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Four-Valued Logics with Infinitely Many Extensions

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Abstract

Here, we we prove that there is a strictly increasing countable chain of finitary relatively finitely-axiomatizable extensions of *({the} truth-singular {version/extension of}) [{the} bounded {expansion of}] first-degree entailments — (TS)[B]FDE*, for short — /"relatively axiomatized by the *Modus Ponens* rule for material implication", in which case the chain does not contain its join, and so this, being a finitary extension of (TS)[B]FDE, is not {relatively} finitely-axiomatizable. ([As a consequence, applying one of our previous works, we immediately get a strictly decreasing chain of finitely-axiomatizable quasi-varieties of bounded De Morgan lattices including the variety of bounded Kleene lattices with non-finitely-axiomatizable intersection.])

Keywords: logic; matrix; extension

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1. Introduction

Appearance of any mathematical object inevitably raises the issue of its connection with other similar ones. Perhaps, the most important relation within *General Logic*, foundations of which go back to [1–5], is the extension/sublogic one between logics reflecting relative deductive strength of different logical systems. On the other hand, thus far, one of most representable non-classical many-valued logics — the four-valued one of first-degree entailments in Relevance logic [6–8] (cf. [9–11]) — has not been studied with regard to the issue involved, though extension lattices of certain *four-valued* expansions [but the bounded one] of it introduced in [10] have already been found — namely, due to [10,12,13]/"{[10,14]|}[11]", it has been known that extensions of the {bounded} implicative/bi-lattice expansion of FDE form a $((12\{+18\})/2)$-element distributive lattice. The principal goal of this work is to fill the above gap[s] up.

The rest of the work is as follows. To avoid extensive specifications of conventional well-known issues underlying the present work, we entirely follow the conventions adopted in [11] tacitly except for making explicit references to specific advanced points. Then, the main results described above are presented in Section 2. Finally, Section 3 is devoted to both a concise summary of principal contributions of the paper and a thorough comparison of them with other related works on non-classical many-valued logics as well as a brief outline of further related work.

2. Main Issues

Given any (non-one-element) $\mathfrak{A} \in [\mathrm{B}]\mathrm{DML}$, $(\Im/\pounds)_{\mathfrak{A}} \triangleq \{a \in A \mid a =/\leqslant^{\mathfrak{A}} \sim^{\mathfrak{A}} a\} \subseteq/= \{b \wedge^{\mathfrak{A}} \sim^{\mathfrak{A}} b \mid b \in A\}[\ni(|\not\ni)\ (\bot(|\top))^{\mathfrak{A}}]$ does ([not])/ include $([\{\top^{\mathfrak{A}}\}/])\{c \in A \mid c \leqslant^{\mathfrak{A}} d \in \pounds^{\mathfrak{A}}\}$.

Let $C^{(\mathsf{T})}$ be the logic of $\mathcal{DB}^{(\mathsf{T})}_{4[,01]} (\triangleq \langle \mathfrak{DM}_{4[,01]}, \{\mathsf{t}\}\rangle)$.

Given any $n \in (\omega \setminus 1)$ and $\diamond \in \Sigma_+$, $\diamond_{1\{+n\}}(x_i)_{i \in (1\{+n\})} \triangleq (\{\diamond_n(x_j)_{j \in n}\diamond\}x_{0\{+n\}}) \in \mathrm{Fm}^{1\{+n\}}_{\Sigma_+}$. Then, any member of $\mathbf{S}(\mathbf{P}(\mathcal{DM}^{(\mathsf{T})}_{4[,01]}))$ is a model of $\mathcal{R}_n \triangleq (\{\vee_n(x_k \wedge \sim x_k)_{k \in n}\} \vdash x_n)$ iff it is a model of $\mathcal{R}_{n+1}[x_{n+l}/x_{n \cdot l}]_{l \in 2}$ if it is a model of $\mathcal{R}_{n+1}$, in which case, by (Theorem 2.8 [11]), the extension $C^{(\mathsf{T})}_n$ of $C^{(\mathsf{T})}$ relatively axiomatized by $\mathcal{R}_n$ is a sub-logic of $C^{(\mathsf{T})}_{n+1}$, and so $\langle C^{(\mathsf{T})}_m \rangle_{m \in (\omega \setminus 1)}$ is a denumerable

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increasing chain of finitary relatively finitely-axiomatizable extensions of $C^{(\intercal)}$ with join, being its finitary extension $C^{(\intercal)}_{\omega}$ relatively axiomatized by $\mathfrak{C}_{\omega} \triangleq \{\mathcal{R}_{\ell} \mid \ell \in (\omega \setminus 1)\}$.

