_2}) + \begin{pmatrix} p_0 \\ -p_1 \\ p_2 \\ -p_3 \end{pmatrix}(\overline{p_0}, -\overline{p_1}, \overline{p_2}, -\overline{p_3}) +$$

$$\begin{pmatrix} p_1 \\ p_0 \\ p_3 \\ p_2 \end{pmatrix}(\overline{p_1}, \overline{p_0}, \overline{p_3}, \overline{p_2}) + \begin{pmatrix} p_0 \\ -p_1 \\ -p_2 \\ p_3 \end{pmatrix}(\overline{p_0}, -\overline{p_1}, -\overline{p_2}, \overline{p_3})$$

$$+\begin{pmatrix} p_3 \\ p_2 \\ p_1 \\ p_0 \end{pmatrix}(\overline{p_3}, \overline{p_2}, \overline{p_1}, \overline{p_0}) + \begin{pmatrix} p_2 \\ -p_3 \\ -p_0 \\ p_1 \end{pmatrix}(\overline{p_2}, -\overline{p_3}, -\overline{p_0}, \overline{p_1}) =$$

$$\begin{pmatrix} p_3\overline{p_3} & p_3\overline{p_2} & -p_3\overline{p_1} & -p_3\overline{p_0} \\ p_2\overline{p_3} & p_2\overline{p_2} & -p_2\overline{p_1} & -p_2\overline{p_0} \\ -p_1\overline{p_3} & -p_1\overline{p_2} & p_1\overline{p_1} & p_1\overline{p_0} \\ -p_0\overline{p_3} & -p_0\overline{p_2} & p_0\overline{p_1} & p_0\overline{p_0} \end{pmatrix} + \begin{pmatrix} p_2\overline{p_2} & -p_2\overline{p_3} & p_2\overline{p_0} & -p_2\overline{p_1} \\ -p_3\overline{p_2} & p_3\overline{p_3} & -p_3\overline{p_0} & p_3\overline{p_1} \\ p_0\overline{p_2} & -p_0\overline{p_3} & p_0\overline{p_0} & -p_0\overline{p_1} \\ -p_1\overline{p_2} & p_1\overline{p_3} & -p_1\overline{p_0} & p_1\overline{p_1} \end{pmatrix}$$

$$+\begin{pmatrix} p_1\overline{p_1} & p_1\overline{p_0} & -p_1\overline{p_3} & -p_1\overline{p_2} \\ p_0\overline{p_1} & p_0\overline{p_0} & -p_0\overline{p_3} & -p_0\overline{p_2} \\ -p_3\overline{p_1} & -p_3\overline{p_0} & p_3\overline{p_3} & p_3\overline{p_2} \\ -p_2\overline{p_1} & -p_2\overline{p_0} & p_2\overline{p_3} & p_2\overline{p_2} \end{pmatrix} + \begin{pmatrix} p_0\overline{p_0} & -p_0\overline{p_1} & p_0\overline{p_2} & -p_0\overline{p_3} \\ -p_1\overline{p_0} & p_1\overline{p_1} & -p_1\overline{p_2} & p_1\overline{p_3} \\ p_2\overline{p_0} & -p_2\overline{p_1} & p_2\overline{p_2} & -p_2\overline{p_3} \\ -p_3\overline{p_0} & p_3\overline{p_1} & -p_3\overline{p_2} & p_3\overline{p_3} \end{pmatrix}$$

$$+\begin{pmatrix} p_1\overline{p_1} & p_1\overline{p_0} & p_1\overline{p_3} & p_1\overline{p_2} \\ p_0\overline{p_1} & p_0\overline{p_0} & p_0\overline{p_3} & p_0\overline{p_2} \\ p_3\overline{p_1} & p_3\overline{p_0} & p_3\overline{p_3} & p_3\overline{p_2} \\ p_2\overline{p_1} & p_2\overline{p_0} & p_2\overline{p_3} & p_2\overline{p_2} \end{pmatrix} + \begin{pmatrix} p_0\overline{p_0} & -p_0\overline{p_1} & -p_0\overline{p_2} & p_0\overline{p_3} \\ -p_1\overline{p_0} & p_1\overline{p_1} & -p_1\overline{p_2} & -p_1\overline{p_3} \\ -p_2\overline{p_0} & p_2\overline{p_1} & p_2\overline{p_2} & -p_2\overline{p_3} \\ p_3\overline{p_0} & -p_3\overline{p_1} & -p_3\overline{p_2} & p_3\overline{p_3} \end{pmatrix}$$

$$+\begin{pmatrix} p_3\overline{p_3} & p_3\overline{p_2} & p_3\overline{p_1} & p_3\overline{p_0} \\ p_2\overline{p_3} & p_2\overline{p_2} & p_2\overline{p_1} & p_2\overline{p_0} \\ p_1\overline{p_3} & p_1\overline{p_2} & p_1\overline{p_1} & p_1\overline{p_0} \\ p_0\overline{p_3} & p_0\overline{p_2} & p_0\overline{p_1} & p_0\overline{p_0} \end{pmatrix} + \begin{pmatrix} p_2\overline{p_2} & -p_2\overline{p_3} & -p_2\overline{p_0} & p_2\overline{p_1} \\ -p_3\overline{p_2} & p_3\overline{p_3} & p_3\overline{p_0} & -p_3\overline{p_1} \\ -p_0\overline{p_2} & p_0\overline{p_3} & p_0\overline{p_0} & -p_0\overline{p_1} \\ p_1\overline{p_2} & -p_1\overline{p_3} & -p_1\overline{p_0} & p_1\overline{p_1} \end{pmatrix}$$

$$= \begin{pmatrix} p_3\overline{p_3} & p_3\overline{p_2} & 0 & 0 \\ p_2\overline{p_3} & p_2\overline{p_2} & 0 & 0 \\ 0 & 0 & p_1\overline{p_1} & p_1\overline{p_0} \\ 0 & 0 & p_0\overline{p_1} & p_0\overline{p_0} \end{pmatrix} + \begin{pmatrix} p_2\overline{p_2} & -p_2\overline{p_3} & 0 & 0 \\ -p_3\overline{p_2} & p_3\overline{p_3} & 0 & 0 \\ 0 & 0 & p_0\overline{p_0} & -p_0\overline{p_1} \\ 0 & 0 & -p_1\overline{p_0} & p_1\overline{p_1} \end{pmatrix}$$

$$+\begin{pmatrix} p_1\overline{p_1} & p_1\overline{p_0} & 0 & 0 \\ p_0\overline{p_1} & p_0\overline{p_0} & 0 & 0 \\ 0 & 0 & p_3\overline{p_3} & p_3\overline{p_2} \\ 0 & 0 & p_2\overline{p_3} & p_2\overline{p_2} \end{pmatrix} + \begin{pmatrix} p_0\overline{p_0} & -p_0\overline{p_1} & 0 & 0 \\ -p_1\overline{p_0} & p_1\overline{p_1} & 0 & 0 \\ 0 & 0 & p_2\overline{p_2} & -p_2\overline{p_3} \\ 0 & 0 & -p_3\overline{p_2} & p_3\overline{p_3} \end{pmatrix}$$

$$+\begin{pmatrix} p_1\overline{p_1} & p_1\overline{p_0} & 0 & 0 \\ p_0\overline{p_1} & p_0\overline{p_0} & 0 & 0 \\ 0 & 0 & p_3\overline{p_3} & p_3\overline{p_2} \\ 0 & 0 & p_2\overline{p_3} & p_2\overline{p_2} \end{pmatrix} + \begin{pmatrix} p_0\overline{p_0} & -p_0\overline{p_1} & 0 & 0 \\ -p_1\overline{p_0} & p_1\overline{p_1} & 0 & 0 \\ 0 & 0 & p_2\overline{p_2} & -p_2\overline{p_3} \\ 0 & 0 & -p_3\overline{p_2} & p_3\overline{p_3} \end{pmatrix}$$

$$+\begin{pmatrix} p_3\overline{p_3} & p_3\overline{p_2} & 0 & 0 \\ p_2\overline{p_3} & p_2\overline{p_2} & 0 & 0 \\ 0 & 0 & p_1\overline{p_1} & p_1\overline{p_0} \\ 0 & 0 & p_0\overline{p_1} & p_0\overline{p_0} \end{pmatrix} + \begin{pmatrix} p_2\overline{p_2} & -p_2\overline{p_3} & 0 & 0 \\ -p_3\overline{p_2} & p_3\overline{p_3} & 0 & 0 \\ 0 & 0 & p_0\overline{p_0} & -p_0\overline{p_1} \\ 0 & 0 & -p_1\overline{p_0} & p_1\overline{p_1} \end{pmatrix}$$

$$T^R(\mathbf{p}) + T_R(\mathbf{p}) + \overline{T^R}(\mathbf{p}) + \overline{T_R}(\mathbf{p}) =$$

$$4(p_0\overline{p_0} + p_1\overline{p_1} + p_2\overline{p_2} + p_3\overline{p_3})\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = 4e(\mathbf{p})I$$

The last operation of taking the value $(p_0\overline{p_0} + p_1\overline{p_1} + p_2\overline{p_2} + p_3\overline{p_3})$ out from under the sign of the integral seems doubtful because of its dependence on the momentum over which the integration is performed. If one closes one's eyes to this, as is generally accepted in the literature, in particular in [9], this relation is taken to be interpreted as a proof of the anti-symmetry of the fermion wave function under the stipulated anticommutation relations. The only situation where this is unquestionably true is when considering in a rest system where boosts are excluded, energy is equal to mass, and invariant to rotations.

It is noteworthy that the antisymmetric treatment, whether or not complex conjugation is considered, yields a diagonal matrix that is invariant in one case but not in the other. It is encouraging to observe that the set of reference spinors remain consistent.

It is crucial to note that the proposed invariant approach cannot be realized within the Minkowski vector space. To achieve this, it is necessary to transition to the spinor space. This reiterates the secondary role of the Minkowski space in comparison to the spinor space.

Dirac's equation can be expressed in both spinor and vector spaces, a fact that led Dirac to discover it. In contrast, the invariant equation can be written in spinor space but not in vector space, which explains why it was unknown.

Let us write down the propagator of the fermionic field and the fermionic field invariant equation of motion using the proposed matrices

$$S^R(\mathbf{p}) = \begin{pmatrix} -p_3 \\ -p_2 \\ p_1 \\ p_0 \end{pmatrix}(p_0, -p_1, p_2, -p_3) - \begin{pmatrix} -p_1 \\ -p_0 \\ p_3 \\ p_2 \end{pmatrix}(p_2, -p_3, p_0, -p_1)$$

$$+\begin{pmatrix} p_1 \\ p_0 \\ p_3 \\ p_2 \end{pmatrix}(p_2, -p_3, -p_0, p_1) - \begin{pmatrix} p_3 \\ p_2 \\ p_1 \\ p_0 \end{pmatrix}(p_0, -p_1, -p_2, p_3)$$

$$S_R(\mathbf{p}) = \begin{pmatrix} p_0 \\ -p_1 \\ p_2 \\ -p_3 \end{pmatrix}(-p_3, -p_2, p_1, p_0) - \begin{pmatrix} p_2 \\ -p_3 \\ p_0 \\ -p_1 \end{pmatrix}(-p_1, -p_0, p_3, p_2)$$

$$+\begin{pmatrix} p_2 \\ -p_3 \\ -p_0 \\ p_1 \end{pmatrix}(p_1, p_0, p_3, p_2) - \begin{pmatrix} p_0 \\ -p_1 \\ -p_2 \\ p_3 \end{pmatrix}(p_3, p_2, p_1, p_0)$$

The equation of motion has the form

$$\left(S^R + \overline{S^R} + S_R + \overline{S_R} - 4(m + \overline{m})I\right)\boldsymbol{\varphi}(\mathbf{x}) = 0$$

where

$$p_0 \to \frac{\partial}{\partial x_1} \equiv \partial_1 \quad p_1 \to -\frac{\partial}{\partial x_0} \equiv -\partial_0 \quad p_2 \to \frac{\partial}{\partial x_3} \equiv \partial_3 \quad p_3 \to -\frac{\partial}{\partial x_2} \equiv -\partial_2$$

$$\overline{p_0} \to \frac{\partial[\,]}{\partial x_1} \equiv \overline{\partial_1} \quad \overline{p_1} \to -\frac{\partial[\,]}{\partial x_0} \equiv -\overline{\partial_0} \quad \overline{p_2} \to \frac{\partial[\,]}{\partial x_3} \equiv \overline{\partial_3} \quad \overline{p_3} \to -\frac{\partial[\,]}{\partial x_2} \equiv -\overline{\partial_2}$$

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$$S^R=\begin{pmatrix}\partial_2\\-\partial_3\\-\partial_0\\\partial_1\end{pmatrix}(\partial_1,\partial_0,\partial_3,\partial_2)-\begin{pmatrix}\partial_0\\-\partial_1\\-\partial_2\\\partial_3\end{pmatrix}(\partial_3,\partial_2,\partial_1,\partial_0)$$

$$+\begin{pmatrix}-\partial_0\\\partial_1\\-\partial_2\\\partial_3\end{pmatrix}(\partial_3,\partial_2,-\partial_1,-\partial_0)-\begin{pmatrix}-\partial_2\\\partial_3\\-\partial_0\\\partial_1\end{pmatrix}(\partial_1,\partial_0,-\partial_3,-\partial_2)$$

$$S_R=\begin{pmatrix}\partial_1\\\partial_0\\\partial_3\\\partial_2\end{pmatrix}(\partial_2,-\partial_3,-\partial_0,\partial_1)-\begin{pmatrix}\partial_3\\\partial_2\\\partial_1\\\partial_0\end{pmatrix}(\partial_0,-\partial_1,-\partial_2,\partial_3)$$

$$+\begin{pmatrix}\partial_3\\\partial_2\\-\partial_1\\-\partial_0\end{pmatrix}(-\partial_0,\partial_1,-\partial_2,\partial_3)-\begin{pmatrix}\partial_1\\\partial_0\\-\partial_3\\-\partial_2\end{pmatrix}(-\partial_2,\partial_3,-\partial_0,\partial_1)$$

The equation is relativistically invariant, respectively we can use the invariant Lagrangian

$$\mathcal{L}=\frac{1}{2}\left[\boldsymbol{\varphi}(\mathbf{x})^T\left(S^R+\overline{S^R}+S_R+\overline{S_R}\right)\boldsymbol{\varphi}(\mathbf{x})-4(m+\overline{m})\boldsymbol{\varphi}(\mathbf{x})^T\boldsymbol{\varphi}(\mathbf{x})\right]$$

to which corresponds the relativistically invariant fermion propagator

$$\mathbf{D}^R(\mathbf{x})=\int\frac{d^4p}{(2\pi)^4}\frac{S^R(\mathbf{p})+\overline{S^R}(\mathbf{p})+S_R(\mathbf{p})+\overline{S_R}(\mathbf{p})+4(m+\overline{m})I}{P^2-m^2}e^{i(p_0x_1-p_1x_0+p_2x_3-p_3x_2+\overline{(\mathbf{p},\mathbf{x})})}$$

The equation can be modified to take into account the electromagnetic potential, the electron charge is taken as a unit

$$p_0\to\partial_1+a_0\qquad p_1\to-\partial_0+a_1\qquad p_2\to\partial_3+a_2\qquad p_3\to-\partial_2+a_3$$

$$S^R=\begin{pmatrix}-(-\partial_2+a_3)\\-(\partial_3+a_2)\\(-\partial_0+a_1)\\(\partial_1+a_0)\end{pmatrix}\big((\partial_1+a_0),-(-\partial_0+a_1),(\partial_3+a_2),-(-\partial_2+a_3)\big)$$

$$-\begin{pmatrix}-(-\partial_0+a_1)\\-(\partial_1+a_0)\\(-\partial_2+a_3)\\(\partial_3+a_2)\end{pmatrix}\big((\partial_3+a_2),-(-\partial_2+a_3),(\partial_1+a_0),-(-\partial_0+a_1)\big)$$

$$+\begin{pmatrix}(-\partial_0+a_1)\\(\partial_1+a_0)\\(-\partial_2+a_3)\\(\partial_3+a_2)\end{pmatrix}\big((\partial_3+a_2),-(-\partial_2+a_3),-(\partial_1+a_0),(-\partial_0+a_1)\big)$$

$$-\begin{pmatrix}(-\partial_2+a_3)\\(\partial_3+a_2)\\(-\partial_0+a_1)\\(\partial_1+a_0)\end{pmatrix}\big((\partial_1+a_0),-(-\partial_0+a_1),-(\partial_3+a_2),(-\partial_2+a_3)\big)$$

$$S_R=\begin{pmatrix}(\partial_1+a_0)\\-(-\partial_0+a_1)\\(\partial_3+a_2)\\-(-\partial_2+a_3)\end{pmatrix}\big(-(-\partial_2+a_3),-(\partial_3+a_2),(-\partial_0+a_1),(\partial_1+a_0)\big)$$

$$-\begin{pmatrix}(\partial_3+a_2)\\-(-\partial_2+a_3)\\(\partial_1+a_0)\\-(-\partial_0+a_1)\end{pmatrix}(-(-\partial_0+a_1),-(\partial_1+a_0),(-\partial_2+a_3),(\partial_3+a_2))$$

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$$+\begin{pmatrix} (\partial_3 + a_2) \\ -(-\partial_2 + a_3) \\ -(\partial_1 + a_0) \\ (-\partial_0 + a_1) \end{pmatrix} \left((-\partial_0 + a_1), (\partial_1 + a_0), (-\partial_2 + a_3), (\partial_3 + a_2)\right)$$

$$-\begin{pmatrix} (\partial_1 + a_0) \\ -(-\partial_0 + a_1) \\ -(\partial_3 + a_2) \\ (-\partial_2 + a_3) \end{pmatrix} \left((-\partial_2 + a_3), (\partial_3 + a_2), (-\partial_0 + a_1), (\partial_1 + a_0)\right)$$

and apply, in particular, to analyze the emission spectrum of the hydrogen-like atom.

Let us look for a representation of the electromagnetic field operator in vector space without first referring to spinor space. Let us define four vectors expressed through the components of the momentum vector

$$\mathbf{U1} = \begin{pmatrix} P_0 \\ -P_1 \\ P_2 \\ P_3 \end{pmatrix} \quad \mathbf{U4} = \begin{pmatrix} P_0 \\ -P_1 \\ -P_2 \\ P_3 \end{pmatrix} \quad \mathbf{V1} = \begin{pmatrix} P_0 \\ P_1 \\ -P_2 \\ -P_3 \end{pmatrix} \quad \mathbf{V4} = \begin{pmatrix} P_0 \\ P_1 \\ P_2 \\ -P_3 \end{pmatrix}$$

Why we have chosen these 4 vectors out of 8 possible combinations of signs of three spatial components? Because they are represented in the previously given table of variants of spinor particles. For these vectors the following relations are valid

$$\mathbf{V1} * \mathbf{U1}^T - \mathbf{U1} * \mathbf{V1}^T + \mathbf{V4} * \mathbf{V1}^T - \mathbf{V1} * \mathbf{V4}^T +$$

$$\mathbf{U4} * \mathbf{V4}^T - \mathbf{V4} * \mathbf{U4}^T + \mathbf{U1} * \mathbf{U4}^T - \mathbf{U4} * \mathbf{U1}^T = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\mathbf{V1} * \mathbf{U1}^T + \mathbf{U1} * \mathbf{V1}^T + \mathbf{V4} * \mathbf{V1}^T + \mathbf{V1} * \mathbf{V4}^T +$$

$$\mathbf{U4} * \mathbf{V4}^T + \mathbf{V4} * \mathbf{U4}^T + \mathbf{U1} * \mathbf{U4}^T + \mathbf{U4} * \mathbf{U1}^T = \begin{pmatrix} 8P_0^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -8P_2^2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\mathbf{V1} * \mathbf{U1}^T + \mathbf{V4} * \mathbf{V1}^T + \mathbf{U4} * \mathbf{V4}^T + \mathbf{U1} * \mathbf{U4}^T = \begin{pmatrix} 4P_0^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -4P_2^2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\mathbf{U1} * \mathbf{V1}^T + \mathbf{V1} * \mathbf{V4}^T + \mathbf{V4} * \mathbf{U4}^T + \mathbf{U4} * \mathbf{U1}^T = \begin{pmatrix} 4P_0^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -4P_2^2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$(\mathbf{U1} * \mathbf{U1}^T + \mathbf{U4} * \mathbf{U4}^T + \mathbf{V1} * \mathbf{V1}^T + \mathbf{V4} * \mathbf{V4}^T) +$$

$$(\mathbf{U1} * \mathbf{V1}^T + \mathbf{V1} * \mathbf{U1}^T + \mathbf{V4} * \mathbf{U4}^T + \mathbf{U4} * \mathbf{V4}^T) = \begin{pmatrix} 8P_0^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$8P_0^2 = \mathbf{U1}^T * \mathbf{U1} + \mathbf{U4}^T * \mathbf{U4} + \mathbf{V1}^T * \mathbf{V1} + \mathbf{V4}^T * \mathbf{V4} + 4M^2$$

$$(\mathbf{U1}^T * \mathbf{U1} + \mathbf{U4}^T * \mathbf{U4} + \mathbf{V1}^T * \mathbf{V1} + \mathbf{V4}^T * \mathbf{V4}) +$$
$$+(\mathbf{U1}^T * \mathbf{V1} + \mathbf{V1}^T * \mathbf{U1} + \mathbf{V4}^T * \mathbf{U4} + \mathbf{U4}^T * \mathbf{V4}) = 8P_0^2$$

$$\mathbf{U1}^T * \mathbf{V1} + \mathbf{V1}^T * \mathbf{U1} + \mathbf{V4}^T * \mathbf{U4} + \mathbf{U4}^T * \mathbf{V4} = 4M^2$$

$$\mathbf{U1}^T * \mathbf{U1} + \mathbf{U4}^T * \mathbf{U4} + \mathbf{V1}^T * \mathbf{V1} + \mathbf{V4}^T * \mathbf{V4} = 8P_0^2 - 4M^2$$

$$\mathbf{U1}^T * \mathbf{V1} = \mathbf{V1}^T * \mathbf{U1} = \mathbf{V4}^T * \mathbf{U4} = \mathbf{U4}^T * \mathbf{V4} = M^2$$

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$$\mathbf{U1}^{\boldsymbol{T}} * \mathbf{U1} = \mathbf{U4}^{\boldsymbol{T}} * \mathbf{U4} = \mathbf{V1}^{\boldsymbol{T}} * \mathbf{V1} = \mathbf{V4}^{\boldsymbol{T}} * \mathbf{V4} = \mathbf{P}^{\boldsymbol{T}} * \mathbf{P} = 2P_0^2 - M^2$$

$$\mathbf{U1}^{\boldsymbol{T}} G\mathbf{U1} = \mathbf{U4}^{\boldsymbol{T}} G\mathbf{U4} = \mathbf{V1}^{\boldsymbol{T}} G\mathbf{V1} = \mathbf{V4}^{\boldsymbol{T}} G\mathbf{V4} = M^2$$

$$M^2 = \mathbf{P}^{\boldsymbol{T}} G\mathbf{P} \equiv (\mathbf{P}, \mathbf{P})$$

$$G = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

$$(\mathbf{U1} - \mathbf{U4}) = \begin{pmatrix} 0 \\ 0 \\ 2P_2 \\ 0 \end{pmatrix} \quad (\mathbf{V1} - \mathbf{V4}) = \begin{pmatrix} 0 \\ 0 \\ -2P_2 \\ 0 \end{pmatrix}$$

$$(\mathbf{U1} + \mathbf{V1}) = \begin{pmatrix} 2P_0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad (\mathbf{U4} + \mathbf{V4}) = \begin{pmatrix} 2P_0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Let us decompose the field into plane waves with operator coefficients and let's find the commutation relations for them. We will use the next notation for the scalar product of vectors

$$(\mathbf{P}, \mathbf{X}) \equiv \mathbf{P}^{\boldsymbol{T}} G\mathbf{X}$$

$$\boldsymbol{\varphi}(\mathbf{X}) = \int \frac{d^4P}{(2\pi)^4}$$

$$\begin{bmatrix} d_1(\mathbf{P})\mathbf{V1}(\mathbf{P}) + b_1(\mathbf{P})\mathbf{U4}(\mathbf{P}) \\ +d_4(\mathbf{P})\mathbf{U1}(\mathbf{P}) + b_4(\mathbf{P})\mathbf{V4}(\mathbf{P}) \end{bmatrix} e^{i(\mathbf{P},\mathbf{X})}$$

$$+$$

$$\begin{bmatrix} b_4^*(\mathbf{P})\mathbf{V1}(\mathbf{P}) + d_4^*(\mathbf{p})\mathbf{U4}(\mathbf{P}) \\ +d_1^*(\mathbf{P})\mathbf{U1}(\mathbf{P}) + b_1^*(\mathbf{P})\mathbf{V4}(\mathbf{p}) \end{bmatrix} e^{-i(\mathbf{P},\mathbf{X})}$$

$$\boldsymbol{\varphi}(\mathbf{X}') = \int \frac{d^4P'}{(2\pi)^4}$$

$$\begin{bmatrix} d_1(\mathbf{P}')\mathbf{V1}(\mathbf{P}') + b_1(\mathbf{P}')\mathbf{U4}(\mathbf{P}') \\ +d_4(\mathbf{P}')\mathbf{U1}(\mathbf{P}') + b_4(\mathbf{P}')\mathbf{V4}(\mathbf{P}') \end{bmatrix} e^{i(\mathbf{P}',\mathbf{X}')}$$

$$+$$

$$\begin{bmatrix} b_4^*(\mathbf{P}')\mathbf{V1}(\mathbf{P}') + d_4^*(\mathbf{P}')\mathbf{U4}(\mathbf{P}') \\ +d_1^*(\mathbf{P}')\mathbf{U1}(\mathbf{P}') + b_1^*(\mathbf{P}')\mathbf{V4}(\mathbf{P}') \end{bmatrix} e^{-i(\mathbf{P}',\mathbf{X}')}$$

$$[\varphi_i(\mathbf{X}), \varphi_j(\mathbf{X}')] = \varphi_i(\mathbf{X})\varphi_j(\mathbf{X}') - \varphi_j(\mathbf{X}')\varphi_i(\mathbf{X}) = \left(\boldsymbol{\varphi}(\mathbf{X})\boldsymbol{\varphi}^T(\mathbf{X}') - \left(\boldsymbol{\varphi}(\mathbf{X}')\boldsymbol{\varphi}^T(\mathbf{X})\right)^T\right)_{ij}$$