Lemma 2.1. *Let $i \in n \in (\omega \setminus 1)$, $\bar{a} \triangleq \langle\langle\{\langle j, \mathsf{b}\rangle\} \cup ((n \setminus \{j\}) \times \{\mathsf{n}\})\rangle_{j \in n}, n \times \{\mathsf{n}\}\rangle$ and $\mathfrak{A} \triangleq \mathfrak{DM}_{4[,01]:n}$ the subalgebra of $\mathfrak{B} \triangleq \mathfrak{DM}^{n}_{4[,01]}$ generated by $I_n \triangleq (\mathrm{img}\,\bar{a}) \subseteq B$. Then, $S_{n,i} \triangleq (£^{\mathfrak{A}} \cap \pi_i^{-1}[\{\underline{\jmath}\}]) = \{a_i\}$.*

Proof. Clearly, $I_n \subseteq (A \cap \mathfrak{I}_{\mathfrak{B}}) = \mathfrak{I}_{\mathfrak{A}} \subseteq £_{\mathfrak{A}} \subseteq £_{\mathfrak{B}} = (2^2 \setminus \{\mathsf{t}\})^n$. In particular, $a_i \in S_{n,i}$. Conversely, given any $X \subseteq B$, let $(C|D)(X) \triangleq (\bigcup_{m \in (\omega \setminus 1)}((\wedge|\vee)^{\mathfrak{B}}_{m}[X^m])) \supseteq X$, in which case $A = (D(C(I_n))[\cup\{\bot^{\mathfrak{B}}, \top^{\mathfrak{B}}\}])$, as $I_n \subseteq \mathfrak{I}_{\mathfrak{B}}$, while $(C(I_n) \setminus I_n) \subseteq \{\mathsf{f}, \mathsf{n}\}^n$, since, for all distinct $\imath, \jmath \in n$, $(a_\imath \wedge^{\mathfrak{B}} a_\jmath) \in \{\mathsf{f}, \mathsf{n}\}^n$, whereas $\{\mathsf{f}, \mathsf{n}\}$ is an ideal of $\mathfrak{D}^2_2$, and so any $b \in S_{n,i}$, being then equal to $\vee^{\mathfrak{B}}_{l}(\bar{c})$, for some $l \in (\omega \setminus 1)$ and $\bar{c} \in C(I_n)^l$ such that $a_i \in (\mathrm{img}\,\bar{c})$, because, otherwise, $\pi_i(b) = \mathsf{b}$ would be in $\{\mathsf{f}, \mathsf{n}\} \not\ni \mathsf{b}$, is equal to a_i, for $\{\mathsf{n}, \underline{\}}^n \ni a_i \leqslant^{\mathfrak{B}} b \in \{\mathsf{f}, \underline{,}\mathsf{n}\}^n$, as required. □

Corollary 2.2. *Let $n \in (\omega \setminus (2\{-1(+1)\})$. Then, $\mathcal{R}_{n-1\{+1\}}$ is {not} true in $\mathcal{D}\langle\mathcal{M}^{(\intercal)}_{4[,01]:n}\rangle \triangleq ((\mathcal{DM}^{(\intercal)}_{4[,01]})^n \restriction DM_{4[,01]:n}) \in \mathrm{Mod}(C^{(\intercal)})$ {under $[x_{\Bbbk}/a_{\Bbbk}]_{\Bbbk \in (n+1)}$}, in which case $\mathcal{D} \in (\mathrm{Mod}(C^{(\intercal)}_{n-1}) \setminus \mathrm{Mod}(C^{(\intercal)}_{n}))$, and so $C^{(\intercal)}_{n-1} \neq C^{(\intercal)}_{n}$. In particular, $\mathcal{DM}^{\intercal}_{4[,01]}$, being isomorphic to the consistent truth-singular subdirect product $\mathcal{DM}_{4[,01]:2}$ of $2 \times \{\mathcal{DM}_{4[,01]}\}$, is a model of C_1.*