$$\boldsymbol{\varphi}(\mathbf{X})\boldsymbol{\varphi}^T(\mathbf{X}') - \left(\boldsymbol{\varphi}(\mathbf{X}')\boldsymbol{\varphi}^T(\mathbf{X})\right)^T =$$

$$= \iint \frac{d^4P}{(2\pi)^4}\frac{d^4P'}{(2\pi)^4}$$

$$\begin{bmatrix} \left(d_1(\mathbf{P})\mathbf{V1}(\mathbf{P})e^{i(\mathbf{P},\mathbf{X})}\right)\left(d_1^*(\mathbf{P}')\mathbf{U1}(\mathbf{P}')e^{-i(\mathbf{P}',\mathbf{X}')}\right)^{\boldsymbol{T}} - \left(\left(d_1(\mathbf{P}')\mathbf{V1}(\mathbf{P}')e^{i(\mathbf{P}',\mathbf{X}')}\right)\left(d_1^*(\mathbf{P})\mathbf{U1}(\mathbf{P})e^{-i(\mathbf{P},\mathbf{X})}\right)^{\boldsymbol{T}}\right)^{T} \\ +\left(b_1(\mathbf{P})\mathbf{U4}(\mathbf{P})e^{i(\mathbf{P},\mathbf{X})}\right)\left(b_1^*(\mathbf{P}')\mathbf{V4}(\mathbf{P}')e^{-i(\mathbf{P}',\mathbf{X}')}\right)^{\boldsymbol{T}} - \left(\left(b_1(\mathbf{P}')\mathbf{U4}(\mathbf{P}')e^{i(\mathbf{P}',\mathbf{X}')}\right)\left(b_1^*(\mathbf{P})\mathbf{v4}(\mathbf{P})e^{-i(\mathbf{P},\mathbf{X})}\right)^{\boldsymbol{T}}\right)^{\boldsymbol{T}} \\ +\left(b_4(\mathbf{P})\mathbf{V4}(\mathbf{P})e^{i(\mathbf{P},\mathbf{X})}\right)\left(b_4^*(\mathbf{P}')\mathbf{V1}(\mathbf{P}')e^{-i(\mathbf{P}',\mathbf{X}')}\right)^{\boldsymbol{T}} - \left(\left(b_4(\mathbf{P}')\mathbf{V4}(\mathbf{P}')e^{i(\mathbf{P}',\mathbf{X}')}\right)\left(b_4^*(\mathbf{P})\mathbf{V1}(\mathbf{P})e^{-i(\mathbf{P},\mathbf{X})}\right)^{\boldsymbol{T}}\right)^{\boldsymbol{T}} \\ +\left(d_4(\mathbf{P})\mathbf{U1}(\mathbf{P})e^{i(\mathbf{P},\mathbf{X})}\right)\left(d_4^*(\mathbf{P}')\mathbf{U4}(\mathbf{P}')e^{-i(\mathbf{P}',\mathbf{X}')}\right)^{\boldsymbol{T}} - \left(\left(d_4(\mathbf{P}')\mathbf{U1}(\mathbf{P}')e^{i(\mathbf{P}',\mathbf{X}')}\right)\left(d_4^*(\mathbf{P})\mathbf{U4}(\mathbf{P})e^{-i(\mathbf{P},\mathbf{X})}\right)^{\boldsymbol{T}}\right)^{T} \end{bmatrix}$$

$$+\iint \frac{d^4P}{(2\pi)^4}\frac{d^4P'}{(2\pi)^4}$$

$$\begin{bmatrix} \left(b_4^*(\mathbf{P})\mathbf{V1}(\mathbf{P})e^{-i(\mathbf{P},\mathbf{X})}\right)\left(b_4(\mathbf{P}')\mathbf{V4}(\mathbf{P}')e^{i(\mathbf{P}',\mathbf{X}')}\right)^{T} - \left(\left(b_4^*(\mathbf{P}')\mathbf{V1}(\mathbf{P}')e^{-i(\mathbf{P}',\mathbf{X}')}\right)\left(b_4(\mathbf{P})\mathbf{V4}(\mathbf{P})e^{i(\mathbf{P},\mathbf{X})}\right)^{T}\right)^{T} \\ +\left(d_4^*(\mathbf{P})\mathbf{U4}(\mathbf{P})e^{-i(\mathbf{P},\mathbf{X})}\right)\left(d_4(\mathbf{P}')\mathbf{U1}(\mathbf{P}')e^{i(\mathbf{P}',\mathbf{X}')}\right)^{T} - \left(\left(d_4^*(\mathbf{P}')\mathbf{U4}(\mathbf{P}')e^{-i(\mathbf{P}',\mathbf{X}')}\right)\left(d_4(\mathbf{P})\mathbf{U1}(\mathbf{P})e^{i(\mathbf{P},\mathbf{X})}\right)^{T}\right)^{T} \\ +\left(d_1^*(\mathbf{P})\mathbf{U1}(\mathbf{P})e^{-i(\mathbf{P},\mathbf{X})}\right)\left(d_1(\mathbf{P}')\mathbf{V1}(\mathbf{P}')e^{i(\mathbf{P}',\mathbf{X}')}\right)^{T} - \left(\left(d_1^*(\mathbf{P}')\mathbf{U1}(\mathbf{P}')e^{-i(\mathbf{P}',\mathbf{X}')}\right)\left(d_1(\mathbf{P})\mathbf{V1}(\mathbf{P})e^{i(\mathbf{P},\mathbf{X})}\right)^{T}\right)^{T} \\ +\left(b_1^*(\mathbf{P})\mathbf{V4}(\mathbf{P})e^{-i(\mathbf{P},\mathbf{X})}\right)\left(b_1(\mathbf{P}')\mathbf{U4}(\mathbf{P}')e^{i(\mathbf{P}',\mathbf{X}')}\right)^{T} - \left(\left(b_1^*(\mathbf{P}')\mathbf{V4}(\mathbf{P}')e^{-i(\mathbf{P}',\mathbf{X}')}\right)\left(b_1(\mathbf{P})\mathbf{U4}(\mathbf{P})e^{i(\mathbf{P},\mathbf{X})}\right)^{T}\right)^{T} \end{bmatrix}$$

$$=\iint \frac{d^4P}{(2\pi)^4}\frac{d^4P'}{(2\pi)^4}$$

$$\begin{bmatrix} d_1(\mathbf{P})d_1^*(\mathbf{P}')\mathbf{V1}(\mathbf{P})\mathbf{U1}^T(\mathbf{P}')e^{i(\mathbf{P},\mathbf{X})}e^{-i(\mathbf{P}',\mathbf{X}')} - d_1(\mathbf{P}')d_1^*(\mathbf{P})\mathbf{U1}(\mathbf{P})\mathbf{V1}^T(\mathbf{P}')e^{i(\mathbf{P}',\mathbf{X}')}e^{-i(\mathbf{P},\mathbf{X})} \\ + b_1(\mathbf{P})b_1^*(\mathbf{P}')\mathbf{U4}(\mathbf{P})\mathbf{V4}^T(\mathbf{P}')e^{i(\mathbf{P},\mathbf{X})}e^{-i(\mathbf{P}',\mathbf{X}')} - b_1(\mathbf{P}')b_1^*(\mathbf{P})\mathbf{V4}(\mathbf{P})\mathbf{U4}^T(\mathbf{P}')e^{i(\mathbf{P}',\mathbf{X}')}e^{-i(\mathbf{P},\mathbf{X})} \\ +b_4(\mathbf{P})b_4^*(\mathbf{P}')\mathbf{V4}(\mathbf{P})\mathbf{V1}^T(\mathbf{P}')e^{i(\mathbf{P},\mathbf{X})}e^{-i(\mathbf{P}',\mathbf{X}')} - b_4(\mathbf{P}')b_4^*(\mathbf{P})\mathbf{V1}(\mathbf{P})\mathbf{V4}^T(\mathbf{P}')e^{i(\mathbf{P}',\mathbf{X}')}e^{-i(\mathbf{P},\mathbf{X})} \\ +d_4(\mathbf{P})d_4^*(\mathbf{P}')\mathbf{U1}(\mathbf{P})\mathbf{U4}^T(\mathbf{P}')e^{i(\mathbf{P},\mathbf{X})}e^{-i(\mathbf{P}',\mathbf{X}')} - d_4(\mathbf{P}')(\mathbf{P})\mathbf{U4}(\mathbf{P})\mathbf{U1}^T(\mathbf{P}')e^{i(\mathbf{P}',\mathbf{X}')}e^{-i(\mathbf{P},\mathbf{X})} \end{bmatrix}$$

$$+\iint \frac{d^4P}{(2\pi)^4}\frac{d^4P'}{(2\pi)^4}$$

$$\begin{bmatrix} b_4^*(\mathbf{P})b_4(\mathbf{P}')\mathbf{V1}(\mathbf{P})\mathbf{V4}^T(\mathbf{P}')e^{-i(\mathbf{P},\mathbf{X})}e^{i(\mathbf{P}',\mathbf{X}')} - b_4^*(\mathbf{P}')b_4(\mathbf{P})\mathbf{V4}(\mathbf{P})\mathbf{V1}^T(\mathbf{P}')\, e^{-i(\mathbf{P}',\mathbf{X}')}e^{i(\mathbf{P},\mathbf{X})} \\ +d_4^*(\mathbf{P})d_4(\mathbf{P}')\mathbf{U4}(\mathbf{P})\mathbf{U1}^T(\mathbf{P}')e^{-i(\mathbf{P},\mathbf{X})}e^{i(\mathbf{P}',\mathbf{X}')} - d_4^*(\mathbf{P}')d_4(\mathbf{P})\mathbf{U1}(\mathbf{P})\mathbf{U4}^T(\mathbf{P}')e^{-i(\mathbf{P}',\mathbf{X}')}e^{i(\mathbf{P},\mathbf{X})} \\ +d_1^*(\mathbf{P})d_1(\mathbf{P}')\mathbf{U1}(\mathbf{P})\mathbf{V1}^T(\mathbf{P}')e^{-i(\mathbf{P},\mathbf{X})}e^{i(\mathbf{P}',\mathbf{X}')} - d_1^*(\mathbf{P}')d_1(\mathbf{P})\mathbf{V1}(\mathbf{P})\mathbf{U1}^T(\mathbf{P}')e^{-i(\mathbf{P}',\mathbf{X}')}e^{i(\mathbf{P},\mathbf{X})} \\ +b_1^*(\mathbf{P})b_1(\mathbf{P}')\mathbf{V4}(\mathbf{P})\mathbf{U4}^T(\mathbf{P}')e^{-i(\mathbf{P},\mathbf{X})}e^{i(\mathbf{P}',\mathbf{X}')} - b_1^*(\mathbf{P}')b_1(\mathbf{P})\mathbf{U4}(\mathbf{P})\mathbf{V4}^T(\mathbf{P}')e^{-i(\mathbf{P}',\mathbf{X}')}e^{i(\mathbf{P},\mathbf{X})} \end{bmatrix}$$

$$=\iint \frac{d^4P}{(2\pi)^4}\frac{d^4P'}{(2\pi)^4}$$

$$\begin{bmatrix} (d_1(\mathbf{P})d_1^*(\mathbf{P}') - d_1^*(\mathbf{P}')d_1(\mathbf{P}))\mathbf{V1}(\mathbf{P})\mathbf{U1}^T(\mathbf{P}')e^{i(\mathbf{P},\mathbf{X})}e^{-i(\mathbf{P}',\mathbf{X}')} \\ +(b_1(\mathbf{P})b_1^*(\mathbf{P}') - b_1^*(\mathbf{P}')b_1(\mathbf{P}))\mathbf{U4}(\mathbf{P})\mathbf{V4}^T(\mathbf{P}')e^{i(\mathbf{P},\mathbf{X})}e^{-i(\mathbf{P}',\mathbf{X}')} \\ +(b_4(\mathbf{P})b_4^*(\mathbf{P}') - b_4^*(\mathbf{P}')b_4(\mathbf{P}))\mathbf{V4}(\mathbf{P})\mathbf{V1}^T(\mathbf{P}')e^{i(\mathbf{P},\mathbf{X})}e^{-i(\mathbf{P}',\mathbf{X}')} \\ +(d_4(\mathbf{P})d_4^*(\mathbf{P}') - d_4^*(\mathbf{P}')d_4(\mathbf{P}))\mathbf{U1}(\mathbf{P})\mathbf{U4}^T(\mathbf{P}')e^{i(\mathbf{P},\mathbf{X})}e^{-i(\mathbf{P}',\mathbf{X}')} \end{bmatrix}$$

$$+\iint \frac{d^4P}{(2\pi)^4}\frac{d^4P'}{(2\pi)^4}$$

$$\begin{bmatrix} (b_4^*(\mathbf{P})b_4(\mathbf{P}') - b_4(\mathbf{P}')b_4^*(\mathbf{P}))\mathbf{V1}(\mathbf{P})\mathbf{V4}^T(\mathbf{P}')e^{-i(\mathbf{P},\mathbf{X})}e^{i(\mathbf{P}',\mathbf{X}')} \\ +(d_4^*(\mathbf{P})d_4(\mathbf{P}') - d_4(\mathbf{P}')d_4^*(\mathbf{P}))\mathbf{U4}(\mathbf{P})\mathbf{U1}^T(\mathbf{P}')e^{-i(\mathbf{P},\mathbf{X})}e^{i(\mathbf{P}',\mathbf{X}')} \\ +(d_1^*(\mathbf{P})d_1(\mathbf{P}') - d_1(\mathbf{P}')d_1^*(\mathbf{P}))\mathbf{U1}(\mathbf{P})\mathbf{V1}^T(\mathbf{P}')e^{-i(\mathbf{P},\mathbf{X})}e^{i(\mathbf{P}',\mathbf{X}')} \\ +(b_1^*(\mathbf{P})b_1(\mathbf{P}') - b_1(\mathbf{P}')b_1^*(\mathbf{P}))\mathbf{V4}(\mathbf{P})\mathbf{U4}^T(\mathbf{P}')e^{-i(\mathbf{P},\mathbf{X})}e^{i(\mathbf{P}',\mathbf{X}')} \end{bmatrix}$$

Let us apply the following commutation relations

$$d_1(\mathbf{P})d_1^*(\mathbf{P}') - d_1^*(\mathbf{P}')d_1(\mathbf{P}) = \delta(\mathbf{P}-\mathbf{P}')$$

$$b_1(\mathbf{P})b_1^*(\mathbf{P}') - b_1^*(\mathbf{P}')b_1(\mathbf{P}) = \delta(\mathbf{P}-\mathbf{P}')$$

$$b_4(\mathbf{P})b_4^*(\mathbf{P}') - b_4^*(\mathbf{P}')b_4(\mathbf{P}) = \delta(\mathbf{P}-\mathbf{P}')$$

$$d_4(\mathbf{P})d_4^*(\mathbf{P}') - d_4^*(\mathbf{P}')d_4(\mathbf{P}) = \delta(\mathbf{P}-\mathbf{P}')$$

$$d_1(\mathbf{P}')d_1^*(\mathbf{P}) - d_1^*(\mathbf{P})d_1(\mathbf{P}') = \delta(\mathbf{P}'-\mathbf{P})$$

$$d_1^*(\mathbf{P})d_1(\mathbf{P}') - d_1(\mathbf{P}')d_1^*(\mathbf{P}) = -\delta(\mathbf{P}'-\mathbf{P})$$

$$b_1^*(\mathbf{P})b_1(\mathbf{P}') - b_1(\mathbf{P}')b_1^*(\mathbf{P}) = -\delta(\mathbf{P}'-\mathbf{P})$$

$$d_4^*(\mathbf{P})d_4(\mathbf{P}') - d_4(\mathbf{P}')d_4^*(\mathbf{P}) = -\delta(\mathbf{P}'-\mathbf{P})$$

$$b_4^*(\mathbf{P})b_4(\mathbf{P}') - b_4(\mathbf{P}')b_4^*(\mathbf{P}) = -\delta(\mathbf{P}'-\mathbf{P})$$

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$$= \iint \frac{d^4P}{(2\pi)^4}\frac{d^4P'}{(2\pi)^4}$$

$$\begin{bmatrix} \delta(\mathbf{P}-\mathbf{P}')\mathbf{V1}(\mathbf{P})\mathbf{U1}^T(\mathbf{P}')e^{i(\mathbf{P},\mathbf{X})}e^{-i(\mathbf{P}',\mathbf{X}')} \\ +\delta(\mathbf{P}-\mathbf{P}')\mathbf{U4}(\mathbf{P})\mathbf{V4}^T(\mathbf{P}')e^{i(\mathbf{P},\mathbf{X})}e^{-i(\mathbf{P}',\mathbf{X}')} \\ +\delta(\mathbf{P}-\mathbf{P}')\mathbf{V4}(\mathbf{P})\mathbf{V1}^T(\mathbf{P}')e^{i(\mathbf{P},\mathbf{X})}e^{-i(\mathbf{P}',\mathbf{X}')} \\ +\delta(\mathbf{P}-\mathbf{P}')\mathbf{U1}(\mathbf{P})\mathbf{U4}^T(\mathbf{P}')e^{i(\mathbf{P},\mathbf{X})}e^{-i(\mathbf{P}',\mathbf{X}')} \end{bmatrix} + \begin{bmatrix} -\delta(\mathbf{P}'-\mathbf{P})\mathbf{V1}(\mathbf{P})\mathbf{V4}^T(\mathbf{P}')e^{-i(\mathbf{P},\mathbf{X})}e^{i(\mathbf{P}',\mathbf{X}')} \\ -\delta(\mathbf{P}'-\mathbf{P})\mathbf{U4}(\mathbf{P})\mathbf{U1}^T(\mathbf{P}')e^{-i(\mathbf{P},\mathbf{X})}e^{i(\mathbf{P}',\mathbf{X}')} \\ -\delta(\mathbf{P}'-\mathbf{P})\mathbf{U1}(\mathbf{P})\mathbf{V1}^T(\mathbf{P}')e^{-i(\mathbf{P},\mathbf{X})}e^{i(\mathbf{P}',\mathbf{X}')} \\ -\delta(\mathbf{P}'-\mathbf{P})\mathbf{V4}(\mathbf{P})\mathbf{U4}^T(\mathbf{P}')e^{-i(\mathbf{P},\mathbf{X})}e^{i(\mathbf{P}',\mathbf{X}')} \end{bmatrix}$$

$$= \int \frac{d^4P}{(2\pi)^4}$$

$$\begin{bmatrix} \mathbf{V1}(\mathbf{P})\mathbf{U1}^T(\mathbf{P})e^{i(\mathbf{P},\mathbf{X})}e^{-i(\mathbf{P},\mathbf{X}')} \\ +\mathbf{U4}(\mathbf{P})\mathbf{V4}^T(\mathbf{P})e^{i(\mathbf{P},\mathbf{X})}e^{-i((\mathbf{P},\mathbf{X}'))} \\ +\mathbf{V4}(\mathbf{P})\mathbf{V1}^T(\mathbf{P})e^{i(\mathbf{P},\mathbf{X})}e^{-i(\mathbf{P},\mathbf{X}')} \\ +\mathbf{U1}(\mathbf{P})\mathbf{U4}^T(\mathbf{P})e^{i(\mathbf{P},\mathbf{X})}e^{-i(\mathbf{P},\mathbf{X}')} \end{bmatrix} + \begin{bmatrix} -\mathbf{V1}(\mathbf{P})\mathbf{V4}^T(\mathbf{P})e^{-i(\mathbf{P},\mathbf{X})}e^{i(\mathbf{P},\mathbf{X}')} \\ -\mathbf{U4}(\mathbf{P})\mathbf{U1}^T(\mathbf{P})e^{-i(\mathbf{P},\mathbf{X})}e^{i(\mathbf{P},\mathbf{X}')} \\ -\mathbf{U1}(\mathbf{P})\mathbf{V1}^T(\mathbf{P})e^{-i(\mathbf{P},\mathbf{X})}e^{i(\mathbf{P},\mathbf{X}')} \\ -\mathbf{V4}(\mathbf{P})\mathbf{U4}^T(\mathbf{P})e^{-i(\mathbf{P},\mathbf{X})}e^{i(\mathbf{P},\mathbf{X}')} \end{bmatrix}$$

$$= \int \frac{d^4P}{(2\pi)^4}$$

$$\begin{bmatrix} \mathbf{V1}(\mathbf{P})\mathbf{U1}^T(\mathbf{P}) \\ +\mathbf{U4}(\mathbf{P})\mathbf{V4}^T(\mathbf{P}) \\ +\mathbf{V4}(\mathbf{P})\mathbf{V1}^T(\mathbf{P}) \\ +\mathbf{U1}(\mathbf{P})\mathbf{U4}^T(\mathbf{P}) \end{bmatrix} e^{i(\mathbf{P},\mathbf{X}-\mathbf{X}')} - \begin{bmatrix} \mathbf{V1}(\mathbf{P})\mathbf{V4}^T(\mathbf{P}) \\ +\mathbf{U4}(\mathbf{P})\mathbf{U1}^T(\mathbf{P}) \\ +\mathbf{U1}(\mathbf{P})\mathbf{V1}^T(\mathbf{P}) \\ +\mathbf{V4}(\mathbf{P})\mathbf{U4}^T(\mathbf{P}) \end{bmatrix} e^{i(\mathbf{P},\mathbf{X}'-\mathbf{X})}$$

$$= \int \frac{d^4P}{(2\pi)^4}\begin{pmatrix} 4P_0^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -4P_2^2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} e^{i(\mathbf{P},\mathbf{X}-\mathbf{X}')} - \int \frac{d^4P}{(2\pi)^4}\begin{pmatrix} 4P_0^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -4P_2^2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} e^{i(\mathbf{P},\mathbf{X}'-\mathbf{X})}$$

$$= \begin{pmatrix} 4P_0^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -4P_2^2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}\int \frac{d^4P}{(2\pi)^4} e^{i(\mathbf{P},\mathbf{X}-\mathbf{X}')} - \begin{pmatrix} 4P_0^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -4P_2^2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}\int \frac{d^4P}{(2\pi)^4} e^{i(\mathbf{P},\mathbf{X}'-\mathbf{X})}$$

$$= \begin{pmatrix} 4P_0^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -4P_2^2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}\delta(\mathbf{X}-\mathbf{X}') - \begin{pmatrix} 4P_0^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -4P_2^2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}\delta(\mathbf{X}'-\mathbf{X}) = 0$$

Here it is taken into account that

$$\mathbf{V1} * \mathbf{U1}^{\mathbf{T}} + \mathbf{V4} * \mathbf{V1}^{\mathbf{T}} + \mathbf{U4} * \mathbf{V4}^{\mathbf{T}} + \mathbf{U1} * \mathbf{U4}^{\mathbf{T}} = \begin{pmatrix} 4P_0^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -4P_2^2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\mathbf{U1} * \mathbf{V1}^{\mathbf{T}} + \mathbf{V1} * \mathbf{V4}^{\mathbf{T}} + \mathbf{V4} * \mathbf{U4}^{\mathbf{T}} + \mathbf{U4} * \mathbf{U1}^{\mathbf{T}} = \begin{pmatrix} 4P_0^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -4P_2^2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

We will consider this relation as a proof of the symmetry of the wave function under the stipulated commutation relations.

Let us find the commutation relations for the wave function and its time derivative, which in this case play the role of canonical momentum

$$[\varphi_i(\mathbf{X}), \dot{\varphi}_j(\mathbf{X}')] = \varphi_i(\mathbf{X})\dot{\varphi}_j(\mathbf{X}') - \dot{\varphi}_j(\mathbf{X}')\varphi_i(\mathbf{X}) = \left(\boldsymbol{\varphi}(\mathbf{X})\dot{\boldsymbol{\varphi}}^T(\mathbf{X}') - \left(\dot{\boldsymbol{\varphi}}(\mathbf{X}')\boldsymbol{\varphi}^T(\mathbf{X})\right)^T\right)_{ij}$$

where

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$$\dot{\varphi}_J(\mathbf{X}) \equiv \frac{\partial \varphi_i(\mathbf{X})}{\partial X_0}$$

$$\boldsymbol{\varphi}(\mathbf{X})\dot{\boldsymbol{\varphi}}^T(\mathbf{X}') - \left(\dot{\boldsymbol{\varphi}}(\mathbf{X}')\boldsymbol{\varphi}^T(\mathbf{X})\right)^T =$$

$$= \iint \frac{d^4P}{(2\pi)^4}\frac{d^4P'}{(2\pi)^4}$$

$$\left[\begin{array}{l}
\left(d_1(\mathbf{P})\mathbf{V1}(\mathbf{P})e^{i(\mathbf{P},\mathbf{X})}\right)\left((-iP_0')d_1^*(\mathbf{P}')\mathbf{U1}(\mathbf{P}')e^{-i(\mathbf{P}',\mathbf{X}')}\right)^{\boldsymbol{T}} - \left(\left((iP_0')d_1(\mathbf{P}')\mathbf{V1}(\mathbf{P}')e^{i(\mathbf{P}',\mathbf{X}')}\right)\left(d_1^*(\mathbf{P})\mathbf{U1}(\mathbf{P})e^{-i(\mathbf{P},\mathbf{X})}\right)^{\boldsymbol{T}}\right)^T \\
+\left(b_1(\mathbf{P})\mathbf{U4}(\mathbf{P})e^{i(\mathbf{P},\mathbf{X})}\right)\left((-iP_0')b_1^*(\mathbf{P}')\mathbf{V4}(\mathbf{P}')e^{-i(\mathbf{P}',\mathbf{X}')}\right)^{\boldsymbol{T}} - \left(\left((iP_0')b_1(\mathbf{P}')\mathbf{U4}(\mathbf{P}')e^{i(\mathbf{P}',\mathbf{X}')}\right)\left(b_1^*(\mathbf{P})\mathbf{V4}(\mathbf{P})e^{-i(\mathbf{P},\mathbf{X})}\right)^{\boldsymbol{T}}\right)^{\boldsymbol{T}} \\
+\left(b_4(\mathbf{P})\mathbf{V4}(\mathbf{P})e^{i(\mathbf{P},\mathbf{X})}\right)\left((-iP_0')b_4^*(\mathbf{P}')\mathbf{V1}(\mathbf{P}')e^{-i(\mathbf{P}',\mathbf{X}')}\right)^{\boldsymbol{T}} - \left(\left((iP_0')b_4(\mathbf{P}')\mathbf{V4}(\mathbf{P}')e^{i(\mathbf{P}',\mathbf{X}')}\right)\left(b_4^*(\mathbf{P})\mathbf{V1}(\mathbf{P})e^{-i(\mathbf{P},\mathbf{X})}\right)^{\boldsymbol{T}}\right)^{\boldsymbol{T}} \\
+\left(d_4(\mathbf{P})\mathbf{U1}(\mathbf{P})e^{i(\mathbf{P},\mathbf{X})}\right)\left((-iP_0')d_4^*(\mathbf{P}')\mathbf{U4}(\mathbf{P}')e^{-i(\mathbf{P}',\mathbf{X}')}\right)^{\boldsymbol{T}} - \left(\left((iP_0')d_4(\mathbf{P}')\mathbf{U1}(\mathbf{P}')e^{i(\mathbf{P}',\mathbf{X}')}\right)\left(d_4^*(\mathbf{P})\mathbf{U4}(\mathbf{P})e^{-i(\mathbf{P},\mathbf{X})}\right)^{\boldsymbol{T}}\right)^T
\end{array}\right]$$

$$+ \iint \frac{d^4P}{(2\pi)^4}\frac{d^4P'}{(2\pi)^4}$$

$$\left[\begin{array}{l}
\left(b_4^*(\mathbf{P})\mathbf{V1}(\mathbf{P})e^{-i(\mathbf{P},\mathbf{X})}\right)\left((iP_0')b_4(\mathbf{P}')\mathbf{V4}(\mathbf{P}')e^{i(\mathbf{P}',\mathbf{X}')}\right)^{\boldsymbol{T}} - \left(\left((-iP_0')b_4^*(\mathbf{P}')\mathbf{V1}(\mathbf{P}')e^{-i(\mathbf{P}',\mathbf{X}')}\right)\left(b_4(\mathbf{P})\mathbf{V4}(\mathbf{P})e^{i(\mathbf{P},\mathbf{X})}\right)^{\boldsymbol{T}}\right)^{\boldsymbol{T}} \\
+\left(d_4^*(\mathbf{P})\mathbf{U4}(\mathbf{P})e^{-i(\mathbf{P},\mathbf{X})}\right)\left((iP_0')d_4(\mathbf{P}')\mathbf{U1}(\mathbf{P}')e^{i(\mathbf{P}',\mathbf{X}')}\right)^{\boldsymbol{T}} - \left(\left((-iP_0')d_4^*(\mathbf{P}')\mathbf{U4}(\mathbf{P}')e^{-i(\mathbf{P}',\mathbf{X}')}\right)\left(d_4(\mathbf{P})\mathbf{U1}(\mathbf{P})e^{i(\mathbf{P},\mathbf{X})}\right)^{\boldsymbol{T}}\right)^{\boldsymbol{T}} \\
+\left(d_1^*(\mathbf{P})\mathbf{U1}(\mathbf{P})e^{-i(\mathbf{P},\mathbf{X})}\right)\left((iP_0')d_1(\mathbf{P}')\mathbf{V1}(\mathbf{P}')e^{i(\mathbf{P}',\mathbf{X}')}\right)^{\boldsymbol{T}} - \left(\left((-iP_0')d_1^*(\mathbf{P}')\mathbf{U1}(\mathbf{P}')e^{-i(\mathbf{P}',\mathbf{X}')}\right)\left(d_1(\mathbf{P})\mathbf{V1}(\mathbf{P})e^{i(\mathbf{P},\mathbf{X})}\right)^{\boldsymbol{T}}\right)^T \\
+\left(b_1^*(\mathbf{P})\mathbf{V4}(\mathbf{P})e^{-i(\mathbf{P},\mathbf{X})}\right)\left((iP_0')b_1(\mathbf{P}')\mathbf{U4}(\mathbf{P}')e^{i(\mathbf{P}',\mathbf{X}')}\right)^{\boldsymbol{T}} - \left(\left((-iP_0')b_1^*(\mathbf{P}')\mathbf{V4}(\mathbf{P}')e^{-i(\mathbf{P}',\mathbf{X}')}\right)\left(b_1(\mathbf{P})\mathbf{U4}(\mathbf{P})e^{i(\mathbf{P},\mathbf{X})}\right)^{\boldsymbol{T}}\right)^T
\end{array}\right]$$

$$= \iint \frac{d^4P}{(2\pi)^4}\frac{d^4P'}{(2\pi)^4}$$

$$\left[\begin{array}{l}
(-iP_0')d_1(\mathbf{P})d_1^*(\mathbf{P}')\mathbf{V1}(\mathbf{P})\mathbf{U1}^T(\mathbf{P}')e^{i(\mathbf{P},\mathbf{X})}e^{-i(\mathbf{P}',\mathbf{X}')} - (iP_0')d_1(\mathbf{P}')d_1^*(\mathbf{P})\mathbf{U1}(\mathbf{P})\mathbf{V1}^T(\mathbf{P}')e^{i(\mathbf{P}',\mathbf{X}')}e^{-i(\mathbf{P},\mathbf{X})} \\
+(-iP_0')b_1(\mathbf{P})b_1^*(\mathbf{P}')\mathbf{U4}(\mathbf{P})\mathbf{V4}^T(\mathbf{P}')e^{i(\mathbf{P},\mathbf{X})}e^{-i(\mathbf{P}',\mathbf{X}')} - (iP_0')b_1(\mathbf{P}')b_1^*(\mathbf{P})\mathbf{V4}(\mathbf{P})\mathbf{U4}^T(\mathbf{P}')e^{i(\mathbf{P}',\mathbf{X}')}e^{-i(\mathbf{P},\mathbf{X})} \\
+(-iP_0')b_4(\mathbf{P})b_4^*(\mathbf{P}')\mathbf{V4}(\mathbf{P})\mathbf{V1}^T(\mathbf{P}')e^{i(\mathbf{P},\mathbf{X})}e^{-i(\mathbf{P}',\mathbf{X}')} - (iP_0')b_4(\mathbf{P}')b_4^*(\mathbf{P})\mathbf{V1}(\mathbf{P})\mathbf{V4}^T(\mathbf{P}')e^{i(\mathbf{P}',\mathbf{X}')}e^{-i(\mathbf{P},\mathbf{X})} \\
+(-iP_0')d_4(\mathbf{P})d_4^*(\mathbf{P}')\mathbf{U1}(\mathbf{P})\mathbf{U4}^T(\mathbf{P}')e^{i(\mathbf{P},\mathbf{X})}e^{-i(\mathbf{P}',\mathbf{X}')} - (iP_0')d_4(\mathbf{P}')d_4^*(\mathbf{P})\mathbf{U4}(\mathbf{P})\mathbf{U1}^T(\mathbf{P}')e^{i(\mathbf{P}',\mathbf{X}')}e^{-i(\mathbf{P},\mathbf{X})}
\end{array}\right]$$

$$+ \iint \frac{d^4P}{(2\pi)^4}\frac{d^4P'}{(2\pi)^4}$$

$$\left[\begin{array}{l}
(iP_0')b_4^*(\mathbf{P})b_4(\mathbf{P}')\mathbf{V1}(\mathbf{P})\mathbf{V4}^T(\mathbf{p}')e^{-i(\mathbf{P},\mathbf{X})}e^{i(\mathbf{P}',\mathbf{X}')} - (-iP_0')b_4^*(\mathbf{P}')b_4(\mathbf{P})\mathbf{V4}(\mathbf{P})\mathbf{V1}^T(\mathbf{P}')\,e^{-i(\mathbf{P}',\mathbf{X}')}e^{i(\mathbf{P},\mathbf{X})} \\
+(iP_0')d_4^*(\mathbf{P})d_4(\mathbf{P}')\mathbf{U4}(\mathbf{P})\mathbf{U1}^T(\mathbf{p}')e^{-i(\mathbf{P},\mathbf{X})}e^{i(\mathbf{P}',\mathbf{X}')} - (-iP_0')d_4^*(\mathbf{P}')d_4(\mathbf{P})\mathbf{U1}(\mathbf{P})\mathbf{U4}^T(\mathbf{P}')e^{-i(\mathbf{P}',\mathbf{X}')}e^{i(\mathbf{P},\mathbf{X})} \\
+(iP_0')d_1^*(\mathbf{P})d_1(\mathbf{P}')\mathbf{U1}(\mathbf{P})\mathbf{V1}^T(\mathbf{p}')e^{-i(\mathbf{P},\mathbf{X})}e^{i(\mathbf{P}',\mathbf{X}')} - (-iP_0')d_1^*(\mathbf{P}')d_1(\mathbf{P})\mathbf{V1}(\mathbf{P})\mathbf{U1}^T(\mathbf{P}')e^{-i(\mathbf{P}',\mathbf{X}')}e^{i(\mathbf{P},\mathbf{X})} \\
+(iP_0')b_1^*(\mathbf{P})b_1(\mathbf{P}')\mathbf{V4}(\mathbf{P})\mathbf{U4}^T(\mathbf{p}')e^{-i(\mathbf{P},\mathbf{X})}e^{i(\mathbf{P}',\mathbf{X}')} - (-iP_0')b_1^*(\mathbf{P}')b_1(\mathbf{P})\mathbf{U4}(\mathbf{P})\mathbf{V4}^T(\mathbf{P}')e^{-i(\mathbf{P}',\mathbf{X}')}e^{i(\mathbf{P},\mathbf{X})}
\end{array}\right]$$

$$= \iint \frac{d^4P}{(2\pi)^4}\frac{d^4P'}{(2\pi)^4}$$

$$\left[\begin{array}{l}
(-iP_0')\left(d_1(\mathbf{P})d_1^*(\mathbf{P}') - d_1^*(\mathbf{P}')d_1(\mathbf{P})\right)\mathbf{V1}(\mathbf{P})\mathbf{U1}^T(\mathbf{P}')e^{i(\mathbf{P},\mathbf{X})}e^{-i(\mathbf{P}',\mathbf{X}')} \\
+(-iP_0')\left(b_1(\mathbf{P})b_1^*(\mathbf{P}') - b_1^*(\mathbf{P}')b_1(\mathbf{P})\right)\mathbf{U4}(\mathbf{P})\mathbf{V4}^T(\mathbf{P}')e^{i(\mathbf{P},\mathbf{X})}e^{-i(\mathbf{P}',\mathbf{X}')} \\
+(-iP_0')\left(b_4(\mathbf{P})b_4^*(\mathbf{P}') - b_4^*(\mathbf{P}')b_4(\mathbf{P})\right)\mathbf{V4}(\mathbf{P})\mathbf{V1}^T(\mathbf{P}')e^{i(\mathbf{P},\mathbf{X})}e^{-i(\mathbf{P}',\mathbf{X}')} \\
+(-iP_0')\left(d_4(\mathbf{P})d_4^*(\mathbf{P}') - d_4^*(\mathbf{P}')d_4(\mathbf{P})\right)\mathbf{U1}(\mathbf{P})\mathbf{U4}^T(\mathbf{P}')e^{i(\mathbf{P},\mathbf{X})}e^{-i(\mathbf{P}',\mathbf{X}')}
\end{array}\right]$$

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$$+\iint \frac{d^4P}{(2\pi)^4}\frac{d^4P'}{(2\pi)^4}$$

$$\begin{bmatrix} (iP_0')(b_4^*(\mathbf{P})b_4(\mathbf{P}') - b_4(\mathbf{P}')b_4^*(\mathbf{P}))\mathbf{V4}(\mathbf{P})\mathbf{V4}^T(\mathbf{P}')e^{-i(\mathbf{P},\mathbf{X})}e^{i(\mathbf{P}',\mathbf{X}')} \\ +(iP_0')(d_4^*(\mathbf{P})d_4(\mathbf{P}') - d_4(\mathbf{P}')d_4^*(\mathbf{P}))\mathbf{U4}(\mathbf{P})\mathbf{U1}^T(\mathbf{P}')e^{-i(\mathbf{P},\mathbf{X})}e^{i(\mathbf{P}',\mathbf{X}')} \\ +(iP_0')(d_1^*(\mathbf{P})d_1(\mathbf{P}') - d_1(\mathbf{P}')d_1^*(\mathbf{P}))\mathbf{U1}(\mathbf{P})\mathbf{V1}^T(\mathbf{P}')e^{-i(\mathbf{P},\mathbf{X})}e^{i(\mathbf{P}',\mathbf{X}')} \\ +(iP_0')(b_1^*(\mathbf{P})b_1(\mathbf{P}') - b_1(\mathbf{P}')b_1^*(\mathbf{P}))\mathbf{V4}(\mathbf{P})\mathbf{U4}^T(\mathbf{P}')e^{-i(\mathbf{P},\mathbf{X})}e^{i(\mathbf{P}',\mathbf{X}')} \end{bmatrix}$$

The commutation relations remain the same

$$d_1(\mathbf{P})d_1^*(\mathbf{P}') - d_1^*(\mathbf{P}')d_1(\mathbf{P}) = \delta(\mathbf{P} - \mathbf{P}')$$
$$b_1(\mathbf{P})b_1^*(\mathbf{P}') - b_1^*(\mathbf{P}')b_1(\mathbf{P}) = \delta(\mathbf{P} - \mathbf{P}')$$
$$b_4(\mathbf{P})b_4^*(\mathbf{P}') - b_4^*(\mathbf{P}')b_4(\mathbf{P}) = \delta(\mathbf{P} - \mathbf{P}')$$
$$d_4(\mathbf{P})d_4^*(\mathbf{P}') - d_4^*(\mathbf{P}')d_4(\mathbf{P}) = \delta(\mathbf{P} - \mathbf{P}')$$
$$d_1^*(\mathbf{P})d_1(\mathbf{P}') - d_1(\mathbf{P}')d_1^*(\mathbf{P}) = -\delta(\mathbf{P}' - \mathbf{P})$$
$$b_1^*(\mathbf{P})b_1(\mathbf{P}') - b_1(\mathbf{P}')b_1^*(\mathbf{P}) = -\delta(\mathbf{P}' - \mathbf{P})$$
$$d_4^*(\mathbf{P})d_4(\mathbf{P}') - d_4(\mathbf{P}')d_4^*(\mathbf{P}) = -\delta(\mathbf{P}' - \mathbf{P})$$
$$b_4^*(\mathbf{P})b_4(\mathbf{P}') - b_4(\mathbf{P}')b_4^*(\mathbf{P}) = -\delta(\mathbf{P}' - \mathbf{P})$$

$$= \iint \frac{d^4P}{(2\pi)^4}\frac{d^4P'}{(2\pi)^4}$$

$$(-iP_0')\begin{bmatrix} \delta(\mathbf{P} - \mathbf{P}')\mathbf{V1}(\mathbf{P})\mathbf{U1}^T(\mathbf{P}')e^{i(\mathbf{P},\mathbf{X})}e^{-i(\mathbf{P}',\mathbf{X}')} \\ +\delta(\mathbf{P} - \mathbf{P}')\mathbf{U4}(\mathbf{P})\mathbf{V4}^T(\mathbf{P}')e^{i(\mathbf{P},\mathbf{X})}e^{-i(\mathbf{P}',\mathbf{X}')} \\ +\delta(\mathbf{P} - \mathbf{P}')\mathbf{V4}(\mathbf{P})\mathbf{V1}^T(\mathbf{P}')e^{i(\mathbf{P},\mathbf{X})}e^{-i(\mathbf{P}',\mathbf{X}')} \\ +\delta(\mathbf{P} - \mathbf{P}')\mathbf{U1}(\mathbf{P})\mathbf{U4}^T(\mathbf{P}')e^{i(\mathbf{P},\mathbf{X})}e^{-i(\mathbf{P}',\mathbf{X}')} \end{bmatrix} + (iP_0')\begin{bmatrix} -\delta(\mathbf{P}' - \mathbf{P})\mathbf{V1}(\mathbf{P})\mathbf{V4}^T(\mathbf{P}')e^{-i(\mathbf{P},\mathbf{X})}e^{i(\mathbf{P}',\mathbf{X}')} \\ -\delta(\mathbf{P}' - \mathbf{P})\mathbf{U4}(\mathbf{P})\mathbf{U1}^T(\mathbf{P}')e^{-i(\mathbf{P},\mathbf{X})}e^{i(\mathbf{P}',\mathbf{X}')} \\ -\delta(\mathbf{P}' - \mathbf{P})\mathbf{U1}(\mathbf{P})\mathbf{V1}^T(\mathbf{P}')e^{-i(\mathbf{P},\mathbf{X})}e^{i(\mathbf{P}',\mathbf{X}')} \\ -\delta(\mathbf{P}' - \mathbf{P})\mathbf{V4}(\mathbf{P})\mathbf{U4}^T(\mathbf{P}')e^{-i(\mathbf{P},\mathbf{X})}e^{i(\mathbf{P}',\mathbf{X}')} \end{bmatrix}$$

$$= \int \frac{d^4P}{(2\pi)^4}$$

$$(-iP_0)\begin{bmatrix} \mathbf{V1}(\mathbf{P})\mathbf{U1}^T(\mathbf{P})e^{i(\mathbf{P},\mathbf{X})}e^{-i(\mathbf{P},\mathbf{X}')} \\ +\mathbf{U4}(\mathbf{P})\mathbf{V4}^T(\mathbf{P})e^{i(\mathbf{P},\mathbf{X})}e^{-i((\mathbf{P},\mathbf{X}'))} \\ +\mathbf{V4}(\mathbf{P})\mathbf{V1}^T(\mathbf{P})e^{i(\mathbf{P},\mathbf{X})}e^{-i(\mathbf{P},\mathbf{X}')} \\ +\mathbf{U1}(\mathbf{P})\mathbf{U4}^T(\mathbf{P})e^{i(\mathbf{P},\mathbf{X})}e^{-i(\mathbf{P},\mathbf{X}')} \end{bmatrix} + (iP_0)\begin{bmatrix} -\mathbf{V1}(\mathbf{P})\mathbf{V4}^T(\mathbf{P})e^{-i(\mathbf{P},\mathbf{X})}e^{i(\mathbf{P},\mathbf{X}')} \\ -\mathbf{U4}(\mathbf{P})\mathbf{U1}^T(\mathbf{P})e^{-i(\mathbf{P},\mathbf{X})}e^{i(\mathbf{P},\mathbf{X}')} \\ -\mathbf{U1}(\mathbf{P})\mathbf{V1}^T(\mathbf{P})e^{-i(\mathbf{P},\mathbf{X})}e^{i(\mathbf{P},\mathbf{X}')} \\ -\mathbf{V4}(\mathbf{P})\mathbf{U4}^T(\mathbf{P})e^{-i(\mathbf{P},\mathbf{X})}e^{i(\mathbf{P},\mathbf{X}')} \end{bmatrix}$$

$$= \int \frac{d^4P}{(2\pi)^4}$$

$$(-iP_0)\begin{bmatrix} \mathbf{V1}(\mathbf{P})\mathbf{U1}^T(\mathbf{P}) \\ +\mathbf{U4}(\mathbf{P})\mathbf{V4}^T(\mathbf{P}) \\ +\mathbf{V4}(\mathbf{P})\mathbf{V1}^T(\mathbf{P}) \\ +\mathbf{U1}(\mathbf{P})\mathbf{U4}^T(\mathbf{P}) \end{bmatrix} e^{i(\mathbf{P},\mathbf{X}-\mathbf{X}')} - (iP_0)\begin{bmatrix} \mathbf{V1}(\mathbf{P})\mathbf{V4}^T(\mathbf{P}) \\ +\mathbf{U4}(\mathbf{P})\mathbf{U1}^T(\mathbf{P}) \\ +\mathbf{U1}(\mathbf{P})\mathbf{V1}^T(\mathbf{P}) \\ +\mathbf{V4}(\mathbf{P})\mathbf{U4}^T(\mathbf{P}) \end{bmatrix} e^{i(\mathbf{P},\mathbf{X}'-\mathbf{X})} =$$

$$(-iP_0)\begin{pmatrix} 4P_0^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -4P_2^2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \int \frac{d^4P}{(2\pi)^2} e^{i(\mathbf{P},\mathbf{X}-\mathbf{X}')} - (iP_0)\begin{pmatrix} 4P_0^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -4P_2^2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \int \frac{d^4P}{(2\pi)^2} e^{i(\mathbf{P},\mathbf{X}'-\mathbf{X})}$$

$$= (-iP_0)\begin{pmatrix} 4P_0^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -4P_2^2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \delta(\mathbf{X} - \mathbf{X}') - (iP_0)\begin{pmatrix} 4P_0^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -4P_2^2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \delta(\mathbf{X}' - \mathbf{X})$$

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$$= -iP_0 \begin{pmatrix} 8P_0^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -8P_2^2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \delta(\mathbf{X} - \mathbf{X}')$$

As one would expect, the field has only two degrees of freedom. This relation is valid for any reference frame, but the values of the momentum components in each of them are different.

Let us calculate the square of the field energy

$$E^2 = \int d^4X\, \boldsymbol{\varphi}^+(\mathbf{X})\boldsymbol{\varphi}(\mathbf{X}) =$$

$$= \int d^4X \iint \frac{d^4P}{(2\pi)^4}\frac{d^4P'}{(2\pi)^4}$$

$$\begin{bmatrix} \begin{bmatrix} d_1^*(\mathbf{P}')\mathbf{V1}^T(\mathbf{P}') + b_1^*(\mathbf{P}')\mathbf{U4}^T(\mathbf{P}') \\ +d_4^*(\mathbf{P}')\mathbf{U1}^T(\mathbf{P}') + b_4^*(\mathbf{P}')\mathbf{V4}^T(\mathbf{P}') \end{bmatrix} e^{-i(\mathbf{P}',\mathbf{X})} \\ + \begin{bmatrix} b_1(\mathbf{P}')\mathbf{V1}^T(\mathbf{P}') + d_1(\mathbf{P}')\mathbf{U4}^T(\mathbf{P}') \\ +b_4(\mathbf{P}')\mathbf{U1}^T(\mathbf{P}') + d_4(\mathbf{P}')\mathbf{V4}^T(\mathbf{P}') \end{bmatrix} e^{i(\mathbf{P}',\mathbf{X})} \end{bmatrix}$$

$$\begin{bmatrix} \begin{bmatrix} d_1(\mathbf{P})\mathbf{V1}^T(\mathbf{P}) + b_1(\mathbf{P})\mathbf{U4}^T(\mathbf{P}) \\ +d_4(\mathbf{P})\mathbf{U1}^T(\mathbf{P}) + b_4(\mathbf{P})\mathbf{V4}^T(\mathbf{P}) \end{bmatrix} e^{i(\mathbf{P},\mathbf{X})} \\ + \begin{bmatrix} b_1^*(\mathbf{P})\mathbf{V1}^T(\mathbf{P}) + d_1^*(\mathbf{P})\mathbf{U4}^T(\mathbf{P}) \\ +b_4^*(\mathbf{P})\mathbf{U1}^T(\mathbf{P}) + d_4^*(\mathbf{P})\mathbf{V4}^T(\mathbf{P}) \end{bmatrix} e^{-(\mathbf{P},\mathbf{X})} \end{bmatrix}$$

$$= \int d^4X \iint \frac{d^4P}{(2\pi)^4}\frac{d^4P'}{(2\pi)^4}$$

$$\begin{bmatrix} \begin{bmatrix} d_1^*(\mathbf{P}')\mathbf{V1}^T(\mathbf{P}') + b_1^*(\mathbf{P}')\mathbf{U4}^T(\mathbf{P}') \\ +d_4^*(\mathbf{P}')\mathbf{U1}^T(\mathbf{P}') + b_4^*(\mathbf{P}')\mathbf{V4}^T(\mathbf{P}') \end{bmatrix} \\ \begin{bmatrix} d_1(\mathbf{P})\mathbf{V1}(\mathbf{P}) + b_1(\mathbf{P})\mathbf{U4}(\mathbf{P}) \\ +d_4(\mathbf{P})\mathbf{U1}(\mathbf{P}) + b_4(\mathbf{P})\mathbf{V4}(\mathbf{P}) \end{bmatrix} \\ e^{-i(\mathbf{P}',\mathbf{X})}e^{i(\mathbf{P},\mathbf{X})} \\ + \begin{bmatrix} b_1(\mathbf{P}')\mathbf{V1}^T(\mathbf{P}') + d_1(\mathbf{P}')\mathbf{U4}^T(\mathbf{P}') \\ +b_4(\mathbf{P}')\mathbf{U1}^T(\mathbf{P}') + d_4(\mathbf{P}')\mathbf{V4}^T(\mathbf{P}') \end{bmatrix} \\ \begin{bmatrix} b_1^*(\mathbf{P})\mathbf{V1}(\mathbf{P}) + d_1^*(\mathbf{P})\mathbf{U4}(\mathbf{P}) \\ +b_4^*(\mathbf{P})\mathbf{U1}(\mathbf{P}) + d_4^*(\mathbf{P})\mathbf{V4}(\mathbf{P}) \end{bmatrix} \\ e^{i(\mathbf{P}',\mathbf{X})}e^{-i(\mathbf{P},\mathbf{X})} \end{bmatrix}$$

$$= \iint \frac{d^4P}{(2\pi)^4}\frac{d^4P'}{(2\pi)^4}$$

$$\begin{bmatrix} \begin{bmatrix} d_1^*(\mathbf{P}')\mathbf{V1}^+(\mathbf{P}') + b_1^*(\mathbf{P}')\mathbf{U4}^T(\mathbf{P}') \\ +d_4^*(\mathbf{P}')\mathbf{U1}^+(\mathbf{P}') + b_4^*(\mathbf{P}')\mathbf{V4}^T(\mathbf{P}') \end{bmatrix} \\ \begin{bmatrix} d_1(\mathbf{P})\mathbf{V1}(\mathbf{P}) + b_1(\mathbf{P})\mathbf{U4}(\mathbf{P}) \\ +d_4(\mathbf{P})\mathbf{U1}(\mathbf{P}) + b_4(\mathbf{P})\mathbf{V4}(\mathbf{P}) \end{bmatrix} \\ \delta(\mathbf{P} - \mathbf{P}') \\ + \begin{bmatrix} b_1(\mathbf{P}')\mathbf{V1}^T(\mathbf{p}') + d_1(\mathbf{P}')\mathbf{U4}^+(\mathbf{P}') \\ +b_4(\mathbf{P}')\mathbf{U1}^T(\mathbf{p}') + d_4(\mathbf{P}')\mathbf{V4}^+(\mathbf{P}') \end{bmatrix} \\ \begin{bmatrix} b_1^*(\mathbf{P})\mathbf{V1}(\mathbf{P}) + d_1^*(\mathbf{P})\mathbf{U4}(\mathbf{P}) \\ +b_4^*(\mathbf{P})\mathbf{U1}(\mathbf{P}) + d_4^*(\mathbf{P})\mathbf{V4}(\mathbf{P}) \end{bmatrix} \\ \delta(\mathbf{P}' - \mathbf{P}) \end{bmatrix}$$