Proof. If $\mathcal{R}_{n-1}$ was not true in $\mathcal{D}$, i.e., there was some $\bar{b} \in £^{n-1}_{\mathfrak{D}} \subseteq (\{\mathsf{f}, \underline{,}\mathsf{n}\}^n)^{n-1}$ such that $\vee^{\mathfrak{D}}_{n-1}(\bar{b}) \in D^{\mathcal{D}} \subseteq \{\langle 1, 1-\ell\rangle \mid \ell \in (2(-1))\}^n$, then, by Lemma 2.1, for each $i \in n$, there would be some $f(i) \in (n-1)$ such that $b_{f(i)} = a_i$, in which case, by the injectivity of $\bar{a}$, $f : n \to (n-1)$ would be injective, and so we would have $n \leqslant (n-1)$. Finally, $A \triangleq DM_{4[,01]:2} = (\Delta_{\Delta_2} \cup ((2^2 \setminus \Delta_2)^2 \setminus \Delta_{2^2 \setminus \Delta_2}))$, in which case $D^{\mathcal{DM}_{4[,01]:2}} = \{\langle \mathsf{t}, \mathsf{t}\rangle\} \not\ni \langle \mathsf{f}, \mathsf{f}\rangle \in A$, while, for each $\ell \in 2$, $\pi_\ell[A] = 2^2$, and so $(\pi_0 \restriction A) \in \hom^{\mathsf{S}}_{\mathsf{S}}(\mathcal{DM}_{4[,01]:2}, \mathcal{DM}^{\intercal}_{4[,01]})$ is injective, ((2.2)[11]) completing the argument. □

This, by (Theorem 2.8, [11]) and the Compactness Theorem [15], eventually yields:

Theorem 2.3. *$\langle C^{(\intercal)}_{n}\rangle_{n \in (\omega \setminus (1(+1)))}$ is a strictly increasing countable chain of extensions of $C^{(\intercal)}$, in which case it does not contain $C^{(\intercal)}_{\omega}$, and so this is not {relatively} finitely-axiomatizable.*

The secondary part of Corollary 2.2 inevitably raises the problem of finding an axiomatization of $C^{\intercal}$ relatively to C to be resolved in the next subsection.

2.1. *Modus Ponens for Material Implication Versus Truth-Singularity*

Let $\sim^{0|1} x_{0\|1} \triangleq (x_{0\|1} || {\sim} x_{0\|1}) \in \mathrm{Fm}^{\{0\|1\}}_{\Sigma_\sim}$ and $\mathcal{R}^{\supset}_{i,j} \triangleq ((\sim^i x_j \vee x_2) \vdash (\sim^i x_{1-j} \vee x_2))$. We start from presenting the following quite immediate observation:

Lemma 2.4. *Let $\Sigma \supseteq \Sigma_\sim$ be a language and $\mathcal{A}$ a $\wedge$-conjunctive Σ-matrix with $(\mathfrak{A} \restriction \Sigma_\sim) \in \mathsf{DML}$. Suppose $\Sigma \setminus \Sigma_\sim$ consists of constants alone. Then, $\supset(\mathcal{A}) = \{\bar{a} \in A^2 \mid \forall b \in A, \forall i, j \in 2 : \mathcal{A} \models \mathcal{R}^{\supset}_{i,j}[x_k/a_k; x_2/b]_{k \in 2}\}$.*

Lemma 2.5. *Let $n \in \omega$, $\overline{\mathcal{A}} \in \mathbf{S}^{\{*\}}_{*}(\mathcal{DM}^{(\intercal)}_{4[,01]})^n$ and $\mathcal{B}$ a consistent subdirect product of $\overline{\mathcal{A}}$. Then, $n \neq 0$, while {$f \triangleq (n \times \{\mathsf{f}\})[= \bot^{\mathfrak{B}}] \in B$, in which case $t \triangleq (n \times \{\mathsf{t}\})[= \top^{\mathfrak{B}}] \in D^{\mathcal{B}} \not\ni f$, and so $\{\langle \mathsf{f}, f\rangle, \langle \mathsf{t}, t\rangle\}$ is an embedding of $\mathcal{DM}^{\langle\intercal\rangle}_{4[,01]} \restriction \Delta_2$ into $\mathcal{B}$, whereas} $\mathcal{B}$ is truth-non-empty iff all members of $\mathrm{img}\,\overline{\mathcal{A}}$ are so.*