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$$= \int \frac{d^4P}{(2\pi)^4} \begin{bmatrix} d_1^*(\mathbf{P})d_1(\mathbf{P})\mathbf{V1}^T(\mathbf{p})\mathbf{V1}(\mathbf{P}) + d_1(\mathbf{P})d_1^*(\mathbf{P})\mathbf{U4}^T(\mathbf{P})\mathbf{U4}(\mathbf{P}) \\ +b_1(\mathbf{P})b_1^*(\mathbf{P})\mathbf{V1}^T(\mathbf{p})\mathbf{V1}(\mathbf{P}) + b_1^*(\mathbf{P})b_1(\mathbf{P})\mathbf{U4}^T(\mathbf{P})\mathbf{U4}(\mathbf{P}) \\ +d_4^*(\mathbf{P})d_4(\mathbf{P})\mathbf{U1}^T(\mathbf{p})\mathbf{U1}(\mathbf{P}) + d_4(\mathbf{P})d_4^*(\mathbf{P})\mathbf{V4}^T(\mathbf{P})\mathbf{V4}(\mathbf{P}) \\ +b_4(\mathbf{P})b_4^*(\mathbf{P})\mathbf{U1}^T(\mathbf{p})\mathbf{U1}(\mathbf{P}) + b_4^*(\mathbf{P})b_4(\mathbf{P})\mathbf{V4}^T(\mathbf{P})\mathbf{V}(\mathbf{P}) \end{bmatrix}$$

$$= \int \frac{d^4P}{(2\pi)^4} E_0^2(\mathbf{P}) \begin{bmatrix} b_1(\mathbf{P})b_1^*(\mathbf{P}) + b_1^*(\mathbf{P})b_1(\mathbf{P}) + d_1^*(\mathbf{P})d_1(\mathbf{P}) + d_1(\mathbf{P})d_1^*(\mathbf{P}) \\ +b_4(\mathbf{P})b_4^*(\mathbf{P}) + b_4^*(\mathbf{P})b_4(\mathbf{P}) + d_4^*(\mathbf{P})d_4(\mathbf{P}) + d_4(\mathbf{P})d_4^*(\mathbf{P}) \end{bmatrix}$$

$$= \int \frac{d^4P}{(2\pi)^4} E_0^2(\mathbf{P}) \begin{bmatrix} (b_1^*(\mathbf{P})b_1(\mathbf{P}) + \delta(\mathbf{0})) + b_1^*(\mathbf{P})b_1(\mathbf{P}) + d_1^*(\mathbf{P})d_1(\mathbf{P}) + (d_1^*(\mathbf{P})d_1(\mathbf{P}) + \delta(\mathbf{0})) \\ +\big(b_4^*(\mathbf{P})b_4(\mathbf{P}) + \delta(\mathbf{0})\big) + b_4^*(\mathbf{P})b_4(\mathbf{P}) + d_4^*(\mathbf{P})d_4(\mathbf{P}) + (d_4^*(\mathbf{P})d_4(\mathbf{P}) + \delta(\mathbf{0})) \end{bmatrix}$$

$$= \int \frac{d^4P}{(2\pi)^4} 2E_0^2(\mathbf{P}) \begin{bmatrix} b_1^*(\mathbf{P})b_1(\mathbf{P}) + d_1^*(\mathbf{P})d_1(\mathbf{P}) \\ +b_4^*(\mathbf{P})b_4(\mathbf{P}) + d_4^*(\mathbf{P})d_4(\mathbf{P}) \end{bmatrix} + \int \frac{d^4P}{(2\pi)^4} 4E_0^2(\mathbf{P})\delta(\mathbf{0})$$

here

$$E_0^2(\mathbf{P}) \equiv \mathbf{V1}^T(\mathbf{P})\mathbf{V1}(\mathbf{P}) = \mathbf{U4}^T(\mathbf{P})\mathbf{U4}(\mathbf{P}) = \mathbf{U1}^T(\mathbf{P})\mathbf{U1}(\mathbf{P}) = \mathbf{V4}^T(\mathbf{P})\mathbf{V4}(\mathbf{P}) =$$

$$= \mathbf{P}^T\mathbf{P} = 2P_0^2 - M^2 = 2P_0^2 - \mathbf{P}^T G\mathbf{P} = 2P_0^2 - (\mathbf{P},\mathbf{P})$$

If we consider the photon field, the mass is zero, so that only the energy of the field remains in the formula. Each summand in brackets under the integral represents the operator of number of particles with a certain reference vector, its action consists in the consecutive application of the annihilation operator and the particle birth operator. The last summand describes the energy of zero-point fluctuations of vacuum. When there is no particle, we have the equality

$$E^2 = \int d^4X\, \boldsymbol{\varphi}^+(\mathbf{X})\boldsymbol{\varphi}(\mathbf{X}) = \int \frac{d^4P}{(2\pi)^4} 4E_0^2(\mathbf{P})\delta(\mathbf{0})$$

In this connection it is logical to use the normalization for the wave operator so that the energy of zero-point fluctuations of vacuum without taking into account the infinite component is unity

$$\frac{\boldsymbol{\varphi}(\mathbf{X})}{2E_0(\mathbf{P})}$$

If the mass is not zero, then we can relate $\mathbf{U1}(\mathbf{P})$ and $\mathbf{V1}(\mathbf{P})$ to the current of electrons with different spins and, respectively, relate $\mathbf{U4}(\mathbf{P})$ and $\mathbf{V4}(\mathbf{P})$ to the current of positrons with different spins.

As we have seen, neither electron current vectors nor electromagnetic field vectors are true vectors. When transforming the coordinate system, the same transformation acts on the components of the momentum vector, from these transformed components in each frame of reference the pseudovectors of the field are formed. But we know that the interaction between current and electromagnetic field is described by an additional term in the Lagrangian of the electrodynamics theory. This term is the scalar product of the current and the electromagnetic potential and it is necessary for this product to be a scalar. But to form a scalar using a metric tensor, two true vectors are needed, and these are not available. There remains only one way to provide the scalar, it is necessary that signs of components in pseudovectors of current and field coincide, then they will compensate each other, and in fact we will get the scalar product of two vectors, and hence we will get a scalar.

Thus, there is a direct connection between the spinor description of the field and its vector description. 16 pseudospinors pass into 4 pseudovectors, moreover, the modulus of the complex mass in spinor space is equal to the mass in vector space. At all this by the value of the phase of a plane wave in spinor space by any direct way it is not possible to calculate the phase of a plane wave in vector space. Hence the assumption arises that operators in spinor space describe nature exactly, while operators in vector space provide only an approximate description. This may partly explain the problems with divergence when integrating in vector space.

To describe the evolution of the field state, we consider the vacuum averaged expression having the sense of the propagator. Let us explain the meaning of operators included in the field decomposition

$$\boldsymbol{\varphi}(\mathbf{X}) = \int \frac{d^4P}{(2\pi)^4} \begin{bmatrix} d_1(\mathbf{P})\mathbf{V1}(\mathbf{P}) + b_1(\mathbf{P})\mathbf{U4}(\mathbf{P}) \\ +d_4(\mathbf{P})\mathbf{U1}(\mathbf{P}) + b_4(\mathbf{P})\mathbf{V4}(\mathbf{P}) \end{bmatrix} e^{i(\mathbf{P},\mathbf{X})} + \begin{bmatrix} b_4^*(\mathbf{P})\mathbf{V1}(\mathbf{P}) + d_4^*(\mathbf{P})\mathbf{U4}(\mathbf{P}) \\ +d_1^*(\mathbf{P})\mathbf{U1}(\mathbf{P}) + b_1^*(\mathbf{P})\mathbf{V4}(\mathbf{P}) \end{bmatrix} e^{-i(\mathbf{P},\mathbf{X})}$$

For example, $d_1(\mathbf{P})$ is an operator of annihilation of a particle with pseudovector $\mathbf{V1}(\mathbf{P})$, similarly, other operators without asterisks annihilate particles with pseudovector which stands in expansion with these operators. Accordingly, the operator $d_1^*(\mathbf{P})d_1(\mathbf{P})$is the operator of the number of particles with pseudovector $\mathbf{V1}(\mathbf{P})$.

Let us define a vacuum state of the field with zero filling numbers of particles of each of four varieties

$$|\Psi_0\rangle \equiv |\Psi_{0000}\rangle$$

by specifying its properties with respect to the action of annihilation operators

$$d_1(\mathbf{P})|\Psi_0\rangle = 0 \qquad d_4(\mathbf{P})|\Psi_0\rangle = 0 \qquad b_1(\mathbf{P})|\Psi_0\rangle = 0 \qquad b_4(\mathbf{P})|\Psi_0\rangle = 0$$
$$\langle\Psi_0^*|d_1^*(\mathbf{P}) = 0 \qquad \langle\Psi_0^*|\, d_4^*(\mathbf{P}) = 0 \qquad \langle\Psi_0^*|b_1^*(\mathbf{P}) = 0 \qquad \langle\Psi_0^*|b_4^*(\mathbf{P}) = 0$$

It follows from these relations that

$$\langle\Psi_0^*|d_1(\mathbf{P})d_1^*(\mathbf{P}')|\Psi_0\rangle = \langle\Psi_0^*|[\, d_1(\mathbf{P}), d_1^*(\mathbf{P}')]|\Psi_0\rangle = \langle\Psi_0^*|\delta(\mathbf{P}-\mathbf{P}')|\Psi_0\rangle$$

Let us construct the amplitude of the field component, which is born at the point with coordinates $\mathbf{X}=\mathbf{0}$ and annihilated at the point with coordinates $\mathbf{X}$

$$\langle\Psi_0^*|\varphi_i(\mathbf{X})\varphi_j(\mathbf{0})|\Psi_0\rangle = (\langle\Psi_0^*|\boldsymbol{\varphi}(\mathbf{X})\boldsymbol{\varphi}^T(\mathbf{0})|\Psi_0\rangle)_{ij}$$

$$\langle\Psi_0^*|\boldsymbol{\varphi}(\mathbf{X})\boldsymbol{\varphi}^T(\mathbf{0})|\Psi_0\rangle =$$

$$\iint \frac{d^4P}{(2\pi)^4}\frac{d^4P'}{(2\pi)^4}\langle\Psi_0^*\left|\begin{bmatrix} d_1(\mathbf{P})\mathbf{V1}(\mathbf{P}) + b_1(\mathbf{P})\mathbf{U4}(\mathbf{P}) \\ +d_4(\mathbf{P})\mathbf{U1}(\mathbf{P}) + b_4(\mathbf{P})\mathbf{V4}(\mathbf{P}) \end{bmatrix}\begin{bmatrix} b_4^*(\mathbf{P}')\mathbf{V1}^T(\mathbf{P}') + d_4^*(\mathbf{P}')\mathbf{U4}^T(\mathbf{P}') \\ +d_1^*(\mathbf{P}')\mathbf{U1}^T(\mathbf{P}') + b_1^*(\mathbf{P}')\mathbf{V4}^T(\mathbf{P}') \end{bmatrix}\right|\Psi_0\rangle\, e^{i(\mathbf{P},\mathbf{X})}$$

$$= \iint \frac{d^4P}{(2\pi)^4}\frac{d^4P'}{(2\pi)^4}\langle\Psi_0^*\left|\begin{bmatrix} d_1(\mathbf{P})d_1^*(\mathbf{P}')\mathbf{V1}(\mathbf{P})\mathbf{U1}^T(\mathbf{P}') + b_1(\mathbf{P})b_1^*(\mathbf{P}')\mathbf{U4}(\mathbf{P})\mathbf{V4}^T(\mathbf{P}') \\ +d_4(\mathbf{P})d_4^*(\mathbf{P}')\mathbf{U1}(\mathbf{P})\mathbf{U4}^T(\mathbf{P}') + b_4(\mathbf{P})b_4^*(\mathbf{P}')\mathbf{V4}(\mathbf{P})\mathbf{V1}^T(\mathbf{P}') \end{bmatrix}\right|\Psi_0\rangle\, e^{i(\mathbf{P},\mathbf{X})}$$

$$= \int \frac{d^4P}{(2\pi)^4}\langle\Psi_0^*\left|\begin{bmatrix} \mathbf{V1}(\mathbf{P})\mathbf{U1}^T(\mathbf{P}) + \mathbf{U4}(\mathbf{P})\mathbf{V4}^T(\mathbf{P}) \\ +\mathbf{U1}(\mathbf{P})\mathbf{U4}^T(\mathbf{P}) + \mathbf{V4}(\mathbf{P})\mathbf{V1}^T(\mathbf{P}) \end{bmatrix}\right|\Psi_0\rangle\, e^{i(\mathbf{P},\mathbf{X})}$$

$$= \int \frac{d^4P}{(2\pi)^4}\langle\Psi_0^*\left|\begin{pmatrix} 4P_0^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -4P_2^2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}\right|\Psi_0\rangle\, e^{i(\mathbf{P},\mathbf{X})}$$

For the reasons given above, let us apply the normalization of the field operator

$$\frac{\boldsymbol{\varphi}(\mathbf{X})}{2E_0(\mathbf{P})}$$

As a result, we get

$$\frac{1}{4E_0^2(\mathbf{P})}\langle\Psi_0^*|\boldsymbol{\varphi}(\mathbf{X})\boldsymbol{\varphi}^T(\mathbf{0})|\Psi_0\rangle =$$

$$\int \frac{d^4P}{(2\pi)^4}\frac{\langle\Psi_0^*|\Psi_0\rangle}{4(2P_0^2 - M^2)}\begin{pmatrix} 4P_0^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -4P_2^2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} e^{i(\mathbf{P},\mathbf{X})} =$$

$$= \int \frac{d^4P}{(2\pi)^4}\frac{\langle\Psi_0^*|\Psi_0\rangle}{2P_0^2 - M^2}\begin{pmatrix} P_0^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -P_2^2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} e^{i(\mathbf{P},\mathbf{X})}$$

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If the mass is zero, this expression is the propagator of the photon.

Note that the matrix entering the propagator has no inverse, so we do not try to find the equation of motion or Lagrangian, they are not necessary in this case, since we have an explicit expression for the field operator. We do not have to worry about following the principles of Lorentzian covariance, gauge invariance, or following ideas of symmetry. Instead, we rely only on the fulfilment of canonical commutation relations for the field operator. The field operator is written identically in any frame of reference, and to pass to another frame it is enough to know how the momentum vector is transformed, which is transformed by exactly the same law as the coordinate vector, which ensures the invariance of the phase of the plane wave. In other words, the field is not a vector but a set of pseudovectors (pseudospinors in spinor space), only momentum and coordinate are vectors (spinor).

We can make our reasoning more intuitively clear if we define the birth and annihilation operators of the field particle

$$\boldsymbol{B}(\mathbf{X}) = \int \frac{d^4P}{(2\pi)^4} \begin{bmatrix} b_4^*(\mathbf{P})\mathbf{V1}(\mathbf{P}) + d_4^*(\mathbf{P})\mathbf{U4}(\mathbf{P}) \\ +d_1^*(\mathbf{P})\mathbf{U1}(\mathbf{P}) + b_1^*(\mathbf{P})\mathbf{V4}(\mathbf{P}) \end{bmatrix} e^{-i(\mathbf{P},\mathbf{X})}$$

$$\boldsymbol{A}(\mathbf{X}) = \int \frac{d^4P}{(2\pi)^4} \begin{bmatrix} d_1(\mathbf{P})\mathbf{V1}(\mathbf{P}) + b_1(\mathbf{P})\mathbf{U4}(\mathbf{P}) \\ +d_4(\mathbf{P})\mathbf{U1}(\mathbf{P}) + b_4(\mathbf{P})\mathbf{V4}(\mathbf{P}) \end{bmatrix} e^{i(\mathbf{P},\mathbf{X})}$$

Let us find the commutation relations between the components of these operators

$$[A_i(\mathbf{X}), B_j(\mathbf{X}')] = A_i(\mathbf{X})B_j(\mathbf{X}') - B_j(\mathbf{X}')A_i(\mathbf{X}) = \left(\boldsymbol{A}(\mathbf{X})\boldsymbol{B}^T(\mathbf{X}') - \left(\boldsymbol{B}(\mathbf{X}')\boldsymbol{A}^T(\mathbf{X})\right)^T\right)_{ij}$$

$$\boldsymbol{A}(\mathbf{X})\boldsymbol{B}^T(\mathbf{X}') - \left(\boldsymbol{B}(\mathbf{X}')\boldsymbol{A}^T(\mathbf{X})\right)^T =$$

$$= \iint \frac{d^4P}{(2\pi)^4}\frac{d^4P'}{(2\pi)^4}$$

$$\begin{bmatrix} d_1(\mathbf{P})d_1^*(\mathbf{P}')\mathbf{V1}(\mathbf{P})\mathbf{U1}^T(\mathbf{P}')e^{i(\mathbf{P},\mathbf{X})}e^{-i(\mathbf{P}',\mathbf{X}')} - d_1^*(\mathbf{P}')d_1(\mathbf{P})\mathbf{V1}(\mathbf{P})\mathbf{U1}^T(\mathbf{P}')e^{-i(\mathbf{P}',\mathbf{X}')}e^{i(\mathbf{P},\mathbf{X})} \\ + b_1(\mathbf{P})b_1^*(\mathbf{P}')\mathbf{U4}(\mathbf{P})\mathbf{V4}^T(\mathbf{P}')e^{i(\mathbf{P},\mathbf{X})}e^{-i(\mathbf{P}',\mathbf{X}')} - b_1^*(\mathbf{P}')b_1(\mathbf{P})\mathbf{U4}(\mathbf{P})\mathbf{V4}^T(\mathbf{P}')e^{-i(\mathbf{P}',\mathbf{X}')}e^{i(\mathbf{P},\mathbf{X})} \\ +b_4(\mathbf{P})b_4^*(\mathbf{P}')\mathbf{V4}(\mathbf{P})\mathbf{V1}^T(\mathbf{P}')e^{i(\mathbf{P},\mathbf{X})}e^{-i(\mathbf{P}',\mathbf{X}')} - b_4^*(\mathbf{P}')b_4(\mathbf{P})\mathbf{V4}(\mathbf{P})\mathbf{V1}^T(\mathbf{P}')\,e^{-i(\mathbf{P}',\mathbf{X}')}e^{i(\mathbf{P},\mathbf{X})} \\ +d_4(\mathbf{P})d_4^*(\mathbf{P}')\mathbf{U1}(\mathbf{P})\mathbf{U4}^T(\mathbf{P}')e^{i(\mathbf{P},\mathbf{X})}e^{-i(\mathbf{P}',\mathbf{X}')} - d_4^*(\mathbf{P}')d_4(\mathbf{P})\mathbf{U1}(\mathbf{P})\mathbf{U4}^T(\mathbf{P}')e^{-i(\mathbf{P}',\mathbf{X}')}e^{i(\mathbf{P},\mathbf{X})} \end{bmatrix}$$

$$= \iint \frac{d^4P}{(2\pi)^4}\frac{d^4P'}{(2\pi)^4} \begin{bmatrix} \left(d_1(\mathbf{P})d_1^*(\mathbf{P}') - d_1^*(\mathbf{P}')d_1(\mathbf{P})\right)\mathbf{V1}(\mathbf{P})\mathbf{U1}^T(\mathbf{P}')e^{i(\mathbf{P},\mathbf{X})}e^{-i(\mathbf{P}',\mathbf{X}')} \\ +\left(b_1(\mathbf{P})b_1^*(\mathbf{P}') - b_1^*(\mathbf{P}')b_1(\mathbf{P})\right)\mathbf{U4}(\mathbf{P})\mathbf{V4}^T(\mathbf{P}')e^{i(\mathbf{P},\mathbf{X})}e^{-i(\mathbf{P}',\mathbf{X}')} \\ +\left(b_4(\mathbf{P})b_4^*(\mathbf{P}') - b_4^*(\mathbf{P}')b_4(\mathbf{P})\right)\mathbf{V4}(\mathbf{P})\mathbf{V1}^T(\mathbf{P}')e^{i(\mathbf{P},\mathbf{X})}e^{-i(\mathbf{P}',\mathbf{X}')} \\ +\left(d_4(\mathbf{P})d_4^*(\mathbf{P}') - d_4^*(\mathbf{P}')d_4(\mathbf{P})\right)\mathbf{U1}(\mathbf{P})\mathbf{U4}^T(\mathbf{P}')e^{i(\mathbf{P},\mathbf{X})}e^{-i(\mathbf{P}',\mathbf{X}')} \end{bmatrix}$$

$$= \iint \frac{d^4P}{(2\pi)^4}\frac{d^4P'}{(2\pi)^4} \begin{bmatrix} \left(\delta(\mathbf{P}-\mathbf{P}')\right)\mathbf{V1}(\mathbf{P})\mathbf{U1}^T(\mathbf{P}')e^{i(\mathbf{P},\mathbf{X})}e^{-i(\mathbf{P}',\mathbf{X}')} \\ +\left(\delta(\mathbf{P}-\mathbf{P}')\right)\mathbf{U4}(\mathbf{P})\mathbf{V4}^T(\mathbf{P}')e^{i(\mathbf{P},\mathbf{X})}e^{-i(\mathbf{P}',\mathbf{X}')} \\ +\left(\delta(\mathbf{P}-\mathbf{P}')\right)\mathbf{V4}(\mathbf{P})\mathbf{V1}^T(\mathbf{P}')e^{i(\mathbf{P},\mathbf{X})}e^{-i(\mathbf{P}',\mathbf{X}')} \\ +\left(\delta(\mathbf{P}-\mathbf{P}')\right)\mathbf{U1}(\mathbf{P})\mathbf{U4}^T(\mathbf{P}')e^{i(\mathbf{P},\mathbf{X})}e^{-i(\mathbf{P}',\mathbf{X}')} \end{bmatrix}$$

$$= \int \frac{d^4P}{(2\pi)^4} \begin{bmatrix} \mathbf{V1}(\mathbf{P})\mathbf{U1}^T(\mathbf{P})e^{i(\mathbf{P},\mathbf{X})}e^{-i(\mathbf{P},\mathbf{X}')} \\ +\mathbf{U4}(\mathbf{P})\mathbf{V4}^T(\mathbf{P})e^{i(\mathbf{P},\mathbf{X})}e^{-i(\mathbf{P},\mathbf{X}')} \\ +\mathbf{V4}(\mathbf{P})\mathbf{V1}^T(\mathbf{P})e^{i(\mathbf{P},\mathbf{X})}e^{-i(\mathbf{P},\mathbf{X}')} \\ +\mathbf{U1}(\mathbf{P})\mathbf{U4}^T(\mathbf{P})e^{i(\mathbf{P},\mathbf{X})}e^{-i(\mathbf{P},\mathbf{X}')} \end{bmatrix}$$

$$= \int \frac{d^4P}{(2\pi)^4} \begin{bmatrix} \mathbf{V1}(\mathbf{P})\mathbf{U1}^T(\mathbf{P}) \\ +\mathbf{U4}(\mathbf{P})\mathbf{V4}^T(\mathbf{P}) \\ +\mathbf{V4}(\mathbf{P})\mathbf{V1}^T(\mathbf{P}) \\ +\mathbf{U1}(\mathbf{P})\mathbf{U4}^T(\mathbf{P}) \end{bmatrix} e^{-i(\mathbf{P},\mathbf{X}-\mathbf{X}')}$$

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$$= \int \frac{d^4P}{(2\pi)^4} \begin{pmatrix} 4P_0^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -4P_2^2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} e^{-i(\mathbf{P},\mathbf{X}-\mathbf{X}')}$$

$$= \begin{pmatrix} 4P_0^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -4P_2^2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \int \frac{d^4P}{(2\pi)^4} e^{-i(\mathbf{P},\mathbf{X}-\mathbf{X}')}$$

$$= \begin{pmatrix} 4P_0^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -4P_2^2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \delta(\mathbf{X}-\mathbf{X}')$$

$$A_i(\mathbf{X})B_j(\mathbf{X}') - B_j(\mathbf{X}')A_i(\mathbf{X}) = \delta(\mathbf{X}-\mathbf{X}') \begin{pmatrix} 4P_0^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -4P_2^2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}_{ij}$$

As we see, the commutation relations are satisfied for the birth and annihilation operators. Let us define the total particle number operator in the form

$$N_{ji}(\mathbf{X}) = B_j(\mathbf{X})A_i(\mathbf{X})$$

$$N_{ji} = \int d^4X\, B_j(\mathbf{X})A_i(\mathbf{X})$$

Let's find the commutator

$$[N_{ji}, B_j(\mathbf{X})] = \int d^4X' \{B_j(\mathbf{X}')A_i(\mathbf{X}')B_j(\mathbf{X}) - B_j(\mathbf{X})B_j(\mathbf{X}')A_i(\mathbf{X}')\} =$$

$$\int d^4X' \{B_j(\mathbf{X}')A_i(\mathbf{X}')B_j(\mathbf{X}) - B_j(\mathbf{X}')B_j(\mathbf{X})A_i(\mathbf{X}')\} =$$

$$\int d^4X' \left\{B_j(\mathbf{X}')\left(A_i(\mathbf{X}')B_j(\mathbf{X}) - B_j(\mathbf{X})A_i(\mathbf{X}')\right)\right\} =$$

$$\int d^4X' \{B_j(\mathbf{X}')\delta(\mathbf{X}'-\mathbf{X})\} \begin{pmatrix} 4P_0^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -4P_2^2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}_{ij} = B_j(\mathbf{X}) \begin{pmatrix} 4P_0^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -4P_2^2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}_{ij}$$

Let's define the vacuum state using the relations

$$d_1(\mathbf{P})|\Psi_0\rangle = 0 \quad b_1(\mathbf{P})|\Psi_0\rangle = 0 \quad d_4(\mathbf{P})|\Psi_0\rangle = 0 \quad b_4(\mathbf{P})|\Psi_0\rangle = 0$$

which implies

$$A_i(\mathbf{X})|\Psi_0\rangle = 0$$

$$N_{ji}|\Psi_0\rangle = \int d^4X\, B_j(\mathbf{X})A_i(\mathbf{X})|\Psi_0\rangle = 0$$

Let's act on vacuum by the birth operator and for the obtained state we find eigenvalues of the particle number operator

$$\begin{pmatrix} 4P_0^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -4P_2^2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}_{ij} B_j(\mathbf{X}) = [N_{ji}, B_j(\mathbf{X})] = N_{ji}B_j(\mathbf{X}) - B_j(\mathbf{X})N_{ji}$$

$$\begin{pmatrix} 4P_0^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -4P_2^2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}_{ij} B_j(\mathbf{X})|\Psi_0\rangle = N_{ji}B_j(\mathbf{X})|\Psi_0\rangle - B_j(\mathbf{X})N_{ji}|\Psi_0\rangle$$

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$$N_{ji}\left(B_j(\mathbf{X})|\Psi_0\rangle\right) = \begin{pmatrix} 4P_0^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -4P_2^2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}_{ij} \left(B_j(\mathbf{X})|\Psi_0\rangle\right)$$

If we apply normalization

$$\frac{A(\mathbf{X})}{2E_0(\mathbf{P})} \quad \frac{B(\mathbf{X})}{2E_0(\mathbf{P})}$$

then the eigenvalues will have the form

$$N_{ji}\left(B_j(\mathbf{X})|\Psi_0\rangle\right) = \frac{1}{2P_0^2 - M^2}\begin{pmatrix} P_0^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -P_2^2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}_{ij} \left(B_j(\mathbf{X})|\Psi_0\rangle\right)$$

Note that in the case of the photon field, the matrix, taking into account the normalization, contains elements whose modulus is less than or equal to ½, since at zero mass $P_2^2 \le P_0^2$.

The fact that for the birth and annihilation operators commutation relations are fulfilled, allows to conclude that quanta of the field obey Bose statistics, therefore a single action of the birth operator increases the number of particles in the field by one, and the action of the annihilation operator decreases this number by one. Hence, by means of these operators it is possible to write the propagator not only for the case when the initial and final states are vacuum, but also for the initial state with an arbitrary number of particles

$$\frac{\langle\Psi_n^*|\boldsymbol{A}(\mathbf{X})\boldsymbol{B}^{\mathrm{T}}(\mathbf{0})|\Psi_n\rangle}{4E_0^2(\mathbf{P})} = \int \frac{d^4P}{(2\pi)^4}\frac{\langle\Psi_n^*|\Psi_n\rangle}{2P_0^2 - M^2}\begin{pmatrix} P_0^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -P_2^2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} e^{i(\mathbf{P},\mathbf{X})}$$

For illustration let us consider a one-particle state

$$|\Psi_1\rangle \equiv B_j(\mathbf{X})|\Psi_0\rangle$$

$$N_{ji}|\Psi_1\rangle = \frac{1}{2P_0^2 - M^2}\begin{pmatrix} P_0^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -P_2^2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}_{ij} |\Psi_1\rangle$$

and act on it with the birth operator. Again, let's take into account

$$\frac{1}{2P_0^2 - M^2}\begin{pmatrix} P_0^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -P_2^2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}_{ij} B_j(\mathbf{X}) = \left[N_{ji}, B_j(\mathbf{X})\right] = N_{ji}B_j(\mathbf{X}) - B_j(\mathbf{X})N_{ji}$$

$$\frac{1}{2P_0^2 - M^2}\begin{pmatrix} P_0^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -P_2^2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}_{ij} B_j(\mathbf{X})|\Psi_1\rangle = N_{ji}B_j(\mathbf{X})|\Psi_1\rangle - B_j(\mathbf{X})N_{ji}|\Psi_1\rangle$$

$$= N_{ji}B_j(\mathbf{X})|\Psi_1\rangle - B_j(\mathbf{X})\frac{1}{2P_0^2 - M^2}\begin{pmatrix} P_0^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -P_2^2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}_{ij} |\Psi_1\rangle$$

The result is

$$N_{ji}B_j(\mathbf{X})|\Psi_1\rangle = 2\frac{1}{2P_0^2 - M^2}\begin{pmatrix} P_0^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -P_2^2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}_{ij} B_j(\mathbf{X})|\Psi_1\rangle$$

The eigenvalue of the particle number operator has increased, instead of a one-particle state we have a two-particle state

$$|\Psi_2\rangle \equiv B_j(\mathbf{X})|\Psi_1\rangle$$

$$N_{ji}|\Psi_2\rangle = 2\frac{1}{2P_0^2 - M^2}\begin{pmatrix} P_0^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -P_2^2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}_{ij} |\Psi_2\rangle$$

Further application of the birth operator increases the number of particles to any value. Now let us find a commutator for the annihilation operator, without taking into account the normalization for the moment

$$[N_{ji}, A_i(\mathbf{X})] = \int d^4X' \{B_j(\mathbf{X}')A_i(\mathbf{X}')A_i(\mathbf{X}) - A_i(\mathbf{X})B_j(\mathbf{X}')A_i(\mathbf{X}')\} =$$

$$\int d^4X' \{B_j(\mathbf{X}')A_i(\mathbf{X})A_i(\mathbf{X}') - A_i(\mathbf{X})B_j(\mathbf{X}')A_i(\mathbf{X}')\} =$$

$$\int d^4X' \left\{\left(B_j(\mathbf{X}')A_i(\mathbf{X}) - A_i(\mathbf{X})B_j(\mathbf{X}')\right) A_i(\mathbf{X}')\right\} =$$

$$\int d^4X' \{-\delta(\mathbf{X}' - \mathbf{X})A_i(\mathbf{X}')\}\begin{pmatrix} 4P_0^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -4P_2^2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}_{ij} = -A_i(\mathbf{X})\begin{pmatrix} 4P_0^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -4P_2^2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}_{ij}$$

The ratios have been taken into account here

$$A_i(\mathbf{X})B_j(\mathbf{X}') - B_j(\mathbf{X}')A_i(\mathbf{X}) = \delta(\mathbf{X} - \mathbf{X}')\begin{pmatrix} 4P_0^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -4P_2^2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}_{ij}$$

$$B_j(\mathbf{X}')A_i(\mathbf{X}) - A_i(\mathbf{X})B_j(\mathbf{X}') = -\delta(\mathbf{X} - \mathbf{X}')\begin{pmatrix} 4P_0^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -4P_2^2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}_{ij}$$

Let's act on the two-particle state by the annihilation operator and for the obtained state we find the eigenvalues of the particle number operator

$$-\begin{pmatrix} 4P_0^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -4P_2^2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}_{ij} A_i(\mathbf{X}) = [N_{ji}, A_i(\mathbf{X})] = N_{ji}A_i(\mathbf{X}) - A_i(\mathbf{X})N_{ji}$$

$$-\begin{pmatrix} 4P_0^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -4P_2^2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}_{ij} A_i(\mathbf{X})|\Psi_2\rangle = N_{ji}A_i(\mathbf{X})|\Psi_2\rangle - A_i(\mathbf{X})N_{ji}|\Psi_2\rangle$$

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$$N_{ji}(A_i(\mathbf{X})|\Psi_2\rangle) = -\begin{pmatrix} 4P_0^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -4P_2^2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}_{ij} A_i(\mathbf{X})|\Psi_2\rangle + 2\begin{pmatrix} 4P_0^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -4P_2^2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}_{ij} (A_i(\mathbf{X})|\Psi_2\rangle)$$

Here, the fact that without taking into account the rationing

$$N_{ji}|\Psi_2\rangle = 2\begin{pmatrix} 4P_0^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -4P_2^2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}_{ij} |\Psi_2\rangle$$

Thus, the annihilation operator reduces the number of particles and puts the field into a single-particle state.

Separate application of the birth and annihilation operators more corresponds to the ideology of second quantization than their use only as a sum, i.e. only as a field operator

$$\boldsymbol{\varphi}(\mathbf{X}) = \boldsymbol{A}(\mathbf{X}) + \boldsymbol{B}(\mathbf{X})$$

In particular, since

$$\langle\Psi_0^*|\boldsymbol{\varphi}(\mathbf{X})\boldsymbol{\varphi}^T(\mathbf{0})|\Psi_0\rangle = \langle\Psi_0^*|\boldsymbol{A}(\mathbf{X})\boldsymbol{B}^T(\mathbf{0})|\Psi_0\rangle$$

then the propagator really acquires the sense of the amplitude of the probability that the particle is born at the origin and annihilated at the point with coordinates $\mathbf{X}$.

Moreover, now the propagator can be not bound to the vacuum state, but can be applied to the field state with arbitrary number of particles n>0. The application of the sum of operators to some state makes sense only in the case when all operators except one give as a result zero. Therefore, at the usual approach we have to work only with the vacuum state so that at calculation of the propagator the annihilation operator gives zero. In our approach this restriction is removed, the operators are not summed, but only multiplied, and they can be applied to a state with any number of particles. For this purpose, let us take into account the following relations

$$\langle\Psi(\mathbf{P},d_1)_n^*|d_1(\mathbf{P})d_1^*(\mathbf{P}')|\Psi(\mathbf{P},d_1)_n\rangle = \langle\Psi(\mathbf{P},d_1)_n^*|d_1^*(\mathbf{P})d_1(\mathbf{P}')|\Psi(\mathbf{P},d_1)_n\rangle$$
$$= \langle\Psi(\mathbf{P},d_1)_n^*|\delta(\mathbf{P}-\mathbf{P}')|\Psi(\mathbf{P},d_1)_n\rangle = \langle\Psi(\mathbf{P})_n^*|\delta(\mathbf{P}-\mathbf{P}')|\Psi(\mathbf{P})_n\rangle$$
$$\langle\Psi(\mathbf{P},b_1)_n^*|b_1(\mathbf{P})b_1^*(\mathbf{P}')|\Psi(\mathbf{P},b_1)_n\rangle = \langle\Psi(\mathbf{P},b_1)_n^*|b_1^*(\mathbf{P})b_1(\mathbf{P}')|\Psi(\mathbf{P},b_1)_n\rangle$$
$$= \langle\Psi(\mathbf{P},b_1)_n^*|\delta(\mathbf{P}-\mathbf{P}')|\Psi(\mathbf{P},b_1)_n\rangle = \langle\Psi(\mathbf{P})_n^*|\delta(\mathbf{P}-\mathbf{P}')|\Psi(\mathbf{P})_n\rangle$$
$$\langle\Psi(\mathbf{P},d_4)_n^*|d_4(\mathbf{P})d_4^*(\mathbf{P}')|\Psi(\mathbf{P},d_4)_n\rangle = \langle\Psi(\mathbf{P},d_4)_n^*|d_4^*(\mathbf{P})d_4(\mathbf{P}')|\Psi(\mathbf{P},d_4)_n\rangle$$
$$= \langle\Psi(\mathbf{P},d_4)_n^*|\delta(\mathbf{P}-\mathbf{P}')|\Psi(\mathbf{P},d_4)_n\rangle = \langle\Psi(\mathbf{P})_n^*|\delta(\mathbf{P}-\mathbf{P}')|\Psi(\mathbf{P})_n\rangle$$
$$\langle\Psi(\mathbf{P},b_4)_n^*|b_4(\mathbf{P})b_4^*(\mathbf{P}')|\Psi(\mathbf{P},b_4)_n\rangle = \langle\Psi(\mathbf{P},b_4)_n^*|b_4^*(\mathbf{P})b_4(\mathbf{P}')|\Psi(\mathbf{P},b_4)_n\rangle$$
$$= \langle\Psi(\mathbf{P},b_4)_n^*|\delta(\mathbf{P}-\mathbf{P}')|\Psi(\mathbf{P},b_4)_n\rangle = \langle\Psi(\mathbf{P})_n^*|\delta(\mathbf{P}-\mathbf{P}')|\Psi(\mathbf{P})_n\rangle$$

$$\langle\Psi_n^*|\boldsymbol{A}(\mathbf{X})\boldsymbol{B}^T(\mathbf{0})|\Psi_n\rangle = \iint \frac{d^4P}{(2\pi)^4}\frac{d^4P'}{(2\pi)^4}$$
$$\langle\Psi(\mathbf{P})_n^*\left|\begin{bmatrix} d_1(\mathbf{P})\mathbf{V1}(\mathbf{P}) + b_1(\mathbf{P})\mathbf{U4}(\mathbf{P}) \\ +d_4(\mathbf{P})\mathbf{U1}(\mathbf{P}) + b_4(\mathbf{P})\mathbf{V4}(\mathbf{P})\end{bmatrix}\begin{bmatrix} b_4^*(\mathbf{P}')\mathbf{V1}^T(\mathbf{P}') + d_4^*(\mathbf{P}')\mathbf{U4}^T(\mathbf{P}') \\ +d_1^*(\mathbf{P}')\mathbf{U1}^T(\mathbf{P}') + b_1^*(\mathbf{P}')\mathbf{V4}^T(\mathbf{P}')\end{bmatrix}\right|\Psi(\mathbf{P})_n\rangle e^{i(\mathbf{P},\mathbf{X})}$$

$$= \iint \frac{d^4P}{(2\pi)^4}\frac{d^4P'}{(2\pi)^4}$$
$$\langle\Psi(\mathbf{P})_n^*\left|\begin{bmatrix} d_1(\mathbf{P})d_1^*(\mathbf{P}')\mathbf{V1}(\mathbf{P})\mathbf{U1}^T(\mathbf{P}') + b_1(\mathbf{P})b_1^*(\mathbf{P}')\mathbf{U4}(\mathbf{P})\mathbf{V4}^T(\mathbf{P}') \\ +d_4(\mathbf{P})d_4^*(\mathbf{P}')\mathbf{U1}(\mathbf{P})\mathbf{U4}^T(\mathbf{P}') + b_4(\mathbf{P})b_4^*(\mathbf{P}')\mathbf{V4}(\mathbf{P})\mathbf{V1}^T(\mathbf{P}')\end{bmatrix}\right|\Psi(\mathbf{P})_n\rangle e^{i(\mathbf{P},\mathbf{X})}$$

$$= \int \frac{d^4P}{(2\pi)^4}\langle\Psi(\mathbf{P})_n^*\left|\begin{bmatrix}\mathbf{V1}(\mathbf{P})\mathbf{U1}^T(\mathbf{P}) + \mathbf{U4}(\mathbf{P})\mathbf{V4}^T(\mathbf{P}) \\ +\mathbf{U1}(\mathbf{P})\mathbf{U4}^T(\mathbf{P}) + \mathbf{V4}(\mathbf{P})\mathbf{V1}^T(\mathbf{P})\end{bmatrix}\right|\Psi(\mathbf{P})_n\rangle e^{i(\mathbf{P},\mathbf{X})}$$

$$= \int \frac{d^4P}{(2\pi)^4} \langle \Psi(\mathbf{P})_n^* \left| \begin{pmatrix} 4P_0^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -4P_2^2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \right| \Psi(\mathbf{P})_n \rangle e^{i(\mathbf{P},\mathbf{X})}$$

$$= \int \frac{d^4P}{(2\pi)^4} \langle \Psi(\mathbf{P})_n^* | \Psi(\mathbf{P})_n \rangle \begin{pmatrix} 4P_0^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -4P_2^2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} e^{i(\mathbf{P},\mathbf{X})}$$

$$= \langle \Psi_n^* | \Psi_n \rangle \int \frac{d^4P}{(2\pi)^4} \begin{pmatrix} 4P_0^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -4P_2^2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} e^{i(\mathbf{P},\mathbf{X})}$$

The assumption used here is that the scalar products $\langle \langle \Psi(\mathbf{P})_n^* | \Psi(\mathbf{P})_n \rangle = \langle \Psi_n^* | \Psi_n \rangle$ are the same for any values of momentum. Taking into account the normalisation

$$\langle \Psi_n^* | \boldsymbol{A}(\mathbf{X}) \boldsymbol{B}^T(\mathbf{0}) | \Psi_n \rangle = \int \frac{d^4P}{(2\pi)^4} \frac{\langle \Psi_n^* | \Psi_n \rangle}{2P_0^2 - M^2} \begin{pmatrix} P_0^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -P_2^2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} e^{i(\mathbf{P},\mathbf{X})}$$

At non-zero number of particles we can change the order of operators and first apply the annihilation operator

$$\langle \Psi_n^* | \boldsymbol{B}(\mathbf{X}) \boldsymbol{A}^T(\mathbf{0}) | \Psi_n \rangle = \int \int \frac{d^4P}{(2\pi)^4} \frac{d^4P'}{(2\pi)^4}$$

$$\langle \Psi(\mathbf{P})_n^* \left| \begin{bmatrix} b_4^*(\mathbf{P})\mathbf{V1}(\mathbf{P}) + d_4^*(\mathbf{P})\mathbf{U4}(\mathbf{P}) \\ +d_1^*(\mathbf{P})\mathbf{U1}(\mathbf{P}) + b_1^*(\mathbf{P})\mathbf{V4}(\mathbf{P}) \end{bmatrix} \begin{bmatrix} d_1(\mathbf{P}')\mathbf{V1}^T(\mathbf{P}') + b_1(\mathbf{P}')\mathbf{U4}^T(\mathbf{P}') \\ +d_4(\mathbf{P}')\mathbf{U1}^T(\mathbf{P}') + b_4(\mathbf{P}')\mathbf{V4}^T(\mathbf{P}') \end{bmatrix} \right| \Psi(\mathbf{P})_n \rangle e^{-i(\mathbf{P},\mathbf{X})}$$

$$= \int \int \frac{d^4P}{(2\pi)^4} \frac{d^4P'}{(2\pi)^4}$$

$$\langle \Psi(\mathbf{P})_n^* \left| \begin{bmatrix} d_1^*(\mathbf{P}) d_1(\mathbf{P}')\mathbf{U1}(\mathbf{P})\mathbf{V1}^T(\mathbf{P}') + b_1^*(\mathbf{P}) b_1(\mathbf{P}')\mathbf{V4}(\mathbf{P})\mathbf{U4}^T(\mathbf{P}') \\ +d_4^*(\mathbf{P}) d_4(\mathbf{P}')\mathbf{U4}(\mathbf{P})\mathbf{U1}^T(\mathbf{P}') + b_4^*(\mathbf{P}) b_4(\mathbf{P}')\mathbf{V1}(\mathbf{P})\mathbf{V4}^T(\mathbf{P}') \end{bmatrix} \right| \Psi(\mathbf{P})_n \rangle e^{-i(\mathbf{P},\mathbf{X})}$$

$$= \int \frac{d^4P}{(2\pi)^4} \langle \Psi(\mathbf{P})_n^* \left| \begin{bmatrix} \mathbf{U1}(\mathbf{P})\mathbf{V1}^T(\mathbf{P}) + \mathbf{V4}(\mathbf{P})\mathbf{U4}^T(\mathbf{P}) \\ +\mathbf{U4}(\mathbf{P})\mathbf{U1}^T(\mathbf{P}) + \mathbf{V1}(\mathbf{P})\mathbf{V4}^T(\mathbf{P}) \end{bmatrix} \right| \Psi(\mathbf{P})_n \rangle e^{-i(\mathbf{P},\mathbf{X})}$$

$$= \int \frac{d^4P}{(2\pi)^4} \langle \Psi(\mathbf{P})_n^* \left| \begin{pmatrix} 4P_0^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -4P_2^2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \right| \Psi(\mathbf{P})_n \rangle e^{-i(\mathbf{P},\mathbf{X})}$$

$$= \int \frac{d^4P}{(2\pi)^4} \langle \Psi(\mathbf{P})_n^* | \Psi(\mathbf{P})_n \rangle \begin{pmatrix} 4P_0^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -4P_2^2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} e^{-i(\mathbf{P},\mathbf{X})}$$

$$= \langle \Psi_n^* | \Psi_n \rangle \int \frac{d^4P}{(2\pi)^4} \begin{pmatrix} 4P_0^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -4P_2^2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} e^{-i(\mathbf{P},\mathbf{X})}$$

After normalization we obtain

$$\langle \Psi_n^* | \boldsymbol{B}(\mathbf{X}) \boldsymbol{A}^T(\mathbf{0}) | \Psi_n \rangle = \int \frac{d^4P}{(2\pi)^4} \frac{\langle \Psi_n^* | \Psi_n \rangle}{2P_0^2 - M^2} \begin{pmatrix} P_0^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -P_2^2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} e^{-i(\mathbf{P},\mathbf{X})}$$

Let's return to the previously used definition of the vacuum state by means of relations

$$d_1(\mathbf{P})|\Psi_0\rangle = 0 \quad b_1(\mathbf{P})|\Psi_0\rangle = 0 \quad d_4(\mathbf{P})|\Psi_0\rangle = 0 \quad b_4(\mathbf{P})|\Psi_0\rangle = 0$$

$$d_1^*(\mathbf{P}) d_1(\mathbf{P})|\Psi_0\rangle = 0 \quad b_1^*(\mathbf{P}) b_1(\mathbf{P})|\Psi_0\rangle = 0 \quad d_4^*(\mathbf{P}) d_4(\mathbf{P})|\Psi_0\rangle = 0 \quad b_4^*(\mathbf{P}) b_4(\mathbf{P})|\Psi_0\rangle = 0$$

which implies

$$A_i(\mathbf{X})|\Psi_0\rangle = 0$$

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$$N_{ji}|\Psi_0\rangle = \int d^4X\, B_j(\mathbf{X})A_i(\mathbf{X})|\Psi_0\rangle = 0$$

As we have seen, the action of the birth operator transforms the zero-particle state into a one-particle state

$$N_{ji}\big(B_j(\mathbf{X})|\Psi_0\rangle\big) = N_{ji}|\Psi_1\rangle = \frac{1}{2P_0^2 - M^2}\begin{pmatrix} P_0^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -P_2^2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}_{ij} |\Psi_1\rangle$$

At that, none of the operators of the number of particles with a particular value of momentum

$$d_1^*(\mathbf{P})d_1(\mathbf{P})|\Psi_1\rangle \quad b_1^*(\mathbf{P})b_1(\mathbf{P})|\Psi_1\rangle \quad d_4^*(\mathbf{P})d_4(\mathbf{P})|\Psi_1\rangle \quad b_4^*(\mathbf{P})b_4(\mathbf{P})|\Psi_1\rangle$$

has no definite meaning, since the particle is only one. In this connection it makes sense not to define the vacuum in such a detailed way, it is enough to define that the vacuum state is characterized by only one condition

$$N_{ji}|\Psi_0\rangle = \int d^4X\, B_j(\mathbf{X})A_i(\mathbf{X})|\Psi_0\rangle = 0$$

At this approach the field energy is not equal to the sum of energies of partial oscillations, accordingly the question about the energy of zero-point oscillations of each oscillator constituting the field is removed. We get rid of the problem of infinite energy of the sum of zero-point vibrations of an infinite number of oscillators.

We would like the propagator to have properties of the Green's function, i.e., to satisfy the Klein-Gordon equation of motion

$$-\left(\frac{\partial^2}{\partial {X_0}^2} - \frac{\partial^2}{\partial {X_1}^2} - \frac{\partial^2}{\partial {X_2}^2} - \frac{\partial^2}{\partial {X_3}^2} + m^2\right) D(\mathbf{X}) = \delta(\mathbf{X})$$

The solution of this equation has the form

$$D(\mathbf{X}) = \int \frac{d^4P}{(2\pi)^4} \frac{e^{i(\mathbf{P},\mathbf{X})}}{P_0^2 - P_1^2 - P_2^2 - P_3^2 - M^2}$$

Therefore, we add the same multiplier to the denominator of the integrand expression

$$\langle\Psi_n^*|\boldsymbol{B}(\mathbf{X})\boldsymbol{A}^T(\mathbf{0})|\Psi_n\rangle = \int \frac{d^4P}{(2\pi)^4} \frac{\langle\Psi_n^*|\Psi_n\rangle}{2P_0^2 - M^2}\begin{pmatrix} P_0^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -P_2^2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \frac{e^{-i(\mathbf{P},\mathbf{X})}}{P_0^2 - P_1^2 - P_2^2 - P_3^2 - M^2}$$

By analogy with the introduced birth and annihilation operators for fields in vector space, let us describe the corresponding operators for fields in spinor space. As an initial one we use the previously described field operator for the fermionic field

$$\boldsymbol{\varphi}(\mathbf{x}) = \int \frac{d^4p}{(2\pi)^4}$$

$$\begin{bmatrix} d_1(\mathbf{p})\mathbf{u1}(\mathbf{p}) + id_2(\mathbf{p})\mathbf{u3}(\mathbf{p}) + ib_2(\mathbf{p})\overline{\mathbf{u2}}(\mathbf{p}) + \ b_1(\mathbf{p})\overline{\mathbf{u4}}(\mathbf{p}) \\ +d_4(\mathbf{p})\mathbf{v1}(\mathbf{p}) + id_3(\mathbf{p})\mathbf{v3}(\mathbf{p}) + ib_3(\mathbf{p})\overline{\mathbf{v2}}(\mathbf{p}) + b_4(\mathbf{p})\overline{\mathbf{v4}}(\mathbf{p}) \end{bmatrix} e^{i(p_0x_1 - p_1x_0 + p_2x_3 - p_3x_2 + \overline{(\mathbf{p},\mathbf{x})})}$$

$$+\begin{bmatrix} b_1^*(\mathbf{p})\overline{\mathbf{u1}}(\mathbf{p}) + ib_2^*(\mathbf{p})\overline{\mathbf{u3}}(\mathbf{p}) + id_2^*(\mathbf{p})\mathbf{u2}(\mathbf{p}) + d_1^*(\mathbf{p})\mathbf{u4}(\mathbf{p}) \\ +b_4^*(\mathbf{p})\overline{\mathbf{v1}}(\mathbf{p}) + ib_3^*(\mathbf{p})\overline{\mathbf{v3}}(\mathbf{p}) + id_3^*(\mathbf{p})\mathbf{v2}(\mathbf{p}) + d_4^*(\mathbf{p})\mathbf{v4}(\mathbf{p}) \end{bmatrix} e^{-i(p_0x_1 - p_1x_0 + p_2x_3 - p_3x_2 + \overline{(\mathbf{p},\mathbf{x})})}$$