Proof. Take any $a \in (B \setminus D^{\mathcal{B}}) \neq \varnothing$, in which case there is some $i \in n$ such that $\pi_i(a) \in \{\mathsf{f}, \mathsf{n}\}$, and so $n \neq 0$. Likewise, for every $j \in n$, $(\pi_j \restriction B) \in \hom(\mathcal{B}, \mathcal{A}_j)$, in which case, for any $b \in D^{\mathcal{B}}$, $\pi_j(b) \in D^{\mathcal{A}_j}$, and so $\mathcal{A}_j$ is truth-non-empty, whenever $\mathcal{B}$ is so. (Consider any $k \in n$, in which case $\pi_k[B] = A_k$, and so there are some $(c|d) \in (B \cap \pi_i^{-1}[A_k \setminus|\cap D^{\mathcal{A}_k}]) \neq \varnothing$. Then, $e_k \triangleq (c \wedge^{\mathfrak{B}} \sim^{\mathfrak{B}} d) \in (B \cap \pi_k^{-1}[\{\mathsf{f}\}])$, in which case $B \ni \wedge^{\mathfrak{B}}_{n}(e_l)_{l \in n} = f$, and so $B \ni \sim^{\mathfrak{B}} f = t$, as required.) □

Corollary 2.6. *$\mathcal{A} \triangleq (\mathcal{DM}_{4[,01]} \restriction \Delta_2)$, being "embedable into the reduction of a sub-matrix of"/"isomorphic to" any consistent truth-non-empty /two-valued /(more specifically, classical) $\mathcal{B} \in \mathrm{Mod}(C)$, is a model of any inferentially consistent extension of C and defines a unique inferentially consistent two-valued {in particular, $\langle\sim$-$\rangle$classical} extension PC of C.*

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Proof. First, any matrix is both consistent and truth-non-empty iff its logic is inferentially consistent. Take any $(a|b) \in (B \cap|\setminus D^{\mathcal{B}}) \neq \varnothing$, in which case $a \neq b$, the sub-matrix $\mathcal{D}$ of $\mathcal{B}$ generated by $\{a, b\}$ being both finitely-generated, consistent and truth-non-empty /"as well as two-valued and non-proper", for $(a|b) \in (B \cap|\setminus D^{\mathcal{B}})$ /"and $2 = |\{a, b\}| \leqslant |B| \leqslant |A| = 2$", so, by (Lemmas 2.7, 3.2 and Example 3.1 [11]), there are some $n \in \omega$, some subdirect product $\mathcal{E}$ of some $\overline{\mathcal{C}} \in \mathbf{S}_*(\mathcal{DM}_{4[,01]})^n$ and some $h \in \hom^{\mathrm{S}}_{\mathrm{S}}(\mathcal{E}, (\Re/)(\mathcal{D}))$, $\mathcal{E}$ being both consistent and truth-non-empty, in view of ((2.2) [11]). Then, by Lemma 2.5, there is some $g \in \hom_{\mathrm{S}}(\mathcal{A}, \mathcal{E})$, in which case, by (Corollary 2.3, Example 3.1 and Lemma 3.2 [11]), $(g \circ h) \in \hom_{\mathrm{S}}(\mathcal{A}, (\Re/)(\mathcal{D}))$ is injective /"and so surjective, as $|A| = 2 = |D|$". Finally, by (Theorem 2.8 [11]), any inferenrially consistent extension of C, being defined by the class of its models subsumed by the one of those of C, has a consistent truth-non-empty model, ((2.2) [11]) completing the argument. □