Let us define the birth and annihilation operators

$$\boldsymbol{b}(\mathbf{x}) = \int \frac{d^4p}{(2\pi)^4}$$

$$\begin{bmatrix} b_1^*(\mathbf{p})\overline{\mathbf{u1}}(\mathbf{p}) + ib_2^*(\mathbf{p})\overline{\mathbf{u3}}(\mathbf{p}) + id_2^*(\mathbf{p})\mathbf{u2}(\mathbf{p}) + d_1^*(\mathbf{p})\mathbf{u4}(\mathbf{p}) \\ +b_4^*(\mathbf{p})\overline{\mathbf{v1}}(\mathbf{p}) + ib_3^*(\mathbf{p})\overline{\mathbf{v3}}(\mathbf{p}) + id_3^*(\mathbf{p})\mathbf{v2}(\mathbf{p}) + d_4^*(\mathbf{p})\mathbf{v4}(\mathbf{p}) \end{bmatrix} e^{-i(p_0x_1 - p_1x_0 + p_2x_3 - p_3x_2 + \overline{(\mathbf{p},\mathbf{x})})}$$

$$\boldsymbol{a}(\mathbf{x}) = \int \frac{d^4p}{(2\pi)^4}$$

$$\begin{bmatrix} d_1(\mathbf{p})\mathbf{u1}(\mathbf{p}) + id_2(\mathbf{p})\mathbf{u3}(\mathbf{p}) + ib_2(\mathbf{p})\overline{\mathbf{u2}}(\mathbf{p}) + \ b_1(\mathbf{p})\overline{\mathbf{u4}}(\mathbf{p}) \\ +d_4(\mathbf{p})\mathbf{v1}(\mathbf{p}) + id_3(\mathbf{p})\mathbf{v3}(\mathbf{p}) + ib_3(\mathbf{p})\overline{\mathbf{v2}}(\mathbf{p}) + b_4(\mathbf{p})\overline{\mathbf{v4}}(\mathbf{p}) \end{bmatrix} e^{i(p_0x_1 - p_1x_0 + p_2x_3 - p_3x_2 + \overline{(\mathbf{p},\mathbf{x})})}$$

Let's find anticommutation relations between components of these operators

$$\{a_i(\mathbf{x}), b_j(\mathbf{x}')\} = a_i(\mathbf{x})b_j(\mathbf{x}') + b_j(\mathbf{x}')a_i(\mathbf{x}) = \left(\boldsymbol{a}(\mathbf{x})\boldsymbol{b}(\mathbf{x}') + \left(\boldsymbol{b}(\mathbf{x}')\boldsymbol{a}^T(\mathbf{x})\right)^T\right)_{ij}$$

$$\boldsymbol{a}(\mathbf{x})\boldsymbol{b}^T(\mathbf{x}') + \left(\boldsymbol{b}(\mathbf{x}')\boldsymbol{a}^T(\mathbf{x})\right)^T =$$

$$= \iint \frac{d^4p}{(2\pi)^4}\frac{d^4p'}{(2\pi)^4}$$

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$$\{a_i(\mathbf{x}), b_j(\mathbf{x}')\} = a_i(\mathbf{x})b_j(\mathbf{x}') + b_j(\mathbf{x}')a_i(\mathbf{x}) = \left(\boldsymbol{a}(\mathbf{x})\boldsymbol{b}(\mathbf{x}') + \left(\boldsymbol{b}(\mathbf{x}')\boldsymbol{a}^T(\mathbf{x})\right)^T\right)_{ij}$$

$$\boldsymbol{a}(\mathbf{x})\boldsymbol{b}^T(\mathbf{x}') + \left(\boldsymbol{b}(\mathbf{x}')\boldsymbol{a}^T(\mathbf{x})\right)^T =$$

$$= \iint \frac{d^4p}{(2\pi)^4}\frac{d^4p'}{(2\pi)^4}$$

$$\begin{bmatrix} d_1(\mathbf{p})\mathbf{u1}(\mathbf{p}) + id_2(\mathbf{p})\mathbf{u3}(\mathbf{p}) + ib_2(\mathbf{p})\overline{\mathbf{u2}}(\mathbf{p}) + b_1(\mathbf{p})\overline{\mathbf{u4}}(\mathbf{p}) \\ +d_4(\mathbf{p})\mathbf{v1}(\mathbf{p}) + id_3(\mathbf{p})\mathbf{v3}(\mathbf{p}) + ib_3(\mathbf{p})\overline{\mathbf{v2}}(\mathbf{p}) + b_4(\mathbf{p})\overline{\mathbf{v4}}(\mathbf{p}) \end{bmatrix}$$

$$\begin{bmatrix} b_1^*(\mathbf{p}')\mathbf{u1}^+(\mathbf{p}') + ib_2^*(\mathbf{p}')\mathbf{u3}^+(\mathbf{p}') + id_2^*(\mathbf{p}')\mathbf{u2}^{\boldsymbol{T}}(\mathbf{p}') + d_1^*(\mathbf{p}')\mathbf{u4}^{\boldsymbol{T}}(\mathbf{p}') \\ +b_4^*(\mathbf{p}')\mathbf{v1}^+(\mathbf{p}') + ib_3^*(\mathbf{p}')\mathbf{v3}^+(\mathbf{p}') + id_3^*(\mathbf{p}')\mathbf{v2}^{\boldsymbol{T}}(\mathbf{p}') + d_4^*(\mathbf{p}')\mathbf{v4}^{\boldsymbol{T}}(\mathbf{p}') \end{bmatrix}$$

$$e^{i(p_0x_1 - p_1x_0 + p_2x_3 - p_3x_2 + \overline{(\mathbf{p},\mathbf{x})})} e^{-i\left(p_0'x_1' - p_1'x_0' + p_2'x_3' - p_3'x_2' + \overline{(\mathbf{p}',\mathbf{x}')}\right)}$$

$$+$$

$$\begin{pmatrix} \begin{bmatrix} b_1^*(\mathbf{p}')\overline{\mathbf{u1}}(\mathbf{p}') + ib_2^*(\mathbf{p}')\overline{\mathbf{u3}}(\mathbf{p}') + id_2^*(\mathbf{p}')\mathbf{u2}(\mathbf{p}') + d_1^*(\mathbf{p}')\mathbf{u4}(\mathbf{p}') \\ +b_4^*(\mathbf{p}')\overline{\mathbf{v1}}(\mathbf{p}') + ib_3^*(\mathbf{p}')\overline{\mathbf{v3}}(\mathbf{p}') + id_3^*(\mathbf{p}')\mathbf{v2}(\mathbf{p}') + d_4^*(\mathbf{p}')\mathbf{v4}(\mathbf{p}') \end{bmatrix} \\ \begin{bmatrix} d_1(\mathbf{p})\mathbf{u1}^{\boldsymbol{T}}(\mathbf{p}) + id_2(\mathbf{p})\mathbf{u3}^{\boldsymbol{T}}(\mathbf{p}) + ib_2(\mathbf{p})\mathbf{u2}^+(\mathbf{p}) + b_1(\mathbf{p})\mathbf{u4}^+(\mathbf{p}) \\ +d_4(\mathbf{p})\mathbf{v1}^{\boldsymbol{T}}(\mathbf{p}) + id_3(\mathbf{p})\mathbf{v3}^{\boldsymbol{T}}(\mathbf{p}) + ib_3(\mathbf{p})\mathbf{v2}^+(\mathbf{p}) + b_4(\mathbf{p})\mathbf{v4}^+(\mathbf{p}) \end{bmatrix} \\ e^{-i\left(p_0'x_1' - p_1'x_0' + p_2'x_3' - p_3'x_2' + \overline{(\mathbf{p}',\mathbf{x}')}\right)} e^{i(p_0x_1 - p_1x_0 + p_2x_3 - p_3x_2 + \overline{(\mathbf{p},\mathbf{x})})} \end{pmatrix}^T$$

$$= \iint \frac{d^4p}{(2\pi)^4}\frac{d^4p'}{(2\pi)^4} \begin{bmatrix} \begin{bmatrix} d_1(\mathbf{p})d_1^*(\mathbf{p}')\mathbf{u1}(\mathbf{p})\mathbf{u4}^{\boldsymbol{T}}(\mathbf{p}') \\ -d_2(\mathbf{p})d_2^*(\mathbf{p}')\mathbf{u3}(\mathbf{p})\mathbf{u2}^{\boldsymbol{T}}(\mathbf{p}') + \cdots \end{bmatrix} \\ e^{i(p_0x_1 - p_1x_0 + p_2x_3 - p_3x_2 + \overline{(\mathbf{p},\mathbf{x})})} e^{-i\left(p_0'x_1' - p_1'x_0' + p_2'x_3' - p_3'x_2' + \overline{(\mathbf{p}',\mathbf{x}')}\right)} \\ + \\ \begin{bmatrix} b_1(\mathbf{p})\overline{b_1^*(\mathbf{p}')\mathbf{u4}}(\mathbf{p})\mathbf{u1}^+(\mathbf{p}') \\ -b_2(\mathbf{p})b_2^*(\mathbf{p}')\overline{\mathbf{u2}}(\mathbf{p})\mathbf{u3}^+(\mathbf{p}') + \cdots \end{bmatrix} \\ e^{i\left(p_0'x_1' - p_1'x_0' + p_2'x_3' - p_3'x_2' + \overline{(\mathbf{p}',\mathbf{x}')}\right)} e^{-i(p_0x_1 - p_1x_0 + p_2x_3 - p_3x_2 + \overline{(\mathbf{p},\mathbf{x})})} \end{bmatrix}$$

$$+ \iint \frac{d^4p}{(2\pi)^4}\frac{d^4p'}{(2\pi)^4} \begin{bmatrix} \begin{bmatrix} b_1^*(\mathbf{p}')b_1(\mathbf{p})\left(\overline{\mathbf{u1}}(\mathbf{p}')\mathbf{u4}^+(\mathbf{p})\right)^{\boldsymbol{T}} \\ -b_2^*(\mathbf{p}')b_2(\mathbf{p})\left(\overline{\mathbf{u3}}(\mathbf{p}')\mathbf{u2}^+(\mathbf{p})\right)^{\boldsymbol{T}} + \cdots \end{bmatrix} \\ e^{-i(p_0x_1 - p_1x_0 + p_2x_3 - p_3x_2 + \overline{(\mathbf{p},\mathbf{x})})} e^{i\left(p_0'x_1' - p_1'x_0' + p_2'x_3' - p_3'x_2' + \overline{(\mathbf{p}',\mathbf{x}')}\right)} \\ + \\ \begin{bmatrix} d_1^*(\mathbf{p}')d_1(\mathbf{p})\left(\mathbf{u4}(\mathbf{p}')\mathbf{u1}^{\boldsymbol{T}}(\mathbf{p})\right)^{\boldsymbol{T}} \\ -d_2^*(\mathbf{p}')d_2(\mathbf{p})\left(\mathbf{u2}(\mathbf{p}')\mathbf{u3}^{\boldsymbol{T}}(\mathbf{p})\right)^{\boldsymbol{T}} + \cdots \end{bmatrix} \\ e^{-i\left(p_0'x_1' - p_1'x_0' + p_2'x_3' - p_3'x_2' + \overline{(\mathbf{p}',\mathbf{x}')}\right)} e^{i(p_0x_1 - p_1x_0 + p_2x_3 - p_3x_2 + \overline{(\mathbf{p},\mathbf{x})})} \end{bmatrix}$$

$$= \iint \frac{d^4p}{(2\pi)^4}\frac{d^4p'}{(2\pi)^4} \begin{bmatrix} \begin{bmatrix} d_1(\mathbf{p})d_1^*(\mathbf{p}')\mathbf{u1}(\mathbf{p})\mathbf{u4}^{\boldsymbol{T}}(\mathbf{p}') + d_1^*(\mathbf{p}')d_1(\mathbf{p})\left(\mathbf{u4}(\mathbf{p}')\mathbf{u1}^{\boldsymbol{T}}(\mathbf{p})\right)^{\boldsymbol{T}} \\ -d_2(\mathbf{p})d_2^*(\mathbf{p}')\mathbf{u3}(\mathbf{p})\mathbf{u2}^{\boldsymbol{T}}(\mathbf{p}') - d_2^*(\mathbf{p}')d_2(\mathbf{p})\left(\mathbf{u2}(\mathbf{p}')\mathbf{u3}^{\boldsymbol{T}}(\mathbf{p})\right)^{\boldsymbol{T}} + \cdots \end{bmatrix} \\ e^{i(p_0x_1 - p_1x_0 + p_2x_3 - p_3x_2 + \overline{(\mathbf{p},\mathbf{x})})} e^{-i\left(p_0'x_1' - p_1'x_0' + p_2'x_3' - p_3'x_2' + \overline{(\mathbf{p}',\mathbf{x}')}\right)} \\ + \\ \begin{bmatrix} b_1(\mathbf{p})\overline{b_1^*(\mathbf{p}')\mathbf{u4}}(\mathbf{p})\mathbf{u1}^+(\mathbf{p}') + b_1^*(\mathbf{p}')b_1(\mathbf{p})\left(\overline{\mathbf{u1}}(\mathbf{p}')\mathbf{u4}^+(\mathbf{p})\right)^{\boldsymbol{T}} \\ -b_2(\mathbf{p})b_2^*(\mathbf{p}')\overline{\mathbf{u2}}(\mathbf{p})\mathbf{u3}^+(\mathbf{p}') - b_2^*(\mathbf{p}')b_2(\mathbf{p})\left(\overline{\mathbf{u3}}(\mathbf{p}')\mathbf{u2}^+(\mathbf{p})\right)^{\boldsymbol{T}} + \cdots \end{bmatrix} \\ e^{i\left(p_0'x_1' - p_1'x_0' + p_2'x_3' - p_3'x_2' + \overline{(\mathbf{p}',\mathbf{x}')}\right)} e^{-i(p_0x_1 - p_1x_0 + p_2x_3 - p_3x_2 + \overline{(\mathbf{p},\mathbf{x})})} \end{bmatrix}$$

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$$= \iint \frac{d^4p}{(2\pi)^4}\frac{d^4p'}{(2\pi)^4}\begin{bmatrix} \begin{bmatrix} d_1(\mathbf{p})d_1^*(\mathbf{p}')\mathbf{u1}(\mathbf{p})\mathbf{u4}^T(\mathbf{p}') + d_1^*(\mathbf{p}')d_1(\mathbf{p})\left(\mathbf{u1}(\mathbf{p})\mathbf{u4}^T(\mathbf{p}')\right) \\ -d_2(\mathbf{p})d_2^*(\mathbf{p}')\mathbf{u3}(\mathbf{p})\mathbf{u2}^T(\mathbf{p}') - d_2^*(\mathbf{p}')d_2(\mathbf{p})\left(\mathbf{u3}(\mathbf{p})\mathbf{u2}^T(\mathbf{p}')\right) + \cdots \end{bmatrix} \\ e^{i(p_0x_1 - p_1x_0 + p_2x_3 - p_3x_2 + \overline{(\mathbf{p},\mathbf{x})})}e^{-i\left(p_0'x_1' - p_1'x_0' + p_2'x_3' - p_3'x_2' + \overline{(\mathbf{p}',\mathbf{x}')}\right)} \\ + \\ \begin{bmatrix} b_1(\mathbf{p})b_1^*(\mathbf{p}')\overline{\mathbf{u4}}(\mathbf{p})\mathbf{u1}^+(\mathbf{p}') + b_1^*(\mathbf{p}')b_1(\mathbf{p})\left(\overline{\mathbf{u4}}(\mathbf{p})\mathbf{u1}^+(\mathbf{p}')\right) \\ -b_2(\mathbf{p})b_2^*(\mathbf{p}')\overline{\mathbf{u2}}(\mathbf{p})\mathbf{u3}^+(\mathbf{p}') - b_2^*(\mathbf{p}')b_2(\mathbf{p})\left(\overline{\mathbf{u2}}(\mathbf{p})\mathbf{u3}^+(\mathbf{p}')\right) + \cdots \end{bmatrix} \\ e^{i\left(p_0'x_1' - p_1'x_0' + p_2'x_3' - p_3'x_2' + \overline{(\mathbf{p}',\mathbf{x}')}\right)}e^{-i(p_0x_1 - p_1x_0 + p_2x_3 - p_3x_2 + \overline{(\mathbf{p},\mathbf{x})})} \end{bmatrix}$$

$$= \int \frac{d^4p}{(2\pi)^4}\begin{bmatrix} \begin{bmatrix} \mathbf{u1}(\mathbf{p})\mathbf{u4}^T(\mathbf{p}) + \mathbf{u4}(\mathbf{p})\mathbf{u1}^T(\mathbf{p}) \\ -\mathbf{u3}(\mathbf{p})\mathbf{u2}^T(\mathbf{p}) - \mathbf{u2}(\mathbf{p})\mathbf{u3}^T(\mathbf{p}) + \cdots \end{bmatrix} \\ e^{i\left(p_0(x_1 - x_1') - p_1(x_0 - x_0') + p_2(x_3 - x_3') - p_3(x_2 - x_2') + \overline{(\mathbf{p},\mathbf{x}-\mathbf{x}')}\right)} \\ + \\ \begin{bmatrix} \overline{\mathbf{u4}}(\mathbf{p})\mathbf{u1}^+(\mathbf{p}) + \overline{\mathbf{u1}}(\mathbf{p})\mathbf{u4}^+(\mathbf{p}) \\ -\overline{\mathbf{u2}}(\mathbf{p})\mathbf{u3}^+(\mathbf{p}) - \overline{\mathbf{u3}}(\mathbf{p})\mathbf{u2}^+(\mathbf{p}) + \cdots \end{bmatrix} \\ e^{-i\left(p_0(x_1 - x_1') - p_1\left(x_0 - x_0' + \overline{(\mathbf{p}',\mathbf{x})}\right) + p_2(x_3 - x_3') - p_3(x_2 - x_2') + \overline{(\mathbf{p},\mathbf{x}-\mathbf{x}')}\right)} \end{bmatrix}$$

$$= \int \frac{d^4p}{(2\pi)^4}\begin{bmatrix} \begin{bmatrix} \mathbf{u1}(\mathbf{p})\mathbf{u4}^T(\mathbf{p}) + \mathbf{u4}(\mathbf{p})\mathbf{u1}^T(\mathbf{p}) + \mathbf{v1}(\mathbf{p})\mathbf{v4}^T(\mathbf{p}) + \mathbf{v4}(\mathbf{p})\mathbf{v1}^T(\mathbf{p}) \\ -\mathbf{u3}(\mathbf{p})\mathbf{u2}^T(\mathbf{p}) - \mathbf{u2}(\mathbf{p})\mathbf{u3}^T(\mathbf{p}) - \mathbf{v3}(\mathbf{p})\mathbf{v2}^T(\mathbf{p}) - \mathbf{v2}(\mathbf{p})\mathbf{v3}^T(\mathbf{p}) \end{bmatrix} \\ e^{i\left(p_0(x_1 - x_1') - p_1(x_0 - x_0') + p_2(x_3 - x_3') - p_3(x_2 - x_2') + \overline{(\mathbf{p},\mathbf{x}-\mathbf{x}')}\right)} \\ + \\ \begin{bmatrix} \overline{\mathbf{u4}}(\mathbf{p})\mathbf{u1}^+(\mathbf{p}) + \overline{\mathbf{u1}}(\mathbf{p})\mathbf{u4}^+(\mathbf{p}) + \overline{\mathbf{v4}}(\mathbf{p})\mathbf{v1}^+(\mathbf{p}) + \overline{\mathbf{v1}}(\mathbf{p})\mathbf{v4}^+(\mathbf{p}) \\ -\overline{\mathbf{u2}}(\mathbf{p})\mathbf{u3}^+(\mathbf{p}) - \overline{\mathbf{u3}}(\mathbf{p})\mathbf{u2}^+(\mathbf{p}) - \overline{\mathbf{v2}}(\mathbf{p})\mathbf{v3}^+(\mathbf{p}) - \overline{\mathbf{v3}}(\mathbf{p})\mathbf{v2}^+(\mathbf{p}) \end{bmatrix} \\ e^{-i\left(p_0(x_1 - x_1') - p_1(x_0 - x_0') + p_2(x_3 - x_3') - p_3(x_2 - x_2') + \overline{(\mathbf{p},\mathbf{x}-\mathbf{x}')}\right)} \end{bmatrix}$$

$$= \int \frac{d^4p}{(2\pi)^4}\left(S^R(\mathbf{p}) + S_R(\mathbf{p})\right)e^{\left(i\left(p_0(x_1 - x_1') - p_1(x_0 - x_0') + p_2(x_3 - x_3') - p_3(x_2 - x_2') + \overline{(\mathbf{p},\mathbf{x}-\mathbf{x}')}\right)\right)}$$

$$+ \int \frac{d^4p}{(2\pi)^4}\left(\overline{S_R}(\mathbf{p}) + \overline{S^R}(\mathbf{p})\right)e^{-\left(i\left(p_0(x_1 - x_1') - p_1(x_0 - x_0') + p_2(x_3 - x_3') - p_3(x_2 - x_2') + \overline{(\mathbf{p},\mathbf{x}-\mathbf{x}')}\right)\right)}$$

$$= \int \frac{d^4p}{(2\pi)^4}\,4\begin{pmatrix} m & 0 & 0 & 0 \\ 0 & m & 0 & 0 \\ 0 & 0 & m & 0 \\ 0 & 0 & 0 & m \end{pmatrix}e^{\left(i\left(p_0(x_1 - x_1') - p_1(x_0 - x_0') + p_2(x_3 - x_3') - p_3(x_2 - x_2') + \overline{(\mathbf{p},\mathbf{x}-\mathbf{x}')}\right)\right)}$$

$$+ \int \frac{d^4p}{(2\pi)^4}\,4\begin{pmatrix} \overline{m} & 0 & 0 & 0 \\ 0 & \overline{m} & 0 & 0 \\ 0 & 0 & \overline{m} & 0 \\ 0 & 0 & 0 & \overline{m} \end{pmatrix}e^{-\left(i\left(p_0(x_1 - x_1') - p_1(x_0 - x_0') + p_2(x_3 - x_3') - p_3(x_2 - x_2') + \overline{(\mathbf{p},\mathbf{x}-\mathbf{x}')}\right)\right)}$$

$$= 4mI\delta(\mathbf{x}' - \mathbf{x}) + 4\overline{m}I\delta(\mathbf{x} - \mathbf{x}')$$

$$\{a_i(\mathbf{x}'), b_j(\mathbf{x})\} = a_i(\mathbf{x}')b_j(\mathbf{x}) + b_j(\mathbf{x})a_i(\mathbf{x}') = 4Re(m)\delta(\mathbf{x}' - \mathbf{x})\delta_{ij}$$

$$\{b_j(\mathbf{x}'), a_i(\mathbf{x})\} = b_j(\mathbf{x}')a_i(\mathbf{x}) + a_i(\mathbf{x})b_j(\mathbf{x}') = 4Re(m)\delta(\mathbf{x} - \mathbf{x}')\delta_{ij}$$

Besides these relations, the following `anti-commutation relations take place between the components of the annihilation and birth operators

$$\{b_i(\mathbf{x}), b_j(\mathbf{x}')\} = b_i(\mathbf{x})b_j(\mathbf{x}') + b_j(\mathbf{x}')b_i(\mathbf{x}) = 0$$
$$\{a_i(\mathbf{x}), a_j(\mathbf{x}')\} = a_i(\mathbf{x})a_j(\mathbf{x}') + a_j(\mathbf{x}')a_i(\mathbf{x}) = 0$$

Let's define operators of the total number of particles in the form

$$N_{ji}(\mathbf{x}) = b_j(\mathbf{x})a_i(\mathbf{x}) \qquad N_{ji} = \int d^4x\, b_j(\mathbf{x})a_i(\mathbf{x})$$

Let's find the commutators

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$$[N_{ji}, b_j(\mathbf{x})] = \int d^4x' \left[b_j(\mathbf{x}')a_i(\mathbf{x}')b_j(\mathbf{x}) - b_j(\mathbf{x})b_j(\mathbf{x}')a_i(\mathbf{x}')\right] =$$
$$\int d^4x' \left[b_j(\mathbf{x}')a_i(\mathbf{x}')b_j(\mathbf{x}) + b_j(\mathbf{x}')b_j(\mathbf{x})a_i(\mathbf{x}')\right] =$$
$$\int d^4x' \left[b_j(\mathbf{x}')\left(a_i(\mathbf{x}')b_j(\mathbf{x}) + b_j(\mathbf{x})a_i(\mathbf{x}')\right)\right] =$$
$$4Re(m)\int d^4x'\, b_j(\mathbf{x}')\delta(\mathbf{x}' - \mathbf{x})\delta_{ij} = 4Re(m)\delta_{ij}b_j(\mathbf{x}) = [N_{ji}, b_j(\mathbf{x})]$$
$$[N_{ji}, a_i(\mathbf{x})] = \int d^4x' \left[b_j(\mathbf{x}')a_i(\mathbf{x}')a_i(\mathbf{x}) - a_i(\mathbf{x})b_j(\mathbf{x}')a_i(\mathbf{x}')\right] =$$
$$\int d^4x' \left[-b_j(\mathbf{x}')a_i(\mathbf{x})a_i(\mathbf{x}') - a_i(\mathbf{x})b_j(\mathbf{x}')a_i(\mathbf{x}')\right] =$$
$$-\int d^4x' \left[\left(b_j(\mathbf{x}')a_i(\mathbf{x}) + a_i(\mathbf{x})b_j(\mathbf{x}')\right)a_i(\mathbf{x}')\right] =$$
$$-4Re(m)\int d^4x'\, \delta(\mathbf{x}' - \mathbf{x})\delta_{ij}a_i(\mathbf{x}') = -4Re(m)\delta_{ij}a_i(\mathbf{x}) = [N_{ji}, a_i(\mathbf{x})]$$

Instead of defining the vacuum state through its properties under the action of annihilation operators

$$d_1(\mathbf{p})|\Psi_0\rangle = d_2(\mathbf{p})|\Psi_0\rangle = b_2(\mathbf{p})|\Psi_0\rangle = b_1(\mathbf{p})|\Psi_0\rangle = 0$$
$$d_4(\mathbf{p})|\Psi_0\rangle = d_3(\mathbf{p})|\Psi_0\rangle = b_3(\mathbf{p})|\Psi_0\rangle = b_3(\mathbf{p})|\Psi_0\rangle = 0$$

which would entail the ratios

$$\boldsymbol{a}(\mathbf{x})|\Psi_0\rangle = 0 \quad N_{ji}|\Psi_0\rangle = 0$$

we will not require from operators all these properties, but we will be limited by a weaker and simpler definition of vacuum, namely, absence of particles in vacuum

$$N_{ji}|\Psi_0\rangle = 0$$

Let's use the found commutator

$$4Re(m)\delta_{ij}b_j(\mathbf{x}) = N_{ji}b_j(\mathbf{x}) - b_j(\mathbf{x})N_{ji}$$
$$4Re(m)\delta_{ij}b_j(\mathbf{x})|\Psi_0\rangle = N_{ji}b_j(\mathbf{x})|\Psi_0\rangle - b_j(\mathbf{x})N_{ji}|\Psi_0\rangle$$
$$N_{ji}b_j(\mathbf{x})|\Psi_0\rangle = 4Re(m)\delta_{ij}b_j(\mathbf{x})|\Psi_0\rangle$$
$$|\Psi_1\rangle \equiv b_j(\mathbf{x})|\Psi_0\rangle$$
$$N_{ji}|\Psi_1\rangle = 4Re(m)\delta_{ij}|\Psi_1\rangle$$

On the obtained one-particle state let's act on the obtained one-particle state by the birth operator again

$$4Re(m)\delta_{ij}b_j(\mathbf{x})|\Psi_1\rangle = N_{ji}b_j(\mathbf{x})|\Psi_1\rangle - b_j(\mathbf{x})N_{ji}|\Psi_1\rangle$$
$$4Re(m)\delta_{ij}b_j(\mathbf{x})|\Psi_1\rangle = N_{ji}b_j(\mathbf{x})|\Psi_1\rangle - 4Re(m)\delta_{ij}b_j(\mathbf{x})|\Psi_1\rangle$$
$$N_{ji}b_j(x)|\Psi_1\rangle = 2(4Re(m)\delta_{ij})b_j(x)|\Psi_1\rangle$$
$$|\Psi_2\rangle \equiv b_j(\mathbf{x})|\Psi_1\rangle$$
$$N_{ji}|\Psi_2\rangle = 2(4Re(m)\delta_{ij})|\Psi_2\rangle$$

We have obtained a state with two particles and we can thus increase the number of particles to infinity. All particles are identical and indistinguishable from each other, each of them is in all allowed states, of which the free field has infinitely many. Electrons in an atom have fewer allowed states, but still any electron occupies all of them equally with the others. This theory describes both electron and positron, the difference between them being only in the sign of the mass, it being convenient to consider that the electron has a negative mass and the positron a positive one.

Similarly, we use the commutator of the annihilation operator

$$-4Re(m)\delta_{ij}a_i(\mathbf{x}) = N_{ji}a_i(\mathbf{x}) - a_i(\mathbf{x})N_{ji}$$
$$-4Re(m)\delta_{ij}a_i(\mathbf{x})|\Psi_2\rangle = N_{ji}a_i(\mathbf{x})|\Psi_2\rangle - a_i(\mathbf{x})N_{ji}|\Psi_2\rangle$$
$$-4Re(m)\delta_{ij}a_i(\mathbf{x})|\Psi_2\rangle = N_{ji}a_i(\mathbf{x})|\Psi_2\rangle - 2(4Re(m)\delta_{ij})a_i(\boldsymbol{x})|\Psi_2\rangle$$
$$N_{ji}a_i(\mathbf{x})|\Psi_2\rangle = 4Re(m)\delta_{ij}a_i(\mathbf{x})|\Psi_2\rangle$$

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Thus, the action of the annihilation operator has transformed the two-particle state into a one-particle state. Using the same calculations, we obtain the result of the annihilation operator action on the one-particle state

$$N_{ji}a_i(\mathbf{x})|\Psi_1\rangle = 0 * a_i(\mathbf{x})|\Psi_1\rangle$$

And in the same way we define the result of its action on the null state

$$N_{ji}a_i(\mathbf{x})|\Psi_0\rangle = -4Re(m)\delta_{ij}a_i(\mathbf{x})|\Psi_0\rangle = 4Re(-m)\delta_{ij}a_i(\mathbf{x})|\Psi_0\rangle$$

We obtain a state with the number of particles minus one, but we see that in fact it is a state with one particle whose mass is negative. Thus, the positron annihilation operator is also the electron birth operator. It destroys positrons until they run out, after which it begins to give birth to electrons. The birth operator, on the contrary, destroys electrons, and when they run out, begins to give birth to positrons. Thus, since there are many electrons in our universe, this operator cannot give birth to positrons because it cannot destroy all electrons due to their number. Moreover, the operator of annihilation of positrons because of the absence of the latter, only gives birth to more and more electrons.

If the mass is zero, then in any state the number of particles is zero, i.e., for example, the electromagnetic field in spinor space, where it should be fermionic, simply has no particles. The absence of particles does not contradict the presence of the field, which is represented by the same 16 spinors, this field obeys Fermi statistics, and it has no charge and can be treated as a Majorana fermion. This field interacts with electrons in spinor space, and the result of the interaction manifests itself in vector space.

With the help of the birth and annihilation operators we can write the propagator for the situation when the initial and final states are states with arbitrary number of particles

$$\langle\Psi_n^*|\boldsymbol{a}(\mathbf{x})\boldsymbol{b}^{\mathrm{T}}(\mathbf{0})|\Psi_n\rangle = \int \frac{d^4p}{(2\pi)^4}\langle\Psi_n^*|\Psi_n\rangle 4Re(m)\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}e^{i(p_0x_1-p_1x_0+p_2x_3-p_3x_2+\overline{(\mathbf{p},\mathbf{x})})}$$

$$\langle\Psi_n^*|\boldsymbol{b}(\mathbf{x})\boldsymbol{a}^{\mathrm{T}}(\mathbf{0})|\Psi_n\rangle = \int \frac{d^4p}{(2\pi)^4}\langle\Psi_n^*|\Psi_n\rangle 4Re(m)\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}e^{-i(p_0x_1-p_1x_0+p_2x_3-p_3x_2+\overline{(\mathbf{p},\mathbf{x})})}$$

We would like the spinor propagator also to have properties of the Green's function, i.e. to satisfy the equations which for this case are given below and which can be combined into one equation by summation

$$\left(\frac{\partial}{\partial x_1}\frac{\partial}{\partial x_2}-\frac{\partial}{\partial x_0}\frac{\partial}{\partial x_3}+m\right)D(\mathbf{x}) = \delta(\mathbf{x})$$

$$\left(\frac{\partial\overline{[\,]}}{\partial\overline{x_1}}\frac{\partial\overline{[\,]}}{\partial\overline{x_2}}-\frac{\partial\overline{[\,]}}{\partial\overline{x_0}}\frac{\partial\overline{[\,]}}{\partial\overline{x_3}}+\overline{m}\right)D(\mathbf{x}) = \delta(\mathbf{x})$$

$$\left(\left(\frac{\partial}{\partial x_1}\frac{\partial}{\partial x_2}-\frac{\partial}{\partial x_0}\frac{\partial}{\partial x_3}\right)+\left(\frac{\partial\overline{[\,]}}{\partial\overline{x_1}}\frac{\partial\overline{[\,]}}{\partial\overline{x_2}}-\frac{\partial\overline{[\,]}}{\partial\overline{x_0}}\frac{\partial\overline{[\,]}}{\partial\overline{x_3}}\right)+m+\overline{m}\right)D(\mathbf{x}) = \delta(\mathbf{x})$$

where the delta function can be represented as

$$\delta(\mathbf{x}) = \int \frac{d^4p}{(2\pi)^4}\, e^{i(p_0x_1-p_1x_0+p_2x_3-p_3x_2+\overline{(p_0x_1-p_1x_0+p_2x_3-p_3x_2)})}$$

The solution of the combined equation has the form

$$D(\mathbf{x}) = \int \frac{d^4p}{(2\pi)^4}\frac{e^{i(p_0x_1-p_1x_0+p_2x_3-p_3x_2+\overline{(\mathbf{p},\mathbf{x})})}}{(p_1p_2-p_0p_3)+(\overline{p_1}\overline{p_2}-\overline{p_0}\overline{p_3})-m-\overline{m}}$$

Therefore, we must add to the denominator of the integrand an appropriate multiplier

$$\langle\Psi_n^*|\boldsymbol{a}(\mathbf{x})\boldsymbol{b}^{\mathrm{T}}(\mathbf{y})|\Psi_n\rangle = \int \frac{d^4p}{(2\pi)^4}\langle\Psi_n^*|\Psi_n\rangle\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}\frac{4Re(m)e^{i\left((\mathbf{p},\mathbf{x}-\mathbf{y})+\overline{(\mathbf{p},\mathbf{x}-\mathbf{y})}\right)}}{(p_1p_2-p_0p_3)+(\overline{p_1}\overline{p_2}-\overline{p_0}\overline{p_3})-Re(m)}$$

$$\langle\Psi_n^*|\boldsymbol{b}(\mathbf{x})\boldsymbol{a}^{\mathrm{T}}(\mathbf{y})|\Psi_n\rangle = \int\frac{d^4p}{(2\pi)^4}\langle\Psi_n^*|\Psi_n\rangle\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}\frac{4Re(m)e^{-i\left((\mathbf{p},\mathbf{x}-\mathbf{y})+\overline{(\mathbf{p},\mathbf{x}-\mathbf{y})}\right)}}{(p_1p_2-p_0p_3)+(\overline{p_1}\overline{p_2}-\overline{p_0}\overline{p_3})-Re(m)}$$

The electron and positron have different mass sign, so their propagators will be different.

Instead of the sum of equations we can use their product, then the corresponding inhomogeneous equation

$$\left(\left(\frac{\partial[\,]}{\partial\overline{x_1}}\frac{\partial[\,]}{\partial\overline{x_2}}-\frac{\partial[\,]}{\partial\overline{x_0}}\frac{\partial[\,]}{\partial\overline{x_3}}\right)\left(\frac{\partial}{\partial x_1}\frac{\partial}{\partial x_2}-\frac{\partial}{\partial x_0}\frac{\partial}{\partial x_3}\right)-m^2\right)D(\mathbf{x})=\delta(\mathbf{x})$$

has a solution

$$D(\mathbf{x})=\int\frac{d^4p}{(2\pi)^4}\frac{e^{i\left(p_0x_1-p_1x_0+p_2x_3-p_3x_2+\overline{(p_0x_1-p_1x_0+p_2x_3-p_3x_2)}\right)}}{(\overline{p_1}\overline{p_2}-\overline{p_0}\overline{p_3})(p_1p_2-p_0p_3)-m^2}$$
$$=\int\frac{d^4p}{(2\pi)^4}\frac{e^{i\left(p_0x_1-p_1x_0+p_2x_3-p_3x_2+\overline{(p_0x_1-p_1x_0+p_2x_3-p_3x_2)}\right)}}{{P_0}^2-{P_1}^2-{P_2}^2-{P_3}^2-m^2}$$

Correspondingly, it is necessary to add to the propagator the denominator in any of these forms, since it was shown earlier that for a free field

$$(p_1p_2-p_0p_3)\overline{(p_1p_2-p_0p_3)}={P_0}^2-{P_1}^2-{P_2}^2-{P_3}^2$$

We can repeat the above calculations, keeping the annihilation operator, but defining the birth operator differently

$$\boldsymbol{a}(\mathbf{x})=\int\frac{d^4p}{(2\pi)^4}$$
$$\begin{bmatrix}d_1(\mathbf{p})\mathbf{u1}(\mathbf{p})+id_2(\mathbf{p})\mathbf{u3}(\mathbf{p})+ib_2(\mathbf{p})\overline{\mathbf{u2}}(\mathbf{p})+\ b_1(\mathbf{p})\overline{\mathbf{u4}}(\mathbf{p})\\+d_4(\mathbf{p})\mathbf{v1}(\mathbf{p})+id_3(\mathbf{p})\mathbf{v3}(\mathbf{p})+ib_3(\mathbf{p})\overline{\mathbf{v2}}(\mathbf{p})+b_4(\mathbf{p})\overline{\mathbf{v4}}(\mathbf{p})\end{bmatrix}e^{i\left(p_0x_1-p_1x_0+p_2x_3-p_3x_2+\overline{(\mathbf{p},\mathbf{x})}\right)}$$
$$\boldsymbol{b}(\mathbf{x})=\int\frac{d^4p}{(2\pi)^4}$$
$$\begin{bmatrix}d_1^*(\mathbf{p})\overline{\mathbf{u1}}(\mathbf{p})-id_2^*(\mathbf{p})\overline{\mathbf{u3}}(\mathbf{p})-ib_2^*(\mathbf{p})\mathbf{u2}(\mathbf{p})+b_1^*(\mathbf{p})\mathbf{u4}(\mathbf{p})\\+d_4^*(\mathbf{p})\overline{\mathbf{v1}}(\mathbf{p})-id_3^*(\mathbf{p})\overline{\mathbf{v3}}(\mathbf{p})-ib_3^*(\mathbf{p})\mathbf{v2}(\mathbf{p})+b_4^*(\mathbf{p})\mathbf{v4}(\mathbf{p})\end{bmatrix}e^{-i\left(p_0x_1-p_1x_0+p_2x_3-p_3x_2+\overline{(\mathbf{p},\mathbf{x})}\right)}$$

As a result, we obtain the anticommutator

$$\{a_i(\mathbf{x}),b_j(\mathbf{x}')\}=a_i(\mathbf{x})b_j(\mathbf{x}')+b_j(\mathbf{x}')a_i(\mathbf{x})=\left(\boldsymbol{a}(\mathbf{x})\boldsymbol{b}(\mathbf{x}')+\left(\boldsymbol{b}(\mathbf{x}')\boldsymbol{a}^T(\mathbf{x})\right)^T\right)_{ij}$$
$$\boldsymbol{a}(\mathbf{x})\boldsymbol{b}^T(\mathbf{x}')+\left(\boldsymbol{b}(\mathbf{x}')\boldsymbol{a}^T(\mathbf{x})\right)^T=4P_0I\delta(\mathbf{x}'-\mathbf{x})+4P_0I\delta(\mathbf{x}-\mathbf{x}')=8P_0\delta(\mathbf{x}-\mathbf{x}')$$
$$P_0=p_0\overline{p_0}+p_1\overline{p_1}+p_2\overline{p_2}+p_3\overline{p_3}$$

As before, using the birth and annihilation operators, we construct propagators for a state with an arbitrary number of particles

$$\langle\Psi_n^*|\boldsymbol{a}(\mathbf{x})\boldsymbol{b}^{\mathrm{T}}(\mathbf{0})|\Psi_n\rangle=\int\frac{d^4p}{(2\pi)^4}\langle\Psi_n^*|\Psi_n\rangle\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}\frac{8P_0e^{i\left(p_0x_1-p_1x_0+p_2x_3-p_3x_2+\overline{(\mathbf{p},\mathbf{x})}\right)}}{(\overline{p_1p_2}-\overline{p_0p_3})(p_1p_2-p_0p_3)-m^2}$$
$$\langle\Psi_n^*|\boldsymbol{b}(\mathbf{x})\boldsymbol{a}^{\mathrm{T}}(\mathbf{0})|\Psi_n\rangle=\int\frac{d^4p}{(2\pi)^4}\langle\Psi_n^*|\Psi_n\rangle\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}\frac{8P_0e^{-i\left(p_0x_1-p_1x_0+p_2x_3-p_3x_2+\overline{(\mathbf{p},\mathbf{x})}\right)}}{(\overline{p_1p_2}-\overline{p_0p_3})(p_1p_2-p_0p_3)-m^2}$$

Now instead of mass the propagator includes energy, therefore such theory is applicable also to the field with zero mass, i.e. it can serve as a model not only for the electron, but also for the electromagnetic field in spinor space. The only problem is that if earlier the action of the annihilation operator on the zero-point state gave a particle with negative mass, now this action gives a particle with negative energy, which makes the interpretation of such theory more difficult.

We can reformulate the above reasoning in a more consistent and logical form. Let us again write down the equations

$$\left(\frac{\partial}{\partial x_1}\frac{\partial}{\partial x_2}-\frac{\partial}{\partial x_0}\frac{\partial}{\partial x_3}+m\right)D(\mathbf{x})=\delta(\mathbf{x})$$

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$$\left(\frac{\partial[\,]}{\partial x_1}\frac{\partial\overline{[\,]}}{\partial x_2}-\frac{\partial[\,]}{\partial x_0}\frac{\partial\overline{[\,]}}{\partial x_3}+\overline{m}\right)D(\mathbf{x})=\delta(\mathbf{x})$$

and put in correspondence with them the birth and annihilation operators, which in this version are conjugate to each other

$$\boldsymbol{a}(\mathbf{x})=\int\frac{d^4p}{(2\pi)^4}$$

$$\begin{bmatrix}d_1(\mathbf{p})\mathbf{u1}(\mathbf{p})+id_2(\mathbf{p})\mathbf{u3}(\mathbf{p})+ib_2(\mathbf{p})\overline{\mathbf{u2}}(\mathbf{p})+\;b_1(\mathbf{p})\overline{\mathbf{u4}}(\mathbf{p})\\+d_4(\mathbf{p})\mathbf{v1}(\mathbf{p})+id_3(\mathbf{p})\mathbf{v3}(\mathbf{p})+ib_3(\mathbf{p})\overline{\mathbf{v2}}(\mathbf{p})+b_4(\mathbf{p})\overline{\mathbf{v4}}(\mathbf{p})\end{bmatrix}\frac{e^{i\left((\mathbf{p},\mathbf{x})+\overline{(\mathbf{p},\mathbf{x})}\right)}}{p_1p_2-p_0p_3-m}$$

$$\boldsymbol{b}(\mathbf{x})=\overline{\boldsymbol{a}(\mathbf{x})}=\int\frac{d^4p}{(2\pi)^4}$$

$$\begin{bmatrix}d_1^*(\mathbf{p})\overline{\mathbf{u1}}(\mathbf{p})-id_2^*(\mathbf{p})\overline{\mathbf{u3}}(\mathbf{p})-ib_2^*(\mathbf{p})\mathbf{u2}(\mathbf{p})+b_1^*(\mathbf{p})\mathbf{u4}(\mathbf{p})\\+d_4^*(\mathbf{p})\overline{\mathbf{v1}}(\mathbf{p})-id_3^*(\mathbf{p})\overline{\mathbf{v3}}(\mathbf{p})-ib_3^*(\mathbf{p})\mathbf{v2}(\mathbf{p})+b_4^*(\mathbf{p})\mathbf{v4}(\mathbf{p})\end{bmatrix}\frac{e^{-i\left(\overline{(\mathbf{p},\mathbf{x})}+(\mathbf{p},\mathbf{x})\right)}}{\overline{p_1}\overline{p_2}-\overline{p_0}\overline{p_3}-\overline{m}}$$

Then the propagator without additional assumptions will have the form

$$D_{ij}(\mathbf{x}-\mathbf{y})=\langle\Psi_n^*|\boldsymbol{a}(\mathbf{x})\boldsymbol{b}^{\mathrm{T}}(\mathbf{y})|\Psi_n\rangle_{ij}=\int\frac{d^4p}{(2\pi)^4}\langle\Psi_n^*|\Psi_n\rangle 8P_0\delta_{ij}\frac{e^{i\left((\mathbf{p},\mathbf{x}-\mathbf{y})+\overline{(\mathbf{p},\mathbf{x}-\mathbf{y})}\right)}}{(p_1p_2-p_0p_3-m)(\overline{p_1}\overline{p_2}-\overline{p_0}\overline{p_3}-\overline{m})}$$

One can even propose to use plane waves in spinor space immediately together with the denominator in any field operators

$$\frac{1}{\sqrt{8P_0}}\frac{e^{i\left((\mathbf{p},\mathbf{x})+\overline{(\mathbf{p},\mathbf{x})}\right)}}{p_1p_2-p_0p_3-m}$$

The considered free field propagators describe the situation when there is a point source with coordinate $\mathbf{x}$ and a point sink with coordinate $\mathbf{y}$. In the general case in the spinor space the distribution of source-stocks ***J*(x)** can be given and the value of

$$W(J)=-\frac{1}{2}\iint d^4x\,d^4y\,J_i(\mathbf{x})D_{ij}(\mathbf{x}-\mathbf{y})J_j(\mathbf{y})$$

which is used for finding the integral over the trajectories and which can be written using the Fourier transform for the spinor space

$$J_i(\mathbf{p})\equiv\int\frac{d^4x}{(2\pi)^4}J_i(\mathbf{x})e^{-i\left((\mathbf{p},\mathbf{x})+\overline{(\mathbf{p},\mathbf{x})}\right)}$$

$$W(J)=-\frac{1}{2}\iint\frac{d^4p}{(2\pi)^4}\overline{J_i(\mathbf{p})}\,8P_0\delta_{ij}\frac{(p_0\overline{p_0}+p_1\overline{p_1}+p_2\overline{p_2}+p_3\overline{p_3})}{(p_1p_2-p_0p_3)(\overline{p_1}\overline{p_2}-\overline{p_0}\overline{p_3})-m^2}J_j(\mathbf{p})$$

In quantum field theory it is customary to calculate a similar quantity

$$W(J)=-\frac{1}{2}\iint d^4X\,d^4Y\,J_i(\mathbf{X})D_{ij}(\mathbf{X}-\mathbf{Y})J_j(\mathbf{Y})$$

in which the coordinates, momenta and the Fourier transform connecting them belong to the vector space. In our opinion, the transition to spinor space, more fundamental than vector space, which is a superstructure over spinor space, can eliminate divergences in calculating integrals in the framework of the formalism of the integral over trajectories. In momentum space the similarity is even more obvious, the kernels of the integrals are the same, the only difference is in the space where the integration takes place and the way of calculating the Fourier transform - either in vector or in spinor space

$$W(J)=-\frac{1}{2}\iint\frac{d^4P}{(2\pi)^4}\overline{J_i(\mathbf{P})}\frac{\delta_{ij}}{P_0^2-P_1^2-P_2^2-P_3^2-m^2}J_j(\mathbf{P})$$

$$W(J)=-\frac{1}{2}\iint\frac{d^4p}{(2\pi)^4}\overline{J_i(\mathbf{p})}\frac{8\delta_{ij}}{P_0^2-P_1^2-P_2^2-P_3^2-m^2}J_j(\mathbf{p})$$

The spinor space has the additional advantage that the integrand is factorised

$$W(J) = -\frac{1}{2}\iint \frac{d^4p}{(2\pi)^4} 8P_0 \frac{\overline{J_\iota(\mathbf{p})}}{(p_1p_2 - p_0p_3 - m)} \delta_{ij} \frac{J_j(\mathbf{p})}{(p_1p_2 - p_0p_3 - m)}$$

This factorization in momentum space looks like a consequence of a more fundamental property of factorization in coordinate space

$$W(J) = -\frac{1}{2}\iint d^4x\, d^4y\, J_i(\mathbf{x}) D_{ij}(\mathbf{x}-\mathbf{y}) J_j(\mathbf{y})$$
$$= -\frac{1}{2}\iint d^4x\, d^4y\, J_i(\mathbf{x}) \langle \Psi_n^* | \boldsymbol{a}(\mathbf{x}) \boldsymbol{b}^{\mathrm{T}}(\mathbf{y}) | \Psi_n \rangle_{ij} J_j(\mathbf{y})$$
$$= -\frac{1}{2}\iint d^4x\, d^4y\, \langle \Psi_n^* \left| J_i(\mathbf{x}) \left(\boldsymbol{a}(\mathbf{x}) \boldsymbol{b}^{\mathrm{T}}(\mathbf{y})\right)_{ij} J_j(\mathbf{y}) \right| \Psi_n \rangle$$
$$= -\frac{1}{2}\langle \Psi_n^* \left| \iint d^4x\, d^4y\, J_i(\mathbf{x}) (\boldsymbol{a}(\mathbf{x}) \boldsymbol{b}^{\mathrm{T}}(\mathbf{y}))_{ij} J_j(\mathbf{y}) \right| \Psi_n \rangle$$

We can assume that first it makes sense to perform integration separately on $\mathbf{x}$ and $\mathbf{y}$, and only then to perform multiplication

$$W(J) = -\frac{1}{2} \langle \Psi_n^* \left| \left(\int d^4x\, \boldsymbol{J}^{\boldsymbol{T}}(\boldsymbol{x})\, \boldsymbol{a}(\boldsymbol{x}) \right) \left(\int d^4y\, \boldsymbol{b}^T(\boldsymbol{y})\, \boldsymbol{J}(\boldsymbol{y}) \right) \right| \Psi_n \rangle$$

Since earlier we have obtained an explicit representation of field operators in both vector and spinor space, we do not need to refer to the equation of motion and the Lagrangian. Proceeding from these representations, we define the birth and annihilation operators, and from them we construct the propagator as a function of relative coordinates. For example, for a spinor space

$$D_{\nu\lambda}(\mathbf{x}-\mathbf{y}) = \int \frac{d^4p}{(2\pi)^4}\, 8P_0 \delta_{\nu\lambda} e^{i\left((\mathbf{p},\mathbf{x}-\mathbf{y}) + \overline{(\mathbf{p},\mathbf{x}-\mathbf{y})}\right)}$$

But we need to take two more steps. The first is to find an equation for which this propagator is an eigenfunction of the relative coordinates. Two equations can be proposed for this role

$$\left(\left(\frac{\partial\overline{[\,]}}{\partial\overline{x_1}} \frac{\partial\overline{[\,]}}{\partial\overline{x_2}} - \frac{\partial\overline{[\,]}}{\partial\overline{x_0}} \frac{\partial\overline{[\,]}}{\partial\overline{x_3}} \right) \left(\frac{\partial}{\partial x_1} \frac{\partial}{\partial x_2} - \frac{\partial}{\partial x_0} \frac{\partial}{\partial x_3} \right) - m^2 \right) g^{\mu\nu} D_{\nu\lambda}(\mathbf{x}) = \delta_\lambda^\mu \delta(\mathbf{x})$$

$$\left(\left(\frac{\partial}{\partial x_1} \frac{\partial}{\partial x_2} - \frac{\partial}{\partial x_0} \frac{\partial}{\partial x_3} \right) - m \right) \left(\left(\frac{\partial\overline{[\,]}}{\partial\overline{x_1}} \frac{\partial\overline{[\,]}}{\partial\overline{x_2}} - \frac{\partial\overline{[\,]}}{\partial\overline{x_0}} \frac{\partial\overline{[\,]}}{\partial\overline{x_3}} \right) - \overline{m} \right) g^{\mu\nu} D_{\nu\lambda}(\mathbf{x}) = \delta_\lambda^\mu \delta(\mathbf{x})$$

It is important that in both cases the eigenvalue is a real value independent of the coordinates. The second step is to ensure that the propagator has the properties of the Green's function, i.e., that the right-hand side of the equation has a delta function. This is necessary so that for an arbitrary distribution of sources we can use this propagator to construct a complete picture of the field propagation. To satisfy these equations, we must include in the solution a normalizing multiplier of the appropriate kind for each equation. This multiplier does not depend on the coordinates, but depends on the momentum. As a result, we obtain the normalized propagator in two variants

$$D_{\nu\lambda}(\mathbf{x}-\mathbf{y}) = \int \frac{d^4p}{(2\pi)^4} \frac{\delta_{\nu\lambda}\, e^{i\left((\mathbf{p},\mathbf{x}-\mathbf{y}) + \overline{(\mathbf{p},\mathbf{x}-\mathbf{y})}\right)}}{(p_1p_2 - p_0p_3)(\overline{p_1}\overline{p_2} - \overline{p_0}\overline{p_3}) - m^2}$$

$$D_{\nu\lambda}(\mathbf{x}-\mathbf{y}) = \int \frac{d^4p}{(2\pi)^4} \frac{\delta_{\nu\lambda}\, e^{i\left((\mathbf{p},\mathbf{x}-\mathbf{y}) + \overline{(\mathbf{p},\mathbf{x}-\mathbf{y})}\right)}}{(p_1p_2 - p_0p_3 - m)(\overline{p_1}\overline{p_2} - \overline{p_0}\overline{p_3} - \overline{m})}$$

In our opinion, the second variant is more preferable, since the denominator consists of two conjugate terms, so we can consider them as an integral part of the birth and annihilation operators, which are also conjugate copies of each other

$$\overline{\boldsymbol{b}(\mathbf{x})} = \boldsymbol{a}(\mathbf{x}) = \int \frac{d^4p}{(2\pi)^4}$$
$$\frac{1}{\sqrt{8P_0}} \frac{e^{i\left((\mathbf{p},\mathbf{x}) + \overline{(\mathbf{p},\mathbf{x})}\right)}}{p_1p_2 - p_0p_3 - m} \begin{bmatrix} d_1(\mathbf{p})\mathbf{u1}(\mathbf{p}) + id_2(\mathbf{p})\mathbf{u3}(\mathbf{p}) + ib_2(\mathbf{p})\overline{\mathbf{u2}}(\mathbf{p}) + b_1(\mathbf{p})\overline{\mathbf{u4}}(\mathbf{p}) \\ +d_4(\mathbf{p})\mathbf{v1}(\mathbf{p}) + id_3(\mathbf{p})\mathbf{v3}(\mathbf{p}) + ib_3(\mathbf{p})\overline{\mathbf{v2}}(\mathbf{p}) + b_4(\mathbf{p})\overline{\mathbf{v4}}(\mathbf{p}) \end{bmatrix}$$

The obtained results allow us to answer the question how the fermion field changes under the action of Lorentz transformations on the coordinates. Exactly, if we move to another frame of reference by rotations and boosts, the coordinate spinor changes. As a consequence, the momentum spinor changes, the components of which are the coefficients of the expansion on the new coordinates,

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and the momentum spinor undergoes exactly the same transformation as the coordinates, so that the phases of all plane waves in spinor space do not change. The components of the new momentum spinor are substituted into the 16 spinors describing the fermion field. Thus, there is no any uniform law of transformation of a spinor of the fermionic field, each of 16 spinors corresponding to the particles forming it, is transformed in its own way.