Corollary 2.7. *Let $n \in \omega$, $\overline{\mathcal{A}} \in \mathbf{S}_*(\mathcal{DM}_{4[,01]})^n$ and $\mathcal{B}$ a (simple) consistent subdirect product of $\overline{\mathcal{A}}$. Then, $\mathcal{B}$ is a model of $\mathcal{MP} \triangleq (\{x_0, \sim x_0 \vee x_1\} \vdash x_1)$ if(f) any of the following equivalent conditions hold:*

(i) *$\mathcal{B}$ is truth-singular;*

(ii) *$D^{\mathcal{B}} = (B \cap \{n \times \{\mathsf{t}\}\})$;*

(iii) *$\mathcal{B} = ((\prod_{i \in n}(\mathcal{DM}^{\mathsf{T}}_{4[,01]} \restriction A_i)) \restriction B)$;*

(iv) *$\mathcal{B} \in \mathbf{S}(\mathbf{P}(\mathcal{DM}^{\mathsf{T}}_{4[,01]}))$.*

In particular, $\mathcal{DM}^{\mathsf{T}}_{4[,01]} \in \mathrm{Mod}(\mathcal{MP})$.

Proof. First, (i)⇒(ii) is by Lemma 2.5 and the inclusion $D^{\mathcal{B}} \supseteq (B \cap \{n \times \{\mathsf{t}\}\})$, (ii/iii/iv)⇒(iii/iv/i) being immediate. Now, assume (i) holds. Consider any $a \in D^{\mathcal{B}}$ and $b \in B$ such that $c \triangleq (\sim^{\mathfrak{B}} a \vee^{\mathfrak{B}} b) \in D^{\mathcal{B}}$, in which case $\mathcal{B}$ is truth-non-empty, and so, by Lemma 2.5, $a = (n \times \{\mathsf{t}\})$. Then, $\sim^{\mathfrak{B}} a = (n \times \{\mathsf{f}\})$, in which case $D^{\mathcal{B}} \ni c = b$, and so $\mathcal{B} \in \mathrm{Mod}(\mathcal{MP})$. (Conversely, assume $\mathcal{B} \in \mathrm{Mod}(\mathcal{MP})$ is truth-non-empty, in which case, by Lemma 2.5, $t \triangleq (n \times \{\mathsf{t}\}) \in D^{\mathcal{B}}$, and so, for all $d \in D^{\mathcal{B}}$, $e \in B$ and $i, j \in 2$, since $\mathcal{B} \models \mathcal{MP}[x_0/d, x_1/e]$, while $(\sim^{\mathfrak{B}} t \vee^{\mathfrak{B}} e) = ((n \times \{\mathsf{f}\}) \vee^{\mathfrak{B}} e) = e \leqslant^{\mathfrak{B}} (\sim^{\mathfrak{B}} d \vee^{\mathfrak{B}} e)$, whereas both $D^{\mathcal{B}} \ni (d|t) \leqslant^{\mathfrak{B}} ((d|t) \vee^{\mathfrak{B}} e)$, by the $\wedge$-conjunctivity of $\mathcal{B}$, ensuing from that of $\mathcal{DM}_{4[,01]}$, we have $\mathcal{B} \models \mathcal{R}^{\supset}_{i,j}[x_0/t, x_1/d, x_2/d]$, d being equal to t, in view of Lemma 2.4 and the simplicity of $\mathcal{B}$. Thus, by the truth-singularity of truth-empty matrices, (ii) holds.) Finally, Corollary 2.2, the consistency of $\mathcal{DM}_{4[,01]}$ and ((2.2) [11]) end the proof. □

This, by ((2.2) and Theorem 2.8 [11]) as well as Corollary 2.2, eventually yields:

Theorem 2.8. *C^{T} is the extension of C relatively axiomatized by $\mathcal{MP}$.*