However, if, following Heisenberg [[12], Chapter 3, Paragraph 1], we index the field components differently

$$\varphi_0(\mathbf{x}) = \xi_{00}(\mathbf{x}) \quad \varphi_1(\mathbf{x}) = \xi_{10}(\mathbf{x}) \quad \overline{\varphi_2}(\mathbf{x}) = \xi_{11}(\mathbf{x}) \quad \overline{\varphi_3}(\mathbf{x}) = -\xi_{01}(\mathbf{x})$$

Then it can appear that this field $\boldsymbol{\xi}$ on the first index will be transformed by three spatial rotations and three boosts, and on the second index it will be transformed by three rotations in isotopic space. In this case the additional quantum number related to the sign of mass may be an isotopic spin.

Let us suggest that the coordinate and momentum spinor spaces can also be indexed in a similar way

$$x_0 = \chi_{00} \quad x_1 = \chi_{10} \quad \overline{x_2} = \chi_{11} \quad \overline{x_3} = -\chi_{01}$$
$$p_0 = \rho_{00} \quad p_1 = \rho_{10} \quad \overline{p_2} = \rho_{11} \quad \overline{p_3} = -\rho_{01}$$

Thus, we are in a space $\boldsymbol{\chi}$ that is subject to three rotations, three boosts, and three isotopic rotations. All of these transformations are equally real, but there is an imbalance due to the luck of isotopic boosts. After all, isotopic rotations, like spatial rotations, are generated by Pauli matrices; these rotations also do not form a group. Therefore, the full isotopic group must also consist of three rotations and three boosts.

Let's rewrite the previously used quantities with new variables

$$p_0x_1 - p_1x_0 + p_2x_3 - p_3x_2 = \rho_{00}\chi_{10} - \rho_{10}\chi_{00} - \overline{\rho_{11}\chi_{01}} + \overline{\rho_{01}\chi_{11}}$$
$$= (\rho_{00}\ \rho_{10})\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\begin{pmatrix}\chi_{00}\\ \chi_{10}\end{pmatrix} + (\overline{\rho_{01}}\ \overline{\rho_{11}})\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\begin{pmatrix}\overline{\chi_{01}}\\ \overline{\chi_{11}}\end{pmatrix}$$
$$m = p_1p_2 - p_0p_3 = \rho_{10}\overline{\rho_{11}} + \rho_{00}\overline{\rho_{01}}$$

$\mathbf{u1}(\mathbf{p}) = \begin{pmatrix}-p_3\\ -p_2\\ p_1\\ p_0\end{pmatrix}$	$\begin{pmatrix}-p_3 & -\overline{p_0}\\ -p_2 & \overline{p_1}\end{pmatrix}$	$\begin{pmatrix}\overline{\rho_{01}} & -\overline{\rho_{00}}\\ -\overline{\rho_{11}} & \overline{\rho_{10}}\end{pmatrix}$
$\mathbf{u4}(\mathbf{p}) = \begin{pmatrix}p_0\\ -p_1\\ p_2\\ -p_3\end{pmatrix}$	$\begin{pmatrix}p_0 & \overline{p_3}\\ -p_1 & \overline{p_2}\end{pmatrix}$	$\begin{pmatrix}\rho_{00} & -\rho_{01}\\ -\rho_{10} & \rho_{11}\end{pmatrix}$
$\overline{\mathbf{u1}}(\mathbf{p}) = \begin{pmatrix}-\overline{p_3}\\ -\overline{p_2}\\ \overline{p_1}\\ \overline{p_0}\end{pmatrix}$	$\begin{pmatrix}-\overline{p_3} & -p_0\\ -\overline{p_2} & p_1\end{pmatrix}$	$\begin{pmatrix}\rho_{01} & -\rho_{00}\\ -\rho_{11} & \rho_{10}\end{pmatrix}$
$\overline{\mathbf{u4}}(\mathbf{p}) = \begin{pmatrix}\overline{p_0}\\ -\overline{p_1}\\ \overline{p_2}\\ -\overline{p_3}\end{pmatrix}$	$\begin{pmatrix}\overline{p_0} & p_3\\ -\overline{p_1} & p_2\end{pmatrix}$	$\begin{pmatrix}\overline{\rho_{00}} & -\overline{\rho_{01}}\\ -\overline{\rho_{10}} & \overline{\rho_{11}}\end{pmatrix}$
$\mathbf{v1}(\mathbf{p}) = \begin{pmatrix}p_1\\ p_0\\ p_3\\ p_2\end{pmatrix}$	$\begin{pmatrix}p_1 & -\overline{p_2}\\ p_0 & \overline{p_3}\end{pmatrix}$	$\begin{pmatrix}\rho_{10} & -\rho_{11}\\ \rho_{00} & -\rho_{01}\end{pmatrix}$
$\mathbf{v4}(\mathbf{p}) = \begin{pmatrix}p_2\\ -p_3\\ -p_0\\ p_1\end{pmatrix}$	$\begin{pmatrix}p_2 & -\overline{p_1}\\ -p_3 & -\overline{p_0}\end{pmatrix}$	$\begin{pmatrix}\overline{\rho_{11}} & -\overline{\rho_{10}}\\ \overline{\rho_{01}} & -\overline{\rho_{00}}\end{pmatrix}$
$\overline{\mathbf{v1}}(\mathbf{p}) = \begin{pmatrix}\overline{p_1}\\ \overline{p_0}\\ \overline{p_3}\\ \overline{p_2}\end{pmatrix}$	$\begin{pmatrix}\overline{p_1} & -p_2\\ \overline{p_0} & p_3\end{pmatrix}$	$\begin{pmatrix}\overline{\rho_{10}} & -\overline{\rho_{11}}\\ \overline{\rho_{00}} & -\overline{\rho_{01}}\end{pmatrix}$

$\overline{\mathbf{v4}}(\mathbf{p}) = \begin{pmatrix} \overline{p_2} \\ -\overline{p_3} \\ -\overline{p_0} \\ \overline{p_1} \end{pmatrix}$	$\begin{pmatrix} \overline{p_2} & -p_1 \\ -\overline{p_3} & -p_0 \end{pmatrix}$	$\begin{pmatrix} \rho_{11} & -\rho_{10} \\ \rho_{01} & -\rho_{00} \end{pmatrix}$
$\mathbf{u3}(\mathbf{p}) = \begin{pmatrix} -p_1 \\ -p_0 \\ p_3 \\ p_2 \end{pmatrix}$	$\begin{pmatrix} -p_1 & -\overline{p_2} \\ -p_0 & \overline{p_3} \end{pmatrix}$	$\begin{pmatrix} -\rho_{10} & -\rho_{11} \\ -\rho_{00} & -\rho_{01} \end{pmatrix}$
$\mathbf{u2}(\mathbf{p}) = \begin{pmatrix} p_2 \\ -p_3 \\ p_0 \\ -p_1 \end{pmatrix}$	$\begin{pmatrix} p_2 & \overline{p_1} \\ -p_3 & \overline{p_0} \end{pmatrix}$	$\begin{pmatrix} \overline{\rho_{11}} & \overline{\rho_{10}} \\ \overline{\rho_{01}} & \overline{\rho_{00}} \end{pmatrix}$
$\overline{\mathbf{u3}}(\mathbf{p}) = \begin{pmatrix} -\overline{p_1} \\ -\overline{p_0} \\ \overline{p_3} \\ \overline{p_2} \end{pmatrix}$	$\begin{pmatrix} -\overline{p_1} & -p_2 \\ -\overline{p_0} & p_3 \end{pmatrix}$	$\begin{pmatrix} -\overline{\rho_{10}} & -\overline{\rho_{11}} \\ -\overline{\rho_{00}} & -\overline{\rho_{01}} \end{pmatrix}$
$\overline{\mathbf{u2}}(\mathbf{p}) = \begin{pmatrix} \overline{p_2} \\ -\overline{p_3} \\ \overline{p_0} \\ -\overline{p_1} \end{pmatrix}$	$\begin{pmatrix} \overline{p_2} & p_1 \\ -\overline{p_3} & p_0 \end{pmatrix}$	$\begin{pmatrix} \rho_{11} & \rho_{10} \\ \rho_{01} & \rho_{00} \end{pmatrix}$
$\mathbf{v3}(\mathbf{p}) = \begin{pmatrix} p_3 \\ p_2 \\ p_1 \\ p_0 \end{pmatrix}$	$\begin{pmatrix} p_3 & -\overline{p_0} \\ p_2 & \overline{p_1} \end{pmatrix}$	$\begin{pmatrix} -\overline{\rho_{01}} & -\overline{\rho_{00}} \\ \overline{\rho_{11}} & \overline{\rho_{10}} \end{pmatrix}$
$\mathbf{v2}(\mathbf{p}) = \begin{pmatrix} p_0 \\ -p_1 \\ -p_2 \\ p_3 \end{pmatrix}$	$\begin{pmatrix} p_0 & -\overline{p_3} \\ -p_1 & -\overline{p_2} \end{pmatrix}$	$\begin{pmatrix} \rho_{00} & \rho_{01} \\ -\rho_{10} & -\rho_{11} \end{pmatrix}$
$\overline{\mathbf{v3}}(\mathbf{p}) = \begin{pmatrix} \overline{p_3} \\ \overline{p_2} \\ \overline{p_1} \\ \overline{p_0} \end{pmatrix}$	$\begin{pmatrix} \overline{p_3} & -p_0 \\ \overline{p_2} & p_1 \end{pmatrix}$	$\begin{pmatrix} -\rho_{01} & -\rho_{00} \\ \rho_{11} & \rho_{10} \end{pmatrix}$
$\overline{\mathbf{v2}}(\mathbf{p}) = \begin{pmatrix} \overline{p_0} \\ -\overline{p_1} \\ -\overline{p_2} \\ \overline{p_3} \end{pmatrix}$	$\begin{pmatrix} \overline{p_0} & -p_3 \\ -\overline{p_1} & -p_2 \end{pmatrix}$	$\begin{pmatrix} \overline{\rho_{00}} & \overline{\rho_{01}} \\ -\overline{\rho_{01}} & -\overline{\rho_{11}} \end{pmatrix}$

Summarizing, we can formulate the following theses. The initial coordinate space is described by complex quantities, which can be represented as a square matrix

$$\chi_{\alpha\beta} = \begin{pmatrix} \chi_{00} & \chi_{01} \\ \chi_{10} & \chi_{11} \end{pmatrix}$$

The field is a superposition of plane waves with complex phase

$$\rho_{00}\chi_{10} - \rho_{10}\chi_{00} - \overline{\rho_{11}\chi_{01}} + \overline{\rho_{01}\chi_{11}} = (\rho_{00}, \rho_{10}) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \chi_{00} \\ \chi_{10} \end{pmatrix} + (\overline{\rho_{01}}, \overline{\rho_{11}}) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \overline{\chi_{01}} \\ \overline{\chi_{11}} \end{pmatrix}$$

where the momentum coefficients of the decomposition are represented as

$$\rho_{\gamma\delta} = \begin{pmatrix} \rho_{00} & \rho_{01} \\ \rho_{10} & \rho_{11} \end{pmatrix}$$

The phase of a plane wave is constructed using two metric tensors of spinor space and therefore does not change if $\chi_{\alpha\beta}$ and $\rho_{\gamma\delta}$ are affected by the same transformation, which is a combination of three rotations and three boosts with arbitrary angles at the first index and a combination of three rotations and three boosts with arbitrary angles at the second index. Any transformation is given by 12 real values representing the angles of the turns and boosts. When we considered a four-component spinor, we made do with 6 angles, since we took the same rotation and boost angles for both indexes. Note also that only under this condition the mass invariance takes place.

Each plane wave in superposition has a multiplier in the form of a matrix

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$$\varepsilon_{\mu\nu} = \begin{pmatrix} \varepsilon_{00} & \varepsilon_{01} \\ \varepsilon_{10} & \varepsilon_{11} \end{pmatrix}$$

which may be any matrix of 16 pulse combinations given in the table, e.g.

$$\begin{pmatrix} \overline{\rho_{01}} & -\overline{\rho_{00}} \\ -\overline{\rho_{11}} & \overline{\rho_{10}} \end{pmatrix}$$

Each of these matrices can be compared to some elementary particle, and at transformation of coordinate and momentum space it is transformed according to some inherent law. The field operator has the form

$$\begin{pmatrix} \xi_{00}(\chi_{\alpha\beta}) & \xi_{01}(\chi_{\alpha\beta}) \\ \xi_{10}(\chi_{\alpha\beta}) & \xi_{11}(\chi_{\alpha\beta}) \end{pmatrix} = \int \frac{d^4\rho_{\gamma\delta}}{(2\pi)^2}$$

$$\begin{bmatrix} d_1(\rho_{\gamma\delta})\begin{pmatrix} \overline{\rho_{01}} & -\overline{\rho_{00}} \\ -\overline{\rho_{11}} & \overline{\rho_{10}} \end{pmatrix} + id_2(\rho_{\gamma\delta})\begin{pmatrix} -\rho_{10} & -\rho_{11} \\ -\rho_{00} & -\rho_{01} \end{pmatrix} + ib_2(\rho_{\gamma\delta})\begin{pmatrix} \rho_{11} & \rho_{10} \\ \rho_{01} & \rho_{00} \end{pmatrix} + b_1(\rho_{\gamma\delta})\begin{pmatrix} \overline{\rho_{00}} & -\overline{\rho_{01}} \\ -\overline{\rho_{01}} & \overline{\rho_{11}} \end{pmatrix} \\ +d_4(\rho_{\gamma\delta})\begin{pmatrix} \rho_{10} & -\rho_{11} \\ \rho_{00} & -\rho_{01} \end{pmatrix} + id_3(\rho_{\gamma\delta})\begin{pmatrix} -\overline{\rho_{01}} & -\overline{\rho_{00}} \\ \overline{\rho_{11}} & \overline{\rho_{10}} \end{pmatrix} + ib_3(\rho_{\gamma\delta})\begin{pmatrix} \overline{\rho_{00}} & \overline{\rho_{01}} \\ -\overline{\rho_{01}} & -\overline{\rho_{11}} \end{pmatrix} + b_4(\rho_{\gamma\delta})\begin{pmatrix} \rho_{11} & -\rho_{10} \\ \rho_{01} & -\rho_{00} \end{pmatrix} \end{bmatrix}$$

$$e^{i\left((\rho_{00}\,\rho_{10})\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} \chi_{00} \\ \chi_{10} \end{pmatrix} + (\overline{\rho_{01}}\,\overline{\rho_{11}})\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} \overline{\chi_{01}} \\ \overline{\chi_{11}} \end{pmatrix}\right)} +$$

$$\begin{bmatrix} b_1^*(\rho_{\gamma\delta})\begin{pmatrix} \rho_{01} & -\rho_{01} \\ -\rho_{11} & \rho_{10} \end{pmatrix} + ib_2^*(\rho_{\gamma\delta})\begin{pmatrix} -\overline{\rho_{10}} & -\overline{\rho_{11}} \\ -\overline{\rho_{00}} & -\overline{\rho_{01}} \end{pmatrix} + id_2^*(\rho_{\gamma\delta})\begin{pmatrix} \overline{\rho_{11}} & \overline{\rho_{10}} \\ \overline{\rho_{01}} & \overline{\rho_{00}} \end{pmatrix} + d_1^*(\rho_{\gamma\delta})\begin{pmatrix} \rho_{00} & -\rho_{01} \\ -\rho_{10} & \rho_{11} \end{pmatrix} \\ +b_4^*(\rho_{\gamma\delta})\begin{pmatrix} \overline{\rho_{10}} & -\overline{\rho_{11}} \\ \overline{\rho_{00}} & -\overline{\rho_{01}} \end{pmatrix} + ib_3^*(\rho_{\gamma\delta})\begin{pmatrix} -\rho_{01} & -\rho_{01} \\ \rho_{11} & \rho_{10} \end{pmatrix} + id_3^*(\rho_{\gamma\delta})\begin{pmatrix} \rho_{00} & \rho_{01} \\ -\rho_{10} & -\rho_{11} \end{pmatrix} + d_4^*(\rho_{\gamma\delta})\begin{pmatrix} \overline{\rho_{11}} & -\overline{\rho_{10}} \\ \overline{\rho_{01}} & -\overline{\rho_{00}} \end{pmatrix} \end{bmatrix}$$

$$e^{-i\left((\rho_{00}\,\rho_{10})\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} \chi_{00} \\ \chi_{10} \end{pmatrix} + (\overline{\rho_{01}}\,\overline{\rho_{11}})\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} \overline{\chi_{01}} \\ \overline{\chi_{11}} \end{pmatrix}\right)}$$

In addition, a complex conjugate version of the phase should be added to both exponents, as was done above, then there would be an imaginary value in the exponent.

For the field $\xi_{\mu\nu}(\chi_{\alpha\beta})$, we can obtain the equation of motion as an equation in partial derivatives on the complex variables $\chi_{\alpha\beta}$ by substituting the derivatives on these variables instead of the derivatives on x_σ in the previously discussed equations.

We can also consider the decomposition of the field by the previously considered plane waves of the form

$$exp[\pm i(p_0x_1 - p_1x_0 + \overline{p_2x_3} - \overline{p_3x_2})(\overline{p_0x_1} - \overline{p_1x_0} + p_2x_3 - p_3x_2)] =$$

$$exp[\pm i(\rho_{00}\chi_{10} - \rho_{10}\chi_{00} - \overline{\rho_{11}\chi_{01}} + \overline{\rho_{01}\chi_{11}})(\overline{\rho_{00}\chi_{10}} - \overline{\rho_{10}\chi_{00}} - \rho_{11}\chi_{01} + \rho_{01}\chi_{11})] =$$

$$exp\left[\pm i\left((\rho_{00}\,\rho_{10})\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} \chi_{00} \\ \chi_{10} \end{pmatrix} + (\overline{\rho_{01}}\,\overline{\rho_{11}})\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} \overline{\chi_{01}} \\ \overline{\chi_{11}} \end{pmatrix}\right)\left((\overline{\rho_{00}}\,\overline{\rho_{10}})\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} \overline{\chi_{00}} \\ \overline{\chi_{10}} \end{pmatrix} + (\rho_{01}\,\rho_{11})\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} \chi_{01} \\ \chi_{11} \end{pmatrix}\right)\right]$$

For the simpler case of a scalar field these plane waves correspond to the Green's function

$$D(\mathbf{x}) = \int \frac{d^4p}{(2\pi)^4} \frac{exp[-i(p_0x_1 - p_1x_0 + \overline{p_2x_3} - \overline{p_3x_2})(\overline{p_0x_1} - \overline{p_1x_0} + p_2x_3 - p_3x_2)]}{i[p_2 - p_0 + p_1 - p_3]}$$

satisfying the equation

$$\left(\left(\frac{\partial}{\partial x_1}\frac{\partial}{\partial x_2} - \frac{\partial}{\partial x_0}\frac{\partial}{\partial x_3}\right) - \left(\frac{\partial}{\partial x_2}\frac{\partial}{\partial x_1} - \frac{\partial}{\partial x_3}\frac{\partial}{\partial x_0}\right)\right)D(\mathbf{x}) = \delta(\mathbf{x})$$

Recall that the transition from spinor space to vector space is performed by transformations

$$P_\mu = \frac{1}{2}\mathbf{p}^\dagger S_\mu \mathbf{p} \qquad X_\mu = \frac{1}{2}\mathbf{x}^\dagger S_\mu \mathbf{x}$$

$$m = p_2p_2 - p_0p_3$$

$$M^2 = P_0P_0 - P_1P_1 - P_2P_2 - P_3P_3$$

$$M^2 = \overline{m}m$$

Lorentz transformations are given by 2×2 matrices with a set of valid rotation angles and boosts

$$n1 = exp\left(-\frac{1}{2}i\alpha1_1\sigma_1\right)exp\left(\frac{1}{2}\beta1_1\sigma_1\right)exp\left(-\frac{1}{2}i\alpha1_2\sigma_2\right)exp\left(\frac{1}{2}\beta1_2\sigma_2\right)exp\left(-\frac{1}{2}i\alpha1_3\sigma_3\right)exp\left(\frac{1}{2}\beta1_3\sigma_3\right)$$

$$n2 = exp\left(-\frac{1}{2}i\alpha2_1\sigma_1\right)exp\left(\frac{1}{2}\beta2_1\sigma_1\right)exp\left(-\frac{1}{2}i\alpha2_2\sigma_2\right)exp\left(\frac{1}{2}\beta2_2\sigma_2\right)exp\left(-\frac{1}{2}i\alpha2_3\sigma_3\right)exp\left(\frac{1}{2}\beta2_3\sigma_3\right)$$

$$N = \begin{pmatrix} n1 & 0 \\ 0 & n2 \end{pmatrix}$$

$$\Lambda_\nu^\mu = \frac{1}{4}Tr[S_\mu N S_\nu N^\dagger]$$

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After acting on both spinors of the Lorentz transformation with 12 arbitrary angles

$$\mathbf{p}' = N\mathbf{p} \qquad \mathbf{x}' = N\mathbf{x}$$

$$P'_\mu = \frac{1}{2}\mathbf{p}'^\dagger S_\mu \mathbf{p}' \qquad X'_\mu = \frac{1}{2}\mathbf{x}'^\dagger S_\mu \mathbf{x}'$$

and corresponding transformations in the vector space

$$\mathbf{P}' = \Lambda\mathbf{P} \qquad \mathbf{X}' = \Lambda\mathbf{X}$$

$$m' = p'_1 p'_2 - p'_0 p'_3$$

$$M'^2 = P'_0 P'_0 - P'_0 P'_0 - P'_0 P'_0 - P'_0 P'_0$$

there is still equality of masses

$$M'^2 = \overline{m'} m'$$

and invariance of the plane wave phase in spinor space

$$p'_0 x'_1 - p'_1 x'_0 + p'_2 x'_3 - p'_3 x'_2 = p_0 x_1 - p_1 x_0 + p_2 x_3 - p_3 x_2$$

$$\left(p'_0 x'_1 - p'_1 x'_0 + \overline{p'_2 x'_3} - \overline{p'_3 x'_2}\right)\left(\overline{p'_0 x'_1} - \overline{p'_1 x'_0} + p'_2 x'_3 - p'_3 x'_2\right)$$
$$= (p_0 x_1 - p_1 x_0 + \overline{p_2 x_3} - \overline{p_3 x_2})(\overline{p_0 x_1} - \overline{p_1 x_0} + p_2 x_3 - p_3 x_2)$$

$$(p'_0 x'_1 - p'_1 x'_0 + p'_2 x'_3 - p'_3 x'_2) + \left(\overline{p'_0 x'_1} - \overline{p'_1 x'_0} + \overline{p'_2 x'_3} - \overline{p'_3 x'_2}\right)$$
$$= (\, p_0 x_1 - p_1 x_0 + p_2 x_3 - p_3 x_2) + (\overline{p_0 x_1} - \overline{p_1 x_0} + \overline{p_2 x_3} - \overline{p_3 x_2})$$

However, at arbitrary 12 angles, the mass is not invariant

$$m' \neq m$$

and the phase of a plane wave in vector space also changes at Lorentz transformations

$$P'_0 X'_0 - P'_1 X'_1 - P'_2 X'_2 - P'_3 X'_3 \neq P_0 X_0 - P_1 X_1 - P_2 X_2 - P_3 X_3$$

And only under the condition of equality of 6 corresponding angles in the transformation matrices, i.e. under equality

$$n1 = n2$$

both these invariance properties are restored.

Thus, a plane wave with invariant phase in spinor space is a more general concept than a plane wave in vector space, although the concept of invariant mass cannot be introduced for it in the general case.

5. Conclusions

An alternative approach to analyze relativistic and quantum effects inherent in charged particles in the presence of an electromagnetic field is proposed. Two ways of describing the electron behavior in the electromagnetic field are considered: by means of the vector equation, which is based on the plane wave model for a free electron, and the spinor equation, which is based on the representation of the electron as a plane wave in spinor space. For both equations, which are valid for a free particle, their applicability to an arbitrary physical situation is postulated, in particular to describe the behavior of a particle in the presence of an electromagnetic field. The presented equations are intended to fulfill the same role as the Schrödinger equation and the Dirac equation. At the same time, in our opinion, the spinor equations more accurately describe the details of the interaction between fields and particles.

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