On the other hand, $\mathcal{DM}^{\langle\mathsf{T}\rangle}_{4[,01]} \restriction \Delta_2$ is the only (consistent) sub-matrix of $\mathcal{DM}^{\mathsf{T}}_{4[,01]}$, being a model of the *Excluded Middle Law* axiom $\mathcal{EM} \triangleq (x_0 \vee \sim x_0)$, because this is {not} true in $\mathcal{DM}^{\mathsf{T}}_{4[,01]}$ under $[x_0/a]$, for all $a \in (\{2^2 \setminus\}\Delta_2)$. Likewise, since, for all $b \in 2^2$, $(\mathcal{DM}_{4[,01]} \models \mathcal{EM}[x_0/b]) \Leftrightarrow (b \neq \mathsf{n})$, a sub-matrix of $\mathcal{DM}_{4[,01]}$ is a model of $\mathcal{EM}$ iff it is a sub-matrix of $\mathcal{DM}_{4[,01]} \restriction \{\mathsf{f}, n, \mathsf{t}\}$, defining the [bounded expansion of the] *logic of paradox* $LP_{[01]}$ [16–18] (viz., the $\Sigma_{\sim}$-fragment of {any normal extension of} Relevance-Mingle $RM_{\{n\}}$ {where $n \in (\omega \setminus 3)$} [19]; cf. (Corollary 4.15 [14])). Finally, any extension of C, satisfying the *Resolution* rule $\mathcal{RS} \triangleq (\{\sim x_0 \vee x_1, x_0 \vee x_1\} \vdash x_1)$, true in $\mathcal{DM}^{\langle\mathsf{T}\rangle}_{4[,01]} \restriction \Delta_2$, satisfies $\mathcal{MP}$. These observations, by ((2.2) and Corollary 2.9 [11]) as well as Corollary 2.6 and Theorem 2.8, immediately imply the following important interesting consequence, yielding a new insight into respective parts of (Corollary 5.3 [9]) and (Theorem 4.13 [14]):

Corollary 2.9. *$PC|LP$ is the axiomatic extension of $C^{\mathsf{T}|}$ relatively axiomatized by $\mathcal{EM}$. In particular, PC is the extension of LP/C relatively axiomatized by $\mathcal{MP}\|\mathcal{RS}$ /"and $\mathcal{EM}$".*

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2.2. *Application to Bounded De Morgan Lattices*

Recall that a *[bounded] Kleene lattice* [more traditionally, a *Kleene algebra*; cf., e.g., [20]] is any [bounded] De Morgan lattice $\mathfrak{A}$ with ideal $\pounds_{\mathfrak{A}}$ of its lattice reduct, their class $[\mathsf{B}]\mathsf{KL}$ being the sub-variety of $[\mathsf{B}]\mathsf{DML}$ relatively axiomatized by the identity ((6) [21]).

Let $\nabla \triangleq \{x \approx \top\}$, in which case $\mathfrak{DM}^{\nabla}_{4,01} \triangleq \mathcal{DM}^{\intercal}_{4,01}$, and so, by the double ()-[]-optional version of Theorem 2.3, (Theorem 3.4 [10]), (Lemma 3.1, Corollary 3.2 and Theorem 3.3 [14]) as well as the Compactness Theorem [15], we immediately get:

Corollary 2.10. $\langle \mathsf{BDML} \cap \mathrm{Mod}(\nabla(\mathcal{R}_n))\rangle_{n\in(\omega\setminus 2)}$ *is a strictly decreasing countable chain of finitely-axiomatizable quasi-varities of bounded De Morgan lattices including* BKL*, in which case it does not cintain its imtersection* $(\mathsf{BDML} \cap \mathrm{Mod}(\nabla\mathfrak{C}_\omega)) \supseteq \mathsf{BKL}$*, and so this is not {relatively} finitely-axiomatizable.*

On the other hand, such is not applicable to the unbounded case, simply because, due to [21], it has been well-known that the lattice of quasi-varieties of De Morgan lattices is finite.

3. Conclusions

Thus, Theorem 2.3 definitely exhausts the issues raised in Section 1. In this connection, it is especially remarkable that this more than profound result, due to the equally profound universal part of the outstanding contribution [14] to *General Algebraic Logic*, in its turn, going back to the fundamental ones [10,12], has been successfully applied to the issue of the lattice of quasi-varieties of bounded De Morgan lattices *including the variety of bounded Kleene lattices*. Perhaps, the most acute problems remained still open are what are the cardinalities and structures of the infinite lattices under consideration. On the other hand, such rather minor points, proving beyond the scopes of the present work, are going to be discussed *in detail* elsewhere.

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